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HÖLDER CONTINUOUS SOLUTIONS OF THE MONGE–AMPÈRE EQUATION ON COMPACT HERMITIAN MANIFOLDS

by Sławomir KOŁODZIEJ & Ngoc Cuong NGUYEN

Dedicated to Jean-Pierre Demailly on the occasion of his 60th birthday

Abstract. — We show that a positive Borel measure of positive finite total mass, on a compact Hermitian manifold, admits a Hölder continuous quasi-plurisubharmonic solution to the Monge–Ampère equation if and only if it is dominated locally by Monge–Ampère measures of Hölder continuous plurisubharmonic functions.

Résumé. — Nous prouvons qu'une mesure de Borel positive avec la masse totale finie, sur une variété hermitienne compacte, admet une solution quasi plurisousharmonique de l'équation de Monge–Ampère si et seulement si elle est dominée localement par des mesures de Monge–Ampère des fonctions plurisousharmoniques continues Höldériennes.

1. Introduction

The analogue of the Calabi–Yau theorem on compact Hermitian manifolds was proven in 2010 by Tosatti and Weinkove [21]. Continuous weak solutions for the right hand side in L^p, $p > 1$ were obtained later by the authors [16]. Here we continue to study weak solutions for more general measures.

Consider a compact Hermitian manifold (X, ω) of dimension n, and a positive Radon measure μ with finite total mass on X. An upper semicontinuous function u on X is called $\omega-$psh if $dd^c u + \omega \geqslant 0$ (as currents). Then we write $u \in \mathrm{PSH}(\omega)$. Our objective is to show that if the complex Monge–Ampère equation has Hölder continuous solutions for μ restricted

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to local charts then it has Hölder continuous solutions globally on X. To be precise we introduce first the following definition.

Definition 1.1. — *We say that μ admits a global Hölder continuous* subsolution *if there exists a Hölder continuous $\omega-$psh function u and $C_0 > 0$ such that*

$$\mu \leqslant C_0(\omega + dd^c u)^n \quad \text{on } X. \tag{1.1}$$

Let us denote by $\mathcal{M}$ the set of all such measures.

To verify the defining condition it is enough to look at μ locally.

Lemma 1.2. — *A measure μ belongs to $\mathcal{M}$ if and only if for every $x \in X$, there exists a neighborhood D of x and a Hölder continuous psh function v on D such that $\mu_{|_D} \leqslant (dd^c v)^n$.*

Proof. — The necessary condition is obvious, so we prove the sufficient condition. Using the strict positivity of ω we can extend a Hölder continuous psh function v defined in a local coordinate chart to the whole space X so that the extension is a Hölder continuous $C\omega-$psh function for some large $C > 0$. Taking a finite cover by coordinate charts and using the partition of unity one easily constructs a global $\omega-$psh function u satisfying (1.1) (see [14] for details of such a construction). □

Our main result can be viewed as a generalization of Demailly et al. [6, Proposition 4.3] from the Kähler to the Hermitian setting.

Theorem 1.3. — *Assume that $0 < \mu(X) < +\infty$. There exists a Hölder continuous ω-psh φ and a constant $c > 0$ solving*

$$(\omega + dd^c\varphi)^n = c\,\mu$$

if and only if μ belongs to $\mathcal{M}$.

Thanks to this theorem the important class of measures having L^p-density, for $p > 1$, admits Hölder continuous solutions. This result was proven in [18, Theorem B] under the extra assumption that the right hand side is strictly positive.

Corollary 1.4. — *Let f be a non-negative function in $L^p(\omega^n)$ for $p > 1$. Assume that $\int_X f\omega^n > 0$. Then there exists a Hölder continuous $\varphi \in \mathrm{PSH}(\omega)$ and a constant $c > 0$ such that*

$$(\omega + dd^c\varphi)^n = cf\omega^n.$$

Proof. — By [16, Theorem 0.1] there exists $\varphi \in \mathrm{PSH}(\omega) \cap C^0(X)$ and a constant $c > 0$ satisfying

$$(\omega + dd^c\varphi)^n = cf\omega^n.$$

Consider a local coordinate chart $B \subset\subset X$ parametrized by the unit ball in $\mathbb{C}^n$. Let χ be a smooth cut-off function such that

$$0 \leqslant \chi \leqslant 1, \quad \chi = 1 \text{ on } B(0, 1/2), \quad \operatorname{supp} \chi \subset\subset B.$$

Find $w \in \mathrm{PSH}(B)$ the solution of the Dirichlet problem for the Monge–Ampère equation:

$$(dd^c w)^n = c\chi f\omega^n, \quad w_{|\partial B} = 0.$$

By the main result of [12] (see also [4]) we get that $w \in C^{0,\alpha}(\bar{B})$ for some α positive depending only on n, p. Therefore, on $B(0, 1/2)$ the right hand side $cf\omega^n$ is dominated by $(dd^c w)^n$. We conclude from Lemma 1.2 and Theorem 1.3 that φ is Hölder continuous. □

Remark 1.5. — Using the recent result from [20] instead of [12] we also can show that if $\mu \in \mathcal{M}$ and $0 \leqslant f \in L^p(d\mu)$ for $p > 1$, then $f d\mu \in \mathcal{M}$. In other words, $\mathcal{M}$ satisfies the L^p−property (see [6]) and the above corollary is a special case.

Another consequence of the main result is the convexity of the range of Monge–Ampère operator acting on Hölder continuous functions.

Corollary 1.6. — *The set*

$$\mathcal{A} := \left\{c \cdot (\omega + dd^c\varphi)^n : \varphi \in \mathrm{PSH}(\omega),\ \varphi \textit{ is Hölder continuous, } c > 0.\right\}$$

is convex.

Proof. — For brevity we use the notation $\omega_\varphi^n := (\omega + dd^c\varphi)^n$. Let $c_1\omega_{\varphi_1}^n$, $c_2\omega_{\varphi_2}^n \in \mathcal{A}$. It is easy to see that

$$\mu := \frac{1}{2}(c_1\omega_{\varphi_1}^n + c_2\omega_{\varphi_2}^n) \leqslant 2^{n-1}(c_1 + c_2)\left(\omega + dd^c\frac{\varphi_1 + \varphi_2}{2}\right)^n.$$

Apply Theorem 1.3 to get that $\omega_\phi^n = c\mu$ for some Hölder continuous ω-psh ϕ and some constant $c > 0$. Therefore, $\mu \in \mathcal{A}$. □

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2. Preliminaries

Let us recall the definition of the Bedford–Taylor capacity. For a Borel set $E \subset X$ put

$$\operatorname{cap}_\omega(E) := \sup \left\{ \int_E \omega_v^n : v \in \mathrm{PSH}(\omega), 0 \leqslant v \leqslant 1 \right\}. \tag{2.1}$$

By [14, p. 52], this capacity is comparable with the local Bedford–Taylor capacity $\operatorname{cap}'_\omega(E)$. Combining this fact with the work of Dinh–Nguyen–Sibony [10] we get the following result.

Lemma 2.1. — *Let $\mu \in \mathcal{M}$. Then, for every compact set $K \subset X$,*

$$\mu(K) \leqslant C \exp\left(\frac{-\alpha_1}{[\operatorname{cap}_\omega(K)]^{\frac{1}{n}}} \right), \tag{2.2}$$

where $C, \alpha_1 > 0$ depend only on X and the Hölder exponent of the global Hölder subsolution.

Corollary 2.2. — *Assume that $\mu \in \mathcal{M}$ and fix $\tau > 0$. Then, there exists $C_\tau > 0$ such that for every compact set $K \subset X$*

$$\mu(K) \leqslant C_\tau \left[\operatorname{cap}_\omega(K)\right]^{1+\tau}. \tag{2.3}$$

The set of measures satisfying this inequality is denoted by $\mathcal{H}(\tau)$.

The proof of the next statement can be found in [8, Theorem 2.1].

Lemma 2.3. — *Let $u \in \mathrm{PSH}(\omega) \cap C^{0,\alpha}(X)$ with $0 < \alpha < 1$. Then there exists a sequence of smooth ω-psh function $\{u_j\}_{j \geqslant 1}$ such that*

$$u_j \to u$$

in $C^{0,\alpha'}(X)$ as $j \to +\infty$, for any $0 < \alpha' < \alpha$.

We need also an estimate which for Kähler manifolds was given in [11].

Proposition 2.4. — *Suppose $\psi \in \mathrm{PSH}(\omega) \cap C^0(X)$ and $\psi \leqslant 0$. Let μ satisfy the inequality* (2.3) *for some $\tau > 0$, i.e. $\mu \in \mathcal{H}(\tau)$. Assume that $\varphi \in \mathrm{PSH}(\omega) \cap C^0(X)$ solves*

$$(\omega + dd^c\varphi)^n = \mu.$$

Then for $\gamma = \frac{1}{1+(n+2)(n+\frac{1}{\tau})}$ and some positive $C > 0$ depending only on τ, ω and $\|\psi\|_\infty$ we have

$$\sup_X (\psi - \varphi) \leqslant C \left\|(\psi - \varphi)_+\right\|_{L^1(d\mu)}^{\gamma}.$$

Proof. — Without loss of generality we may assume that $-1 \leqslant \psi \leqslant 0$. Put

$$U(\varepsilon, s) = \{\varphi < (1-\varepsilon)\psi + \inf_X [\varphi - (1-\varepsilon)\psi] + s\},$$

where $0 < \varepsilon < 1$ and $s > 0$.

LEMMA 2.5. — *For*

$$0 < s \leqslant \frac{1}{3} \min\left\{\varepsilon^n, \frac{\varepsilon^3}{16B}\right\}, \quad 0 < t \leqslant \frac{4}{3}(1-\varepsilon) \min\left\{\varepsilon^n, \frac{\varepsilon^3}{16B}\right\}$$

we have

$$t^n \operatorname{cap}_\omega(U(\varepsilon, s)) \leqslant C \left[\operatorname{cap}_\omega(U(\varepsilon, s+t))\right]^{1+\tau},$$

where C is a dimensional constant.

Proof of Lemma 2.5. — By [16, Lemma 5.4]

$$t^n \operatorname{cap}_\omega(U(\varepsilon, s)) \leqslant C \int_{U(\varepsilon, s+t)} \omega_\varphi^n, \tag{2.4}$$

The lemma now follows from (2.3). □

LEMMA 2.6. — *Fix $0 < \varepsilon < 3/4$ and $\varepsilon_B := \frac{1}{3}\min\{\varepsilon^n, \frac{\varepsilon^3}{16B}\}$. Then, there exists a contant $C_\tau = C(\tau, \omega)$ such that for $0 < s < \varepsilon_B$,*

$$s \leqslant C_\tau \left[\operatorname{cap}_\omega(U(\varepsilon, s))\right]^{\frac{\tau}{n}}.$$

Proof of Lemma 2.6. — Let us use the notation

$$a(s) := \left[\operatorname{cap}_\omega(U(\varepsilon, s))\right]^{\frac{1}{n}}.$$

It follows easily from (2.4) that

$$ta(s) \leqslant C \left[a(s+t)\right]^{1+\tau}.$$

This is the inequality [17, (3.6)]. The arguments that follow in that paper complete the proof of the present lemma. □

To finish the proof of the proposition we proceed as in [17, Theorem 3.11]. One needs to estimate

$$-S := \sup_X (\psi - \varphi) > 0$$

in terms of $\|(\psi-\varphi)_+\|_{L^1(\mathrm{d}\mu)}$ as in the Kähler case [13]. Suppose that

$$\|(\psi-\varphi)_+\|_{L^1(\mathrm{d}\mu)} \leqslant \varepsilon^a \tag{2.5}$$

for $0 < \varepsilon << 3/4$ and $a = \frac{1}{\gamma}$. Let

$$\hbar(s) := (s/C_\tau)^{\frac{1}{\tau}}$$

be the inverse function of $C_\tau s^\tau$. Consider sublevel sets $U(\varepsilon, t) = \{\varphi < (1-\varepsilon)\psi + S_\varepsilon + t\}$, where $S_\varepsilon = \inf_X[\varphi - (1-\varepsilon)\psi]$. It is clear that

$$S - \varepsilon \leqslant S_\varepsilon \leqslant S. \tag{2.6}$$

Therefore, $U(\varepsilon, 2t) \subset \{\varphi < \psi + S + \varepsilon + 2t\}$. Then, $(\psi - \varphi)_+ \geqslant |S| - \varepsilon - 2t > 0$ for $0 < t < \varepsilon_B$ and $0 < \varepsilon < |S|/2$ on the latter set (if $|S| \leqslant 2\varepsilon$ then we are done).

By (2.4) we have

$$\begin{aligned} \mathrm{cap}_\omega(U(\varepsilon,t)) &\leqslant \frac{C}{t^n}\int_{U(\varepsilon,2t)} \mathrm{d}\mu \leqslant \frac{C}{t^n}\int_X \frac{(\psi-\varphi)_+}{(|S|-\varepsilon-2t)}\mathrm{d}\mu \\ &\leqslant \frac{C\|(\psi-\varphi)_+\|_{L^1(\mathrm{d}\mu)}}{t^n(|S|-\varepsilon-2t)}. \end{aligned}$$

Moreover, by Lemma 2.6

$$\hbar(t) \leqslant [\mathrm{cap}_\omega(U(\varepsilon,t))]^{\frac{1}{n}}.$$

Combining these inequalites, we obtain

$$(|S| - \varepsilon - 2t) \leqslant \frac{C\|(\psi-\varphi)_+\|_{L^1(\mathrm{d}\mu)}}{t^n[\hbar(t)]^n}.$$

Therefore, using (2.5),

$$\begin{aligned} |S| &\leqslant \varepsilon + 2t + \frac{C\|(\psi-\varphi)_+\|_{L^1(\mathrm{d}\mu)}}{t^n[\hbar(t)]^n} \\ &\leqslant 3\varepsilon + \frac{C\varepsilon^a}{t^n[\hbar(t)]^n}. \end{aligned}$$

Recall that $\varepsilon_B = \frac{1}{3}\min\{\varepsilon^n, \frac{\varepsilon^3}{16B}\}$. So, taking $t = \varepsilon_B/2 \geqslant \varepsilon^{n+2}$ we have

$$\hbar(t) = \left(\frac{t}{C_\tau}\right)^{1/\tau} \geqslant C\varepsilon^{(n+2)/\tau}.$$

With our choice of a

$$\frac{\varepsilon^a}{\varepsilon^{n(n+2)+\frac{(n+2)}{\tau}}} = \varepsilon.$$

Hence $|S| \leqslant C\varepsilon$ with $C = C(\tau, \omega)$. Thus,

$$\sup_X(\psi - \varphi) \leqslant C\|(\psi-\varphi)_+\|^{\frac{1}{a}}_{L^1(\mathrm{d}\mu)}.$$

This is the desired stability estimate. □

Following [5] we consider $\rho_\delta\varphi$- the regularization of the ω-psh function φ defined by

$$\rho_\delta\varphi(z) = \frac{1}{\delta^{2n}}\int_{\zeta\in T_zX} \varphi(\exp h_z(\zeta))\rho\left(\frac{|\zeta|^2_\omega}{\delta^2}\right)\mathrm{d}V_\omega(\zeta),\ \ \delta > 0; \tag{2.7}$$

where $\zeta \to \exp h_z(\zeta)$ is the (formal) holomorphic part of the Taylor expansion of the exponential map of the Chern connection on the tangent bundle of X associated to ω, and the modifier $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ is given by

$$\rho(t) = \begin{cases} \frac{\eta}{(1-t)^2} \exp(\frac{1}{t-1}) & \text{if } 0 \leqslant t \leqslant 1, \\ 0 & \text{if } t > 1 \end{cases}$$

with the constant η chosen so that

$$\int_{\mathbb{C}^n} \rho(\|z\|^2)\, \mathrm{d}V(z) = 1, \tag{2.8}$$

where $\mathrm{d}V$ denotes the Lebesgue measure in $\mathbb{C}^n$).

The proof of the following variation of [5, Proposition 3.8] and [2, Lemma 1.12] was given in [18].

Lemma 2.7. — *Fix $\varphi \in \mathrm{PSH}(\omega) \cap L^\infty(X)$. Define the Kiselman–Legendre transform with level $b > 0$ by*

$$\Phi_{\delta,b}(z) = \inf_{t\in[0,\delta]} \left(\rho_t\varphi(z) + Kt^2 + Kt - b \log \frac{t}{\delta} \right), \tag{2.9}$$

Then for some positive constant K depending on the curvature, the function $\rho_t\varphi + Kt^2$ is increasing in t and the following estimate holds:

$$\omega + \mathrm{dd}^c \Phi_{\delta,b} \geqslant -(Ab + 2K\delta)\, \omega, \tag{2.10}$$

where A is a lower bound of the negative part of the Chern curvature of ω.

The next lemma is essentially proven in [6, Theorem 4.3] or [9, Lemma 3.3, Proposition 4.4]. The adaption of those proofs to the case of compact Hermitian manifolds is straightforward.

Lemma 2.8. — *Let $\mu \in \mathcal{M}$ and $\varphi \in \mathrm{PSH}(\omega) \cap L^\infty(X)$. Then, there exists $0 < \alpha_1 < 1$ such that*

$$\|\rho_\delta\varphi - \varphi\|_{L^1(\mathrm{d}\mu)} \leqslant C\delta^{\alpha_1}. \tag{2.11}$$

3. Proof of Theorem 1.3

The necessary condition follows easily. It remains to prove the other one. As $\mu \in \mathcal{M}$ there exists $u \in \mathrm{PSH}(\omega) \cap C^{0,\alpha_0}(X)$ with $0 < \alpha_0 \leqslant 1$, and $C_0 > 0$ such that

$$\mu \leqslant C_0(\omega + \mathrm{dd}^c u)^n. \tag{3.1}$$

Using Radon-Nikodym's theorem, we write $\mu = C_0 h\omega_u^n$ for a Borel measurable function $0 \leqslant h \leqslant 1$. Let u_j be the smooth approximation of u as in Lemma 2.3 and denote

$$\mu_j := C_0 h\omega_{u_j}^n.$$

Then μ_j converges weakly to μ as $j \to +\infty$. Using [16, Theorem 0.1] we find $\varphi_j \in \mathrm{PSH}(\omega) \cap C^0(X)$ with normalisation $\sup_X \varphi_j = 0$, and $c_j > 0$ satisfying

$$\omega_{\varphi_j}^n = c_j \mu_j. \tag{3.2}$$

The first thing we need to show is the following.

Claim 3.1. — *There is a uniform constant $C_1 > 0$ such that $1/C_1 < c_j < C_1$.*

Proof. — Since $\mu(X) > 0$, it follows that $\int_X h\omega_u^n > 0$. Therefore, $\int_X h^{\frac{1}{n}}\omega_u^n > 0$. By the Bedford–Taylor convergence theorem [1] we know that $\omega_{u_j}^n$ converges weakly to ω_u^n. Since $C^0(X)$ is dense in $L^1(X, \omega_u^n)$, we have

$$\int_X h^{\frac{1}{n}}\omega_{u_j}^n > C$$

for some uniform $C > 0$. Applying the mixed forms type inequality (see [14], [19]) one obtains

$$\omega_{\varphi_j} \wedge \omega_{u_j}^{n-1} \geqslant \left[\frac{\omega_{\varphi_j}^n}{\omega_{u_j}^n}\right]^{\frac{1}{n}} \omega_{u_j}^n = (c_j C_0 h)^{\frac{1}{n}}\omega_{u_j}^n.$$

On the other hand,

$$\begin{aligned}\int_X \omega_{\varphi_j} \wedge \omega_{u_j}^{n-1} &= \int_X \omega \wedge \omega_{u_j}^{n-1} + \int_X \mathrm{dd}^c\varphi_j \wedge \omega_{u_j}^{n-1} \\ &= \int_X \omega \wedge \omega_{u_j}^{n-1} + \int_X \varphi_j \mathrm{dd}^c(\omega_{u_j}^{n-1}) \\ &\leqslant \int_X \omega \wedge \omega_{u_j}^{n-1} + B\int_X |\varphi_j|(\omega^2 \wedge \omega_{u_j}^{n-2} + \omega^3 \wedge \omega_{u_j}^{n-3}),\end{aligned}$$

where B is a constant depending only on ω (see e.g. [7] for details). Since $\|u_j\|_\infty < C$ and $\sup_X \varphi_j = 0$, it follows from the Chern–Levine–Nirenberg type inequality ([19, Proposition 1.1]) that the right hand side is uniformly bounded. Thus,

$$\int_X \omega_{\varphi_j} \wedge \omega_{u_j}^{n-1} \leqslant C.$$

Combining the above inequalities we get

$$c_j < C_1 := \frac{\int_X \omega_{\varphi_j} \wedge \omega_{u_j}^{n-1}}{\int_X (C_0 h)^{\frac{1}{n}} \omega_{u_j}^n} < +\infty.$$

Hence, by [16, Lemma 5.9] we also have

$$c_j > 1/C_1,$$

increasing C_1 if necessary. Thus, Claim 3.1 is proven. □

Thanks to Lemma 2.1, Lemma 2.3 and Claim 3.1 measures μ_j satisfy the volume-capacity inequality (2.2) with a uniform constant. Thus by [16, Corollary 5.6] we have $\|\varphi_j\|_\infty < C_2$. Passing to a subsequence one may assume that $\{\varphi_j\}$ is a Cauchy sequence in $L^1(\omega^n)$, and $\{c_j\}$ converges. Set

$$\varphi := (\limsup_j \varphi_j)^*, \quad c = \lim_j c_j. \tag{3.3}$$

Again passing to a subsequence if necessary we can also assume that

$$\varphi_j \to \varphi \quad \text{in } L^1(\omega^n) \quad \text{as } j \to \infty. \tag{3.4}$$

Lemma 3.2. — *We have*

$$\int_X |\varphi_k - \varphi| \omega_{u_j}^n \to 0 \quad \textit{as } \min\{j,k\} \to \infty.$$

Proof. — Using the uniform boundedness of $\|\varphi_j\|_\infty$, $\|u_j\|_\infty$ and the argument in Cegrell [3, Lemma 5.2] (it's a version of Vitali's convergence theorem) we get that

$$\int_X |\varphi_k - \varphi| \omega_u^n \to 0 \quad \text{as } k \to \infty. \tag{3.5}$$

Indeed, we first have $\int_X (\varphi_k - \varphi)\omega_u^n \to 0$ as $k \to \infty$. Moreover, all functions are negative and so we get the result.

We shall prove the lemma by the contradiction argument. Assume that there exist subsequences, still denoted by $\{\varphi_k\}_{k\geqslant 1}^\infty$, $\{u_j\}_{j\geqslant 1}^\infty$, and $\delta > 0$ such that

$$\int_X |\varphi_k - \varphi| \omega_{u_j}^n > \delta.$$

Let $a > 0$ be small. By Hartogs' lemma there exists k_0 such that

$$\varphi_k \leqslant \varphi + a \quad \forall\, k \geqslant k_0.$$

If we choose a small enough, then for $k \geqslant k_0$ and $j \geqslant 1$,

$$\int_X (\varphi - \varphi_k) \omega_{u_j}^n \geqslant \delta/2. \tag{3.6}$$

Next, we are going to show that

$$E_{jk} := \int_X (\varphi - \varphi_k)\omega_{u_j}^n - \int_X (\varphi - \varphi_k)\omega_u^n \to 0 \tag{3.7}$$

as $\min\{j,k\} \to +\infty$. Indeed,

$$E_{jk} = \int_X (\varphi - \varphi_k) dd^c(u_j - u) \wedge \sum_{p=0}^{n-1} \omega_{u_j}^p \wedge \omega_u^{n-1-p}.$$

Let us denote by $T_p(j)$ the current $\omega_{u_j}^p \wedge \omega_u^{n-p-1}$. Then

$$\begin{aligned} dd^c\left[(\varphi - \varphi_k)T_p(j)\right] &= dd^c(\varphi - \varphi_k) \wedge T_p(j) + d(\varphi - \varphi_k) \wedge d^c T_p(j) \\ &\quad - d^c(\varphi - \varphi_k) \wedge dT_p(j) + (\varphi - \varphi_k) dd^c T_p(j) \\ &=: S_1 + S_2 + S_3 + S_4. \end{aligned}$$

By integration by parts

$$\begin{aligned} E_{jk} &= \int_X (u_j - u) dd^c\left[(\varphi - \varphi_k)T_p(j)\right] \\ &= \int_X (u_j - u)(S_1 + S_2 + S_3 + S_4). \end{aligned} \tag{3.8}$$

Now we shall estimate each term in the right hand side. First, since $S_1 = (\omega_\varphi - \omega_{\varphi_k}) \wedge T_p(j)$,

$$\left|\int_X (u - u_j) S_1\right| \leqslant \|u - u_j\|_\infty \left(\int_X (\omega_\varphi + \omega_{\varphi_k}) \wedge T_p(j)\right) \to 0 \tag{3.9}$$

as $j \to +\infty$.

Next, we estimate $\int_X (u - u_j) S_2$. As $d^c T_p(j) = d^c\omega \wedge T_p'(j)$, where $T_p'(j)$ is a sum of terms of the form $C_3 \omega_{u_j}^k \wedge \omega_u^q$ (the constant C_3 depending only on n, p), we apply the Cauchy–Schwarz inequality [19, Proposition 1.4] to get that

$$\begin{aligned} &\left|\int_X (u - u_j) d\varphi \wedge S_2\right| \\ &\leqslant C\|u - u_j\|_\infty \left[\int_X d\varphi \wedge d^c\varphi \wedge \omega \wedge T_p'(j)\right]^{\frac{1}{2}} \left[\int_X \omega^2 \wedge T_p'(j)\right]^{\frac{1}{2}}. \end{aligned} \tag{3.10}$$

Moreover,

$$\begin{aligned} 2\int_X d\varphi \wedge d^c\varphi \wedge T_p'(j) &= \int_X dd^c\varphi^2 \wedge T_p'(j) - \int_X 2\varphi\omega_\varphi \wedge T_p'(j) \\ &\quad + 2\int_X \omega \wedge T_p'(j) \\ &\leqslant C\left(\int_X \omega^n + \|\varphi\|_\infty^n \|u_j\|_\infty^n \|u\|_\infty^n\right), \end{aligned} \tag{3.11}$$

where in the last inequality we used [19, Proposition 1.5]. Therefore, we conclude the right hand side of the previous inequality tends to 0 as $j \to +\infty$. Similar estimates are also applied to the remaining terms with S_3, S_4. Thus we have shown that $E_{jk} \to 0$ as $\min\{j,k\} \to +\infty$.

Combining (3.5), (3.6), and (3.7) we get a contradiction. The lemma thus follows. □

3.1. Existence of a continuous solution

Notice that

$$\int_X |\varphi_j - \varphi_k|\omega_{u_j}^n \leqslant \int_X |\varphi_j - \varphi|\omega_{u_j}^n + \int_X |\varphi_k - \varphi|\omega_{u_j}^n \to 0$$

as $\min\{j,k\} \to +\infty$. Therefore, using Lemma 3.2 and the argument in [16, Theorem 5.8] we get that $\{\varphi_j\}_{j\geqslant 1}$ is a Cauchy sequence in $C^0(X)$. Thus,

$$\varphi = \lim_j \varphi_j \quad \text{ in } C^0(X).$$

We conclude that $\varphi \in \mathrm{PSH}(\omega) \cap C^0(X)$ and it solves

$$\omega_\varphi^n = c\,\mu, \tag{3.12}$$

where c is defined in (3.3).

3.2. Hölder continuity of the solution

We shall show that the solution φ obtained in (3.12) is Hölder continuous. Fix $\tau > 0$ and set

$$\alpha = \min\left\{\frac{1}{1+(n+2)(n+\frac{1}{\tau})}, \alpha_1\right\},$$

where α_1 is given in Lemma 2.8. By Corollary 2.2 $\mu \in \mathcal{H}(\tau)$ and then Proposition 2.4 holds with $\gamma = \alpha$.

Consider the regularization of φ as in (2.7). As explained in [15] and [6] the result follows as soon as we show that

$$\rho_t\varphi - \varphi \leqslant Ct^{\alpha\alpha_1}$$

for t small enough.

It follows from Lemma 2.7 that

$$\begin{aligned}\varphi \leqslant \Phi_{\delta,b} &\leqslant \rho_\delta\varphi + K(\delta+\delta^2).\\ &\leqslant \rho_\delta\varphi + 2K\delta.\end{aligned}$$

Choose the level $b = (\delta^\alpha - 2K\delta)/A = O(\delta^\alpha)$ so that

$$Ab + 2K\delta = \delta^\alpha. \tag{3.13}$$

After fixing the level b, we write

$$\Phi_\delta := (1-\delta^\alpha)\Phi_{\delta,b}. \tag{3.14}$$

Then, by Lemma 2.7

$$\omega + dd^c\Phi_\delta \geqslant \delta^{2\alpha}\omega. \tag{3.15}$$

Since $-C_4 \leqslant \varphi \leqslant 0$ and $\rho_\delta\varphi \leqslant 0$ one obtains

$$\Phi_\delta \leqslant (1-\delta^\alpha)(\rho_\delta\varphi + K\delta + K\delta^2) \leqslant 2K\delta. \tag{3.16}$$

It follows that

$$\Phi_\delta \leqslant C_4\delta^\alpha \tag{3.17}$$

for $\delta \leqslant \delta_0$ small. Therefore, by (3.16) and (3.17) we have

$$\Phi_\delta - \varphi \leqslant C_4\delta^\alpha + (1-\delta^\alpha)(\rho_\delta\varphi + K\delta + K\delta^2 - \varphi). \tag{3.18}$$

Next, the stability estimate Proposition 2.4 applied for $\Phi_\delta - C_4\delta^\alpha$ and φ, and $\gamma = \alpha$ give us that

$$\begin{aligned}\sup_X(\Phi_\delta - \varphi) &\leqslant C_5\|\max\{\Phi_\delta - \varphi - C_4\delta^\alpha, 0\}\|^\alpha_{L^1(d\mu)} + C_4\delta^\alpha \\ &\leqslant C_5\|\rho_\delta\varphi + K\delta + K\delta^2 - \varphi\|^\alpha_{L^1(d\mu)} + C_4\delta^\alpha,\end{aligned}$$

where we used (3.18) for the second inequality. Hence, using Lemma 2.8, we conclude that

$$\Phi_\delta - \varphi \leqslant C_6\delta^{\alpha\alpha_1}. \tag{3.19}$$

For a fixed point z, the minimum in the definition of $\Phi_{\delta,b}(z)$ is realized for some $t_0 = t_0(z)$. Then, (3.14) and (3.17) imply

$$(1-\delta^\alpha)\left(\rho_{t_0}\varphi + Kt_0 + Kt_0^2 - b\log\frac{t_0}{\delta} - \varphi\right) \leqslant C_6\delta^\alpha.$$

Since $\rho_t\varphi + Kt^2 + Kt - \varphi \geqslant 0$, we have

$$b(1-\delta^\alpha)\log\frac{t_0}{\delta} \geqslant -C_6\delta^\alpha.$$

Combining this with $b \geqslant \delta^\alpha/(2A)$, one gets that

$$t_0(z) \geqslant \delta\kappa \quad \text{for } \kappa = \exp\left(-\frac{2AC_6}{(1-\delta_0^\alpha)}\right), \tag{3.20}$$

where δ_0 is fixed, and κ is a uniform constant.

Now, we are ready to conclude the proof. Since $t_0 = t_0(z) \geqslant \delta\kappa$ and $t \mapsto \rho_t\varphi + Kt^2$ is increasing,

$$\begin{aligned}\rho_{\kappa\delta}\varphi(z) + K(\delta\kappa)^2 + K\delta\kappa - \varphi(z) &\leqslant \rho_{t_0}\varphi(z) + Kt_0^2 + Kt_0 - \varphi(z)\\ &= \Phi_{\delta,b}(z) - \varphi(z)\\ &= \frac{\delta^\alpha}{1-\delta^\alpha}\Phi_\delta + (\Phi_\delta - \varphi).\end{aligned}$$

Combining this, (3.17) and (3.19) we get that

$$\rho_{\kappa\delta}\varphi(z) - \varphi(z) \leqslant C_7\delta^{\alpha\alpha_1}.$$

The desired estimate follows by rescaling $\delta := \kappa\delta$ and increasing C_7.

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