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BOUNDEDNESS OF THE NUMBER OF NODAL DOMAINS FOR EIGENFUNCTIONS OF GENERIC KALUZA–KLEIN 3-FOLDS

by Junehyuk JUNG & Steve ZELDITCH (*)

ABSTRACT. — This article concerns the number of nodal domains of eigenfunctions of the Laplacian on special Riemannian 3-manifolds, namely nontrivial principal S^1 bundles $P \rightarrow X$ over Riemann surfaces equipped with certain S^1 invariant metrics, the Kaluza–Klein metrics. We prove for generic Kaluza–Klein metrics that any Laplacian eigenfunction has exactly two nodal domains unless it is invariant under the S^1 action.

We also construct an explicit orthonormal eigenbasis on the flat 3-torus \mathbb{T}^3 for which every non-constant eigenfunction has two nodal domains.

RÉSUMÉ. — Cet article concerne le nombre de domaines nodaux des fonctions propres du Laplacien sur des variétés Riemanniennes Kaluza–Klein en dimension trois, à savoir des variétés qui sont des fibrés S^1 -principaux $P \rightarrow X$ sur des surfaces de Riemann équipées avec une métrique S^1 -invariante de type Kaluza–Klein. Pour des métriques génériques de ce type, on prouve que chaque fonction propre possède exactement deux domaines nodaux, sauf si elle est invariante par l'action de S^1 .

On construit aussi une base orthonormale de fonctions propres explicites du tore plat \mathbb{T}^3 pour que chaque fonction propre non constante possède exactement deux domaines nodaux.

1. Introduction

This article is concerned with the number of nodal domains of eigenfunctions of the Laplacian on certain 3-dimensional compact smooth Riemannian manifolds (P, G) . The manifolds are $S^1 = SO(2)$ bundles $\pi : P \rightarrow X$ over a Riemannian surface (X, g) , and G is assumed to be a Kaluza–Klein metric *adapted* to π , i.e., G is invariant under the free S^1 action

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on P and there exists a splitting $TP = H(P) \oplus V(P)$ of TP so that $d\pi : H_p(P) \rightarrow T_{\pi(p)}X$ is isometric and so the fibers are geodesics. Thus, $\pi : P \rightarrow X$ is a special kind of Riemannian submersion with totally geodesic fibers in the sense of [3] (see Definition 1.1 and Definition 6.3 of the present article). The S^1 action commutes with the Laplacian Δ_G of the Kaluza–Klein metric G and one may separate variables to obtain an orthonormal basis of joint eigenfunctions $\phi_{m,j}$,

$$(1.1) \quad \Delta_G \phi_{m,j} = -\lambda_{m,j} \phi_{m,j}, \quad \frac{\partial}{\partial \theta} \phi_{m,j} = im \phi_{m,j}.$$

Our focus is on the nodal sets of the real or imaginary parts of

$$(1.2) \quad \phi_{m,j} = u_{m,j} + iv_{m,j}$$

and on particularly on the number of their nodal domains. Since Δ_G is a real operator, the real and imaginary parts (1.2) satisfied the modified eigenvalue system,

$$\begin{cases} \Delta_G u_{m,j} = -\lambda_{m,j} u_{m,j}, \\ \Delta_G v_{m,j} = -\lambda_{m,j} v_{m,j}, \\ \frac{\partial}{\partial \theta} u_j = mv_j, \frac{\partial}{\partial \theta} v_j = -mu_j. \end{cases}$$

Our main result (Theorem 1.5) is that when 0 is a regular value of $\phi_{m,j}$ for all (m, j) , then for $m \neq 0$, the nodal sets of $u_{m,j}$, resp. $v_{m,j}$, are connected and there exist exactly 2 corresponding nodal domains. The case $m = 0$ is special because $\phi_{0,j}$ is then real valued and is a pullback from the base X ; in this case, the number of connected components of the nodal set (and the number of nodal domains) is the same as for the corresponding eigenfunction on X . Theorem 1.4 shows that it is a generic property of Kaluza–Klein metrics on S^1 bundles over Riemann surfaces that 0 is indeed a regular value of $\phi_{m,j}$ for all (m, j) . The precise statement requires a discussion of the geometric data underlying a Kaluza–Klein metric and how we allow it to vary when defining “genericity”. An introductory discussion of the Kaluza–Klein metrics of this article is given in Section 1.1 and a more detailed discussion is given in Section 4 (see Theorem 4.1 and Lemma 4.9).

1.1. Adapted Kaluza–Klein metrics

We now define Kaluza–Klein metrics on a three-dimensional manifold P which is an S^1 bundle over a (usually) compact Riemannian surface X . In our main results, P is the unit co-circle bundle of an ample complex

holomorphic line bundle $L \rightarrow X$. Thus, $P = S^3$ or $P = U(S^2)$ (the unit tangent bundle) when $X = S^2$, and $P = S^*X$ (the unit conormal bundle) when the genus $g \geq 2$. When $g = 1$, P can be the unit circle bundle of the theta line bundle over \mathbb{T}^2 , or it may be the trivial circle bundle $S^*\mathbb{T}^2$ (see Section 9.1). Other examples where X has constant curvature are discussed in Section 9.

A Kaluza–Klein metric is determined by the following data:

- (i) A surface X equipped with a Riemannian metric g and a complex structure J ,
- (ii) A nontrivial complex holomorphic line bundle $L \rightarrow X$ over a surface,
- (iii) A Hermitian metric h on L ,
- (iv) A complex structure J_L on L ,
- (v) An h -compatible connection ∇ on L .

In this article, we fix J , L and J_L and only vary the data (g, h, ∇) . The unitary frame bundle for the Hermitian metric h is defined by

$$P_h = \{(z, \lambda) \in L^* : h_z^*(\lambda) = 1\}.$$

The connection ∇ induces a connection 1-form on P_h and a splitting $TP_h = H(P_h) \oplus V(P_h)$ into horizontal and vertical spaces; see Section 2 for background.

DEFINITION 1.1. — *The Kaluza–Klein metric on P_h is the $U(1)$ -invariant metric G such that the horizontal space $H_p := \ker d\pi$ is isometric to $T_{\pi(p)}X$, so that $V = \mathbb{R}\frac{\partial}{\partial\theta}$ is orthogonal to H and is invariant under the natural S^1 action and so that the orbits of the S^1 action are unit speed vertical geodesics.*

The data (g, h, ∇) determines a horizontal Laplacian Δ_H , a vertical Laplacian $\frac{\partial^2}{\partial\theta^2}$, and their sum, the Kaluza–Klein Laplacian (1.1),

$$(1.3) \quad \Delta_G = \Delta_H + \frac{\partial^2}{\partial\theta^2}.$$

As is well-known, sections s of powers L^m of a complex line bundle L lift to equivariant functions $\hat{s} : L^* \rightarrow \mathbb{C}$ on the dual line bundle by the formula $\hat{s}(z, \lambda) = \lambda(s(z))$ where a point of L^* is denoted by $\lambda \in L_z^*$. Under this identification, the horizontal Laplacian is equivalent to the Bochner Laplacians $\nabla_m^* \nabla_m$ on sections of L^m . Thus, equivariant eigenfunctions of Δ_G of weight m on M are lifts of eigensections of $\nabla_m^* \nabla_m$. See Section 3 and Lemma 6.6 for details. In proving genericity theorems it is easier to

work downstairs on X . But the nodal results pertain to the equivariant eigenfunctions on P_h .

Remark 1.2. — Not all S^1 invariant metrics on SX are adapted Kaluza–Klein metrics (see Section 6.1). Many of the techniques of this paper extend to general S^1 -invariant metrics on principal S^1 bundles over manifolds of all dimensions. For simplicity of exposition we restrict to dimension 3.

1.2. Nodal sets

We thus have two versions of the eigenfunctions of the Kaluza–Klein Δ_G , first as scalar complex valued equivariant eigenfunctions on P_h and second as complex eigensections on X . In each version we have a nodal set, and we use the base nodal set on X to analyse the nodal set on P_h .

We denote the eigensection corresponding to $\phi_{m,j}$ as $f_{m,j}e_L^m$ in a local holomorphic frame. We mainly consider $L = K_X$ and then we write the section as $f_{m,j}(z)(dz)^m$. Let

$$\Re f_{m,j} = a_{m,j}(z), \quad \Im f_{m,j} = b_{m,j}(z).$$

Then,

$$f_{m,j}(z)e^{-im\theta} = (a_{m,j}(z) + ib_{m,j}(z))(\cos m\theta - i \sin m\theta),$$

so that with $\phi_{m,j} = u_{m,j} + iv_{m,j}$,

$$(1.4) \quad \begin{cases} u_{m,j} = a_{m,j} \cos m\theta + b_{m,j} \sin m\theta, \\ v_{m,j} = b_{m,j} \cos m\theta - a_{m,j} \sin m\theta. \end{cases}$$

See Section 6.2 for more details.

We denote by $\mathcal{Z}_{f_{m,j}}$ the zero set of the eigensection $f_{m,j}e_L^m$ on X :

$$\mathcal{Z}_{f_{m,j}} = \{z \in X : f_{m,j}(z) = 0\}.$$

It is easy to see that the zero set $\mathcal{Z}_{\phi_{m,j}}$ of $\phi_{m,j}$ is the inverse image of $\mathcal{Z}_{f_{m,j}}$ under the natural projection π :

$$\mathcal{Z}_{\phi_{m,j}} = \pi^{-1}\mathcal{Z}_{f_{m,j}}.$$

Usually we study the nodal sets of the real and imaginary parts of the lift, not to be confused with the lifts of the real and imaginary parts of the local expression $f_{m,j}$ of the section (since the frame e_L^m must also be taken into account). In general, it is not obvious whether or not the zero set of $f_{m,j}$ is discrete in X .

We denote the nodal sets of the real, resp. imaginary parts, of the lift by $\mathcal{N}_{\Re\phi_{m,j}}\{p \in P_h : \Re\phi_{m,j}(p) = 0\}$, resp. $\mathcal{N}_{\Im\phi_{m,j}} = \{p \in P_h : \Im\phi_{m,j}(p) = 0\}$.

The analysis is the same for real and imaginary parts and we generally work with the imaginary part, following the tradition for quadratic differentials.

Perhaps the most familiar setting for such Kaluza–Klein metrics and lifts of m -differentials to equivariant sections is that of hyperbolic surfaces of finite area. The reader familiar with Maass forms and operators may want to compare Kaluza–Klein notions with those of $SL(2, \mathbb{R})$ -theory in Section 9.3.

1.3. Statement of results

Let (X, J, g) be a Riemannian surface with complex structure J . Let (L, h) be a Hermitian holomorphic line bundle over X , and let $P_h \subset L^* \rightarrow X$ be the principal S^1 bundle associated to h . Let ∇_h be an h compatible connection on L , and let $G = G(g, h, \nabla_h)$ be the associated Kaluza–Klein metric on P_h .

As mentioned above, weight 0 (invariant) eigenfunctions are special because they are real-valued (once they are multiplied by a suitable constant). We therefore separate the case $m = 0$ from the remainder of the discussion, and state the obvious (but interesting)

PROPOSITION 1.3. — *For $m = 0$, invariant eigenfunctions ($m = 0$) of Δ_G are lifts $\pi^*\psi_j$ of eigenfunctions ψ_j of Δ_g on the base X , and the nodal set of $\pi^*\psi_j$ is the inverse image under π of the nodal set of ψ_j . The number of nodal domains of $\pi^*\psi_j$ equals the number of nodal domains of ψ_j .*

Indeed, their nodal sets are inverse images of nodal sets on the base. Hence the number of nodal domains of “invariant” Kaluza–Klein eigenfunctions is the number for the corresponding eigenfunction on the base. Henceforth we always assume $m \neq 0$.

To prepare for our main result when $m \neq 0$, we first state a result on generic properties of equivariant Kaluza–Klein eigenfunctions. By “generic” properties of Kaluza–Klein metrics, we mean properties of residual sets in a suitable C^k space of the data (g, h, ∇) , often when only one component is varied and the others are fixed. The generic properties of concern in this article hold for many choices of Banach spaces of data, which could be suitable C^k spaces or a Sobolev spaces H^s . Our basic reference for genericity properties of eigenvalues/eigenfunctions in [20], and the reader

is referred there for background. Somewhat surprisingly, generic properties of eigensections of Bochner–Kodaira operators on complex line bundles do not seem to have been studied before.

The most general genericity results are stated in Theorem 4.1 in Section 4. In these results, we define “admissible data” such as (g, h, ∇) , and prove the main genericity properties as different components of the data are varied. Since the general result requires some definitions from Sections 2–3, we only state the most elementary result here in the case where $L = T^*X$ is the canonical bundle, and where only the metric g on X is varied. But we also require that the Hermitian metric h on K is induced by g and that the connection ∇_h is the Chern (or equivalently, Riemannian) connection. Of course, K is not ample when $g = 0, 1$ but we are considering eigensections of Bochner–Kodaira Laplacians, not holomorphic sections, so ample-ness is not particularly relevant. Moreover, the proofs also work for K^{-1} , so that one could replace T^*S^2 by the ample line bundle TS^2 or $\mathcal{O}(1) \rightarrow \mathbb{C}P^1$ and obtain similar results.

THEOREM 1.4. — *Let (X, J) be a Riemann surface, and let $L = K^m$. We consider Riemannian metrics g in the conformal class associated to J . We assume that h is the Hermitian metric induced by g and that ∇ is the Levi-Civita connection. Then, for generic metrics g in the class of J on X ,*

- (1) *the spectrum of each Bochner Laplacian $\nabla_{g,h}^*$ on $C^k(X, L^m)$ is simple (i.e. of multiplicity 1). Thus, the multiplicity of the eigenvalue $\lambda = \lambda_{m,j}$ of Δ_G is 1 if $m = 0$, and 2 if $m \neq 0$.*
- (2) *Every eigenfunction is a joint eigenfunction of Δ_G and $\frac{\partial^2}{\partial \theta^2}$.*
- (3) *all of the eigensections $f_{m,j}$ have isolated zeros and zero is a regular value. In particular, $\mathcal{Z}_{f_{j,m}}$ is a finite set of points;*
- (4) *If we lift sections to equivariant eigenfunctions ϕ , then $\Re\phi$ and $\Im\phi$ have zero as a regular value.*

An important consequence of (1)–(2) of Theorem 1.4 is that, for generic data, all real Kaluza–Klein eigenfunctions are real/imaginary parts of equivariant eigenfunctions. Therefore, the results we prove for equivariant eigenfunctions hold for all possible eigenfunctions. Theorem 4.1 states similar results for three other types of variations of the data.

We can now state our main result. The first statement repeats (1)–(2) of Theorem 1.4 for the sake of clarity.

THEOREM 1.5. — *Suppose that the data (g, h, ∇) of the Kaluza–Klein metric satisfies the generic properties of Theorem 1.4. Then,*

- (1) The eigenspace of Δ_G corresponding to $\lambda = \lambda_{m,j} = \lambda_{-m,j}$ is spanned by $\phi_{m,j}$ and $\phi_{-m,j} = \overline{\phi_{m,j}}$. In particular, any real eigenfunction with the eigenvalue $\lambda_{m,j}$ is a constant multiple of $T_\theta(\Re\phi_{m,j})$, where T_θ is the S^1 action on P parameterized by θ .
- (2) For $m \neq 0$, the nodal sets of $\Re\phi_{m,j}$ are connected.
- (3) For $m \neq 0$, the number of nodal domains of $\Re\phi_{m,j}$ is 2.

Note from Weyl law that $\#\{\lambda_{j,0} < \Lambda\} \sim \Lambda$ and that $\#\{\lambda_{m,j} < \Lambda\} \sim \Lambda^{3/2}$. Therefore as an immediate consequence of Theorem 1.5, we have the following:

COROLLARY 1.6. — *Let $P \rightarrow X$ be a non-trivial principal S^1 bundle with a generic Kaluza–Klein metric. For any given orthonormal eigenbasis, almost all (i.e., along a subsequence of density one) eigenfunctions have exactly two nodal domains.*

The density one subsequence is of course the one with $m \neq 0$. Theorem 1.5 furnishes the first example of Riemannian manifolds of dimension > 2 for which the number of nodal domains and connected components of the nodal set have been counted precisely. The results for $m \neq 0$ may seem rather surprising, since in dimension 2 the only known sequences of eigenfunctions with a bounded number of nodal domains are those constructed in an ingenious way by H. Lewy on the standard S^2 [16] and those of Stern on a flat torus [4, 18]. In those cases, the separation-of-variables eigenfunctions have connected nodal sets but the complement of the nodal set has many components, i.e., nodal domains, saturating the Courant bound that the number of nodal domains of the j th eigenfunction (in order of increasing eigenvalue) is j . In the Kaluza–Klein case, all eigenfunctions for generic Kaluza–Klein metrics are separation-of-variables eigenfunctions and have connected nodal sets. But the connectivity is of a different kind than in dimension two and by a simple argument it induces connectivity of nodal domains.

Remark 1.7. — When $P \rightarrow X$ is trivial, and $P \cong S^1 \times X$ is endowed with the product metric, we have $\phi_{m,j} = \psi_j e^{im\theta}$ where ψ_j is an eigenfunction of Δ_g on the base X . Hence $\Re\phi_{m,j} = \psi_j \cos m\theta$ has many nodal domains, and the last statement in Theorem 1.5 fails. Hence, the “generic” set of metrics is not the full set of metrics. See Section 9.1 for flat tori for a product setting where the nodal results above do hold.

1.4. Outline of the proof

The nodal set of the lift real and imaginary parts of the lift $\phi_{m,j}$ of $f_{m,j}e_L^m$ is very different over nodal versus non-nodal points of $f_{m,j}$. Let

$$(1.5) \quad \Sigma := \pi^{-1} \mathcal{Z}_{f_{m,j}}$$

be the inverse image of the base nodal points. It is a union of fibers and is a finite union of fibers if and only if $f_{m,j}$ has a finite number of zeros. We refer to Σ as the “singular fibers” or singular set. We denote by $X \setminus \mathcal{Z}_{f_{m,j}}$ the punctured Riemann surface in which the zero set of $f_{m,j}$ is deleted. A key statement in the nodal analysis is the following:

PROPOSITION 1.8. — *For $m \neq 0$, the maps*

$$\pi : \mathcal{N}_{\Re\phi_{m,j}} \setminus \Sigma \rightarrow X \setminus \mathcal{Z}_{f_{m,j}}, \quad \mathcal{N}_{\Im\phi_{m,j}} \setminus \Sigma \rightarrow X \setminus \mathcal{Z}_{f_{m,j}}$$

is an m -fold covering space.

It follows that the topology of the nodal set is entirely determined by the combinatorics of gluing the sheets along the singular fibers. In fact, the gluing is rather simple and easily yields the following

THEOREM 1.9. — *For all $m \neq 0$, the nodal set $\mathcal{N}_{\Re\phi_{m,j}}$ is connected.*

To count nodal domains, we need to make the assumption that there are just a finite number of zeros of $f_{m,j}$ and that at least one of them is regular.

When the zero set is transverse to the zero section, then the sum of the indices of the zeros is the first Chern class of K^m , and in particular is non-empty when the genus of X is $\neq 0$, i.e., when X is not a torus.

For metrics satisfying Theorem 1.4 we prove Theorem 1.5 by using Proposition 1.8 together with some geometric observations on how the sheets fit together at the singular fibers. This is done using a Bers type local analysis of the eigensections (Section 5) and some geometric/combinatorial arguments in Section 8.

To put the nodal results into context, it is proved in varying degrees of generality in [8, 9, 11, 12, 13, 14, 17, 22] that in dimension 2, the number of nodal domains of an orthonormal basis $\{u_j\}$ of Laplace eigenfunctions on certain surfaces with ergodic geodesic flow tends to infinity with the eigenvalue along almost the entire sequence of eigenvalues. By the first item of Theorem 1.5, the same is true for their lifts to the unit tangent bundle SX as invariant eigenfunctions of the Kaluza–Klein metric. But for higher weight eigenfunctions, the situation is virtually the opposite and the number of nodal domains is bounded.

Remark 1.10. — Note that the geodesic flow on P with a Kaluza–Klein metric never is ergodic. To see this, observe that the hypersurface

$$\{(x, v) \in SP : v \perp \partial_\theta\} \subset SP$$

is invariant under the geodesic flow, and it divides SP into two subsets of positive measure:

$$\{(x, v) \in SP : \langle v, \partial_\theta \rangle_G > 0\} \quad \text{and} \quad \{(x, v) \in SP : \langle v, \partial_\theta \rangle_G < 0\}.$$

1.4.1. Surfaces of constant curvature

Metrics g of constant curvature with their associated Hermitian metrics and connections on $K \rightarrow X$ are not generic. But they are of special interest, so we comment on what we are able to prove about them. Note that the standard metric on \mathbb{T}^3 is Kaluza–Klein, as is the standard metric on S^3 or $SO(3) = U(S^2)$. The standard metric on the unit tangent bundle $\mathrm{PSL}(2, \mathbb{R})/\Gamma$ over a hyperbolic surface is Lorentzian, but if one changes the sign of the vertical Laplacian it is also Kaluza–Klein.

Perhaps surprisingly, the results of Theorem 1.5 are valid for some orthonormal bases of eigenfunctions on flat 3-tori.

THEOREM 1.11. — *On the flat 3 torus \mathbb{T}^3 , one can find an orthonormal eigenbasis for which all nonconstant eigenfunctions have two nodal domains.*

Next we turn to hyperbolic metrics on a surface X of genus $g \geq 2$. Then the total space $P_h = \mathrm{PSL}(2, \mathbb{R})/\Gamma$ and the equivariant eigenfunctions of the Kaluza–Klein Laplacian Δ are the same as joint eigenfunctions of the generator W of K and of the Casimir operator Ω . When the weight m is fixed, one may separate variables and obtain a Maass Laplacian D_m on smooth sections of a complex line bundle $\pi : K^m \rightarrow X$, namely the bundle of m -differentials of type $(dz)^m$, and are the usual weight m automorphic Maass eigendifferentials $f_{m,j}(z)(dz)^m$,

$$D_m f_{m,j}(z) = s(1-s)f_{m,j}(z),$$

of the Maass Laplacians

$$D_m = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - 2imy \frac{\partial}{\partial x}.$$

Unfortunately, we are not able to verify that (any) Maass eigendifferentials have 0 as a regular value, i.e. the generic conditions needed for Theorem 1.5. Indeed, we do not know how to prove that (any) eigenfunctions $\phi_{0,j}$ of the

hyperbolic Laplacian have a discrete set of critical points, much less that 0 is a regular value of $d\phi_{0,j}$.

Regarding spherical harmonics on S^3 we will be brief, because we study random equivariant spherical harmonics in detail in a forthcoming article [15]. The round metric is a Kaluza–Klein metric, but of course not a generic one. In Section 9.2 we show that the joint eigenfunctions $Y_N^{m_1, m_2}$ of Δ and of two commuting S^1 actions have very different nodal sets from the ones in Theorem 1.5. On the other hand, in [15] we show that “random” linear combinations of such $Y_N^{m_1, m_2}$ with m_1 fixed do satisfy the results of Theorem 1.5 and the nodal sets of their real, resp. imaginary parts, have just one nodal component. We also find their expected Euler characteristic. These results were motivated by the numerical discovery of Barnett et. al. that 3D random spherical harmonics on S^3 of fixed degree, the nodal set contains one “giant component” and many small components [2]. We are fixing the weight m_1 and therefore do not work with general random spherical harmonics. But in the N dimensional subspaces where m_1 is fixed with $|m_1| \leq N, 2|N - m_1|$, the nodal sets of the real and imaginary parts are connected and divide S^3 into just two components. For further discussion we refer to [15].

2. Geometric background

In this section we discuss the geometric data that goes into the construction of Kaluza–Klein metrics, which are defined in Definition 1.1. They are also the data needed to define Bochner Laplacians $\nabla^* \nabla$ and Kaluza–Klein Laplacians Δ_G . We plan to vary the data and study perturbation theory of eigenvalues and eigensections in Section 4.

2.1. Riemannian metrics on X and Hermitian metrics on L

Let (X, J, g) denote a Riemann surface with complex structure J and Riemannian metric g . We write $g_{1\bar{1}} = g(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}})$ and $g^{1\bar{1}} = g^*(dz, d\bar{z})$, where g^* is the dual metric. The complex structure gives a decomposition of $T^*X \otimes \mathbb{C} = T^{*(1,0)} \oplus T^{*(0,1)}$ into $(1,0)$ resp. $(0,1)$ parts. We denote the area form of g by

$$dA_g = \omega = ig_{1\bar{1}} dz \wedge d\bar{z},$$

where the Kähler form ω is the $(1,1)$ form defined by $g_J(X, Y) = \omega(JX, Y)$. Locally there exists a Kähler potential ψ defined up to a constant by

$dd^c\phi = \omega$. Here $d^c = \frac{1}{2i}(\partial - \bar{\partial})$. Then, $\omega_h = i\partial\bar{\partial}\log h = dd^c\phi = \Delta_g\phi L(dz)$, where $L(dz) = idz \wedge d\bar{z}$.

2.1.1. The space of isometry classes of metrics on surfaces

It is well-known that the space of Riemannian metric tensors on a manifold X splits as a product $\text{Vol}(X) \times \text{Met}_\mu(X)$ of volume forms times metrics with a fixed volume form. Thus, one may separately consider metrics with a fixed volume form and conformal classes of metrics.

Choice of a complex structure J on X is equivalent to choice of a conformal class $\text{Conf}(g_0)$ of metrics. The moduli space of conformal classes is the same as the $(3g - 3)$ -dimensional moduli space \mathfrak{M}_g of complex structures on X . In each conformal class, we may pick a background metric g_0 and represent other metrics in the form

$$\text{Conf}(g_0) = \{e^{2\sigma}g_0 : \sigma \in C^\infty(X)\}.$$

If we fix a complex structure J , then the Riemannian metrics in the corresponding conformal class are Kähler metrics, and may be parameterized by their Kähler potentials ϕ , where the area form ω_ϕ of the Kähler metric is related to that of the reference metric by

$$\omega_\phi = \omega_0 + i\partial\bar{\partial}\phi.$$

We let

$$\mathcal{K}_\omega := \{\omega_\phi := \omega_0 + i\partial\bar{\partial}\phi > 0\}$$

which may be identified with an open set in $C^\infty(X)$. The Liouville field and Kähler potential are related by

$$e^{2\sigma} = 1 - \frac{1}{2}\Delta_0\phi,$$

where Δ_0 is the Laplacian of g_0 . The only difference in the two parameterizations of conformal metrics is that the area of metrics in \mathcal{K}_ω is fixed while it may vary in $\text{Conf}(g_0)$. Thus $\text{Conf}(g_0) \cong \mathcal{K}_\omega \times \mathbb{R}$.

In the case of Riemann surfaces, the area form is the symplectic form associated to a Kähler metric. Given a complex structure J , the Kähler metric g_J can be recovered from its area form ω by the formula $g_J(X, Y) = \omega(X, JY)$. Hence isometry classes of Kähler metrics with a fixed area form are parameterized by \mathfrak{M}_g .

2.2. Complex line bundles $L \rightarrow X$, connections and curvature

If we fix a complex structure J and holomorphic line bundle $L \rightarrow X$, then a Hermitian metric h on L is determined by the length of a local holomorphic frame e_L (i.e., a local holomorphic nonvanishing section) of L over an open set $U \subset M$ by $e^{-\psi} = \|e_L\|_h^2$, where $\|e_L\|_h = h(e_L, e_L)^{1/2}$ denotes the h -norm of e_L .

2.2.1. Connections

In the real setting, a connection on a vector bundle E defines a covariant derivative

$$\nabla : C^\infty(X, E) \rightarrow C^\infty(X, E \otimes T^*X).$$

In our complex setting, we assume L is a holomorphic Hermitian line bundle, i.e., we equip L with a complex structure J_L , a connection ∇ , and a Hermitian metric h . In a local frame e_L it is defined by $\nabla e_L = \alpha \otimes e_L$. We consider several types of compatibility conditions between this data:

- An h -connection ∇_h is one compatible with h . In a unitary frame, the connection 1-form is $i\mathbb{R}$ valued and is denoted by $i\alpha$. We denote the space of h -compatible connections by \mathcal{A}_h .
- Or a J_L -compatible connection. In a holomorphic frame e_L the connection 1-form α is of type $(1, 0)$. We denote the space of J_L -compatible connections by $\mathcal{A}_\mathbb{C}$.
- The unique Chern connection $\alpha_{J_L, h}$ which is compatible with both J_L, h .

This data induces:

- The Hermitian metric h induces the principal bundle of h -unitary frames $P_h = \{(z, \lambda) \in L^* : |\lambda|_h = 1\}$.
- An h -compatible connection $\nabla \in \mathcal{A}_h$ induces a real 1-form α on P_h .
- Connections $\mathcal{A}_\mathbb{C}$ determine complex-valued 1-forms on L^* .

The connection 1-form in the frame e_L is given by

$$\nabla e_L = \alpha \otimes e_L.$$

We denote the $(1, 0)$ resp. $(0, 1)$ parts of ∇ by $\nabla^{1,0}$, resp. $\nabla^{0,1}$.

Suppose that $\nabla \in \mathcal{A}_\mathbb{C}$. Then if $s = fe$ with e a local holomorphic frame,

$$\begin{cases} \nabla^{(1,0)}(fe) = (\partial f + \alpha f) \otimes e, \\ \nabla^{(0,1)}(fe) = \bar{\partial} f \otimes e. \end{cases}$$

The holomorphic line bundle L also has a natural Cauchy–Riemann operator,

$$\bar{\partial}_L : C^\infty(M, L) \rightarrow C_1^\infty(M, L).$$

In a local holomorphic frame e , we write a smooth section $s = fe$ and then

$$\bar{\partial}_L s = \bar{\partial}f \otimes e.$$

It is well-defined since if e' is another holomorphic frame and $e = ge'$, then $s = fge'$ and $\bar{\partial}_L s = \bar{\partial}f \otimes ge' = \bar{\partial}f \otimes e$.

The Chern connection ∇ associated to the Hermitian metric h is the unique metric connection

$$\nabla : C^\infty(X, L) \rightarrow C_1^\infty(M, X \otimes T^*X)$$

whose connection 1-form in a holomorphic frame e_L has type $(1, 0)$. The connection 1-form is given by $\nabla e_L = \alpha \otimes e_L$ with $\alpha = \partial \log |h|$.

The metric g^* is a Hermitian metric on K . Any Hermitian metric h on a line bundle L induces metrics $h^m = e^{-m\phi}$ on the tensor powers L^m in the local frame e_L^m . The Hermitian metric and complex structure determine a Chern connection $\partial \log h$ whose curvature 2-form Θ_h is given locally by

$$\Theta_h = -\partial\bar{\partial} \log \|e_L\|_h^2,$$

and we say that (L, h) is positive if the (real) 2-form $\frac{\sqrt{-1}}{2}\Theta_h$ is positive.

2.3. Curvature form

Given a connection ∇ on L and a vector field V on X , the covariant derivative of a section s is defined by $\nabla_V s = \langle \nabla s, V \rangle$. The curvature is the 2-form Ω^∇ defined by $\Omega^\nabla(V, W) = [\nabla_V, \nabla_W] - \nabla_{[V, W]}$. If e_L is a local frame and $\nabla e_L = \alpha \otimes e_L$ then $\Omega^\nabla = d\alpha$.

2.4. Examples

- Let F^*X be the unit co-frame bundle X , consisting of orthonormal frames of T^*X . Then $S^m F^*X$ is the bundle of real m -differentials, i.e., homogeneous polynomials of degree m in dx, dy or $dz, d\bar{z}$.
- When X is given a complex structure, we may decompose $T^*X \otimes \mathbb{C} = T^{*(1,0)} \oplus T^{*(0,1)}$ into co-vectors $f dz$ of type $(1, 0)$ and $g d\bar{z}$ of type $(0, 1)$. The holomorphic co-tangent bundle is usually denoted by $K = K_X = T^{*(1,0)}$ and is called the canonical bundle. Its tensor powers K^m are bundles of differentials of type $f(dz)^m$ with $f(z, \bar{z})$ a smooth function.

- When the genus is 0, i.e., $X = S^2$, K_X is a negative line bundle and has no holomorphic sections. The associated circle bundle of frames is $SO(3) \cong \mathbb{R}P^3 = S^3 / \pm 1$.
- When the genus is 1, then K_X and T^*X are trivial and $F_h \cong \mathbb{T}^2 \times S^1$.
- When the genus is ≥ 1 we may twist L by a flat line bundle. This is not particularly relevant for this article except that we usually ignore this additional degree of freedom. There is an ample line bundle $L \rightarrow \mathbb{T}^2$ whose holomorphic sections are theta functions. The associated principal S^1 bundle is the reduced Heisenberg group, the quotient of the simply connected Heisenberg group by the integer lattice.
- When the genus is ≥ 2 then $X = \mathbb{H}^2 / \Gamma$ where $\Gamma \subset \text{PSL}(2, \mathbb{R})$. The associated S^1 bundle is $SL_2(\mathbb{R}) / \Gamma$. K_X is ample and for m large there are many holomorphic sections of K_X^m . This is only significant in this article when we discuss splitting eigenspaces.

2.5. Canonical bundle and m -differentials

Let $K \rightarrow X$ denote the canonical bundle $T^{*(1,0)}X$ of $(1, 0)$ forms fdz . Up to twisting by a flat line bundle, it is the unique ample line bundle on X . Hence there exists a Hermitian metric h on K with curvature form $i\partial\bar{\partial} \log h = \omega$. This should be distinguished from the curvature of g , which is given by

$$dd^c \log g^{1\bar{1}} = K\omega_h, \text{ where } K \text{ is the scalar curvature.}$$

In terms of the Hermitian metric $h = e^{-\phi_0}$ on K , $|dz|_g = e^{-\phi_0} = g^{1\bar{1}}$. Also,

$$\partial_z \log \omega_0 = \partial_z \log(1 - \Delta_0 \phi), \quad \omega_\phi = (\omega_0 + dd^c \phi) = ((1 - \Delta_0)\phi)\omega_0.$$

We now regard $g(X, Y) = \omega(X, JY)$ as a Kähler metric. The co-metric $*$ defines metric coefficients on $T^*X \otimes \mathbb{C}$ by extending g^* by complex linearity and induces the Hermitian metric,

$$\|dz\|_{g^*} = g^{1\bar{1}},$$

on K_X . The curvature $(1, 1)$ form is therefore $\partial\bar{\partial} \log g^{1\bar{1}}$.

The associated $(1, 1)$ form ω_h is positive if the genus is ≥ 2 and if g is a metric of negative curvature K . It is negative if the genus is 0 and the metric g is of positive curvature. When the genus is 0 there do not exist Hermitian metrics with strictly positive or negative curvature forms.

When $\dim X = 2$ we write the area form as $dA_g = \sqrt{g}dx$ or as ω_g . The metric g induces metrics g on $TX, T^*X, T^{*(1,0)}X = K_X$ and on powers such as K^m . On K^m , the Hermitian metric induced by g is $\|dz\|_g^2 = e^{-\phi} = g^{1\bar{1}}$ so $\phi = -\log g^{1\bar{1}}$. The Chern connection on K is the same as the Riemannian connection.

Consider the complex line vector bundles $T_X^{1,0}$ and (TX, J) . They are isomorphic under the map

$$\xi : T_X \rightarrow T_X^{1,0}, \quad v \rightarrow \frac{1}{2}(v - iJv).$$

LEMMA 2.1. — *Let (X, g) be a Kähler manifold. Under the isomorphism $\xi \in T_X \rightarrow T_X^{1,0}$, the Chern connection D on the holomorphic tangent bundle $T^{1,0}$ is the Levi-Civita connection ∇ .*

2.6. Orthonormal frame bundles and m -differentials

If (X, g) is a Riemannian surface, and F is the $SO(2)$ -bundle of orthonormal frames of T^*X , then g determines a Riemannian connection on F . Similarly, if we fix a complex structure and define the principal S^1 bundle $F_h \subset TX$ associated to $K_X = T^{*(1,0)}$, then g determines a Hermitian metric on K_X . We first discuss the real geometry and then the complex geometry.

The metric $g = \sum_{i,j=1}^2 g_{ij}dx_i \otimes dx_j$ on TX induces a co-metric g^* on T^*X , usually denoted by raised indices. It then induces metrics $g^{\otimes n}$ on powers $S^m T^*X$.

There always exists a basis of *basic* or horizontal 1-forms at a frame (μ_1, μ_2) such that

$$(\omega_j)_{x, \mu_1, \mu_2}(v) = \mu_j(\pi_*v).$$

The Riemannian connection 1-form α is defined by the equations,

$$\begin{cases} \langle \alpha, \frac{\partial}{\partial \theta} \rangle = 1, \\ d\omega_1 = \alpha \wedge \omega_2, \\ d\omega_2 = -\alpha \wedge \omega_1, \\ d\alpha = K\omega_1 \wedge \omega_2, \end{cases}$$

where K is the scalar curvature. Dually, there exist vector fields ξ_1, ξ_2 so that

$$\begin{cases} [\frac{\partial}{\partial \theta}, \xi_1] = \xi_2 \\ [\frac{\partial}{\partial \theta}, \xi_2] = -\xi_1, \\ [\xi_1, \xi_2] = K \frac{\partial}{\partial \theta} \end{cases}$$

Then define

$$E^\pm = \frac{1}{2}(\xi_1 \mp \xi_2).$$

Then,

$$\begin{cases} [\frac{\partial}{\partial\theta}, E^+] = E^+ \\ [\frac{\partial}{\partial\theta}, E^-] = -E^-, \\ [E^+, E^-] = \frac{i}{2}K \frac{\partial}{\partial\theta} \end{cases}$$

In the frame $\{\xi_1, \xi_2, \frac{\partial}{\partial\theta}\}$ the volume form is $\omega_1 \wedge \omega_2 \wedge \alpha$. The vector fields ξ_1, ξ_2 are also of unit length since they are defined to be horizontal lifts of a unit frame.

2.7. Hilbert spaces of sections

Let $(L, h) \rightarrow X$ be a Hermitian holomorphic line bundle. We thus have a pair of metrics, h resp. g (with Kähler form ω_ϕ) on L resp. TX .

To each pair (h, g) of metrics we associate Hilbert space inner products $\text{Hilb}_m(h, g)$ on sections $s \in L^2_{m\psi}(X, L^m)$ of the form

$$\|s\|_{h^m}^2 := \int_X |s(z)|_{h^m}^2 \omega_g,$$

where $|s(z)|_{h^m}^2$ is the pointwise Hermitian norm-squared of the section s in the metric h^m . In a local holomorphic frame e_L , we write

$$\|e_L\|_h^2 = e^{-\psi}.$$

In local coordinates z and the local frame e_L^m of L^m , we may write $s = f e_L^m$ and then

$$|s(z)|_{h^m}^2 = |f(z)|^2 e^{-m\psi(z)}.$$

Henceforth we write

$$\|f e_L^m\|_{h^m}^2 := \int_X |f(z)|^2 e^{-m\psi} \omega_\phi.$$

In the special case where $L = K_X$, we may use the frame dz in a local holomorphic coordinate z . In the local frames $(dz)^m$ of K^m we may write sections as $s = f(dz)^m$ and then $|s(z)|_{h^m}^2 = |f(z)|^2 e^{-m\psi(z)}$ and then,

$$\|f(dz)^m\|_{h^m}^2 := \int_X |f(z)|^2 e^{-m\psi} dV_g,$$

where $dV_g = \omega_\phi$ is the area form of g .

3. Bochner Laplacians on line bundles

In this section, we give explicit local formulae for Bochner Laplacians. Bochner Laplacians $\nabla_{h,g}^* \nabla$ on $L^2(X, L)$ equipped with the data

$$(g, h, J, J_L, \nabla),$$

where $(L, h) \rightarrow X$ is a Hermitian holomorphic line bundle, g is a metric on X , ∇ is a connection on L . In a local frame e_L of L , with $s = fe_L$, the inner product $\text{Hilb}(g, h)$ on $L^2(X, L)$ takes the form,

$$\|s\|_{\text{Hilb}(g,h)}^2 = \int_X |f|^2 e^{-\psi} dV_g, \quad (\text{where } \|e_L(z)\|_h^2 = e^{-\psi(z)}).$$

The inner product on $L^2(X, L \otimes T^*X)$ has the form,

$$\|s \otimes \eta\|_{\text{Hilb}(g,h)}^2 = \int_X |f|^2 \|\eta\|_g^2 e^{-\psi} dV_g, \quad (\text{where } \|e_L(z)\|_h^2 = e^{-\psi(z)}).$$

With no loss of generality, we fix J on X and assume that (g, J, ω) is a Kähler metric with $g(X, Y) = \omega(X, JY)$. Then $g(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}) = 0 = g(\frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial \bar{z}}) = 0$. There is only one metric coefficient, $g^{1\bar{1}} = G(dz, d\bar{z})$. It is a Hermitian metric on $T^{1,0}X$ and is compatible with J .⁽¹⁾ We also denote the Riemannian volume form by $dV_g = \omega = dd^c \log g^{1\bar{1}}$.

Remark 3.1. — Notational remark: We use G rather than g^{-1} or g^* for the dual co-metric on 1-forms, because it is a convenient notation for later variations.

The Bochner Laplacian is the Laplacian on $L^2(X, L)$ determined by the quadratic form,

$$(3.1) \quad q_{g,h,\nabla}(s) = \int_X |\nabla s|_{h \otimes g}^2 dV_g = \langle \nabla_{g,h}^* \nabla s, s \rangle_{\text{Hilb}(g,h)}.$$

Throughout we assume that g is J -compatible. In a local frame e_L of L , with $s = fe_L$, $\nabla(fe_L) = (df + f\alpha) \otimes e_L$ and with $\|e_L(z)\|_h^2 = e^{-\psi(x)}$, and the quadratic form is given by

$$q_{g,h,\nabla}(fe_L) = \int_X |df + f\alpha|_g^2 e^{-\psi} dV_g.$$

The adjoints are taken with respect to the volume form $e^{-\psi} dV_g$.

We give local formulae for $\nabla_{h,g}^* \nabla$ under several assumptions on ∇ and in correspondingly adapted frames (equivalently, choosing a gauge for ∇):

⁽¹⁾ Although $\dim M = 2$, we use the notation dV_g and the term “volume form” to avoid clashing with the notation A for connections and area form.

- (i) ∇ is h -compatible (see Section 3.1); in this case, we compute in a local unitary frame. Fixing h is equivalent to fixing the principal S^1 bundle $P_h \rightarrow X$, and varying the connection 1-forms $\alpha \in \mathcal{A}_h$ on P_h (with fixed g).
- (ii) $\nabla \in \mathcal{A}_{\mathbb{C}}$ is compatible with a fixed complex structure J_L on L (see Section 3.2); in this case we compute in a local holomorphic frame. In the next section we fix J_L and vary h (with fixed g).
- (iii) ∇ is compatible with both (h, J_L) , hence is the Chern connection; see Section 3.3. The $(1, 0)$ -part of the connection has the form $\alpha = \partial \log h$ and is parameterized by the Hermitian metric h on L ; in the next section we consider its variation with h (with fixed g).
- (iv) When $L = K_X$ is the canonical line bundle, we let h be the Hermitian metric induced by g and let ∇ be the Levi-Civita connection. This is a special case of an h -compatible connection but is special because h is induced by g . Moreover, the Riemannian connection w.r.t. g is the Chern connection for the Hermitian metric g . In a holomorphic frame dz the connection form is $\partial \log g^{1\bar{1}} = \partial \log G(dz, d\bar{z})$. Also, $dV_g = g^{1\bar{1}} dz \wedge d\bar{z}$. In the next section, we vary this connection by varying g on X .

There exist many formulae for Bochner Laplacians in the literature (see for instance [5]), but they often make assumptions on the compatibility of the connection with other data (the Hermitian metric or complex structure) and we need explicit dependence on the compatibility conditions so that we can perturb some of the data while holding others fixed. We therefore go through the calculations with explicit assumptions on the compatibility of ∇ with the data (g, h, ∇, J, J_L) .

We also recall the general identities, $d^*(f\alpha) = -*d*(f\alpha) = -*d(f*\alpha) = -*df \wedge *\alpha + fd^*\alpha$. Note that $G(\eta, \zeta)\omega = \eta \wedge *\zeta$. Hence, $-*df \wedge *\alpha = -*G(df, \alpha)\omega = -G(df, \alpha)$ since $*\omega = 1$.

3.1. Calculation in a unitary frame

In this section we assume that ∇ is compatible with h . We recall from Section 2 that on a Hermitian line bundle (L, h) , the set \mathcal{A}_h of connections on L which are compatible with the Hermitian metric is the affine space $\{A_\alpha = A_0 + \alpha : \alpha \in \Omega^1(X)\}$ where A_0 is a fixed background connection and $\Omega^1(X)$ are the real 1-forms on X . The Hermitian metric determines the principal $U(1)$ bundle P_h of unitary frames of L and as before A_1 determines

a connection 1-form α_1 on P_h . On the base X , the connection 1-form is $i\mathbb{R}$ -valued in a unitary frame and we write it as $i\alpha$ with a real-valued α .

PROPOSITION 3.2. — *Let (X, g) be a Riemannian manifold and let (L, h) be a Hermitian line bundle with h -compatible connection ∇ . Let $\nabla(fe_L) = (df + if\alpha) \otimes e_L$ with $\alpha \in \mathbb{R}$ in a unitary frame e_L . Then,*

$$\nabla^* \nabla(fe_L) = (-\Delta_g f - 2iG(df, \alpha) + ifd_g^* \alpha + G(\alpha, \alpha)f) e_L.$$

where Δ_g is the scalar Laplace operator.

Proof. — In a unitary frame, $|e_L|_h^2 = e^{-\psi} = 1$ and this factor drops out. We leave it in until the last step for purposes of later comparison to other frames. Since $\nabla(fe_L) = df \otimes e_L + if\alpha e_L$, and by (3.1),

$$q_{g,h,\nabla}(s) = \int_X |df + if\alpha|_g^2 e^{-\psi} dV_g.$$

Note that

$$|df + if\alpha|_g^2 = G(df + if\alpha, \overline{df + if\alpha}) = |df|_g^2 + 2\Re \bar{f}G(df, -i\alpha) + G(\alpha, \alpha)|f|^2$$

is the Hermitian norm-squared, so

$$\begin{aligned} q_{g,h,\nabla}(s) &= \int_X (|df|_g^2 + 2\Re \bar{f}G(df, -i\alpha) + G(\alpha, \alpha)|f|^2) e^{-\psi} dV_g \\ &= \int_X (-2\Re \bar{f}G(df, i\alpha) + G(\alpha, \alpha)|f|^2) e^{-\psi} dV_g - \int_X \bar{f}d_g^*(e^{-\psi} df) dV_g \\ &= \int_X (-2\Re \bar{f}G(df, i\alpha) + G(\alpha, \alpha)|f|^2) dV_g - \int_X \bar{f}(d_g^* df) dV_g, \end{aligned}$$

where in the last line we use that $\psi = 0$ in a unitary frame. Since α is real-valued,

$$-2\Re \bar{f}G(df, i\alpha) = -i(\bar{f}G(df, \alpha) - fG(d\bar{f}, \alpha)).$$

Recall that $d_g^*(f\alpha) = -*d*(f\alpha) = -G(df, \alpha) + fd_g^*\alpha$. Replacing $iG(d\bar{f}, \alpha)$ by $-i(d_g^*(\bar{f}\alpha) - \bar{f}d_g^*\alpha)$ and integrating the d_g^* by parts gives

$$\begin{aligned} \int_X (-i\bar{f}G(df, \alpha) - i\bar{f}G(df, \alpha) + i|f|^2 d_g^* \alpha + G(\alpha, \alpha)|f|^2) dV_g \\ - \int_X \bar{f}(d_g^* df) dV_g, \end{aligned}$$

Thus, we get

$$\nabla^* \nabla f = -\Delta_g f - 2iG(df, \alpha) + ifd_g^* \alpha + G(\alpha, \alpha)f. \quad \square$$

3.2. Holomorphic line bundles: J_L -compatible connections.

In this section we give a local formula for the Bochner Laplacian when $L \rightarrow X$ is a holomorphic line bundle and ∇ is compatible with the complex structure. Thus, complex structures J on X and J_L on L are fixed. In a holomorphic frame, $\|e_L\| = e^{-\psi} \neq 1$ and $\nabla e_L = \alpha \otimes e_L$, where α is of type $(1, 0)$. We write $\nabla = \partial^\nabla + \bar{\partial}^\nabla$ for the decomposition of a connection into its $(1, 0)$ resp. $(0, 1)$ parts, with $\partial^\nabla = \nabla^{(1,0)}$, $\bar{\partial}^\nabla = \nabla^{(0,1)}$.

The Bochner–Kodaira identity relates $\nabla^* \nabla$ to $\bar{\partial}_L^* \bar{\partial}_L$, where

$$\begin{cases} \partial_L(fe_L) := \nabla^{(1,0)}(fe_L) = (\partial f + \alpha f) \otimes e_L, \\ \bar{\partial}_L(fe_L) := \nabla^{(0,1)}(fe_L) = \bar{\partial} f \otimes e_L. \end{cases}$$

The analogue of Proposition 3.2 is

PROPOSITION 3.3. — *If ∇ is compatible with J_L with connection 1-form $\nabla e_L = \alpha \otimes e_L$ with α of type $(1, 0)$ in the holomorphic frame e_L , then $\nabla^* \nabla(fe_L) = (-\Delta_g f + G(d\psi + \alpha, df) - fG(d\psi, \alpha) + f d^* \alpha + G(\alpha, \bar{\alpha})f) e_L$.*

Proof. — The proof is similar to that of Proposition 3.2, with two differences:

- (i) we use a holomorphic frame rather than a unitary frame and $|e_L|_h^2 = e^{-\psi}$ is not equal to 1;
- (ii) α is of type $(1, 0)$ rather being $i\mathbb{R}$ -valued.

Note that

$$|df + f\alpha|_g^2 = G(df + f\alpha, \overline{df + f\alpha}) = |df|_g^2 + 2\Re \bar{f}G(df, \bar{\alpha}) + G(\alpha, \bar{\alpha})|f|^2$$

By (3.1), and integrating by parts the $|df|_g^2$ term, and with $|\cdot|^2$ denoting the Hermitian metric, we get

$$\begin{aligned} & q_{g,h,\nabla}(s) \\ &= \int_X |df + f\alpha|_g^2 e^{-\psi} dV_g \\ &= \int_X (|df|_g^2 + 2\Re \bar{f}G(df, \bar{\alpha}) + G(\alpha, \bar{\alpha})|f|^2) e^{-\psi} dV_g \\ &= \int_X (2\Re \bar{f}G(df, \bar{\alpha}) + G(\alpha, \bar{\alpha})|f|^2) e^{-\psi} dV_g - \int_X \bar{f}d_g^*(e^{-\psi} df) dV_g \\ &= \int_X (2\Re \bar{f}G(df, \bar{\alpha}) + G(\alpha, \bar{\alpha})|f|^2) e^{-\psi} dV_g - \int_X \bar{f}(d_g^* df - G(d\psi, df)) e^{-\psi} dV_g \end{aligned}$$

Further,

$$2\Re \bar{f}G(df, \bar{\alpha}) = \bar{f}G(df, \bar{\alpha}) + fG(d\bar{f}, \alpha).$$

We simplify the $fG(d\bar{f}, \alpha)$ term using that $d^*(\bar{f}\alpha) = - * d * (\bar{f}\alpha) = -G(d\bar{f}, \alpha) + f d^* \alpha$ so that $G(d\bar{f}, \alpha) = -d^*(\bar{f}\alpha) + \bar{f} d^* \alpha$. Integrating the d^* by parts gives

$$\begin{aligned} & \int_X (2\Re \bar{f} G(df, \bar{\alpha}) + G(\alpha, \bar{\alpha})|f|^2) e^{-\psi} dV_g \\ &= \int_X (\bar{f} G(df, \bar{\alpha}) + f G(d\bar{f}, \alpha) + G(\alpha, \bar{\alpha})|f|^2) e^{-\psi} \\ &= \int_X (\bar{f} G(df, \bar{\alpha}) - f d^*(\bar{f}\alpha) + |f|^2 d^* \alpha + G(\alpha, \bar{\alpha})|f|^2) e^{-\psi} dV_g \\ &= \int_X (\bar{f} G(df, \bar{\alpha}) - \bar{f} G(df, \alpha) - |f|^2 (G(d\psi, \alpha) \\ & \qquad \qquad \qquad + |f|^2 d^* \alpha + G(\alpha, \bar{\alpha})|f|^2) e^{-\psi} dV_g. \end{aligned}$$

Combining with the term $-\int_X \bar{f} (d_g^* df - G(d\psi, df)) e^{-\psi} dV_g$, we get $\nabla^* \nabla f = -\Delta_g f + G(d\psi + \bar{\alpha} - \alpha, df) - f G(d\psi, \alpha) + f d^* \alpha + G(\alpha, \bar{\alpha})f$. \square

3.3. Chern connection

In this section we assume ∇ is both h -compatible and J_L -compatible, i.e., that it is the Chern connection with connection 1-form $\partial\psi$. One can then compute $\nabla^* \nabla$ using the relation

$$(3.2) \qquad \nabla^* \nabla = 2\bar{\partial}^{\nabla^*} \bar{\partial}^{\nabla} + i * d\alpha$$

between the Kodaira and Bochner Laplacians.

Note that $d\psi$ is real and $\partial\psi = \alpha$, so $d\psi = \alpha + \bar{\alpha}$ and $G(d\psi + \alpha - \bar{\alpha}, df) = G(\alpha, \bar{\partial}f)$ above. Also, $G(d\psi, \alpha) = G(\bar{\alpha}, \alpha)$, so the terms $-fG(d\psi, \alpha) + G(\alpha, \bar{\alpha})f$ cancel and from the preceding Proposition we get

$$\nabla^* \nabla (fe_L) = -\Delta_g f + G(\alpha, \bar{\partial}f) + f d_g^* \alpha.$$

We now prove this directly.

PROPOSITION 3.4. — *Let ∇ be the Chern connection for (L, h) . Then,*

$$\nabla^* \nabla (fe_L) = (-\Delta_g f + G(\partial\psi, \bar{\partial}f) + f(i * \Omega^\nabla))e_L.$$

Proof. — Using (3.2) and $\bar{\partial}^{\nabla}(fe_L) = (\bar{\partial}f \otimes e_L)$, we have

$$\langle \bar{\partial}_L^* \bar{\partial}_L s, s \rangle_h := \langle \bar{\partial}_L s, \bar{\partial}_L s \rangle_{h \otimes g}.$$

We rewrite $G(\bar{\partial}f, \partial f)$ term using that $d^*(\bar{f}\partial f) = - * d * (\bar{f}\partial f) = -G(d\bar{f}, \partial f) + \bar{f}d^*\partial f$ so that $G(\bar{\partial}f, \partial f) = -d^*(\bar{f}\partial f) + \bar{f}d^*\partial f$. Integrating the d^* by parts gives

$$\begin{aligned} \int_X G(\bar{\partial}f, \partial f)e^{-\psi}\omega_g &= \int_X (-d^*(\bar{f}\partial f) + \bar{f}d^*\partial f)e^{-\psi}\omega_g \\ &= \int_X (G(\partial f, d\psi) - \Delta f)\bar{f}e^{-\psi}\omega_g. \end{aligned}$$

Adding the curvature term adds $f(i * \Omega^\nabla)$. □

Remark 3.5. — Proposition 3.4 and (3.2) are consistent by the following calculation: If $\alpha = \partial\psi$ is a Chern connection 1-form, then

$$(d_g^*\alpha)\omega_g = \Omega^\nabla = i\bar{\partial}\bar{\partial}\psi = (\Delta_g\psi)\omega_g, \quad d_g^*\alpha = \Delta_g\psi.$$

Indeed, in terms of the Hermitian inner product,

$$\begin{aligned} \langle d_g^*\alpha, f \rangle_{L^2} &= \langle \alpha, df \rangle_{L^2} = \langle \alpha, \partial f \rangle_{L^2} = \int_X \partial\alpha \cdot \bar{\partial}\bar{f}\omega_g = i \int_X \frac{\partial\psi}{\partial z} \frac{\bar{\partial}\bar{f}}{\partial\bar{z}} dzd\bar{z} \\ &= -i \int_X \frac{\partial^2\psi}{\partial z\partial\bar{z}} \bar{f} dzd\bar{z} \\ &= -\langle \Delta\psi, f \rangle_{L^2}. \end{aligned}$$

3.4. Canonical bundle: $L = K$ and ∇ is the Riemannian connection

Let z be a local holomorphic coordinate and let dz be the associated section of K . Differentials of type $(dz)^m$ are sections of K^m , the m -th power of the canonical bundle. The Riemannian metric on X induces a Hermitian metric h^m on K^m , namely $|dz|_h = |dz|_g$ where g is the cometric. $dz = dx + i dy$ and at $x + iy$, $|dz| = y$.

The metric g on TX endows a Hermitian metric g^* on K and the associated Riemannian connection ∇_g is the Chern connection with connection 1-form $\alpha = -\partial \log g^{1\bar{1}}$ in the frame dz . For simplicity of notation we write $\phi = -\log g^{1\bar{1}}$. It induces connections and Hermitian metrics on K^m with connection 1-forms $m\alpha$. The associated Bochner Laplacian $\nabla_{m,g}^* \nabla_{m,g}$ on K^m corresponds to the quadratic form

$$\begin{aligned} q_{m,g}(s) &= \int_X |\nabla_{m,g}s|_{m,g}^2 \omega_g = \int_X |df + mf\partial\phi|^2 \|(dz)^m\|_{g^m}^2 \omega_g \\ &= \int_X |df + mf\partial\phi|^2 e^{-m\phi} \omega_g. \end{aligned}$$

Note that $\omega_g = \frac{i}{2}g_{1\bar{1}}dz \wedge d\bar{z}$ and the Laplacian on scalar functions is given by $\Delta_0 f = g^{1\bar{1}} \frac{\partial^2 f}{\partial z \partial \bar{z}}$.

PROPOSITION 3.6. — *Let ∇_m be the Chern connection for (K^m, g^{*m}) . Then,*

$$\nabla_m^* \nabla_m (f(dz)^m) = (-\Delta_g f + mG(\partial\phi, \bar{\partial}f) + mKf)(dz)^m.$$

Proof. — This follows from Proposition 3.4. We give a direct proof. By the Bochner–Kodaira formula (3.2), it suffices to prove

$$(3.3) \quad \bar{\partial}_m^* \bar{\partial}_m (f(dz)^m) = \left(g^{1\bar{1}} \frac{\partial^2 f}{\partial z \partial \bar{z}} - m \left(\frac{\partial f}{\partial \bar{z}} g^{1\bar{1}} \right) \frac{\partial \phi}{\partial z} \right) (dz)^m,$$

where $\phi(z) = -\log |dz|_g = -\log g^{1,\bar{1}}$.

As above, we calculate the adjoint to be

$$\begin{aligned} \bar{\partial}_m^* (f(dz)^m \otimes d\bar{z}) &= \left(e^{m\phi} \omega_h^{-1} \frac{\partial}{\partial z} \left(f(z) g^{1\bar{1}} e^{-m\phi} \omega_h \right) \right) \\ &= g^{1\bar{1}} \frac{\partial f}{\partial z} - m f(z) g^{1\bar{1}} \frac{\partial \phi}{\partial z}. \end{aligned}$$

It follows that

$$\begin{aligned} \bar{\partial}_m^* \bar{\partial}_m (f(dz)^m) &= \bar{\partial}_m^* \left(\frac{\partial f}{\partial \bar{z}} (dz)^m \otimes (d\bar{z}) \right) \\ &= \left(e^{m\phi} \omega_g^{-1} \frac{\partial}{\partial z} \left(\frac{\partial f}{\partial \bar{z}} g^{1\bar{1}} e^{-m\phi} \omega_g \right) \right) \\ &= g^{1\bar{1}} \frac{\partial^2 f}{\partial z \partial \bar{z}} - m \left(\frac{\partial f}{\partial \bar{z}} g^{1\bar{1}} \right) \frac{\partial \phi}{\partial z} + \frac{\partial f}{\partial \bar{z}} g^{1\bar{1}} \frac{\partial \log g^{1\bar{1}}}{\partial z} + \left(\frac{\partial f}{\partial \bar{z}} g^{1\bar{1}} \right) \frac{\partial \log \omega_g}{\partial z} \\ &= g^{1\bar{1}} \frac{\partial^2 f}{\partial z \partial \bar{z}} - m \left(\frac{\partial f}{\partial \bar{z}} g^{1\bar{1}} \right) \frac{\partial \phi}{\partial z}, \end{aligned}$$

where we used $\frac{\partial \log g^{1\bar{1}}}{\partial z} + \frac{\partial \log \omega_h}{\partial z} = 0$. □

4. Perturbation theory and genericity

In this section we prove generic properties of the eigenvalues and eigensections of Bochner Laplacians $\nabla_{g,h}^* \nabla$ on complex holomorphic Hermitian line bundles $(L, h) \rightarrow X$. Our ultimate goal is to deduce generic properties of Kaluza–Klein Laplacians on the principal $U(1)$ frame bundles $P_h \rightarrow X$ associated to h . First we discuss generic properties of Bochner Laplacians

on the line bundles and then we draw conclusions for the Kaluza–Klein Laplacians. We prove that for generic data (g, h, ∇) (with fixed (J_L, J)), eigenvalues of Bochner Laplacians $\nabla_{g,h}^* \nabla$ are simple (multiplicity one) and all eigensections intersect the zero section transversally (i.e., have 0 as a regular value). This immediately implies that for the associated Kaluza–Klein Laplacians Δ_G on P_h , all joint eigenfunctions of the $U(1)$ action and Δ_G have simple joint spectrum and have 0 as a regular value. In Section 4.6, we discuss the multiplicity of the spectrum of Δ_G , hence proving a part of Theorem 1.5.

The main result of this section is:

THEOREM 4.1. — *For generic “admissible data” described below, and for every m , the spectrum of each Bochner Laplacians $\nabla_{g,h}^* \nabla$ on $C^k(X, L^m)$ is simple and all of its eigensections have zero as a regular value. Moreover, if we lift sections to equivariant eigenfunctions ϕ , then $\Re\phi$ and $\Im\phi$ have zero as a regular value.*

The generic admissible data is of the following kinds:

- (1) We fix h, g and vary the connection ∇ in \mathcal{A}_h . Fixing h is equivalent to fixing the principal $U(1)$ bundle $P_h \rightarrow X$, and varying the connection 1-forms.
- (2) We fix (J, J_L, g) and vary both h and ∇ , assuming that $\nabla \in \mathcal{A}_C$ is compatible with J_L on L but not necessarily with h .
- (3) We fix (g, J_L, J) and vary (h, ∇) assuming that ∇ is compatible with both (h, J_L) , hence is the Chern connection of (L, J_L, h) .
- (4) We fix $L = K^m$ and also fix J and vary g in the conformal class associated to J . We assume that h is the Hermitian metric induced by g and that ∇ is the Levi-Civita connection.

The proofs in each of the cases are given in separate sections.

Note that the functions relevant to this article are smooth sections of a complex line bundle L , and may locally be represented as complex valued functions u . We will prove that $u : M \rightarrow \mathbb{C}$ has zero as a regular value, i.e., that $du_p = d\Re u + id\Im u$ is surjective. It follows that $\Re u, \Im u$ are independent and nowhere vanishing on their zero sets, and that each has zero as a regular value

4.1. The Uhlenbeck framework

To study generic properties of the spectrum, we follow [20] and work with C^r spaces of metrics and connections. We use the following notation:

- We denote by $\mathcal{G}^r(X)$ the Banach space of C^r metrics on X . Since X is a surface and we usually fix the complex structure J , we only work with C^r metrics in the associated conformal class $\text{Conf}(J)$ and represent them in the usual Weyl gauge $g = e^\rho g_0$ relative to a fixed background metric $g_0 \in \text{Conf}(J)$. Thus, we may identify $\mathcal{G}^r(X) \cong C^r(X)$. We may also fix the area of the metrics with no loss of generality and then $\text{Conf}(J)$ may be identified with the space \mathcal{K}_ω of Kähler metrics on X in a fixed cohomology class. This is simply a different choice of gauge in which we write the Kähler forms as $\omega_\phi = \omega_0 + i\partial\bar{\partial}\phi$ and use the potentials ϕ rather than the Weyl gauge u to parameterize metrics.
- We denote by $\mathcal{H}^r(L)$ the Banach space of C^r Hermitian metrics on L . Once we fix a local frame e_L we may identify $h \in \mathcal{H}^r(L)$ with the function ψ such that $\|e_L(z)\|_h^2 = e^{-\psi(z)}$, and $\mathcal{H}^r(L)$ is then equivalent to $C^r(X)$ except of course that the identification is frame dependent and the frame is only local (defined on the complement of a smooth closed curve in X , e.g.).
- We denote by $\mathcal{A}^r(L)$ the space of connections with C^r connection forms. As before, we also denote by \mathcal{A}_h^r , resp. $\mathcal{A}_\mathbb{C}^r$, the h -compatible (resp. J_L -compatible) C^r connections.
- We denote by $C^r(X, L)$ the C^r sections of L . We also denote by $H^s(X, L)$ the Sobolev space of sections with s derivatives in L^2 .

We define

$$\Phi_L : \mathcal{G}^r(X) \times \mathcal{H}^r(L) \times \mathcal{A}^r(L) \times H^2(X, L) \times \mathbb{C} \rightarrow L^2(X, L),$$

by

$$\Phi_L(g, h, \nabla, s, \lambda) = (\nabla_{g,h}^* \nabla - \lambda)s.$$

Here, the eigenvalue parameter λ in the domain is allowed to be complex even though at zeros of Φ_L it is always real. This does not change the arguments in [20] but is needed so that λs spans the eigenspace when s is an eigensection. In [20] the eigenfunctions were real-valued, so this issue did not arise.

Recall that a linear map between Banach spaces is *Fredholm* if it has closed image and finite dimensional kernel and cokernel. The index of a Fredholm operator is the difference of the dimensions of its kernel and cokernel. A nonlinear map $\Phi : N \rightarrow Y$ of Banach manifolds is Fredholm if its derivative $D\Phi_n$ is Fredholm for every $n \in N$.

Our first goal, roughly speaking, is to prove that Φ is a Fredholm map of index 0, i.e., to prove surjectivity of the differentials $D_2\Phi$ from tangent

spaces of

$$Q = \{(g, h, \nabla, s, \lambda) : \Phi_L(g, h, \nabla, \lambda) = 0\}$$

to $L^2(X, L)$. It is sufficient to pick the relevant types of frames and calculate the Bochner Laplacians in the frame as in Section 3.

Regarding the surjectivity, we need to prove density of the image and that the image is closed. Some care needs to be taken because sections of complex line bundles are “vector-valued”, i.e., have two real components. As explained in [6], there are pitfalls to avoid when generalizing the arguments of [20] to the vector-valued case. But sections of line bundles are locally complex-valued functions and are essentially scalar functions, albeit with scalars in \mathbb{C} .

4.2. Uhlenbeck’s argument

We briefly review Uhlenbeck’s proof that for generic metrics on compact C^r Riemannian manifolds, all eigenvalues are simple and all eigenfunctions have 0 as a regular value.

Her framework is quite general and therefore uses the notation B for the relevant space of metrics or other geometric data, and L_b for the Laplacian associated to b . The relevant functions are denoted by u and the space of such functions on a manifold M is denoted by $C^k(M)$, even though they could be sections of a bundle over M . Then define

$$\Phi(u, \lambda, b) = (L_b + \lambda)u,$$

and put

- $Q := \{(u, \lambda, g) \in C^k(X) \times \mathbb{R}_+ \times B : \Phi(u, \lambda, b) = 0\}$.
- $\alpha : Q \times M \rightarrow \mathbb{C} : \alpha(u, \lambda, b, x) = u(x)$.
- $\beta : Q \times M \rightarrow T^*M : \beta(u, \lambda, b, x) = \nabla u(x)$.

Then,

$$T_{u,\lambda,b}Q = \left\{ (v, \eta, s) \in H^{1,0}(X) \times \mathbb{C} \times T_bB : \begin{aligned} \int_X uv dV_g &= 0, \\ (L_b + \lambda)v + \eta u + D_2\phi s &= 0 \end{aligned} \right\}.$$

We often write

$$v = \dot{u}, \quad \eta = \dot{\lambda}, \quad D_2(\Phi)s = \lambda \dot{\Delta}u, \quad (\Delta + \lambda)\dot{u} + (\dot{\Delta} + \dot{\lambda})u = 0.$$

Further, let $D_1\alpha$ denote the derivative of α along Q . Then,

$$D_{(u,\lambda,b)}\alpha(v, 0, c, 0) = v(x) = \dot{u}(x).$$

Also define J to be the image of $D_2\Phi$,

$$J = \text{Im } D_2\Phi_{(u,\lambda,b)} = \{\dot{\Delta}u : \dot{\Delta} \text{ is a variation of } \Delta \text{ along a curve of metrics}\}.$$

We use the following “abstract genericity” result of [20, Theorem 1] and [20, Lemmas 2.7–2.8].

THEOREM 4.2. — *Assume that Φ is C^k and has zero as a regular value. Then the eigenspaces of L_b are one-dimensional. If additionally, $\alpha : Q \times M \rightarrow \mathbb{C}$ has zero as a regular value, then additionally*

$$\{b \in B : \text{the eigenfunctions of } L_b \text{ have } 0 \text{ as a regular value}\}$$

is residual in B .

The key proposition is the following procedure for verifying the first hypothesis of Theorem 4.2. (see [20, Proposition 2.10]).

PROPOSITION 4.3. — *Let $J = \text{im } D_2\Phi$ and assume that for $W \in L^1(M)$ and $W \in C^2(M - \{y\})$, the property $\int_M W(x)j(x)d\mu_x = 0$ for all $j \in J$ implies $W = 0$. Then ϕ is C^k and has zero as a regular value.*

For the sake of completeness, we briefly review the main steps in proving Theorem 4.2: The main input are two transversality theorems. The first is: Let $\phi : H \times B \rightarrow E$ be a C^k map where H, B, E are Banach manifolds. If 0 is a regular value of ϕ and $\phi_b(\cdot) := \phi(\cdot, b)$ is a Fredholm map of index $< k$, then the set $\{b \in B : 0 \text{ is a regular value of } \phi_b\}$ is residual in B .

The second statement follows from [20, Lemma 2.7]: Let $\pi : Q \rightarrow B$ be a C^k Fredholm map of index 0. Then if $f : Q \times X \rightarrow Y$ is a C^k map for k sufficiently large and if f is transverse to Y' then $\{b \in B : f_b := f|_{\pi^{-1}(b)} \text{ is transverse to } Y'\}$ is residual in B . Let

$$\alpha : f^{-1}(Y') \rightarrow B \text{ be } \alpha : f^{-1}(Y') \subset Q \rightarrow B.$$

LEMMA 4.4. — *The eigenfunctions of L_b have zero as a regular value if b is a regular value of π and if 0 is a regular value of $\alpha|_{\pi^{-1}(b)} \times M := \alpha_b$.*

Eigenfunctions and eigenvalues move continuously under perturbations of the operator. So it is easy to show that the set of metrics with for which the j th eigenvalue is simple is open. The difficulty is to prove that this set is dense.

To prove the first statement in Theorem 4.1 we need to verify the hypotheses of Theorem 4.2 and therefore need to prove Proposition 4.3, i.e., to determine the range of $D_2\phi$.

PROPOSITION 4.5. — *For each of the admissible types of perturbation, $D_2\Phi_m$ is surjective from $T_{(u,\lambda,\phi)}Q_m \rightarrow C^{k-2}$.*

4.3. Base metric variations

In this section we fix (h, ∇, J, J_L) and vary only $g = e^\rho$. Equivalently, we consider Kaluza–Klein metrics on a fixed $U(1)$ bundle $P_h \rightarrow M$ with a fixed connection α and vary the base metric g .

PROPOSITION 4.6. — *Suppose that $(L, h, J) \rightarrow X$ is a Hermitian holomorphic line bundle with h -compatible connection ∇ . Let $\nabla(fe_L) = (df + if\alpha) \otimes e_L$ with $\alpha \in \mathbb{R}$ in a unitary frame e_L . Then for generic Riemannian metrics $g = e^\rho g_0$ in the conformal class of J , all of the eigenvalues of $\nabla_{g,h}^* \nabla_h$ are simple and all of the eigensections have 0 as a regular value.*

Proof. — By Proposition 3.2,

$$\nabla^* \nabla(fe_L) = (-\Delta_g f - 2iG(df, \alpha) + ifd_g^* \alpha + G(\alpha, \alpha)f) e_L.$$

where $\Delta_g f$ is the scalar Laplace operator, where $g = e^\rho$.

Taking the variation δ with respect to ρ (and designating the variation with a dot),

$$\delta \nabla^* \nabla(fe_L) = (-\dot{\Delta}_g f - 2i\dot{G}(df, \alpha) + if\dot{d}_g^* \alpha + \dot{G}(\alpha, \alpha)f) e_L.$$

But each term is conformal to that of g with conformal factor $e^{-\rho}$. Hence

$$\begin{aligned} \delta \nabla^* \nabla(fe_L) &= -\frac{1}{2} \rho \Delta f(x) - 2i\rho G(df, \alpha) + (i\rho d_g^* \alpha + \rho G(\theta, \theta)) fe_L \\ &= \rho \nabla^* \nabla(fe_L). \end{aligned}$$

If $\nabla^* \nabla(fe_L)u = -\lambda u$ then

$$\delta_g \nabla^* \nabla(fe_L)u(x) = -\lambda \rho u.$$

To prove that the image of $D_2\Phi_L$ is dense we argue by contradiction and suppose that there exists $W \in L^2(X, L)$ such that

$$\int_X \delta_g \nabla^* \nabla h(fe_L, W(z))_h dV_g = 0$$

for all ρ . But this implies that $\int_X \rho(fe_L, W)_h dV_g = 0$ for all ρ , then $W = 0$. Write $W = Fe_L$ so that the integral becomes, $\int_X \rho f \bar{F} e^{-\psi} dV_g = 0$ for all ρ . This is only possible if $f \bar{F} e^{-\psi} = 0$. But f and $e^{-\psi}$ can only vanish on a set of measure zero, so $F \equiv 0$ almost everywhere. The image is closed because $\nabla^* \nabla$ is a Fredholm operator. □

4.4. Varying the Hermitian connection

In this section we fix g, h and vary $\nabla \in \mathcal{A}_h$. In the application to Kaluza–Klein metrics, P_h is fixed and the base and vertical metrics are fixed and only the splitting into horizontal and vertical is varied.

We recall from Section 2 that some of the variations are “trivial”, i.e., are within a gauge equivalence class. Bochner Laplacians with gauge-equivalent connections are unitary equivalent by a gauge transformation, i.e., they have the same spectrum and their eigensections are related by a gauge transformation. Viewed in terms of line bundles over X , gauge equivalent connection forms are connection forms of a single connection in two different unitary frames, hence differ by a gauge transformation $e^{i\theta} \in \text{Map}(X, S^1)$ taking $e_L \rightarrow e^{i\theta}e_L$. The connection 1-form then changes by $id\theta \in \Omega^1(X, \mathbb{R})$. A unique representative of a gauge equivalence class is defined by the Coulomb gauge $d^*a = 0$.

PROPOSITION 4.7. — *Suppose that $(L, h, J) \rightarrow X$ is a Hermitian holomorphic line bundle and let $\nabla \in \mathcal{A}_h$ be given by $\nabla(fe_L) = (df + if\alpha) \otimes e_L$ with $\alpha \in \mathbb{R}$ in a unitary frame e_L . Suppose that L is non-flat, or if it is flat, that $d\alpha \neq 0$. Then for generic gauge equivalence classes $\alpha \in \mathcal{A}_h \cong \Omega^1(X)$, all of the eigenvalues of $\nabla_{g,h}^* \nabla_h$ are simple and all of the eigensections have 0 as a regular value.*

Proof. — Again by Proposition 3.2,

$$\nabla^* \nabla(fe_L) = (-\Delta_g f - 2iG(df, \alpha) + ifd_g^* \alpha + G(\alpha, \alpha)f) e_L,$$

where $\Delta_g f$ is the scalar Laplace operator. Taking the variation with respect to α gives,

$$\delta \nabla^* \nabla(fe_L) = (-2iG(df, \dot{\alpha}) + ifd_g^* \dot{\alpha} + 2G(\dot{\alpha}, \alpha)f) e_L.$$

If the image is not dense, there exists $W = Fe_L$ so that

$$\int_X (-2iG(df, \dot{\alpha}) + ifd_g^* \dot{\alpha} + 2G(\dot{\alpha}, \alpha)f) \bar{F} e^{-\psi} dV_g = 0,$$

for all $\dot{\alpha} \in \Omega^1(X)$. We integrate d_g^* by parts to get,

$$\int_X ((-2iG(df, \dot{\alpha}) + 2G(\dot{\alpha}, \alpha)f) \bar{F} + iG(\dot{\alpha}, d(f\bar{F}))) e^{-\psi} dV_g = 0.$$

We may assume that the frame e_L is unitary so that $\psi = 0$. If $\beta \in \Omega^1(M, \mathbb{C})$ and $\int_X G(\beta, \nu) dV_g = 0$ for all $\nu \in \Omega^1(M, \mathbb{R})$, then $\beta = 0$. Indeed, we may consider ν of the types $\nu = \nu_1 dx, \nu_2 dy$ separately to get orthogonality of

the components β_j with ν_j . This reduces matters to the fact that if u, v are complex-valued and $\int uv dV_g = 0$ for all v , then $u \equiv 0$. We conclude that

$$(-2idf + 2\alpha f)\bar{F} + id(f\bar{F}) = 0 \iff (-idf + 2\alpha f)\bar{F} + ifd\bar{F} = 0.$$

On any open set U where $f, F \neq 0$ we may divide by $if\bar{F}$ and write the solution as,

$$\frac{d\bar{F}}{\bar{F}} = -\left(-\frac{df}{f} + 2i\alpha\right).$$

This implies that

$$d \log \frac{\bar{F}}{f} = -2i\alpha \implies d\alpha = 0$$

on a dense open set and since $\alpha \in C^\infty$, it is everywhere closed and hence the curvature of (L, h) is zero. This is impossible unless L is a topologically trivial line bundle, and the contradiction implies that $F \equiv 0$ except when $d\alpha = 0$. □

4.5. Proof of Theorem 4.1

As mentioned in the previous section, eigenfunctions move continuously under perturbations of the operator. So it is easy to show that the set of metrics with for which the j th eigenvalue is simple is open. The difficulty is to prove that this set is dense.

To prove the first statement in Theorem 4.1 we need to verify the hypotheses of Theorem 4.2 and therefore need to prove Proposition 4.3, i.e., to determine the range J of $D_2\phi$.

To complete the proof of Theorem 4.1 it suffices to prove:

PROPOSITION 4.8. — *For each m , $D_1\alpha_m$ is surjective to \mathbb{C} .*

Proof. — Let $G_{m,\lambda}(z, w)$ be the kernel of the Green’s function $G_{m,\lambda} : [\ker(D_m + \lambda)]^\perp \rightarrow [\ker(D_m + \lambda)]^\perp$ for $D_m(g) + \lambda$ for a given background metric g . As above, one may use the Hermitian metric h on K or the associated Kähler metric $g = \omega_J$ as the parameter space of metrics.

We need to show that for each $x \in M$,

$$\alpha_m : Q \times \{x\} \rightarrow \mathbb{C} : \alpha(u, \lambda, g, x) = u(x)$$

has $0 \in \mathbb{C}$ as a regular value, i.e., that

$$D_1\alpha(\cdot, x) : T_{u,\lambda,b}(Q) \rightarrow \mathbb{C}, \quad D_1\alpha(\cdot, x)_{(u,\lambda,g)}(\delta u(x), 0, c, 0) = \delta u(x)$$

is surjective to \mathbb{C} , where D_1 is the differential along Q with $x \in M$ held fixed. Since x is fixed we may use a local coordinate z and frame $(dz)^m$

as above and identify local sections of K^m with complex-valued functions $u : U \rightarrow \mathbb{C}$, where U is an open set containing x .

The constraint equation for $(v, 0, c, 0) \in T_{(u, \lambda, b)}^* Q$ is

$$(D_m(g) + \lambda)v + (\dot{D}_m(g) + \dot{\lambda})u = 0,$$

and we can solve for $v \perp \ker(D_m(g) + \lambda)$ as

$$v(x) = - \int_M G_{m, \lambda}(x, y) \Pi_\lambda^\perp [(\dot{D}_m + \dot{\lambda})u](y) dV(y).$$

By Proposition 4.5, the range of $D_2\Phi$, i.e., the set of functions $[(\dot{D}_m + \dot{\lambda})u]$, spans L_0^2 . Therefore, the image $\Pi_\lambda^\perp [(\dot{D}_m + \dot{\lambda})u]$ spans $[\ker(D_m + \lambda)]^\perp$. It follows that the possible values of v are all functions of the form,

$$v(x) = \int_M G_{m, \lambda}(x, y) f(y) dV(y),$$

where $f \perp \ker(D_m(g) + \lambda)$. Thus, $D_1\alpha$ is surjective to \mathbb{C} unless for all $j \perp \ker(D_m(g) + \lambda)$, either the real or imaginary parts of

$$G_{m, \lambda}(j)(x) = \int_M G_{m, \lambda}(x, y) j(y) dV(y)$$

vanish (or both) for every such j .

Since $j = [D_m(g) + \lambda]f$ where $\int f = 0$ we would get the absurd conclusion that

$$f(x) = 0, \quad \forall f \perp \ker(D_m(g) + \lambda).$$

Equivalently,

$$G_{m, \lambda}(x, y) + u_\lambda(x) = 0.$$

This is not possible and the contradiction ends the proof. \square

4.6. Multiplicity of the spectrum of Δ_g

We begin by observing that $\lambda_{m, j} = \lambda_{-m, j}$ and that $\phi_{-m, j} = \overline{\phi_{m, j}}$. Then any real eigenfunction which is a linear combination of $\phi_{m, j}$ and $\phi_{-m, j}$ is

$$\Re(e^{i\theta_0} \phi_{m, j})$$

for some constant θ_0 . In local coordinates, $\phi_{m, j}$ is $\tilde{\phi}(z)e^{im\theta}$, and therefore we see that

$$\Re(e^{i\theta_0} \phi_{m, j}) = \Re(T_{\theta_0} \phi_{m, j}) = T_{\theta_0} \Re(\phi_{m, j}).$$

For m_1 and m_2 such that $|m_1| \neq |m_2|$, we argue that $\lambda_{m_1, j_1} \neq \lambda_{m_2, j_2}$ is satisfied for an open dense subset of metric G . This immediately implies the first, third, and the fourth statement of Theorem 1.5. Note that the

eigenvalue moves continuously with respect to G . So it is sufficient to prove that

LEMMA 4.9. — *Let $P \rightarrow X$ be a non-trivial principal S^1 bundle. Fix integers m_1 and m_2 such that $|m_1| \neq |m_2|$. Among all S^1 -invariant metric G on P , G satisfying $\lambda_{m_1, j_1} \neq \lambda_{m_2, j_2}$ is dense.*

Proof. — The deformation of the base of the Kaluza–Klein metric does not touch the vertical operator $\frac{\partial}{\partial \theta}$ and therefore the first order perturbation equations for infinitesimal deformations of the base metric g gives,

$$(4.1) \quad (\dot{\Delta}_H + \dot{\lambda}_{m,j})\phi_{m,j} = (\Delta_H + \lambda_{m,j} + m^2)\dot{\phi}_{m,j}$$

Taking the inner product with $\phi_{m,j}$ gives

$$(4.2) \quad -\dot{\lambda}_{m,j} = \langle \dot{\Delta}_H \phi_{m,j}, \phi_{m,j} \rangle.$$

If there exist weights $m_1 \neq m_2$ for which we cannot split the eigenvalue $\lambda_{m_1, j_1} = \lambda_{m_2, j_2}$ then for all infinitesimal base perturbations ρ we get

$$(4.3) \quad \dot{\lambda}_{m_1, j_1} = \dot{\lambda}_{m_2, j_2} \iff \langle \dot{\Delta}_H \phi_{m_1, j_1}, \phi_{m_1, j_1} \rangle = \langle \dot{\Delta}_H \phi_{m_2, j_2}, \phi_{m_2, j_2} \rangle.$$

Write $\phi_{m,j} = f_{m,j}(dz)^m$. Differentiation of the eigenvalue equation therefore gives the well-known formula

$$\langle \dot{D}_{m_1} f_{m_1, j_1}, f_{m_1, j_1} \rangle = \langle \dot{D}_{m_2} g_{m_2, j_2}, g_{m_2, j_2} \rangle$$

for every variation of g , where the inner product is that of g_0 .

Recall from previous section that $\dot{\Delta}_H = \rho \Delta_H$. Because $-\Delta_H \phi_{m,j} = (\lambda_{m,j} - m^2)\phi_{m,j}$ we have for any $\rho \in C^\infty(X)$,

$$\begin{aligned} (\lambda_{m_1, j_1} - m_1^2) \int_X \rho |f_{m_1, j_1}|^2 e^{-m_1 \phi} dA_0 \\ = (\lambda_{m_2, j_2} - m_2^2) \int_X \rho |f_{m_2, j_2}|^2 e^{-m_2 \phi} dA_0. \end{aligned}$$

Thus,

$$(\lambda_{m_1, j_1} - m_1^2) |f_{m_1, j_1}|^2 e^{-m_1 \phi} = (\lambda_{m_2, j_2} - m_2^2) |f_{m_2, j_2}|^2 e^{-m_2 \phi}.$$

Integrating both sides against dV_g and using that both eigenfunctions are L^2 normalized gives

$$(\lambda_{m_1, j_1} - m_1^2) = (\lambda_{m_2, j_2} - m_2^2), \quad \text{i.e., } |m_1| = |m_2|. \quad \square$$

5. Local structure of eigensections at zeros

To study the nodal sets of real and imaginary parts of Kaluza–Klein Laplacians, we first study the zeros of the associated sections of the line bundles. For simplicity of exposition, we assume that $L = K$ and describe the zero sets of eigen- m -differentials. Essentially the same discussion is valid for other line bundles.

We follow the notation and terminology in the theory of holomorphic quadratic differentials, even though our eigendifferentials are C^∞ , usually not holomorphic and of general weight m . Following a standard terminology for quadratic differentials, we call a point z such that $f_{m,r_j}(z) \neq 0$ a “regular point” and a point where $f_{m,r_j}(z) = 0$ a “critical point” or a “singular point”.

After the first version of this article was written, we located some recent articles generalizing the geometric properties of quadratic differentials on Riemann surfaces to C^∞ higher order differentials [7, 1] and to other line bundles. We now use the terminology and results of these articles but have retained some from our first version since it is important for us to lift to P_h .

5.1. Trajectories of eigen-differentials.

The real and imaginary parts of the eigendifferentials $\omega_{m,j} = f_{m,j}(z)(dz)^m$ are called binary differentials of degree m and the equation for the zero set of $\Im\omega_{m,j}$ is called a binary differential equation of degree m [7]. It is traditional to consider the nodal set $\Im f_{m,j}(z)(dz)^m = 0$. If there exist exactly m solutions at a regular point where $\omega_{m,j}(z) \neq 0$ then $\omega_{m,j}$ is called *totally real* in [7]. Our m -differentials are of a special type since they are real and imaginary parts of $f_{m,j}(z)(dz)^m$ and therefore only have terms of the form $(dz)^m$ or $(d\bar{z})^m$. The following is the key input into Proposition 1.8.

LEMMA 5.1. — $\Im f_{m,j}(dz)^m$ is a totally real m -differential. At a regular point z , there exist m distinct solutions v of $\Im f_{m,j}(dz)^m(v) = 0$ in $T_z X$.

Proof. — If $v = (\cos \phi, \sin \phi)$, then in the notation of (1.4), the equation is

$$(a_{m,j}\mathfrak{c}_m - b_{m,j}\mathfrak{s}_m)(\cos \theta, \sin \theta) = 0.$$

Here $\mathfrak{c}_m = \Re(\cos \theta + i \sin \theta)^m = \cos m\theta$, and the equation is

$$a_{m,j}(z) \cos m\theta - b_{m,j}(z) \sin m\theta = 0 \iff \tan m\theta = \frac{a_{m,j}}{b_{m,j}},$$

where $a_{m,j}, b_{m,j} \in \mathbb{R}$ and where we assume with no loss of generality that $b_{m,j} \neq 0$. Since the principal branch of $\tan^{-1} : \mathbb{R} \rightarrow (-\pi/2, \pi/2)$ is one-to-one, there exists precisely one solution θ_0 of $\tan m\theta = \frac{a_{m,j}}{b_{m,j}}$ with $m\theta \in (-\pi/2, \pi/2)$, namely the principal branch of $\tan^{-1}(\frac{1}{m} \frac{a_{m,j}}{b_{m,j}})$. Since $\tan \theta$ is π -periodic, $\tan m\theta$ is $\frac{\pi}{m}$ -periodic, and the full set of solutions is $\theta_0 + k\frac{\pi}{m}$ with $k = 0, \dots, m - 1$. \square

The kernel of $\Im f_{m,j}(dz)^m$ defines a smooth m -valued distribution on X with singularities where $\omega_{m,j} = f_{m,j}(dz)^m = 0$. The m line fields defines a web of m transverse singular foliations, whose leaves are called the *trajectories*:

DEFINITION 5.2. — *The trajectories of the m differential $f_{m,j}(dz)^m$ are the integral curves of the kernel of $\Im f_{m,j}(dz)^m$, i.e. the trajectories are the (smooth) curves $\gamma(t)$ in X along which $\Im \phi_{m,j}(\gamma(t), \gamma'(t)) = 0$.*

Remark 5.3. — A trajectory in this sense of this article is called a “horizontal trajectory” in [19, Definition 5.5.3]. They are illustrated in [19, Section 7] for holomorphic quadratic differentials. Illustrations of webs for higher order real differentials can be found in [7].

Trajectories downstairs on X lift to P_h by their tangent vectors. A trajectory $\gamma_{z_0, \theta_0}(t)$ downstairs is a smooth curve along which

$$\Im(\phi_{m,j}(\gamma_{z_0, \theta_0}(t), \dot{\gamma}_{z_0, \theta_0}(t))) = 0.$$

It lifts to a smooth curve $(\gamma_{z_0, \theta_0}(t), \dot{\gamma}_{z_0, \theta_0}(t))$ in the nodal set upstairs. Since $d\pi$ is an isomorphism, the trajectories are special curves on the nodal set $\Im \phi_{m,j} = 0$.

5.2. Non-degenerate singular points

The structure of the trajectories through a singular (zero) may be complicated in general if no conditions are placed on the degeneracy of the zeros. The purpose of Theorem 4.1 is to allow us to assume that the zeros are of first order, so that they are isolated and non-degenerate.

The structure of the trajectories of a totally real m -differential near an isolated singular point is discussed in [7]. As with vector fields, the key topological invariant of the singular point is its *index*.

DEFINITION 5.4. — *The index of a singular point z_0 where*

$$f_{m,j}(z_0)(dz)^m = 0$$

is related to the degree of the circle map defined by

$$\delta(t) = z_0 + re^{it} \rightarrow \frac{f_{m,j}(\delta(t))}{|f_{m,j}(\delta(t))|}$$

on a small circle around z_0 to S^1 by

$$\text{ind}(z_0) = \frac{\pm 1}{m} \deg \frac{f_{m,j}(\delta(t))}{|f_{m,j}(\delta(t))|}.$$

Equivalently, in a small circle C around z_0 , choose a unit vector $X(0) \in \ker \Im f_{m,j}(dz)^m|_{C(0)}$ where $C(t) : [0, 2\pi] \rightarrow X$ is a constant speed parametrization of C and let $\ell = L(C)$ be its length. Let $X(t)$ be a smooth extension of $X(0)$ along $C(t)$. After a complete turn, $X(2\pi)$ must be one of the $2m$ solutions of $\omega(X) = 0$. After $2m$ turns $X(2m\ell) = 0$. Let $\theta(t)$ be a smooth determination of the angle between the tangent line to C and $X(t)$. Then $\theta(2m\ell)$ and $\theta(0)$ differ by an integer multiple of 2π . The index of z_0 is defined by

$$\text{ind}(\omega, z_0) = \frac{\theta(2\pi m\ell) - \theta(0)}{4\pi m}.$$

Thus, the index has the form $\frac{s}{2m}$ with $s \in \mathbb{Z}$. The following Lemma shows that singular points must exist when the genus of X is non-zero.

LEMMA 5.5. — *If $f_{m,j}(dz)^m$ has isolated non-degenerate zeros, then the sum of the indices of the zeros is the Chern class of K_X^m .*

LEMMA 5.6. — *If z_0 is a non-degenerate singular point (zero of order 1) of $f_{m,j}(dz)^m$, then $\text{ind}(\omega_{m,j}, z_0) = \frac{\pm 1}{m}$.*

Proof. — This follows from the fact that $f_{m,j}$ is linear in this case and hence the degree of the associated circle map is ± 1 . \square

PROPOSITION 5.7. — *For a generic Riemannian metric g on X , all singular points of all eigendifferentials of $\nabla^* \nabla$ on K^m have index $\frac{\pm 1}{m}$ for all $m \neq 0$.*

Proof. — It is part of Theorem 4.1, all singular points are non-degenerate. To prove this it suffices to show that the coefficients $f_{m,j}$ are linear near each singular point. This follows from the Bers local formula for eigensections around a zero. We use Proposition 3.3 to Taylor expand the operator

$$D_m = \nabla_m^* \nabla_m = 2g^{1\bar{1}} \frac{\partial^2}{\partial z \partial \bar{z}} f - 2m \left[\frac{\partial f}{\partial \bar{z}} g^{1\bar{1}} \right] \frac{\partial \phi}{\partial z} + Kf,$$

around a nodal point.

Let p be a nodal point of $f_{m,j}$. We Taylor expand the coefficients in Kähler normal coordinates for (J, g) in a disc $z \in D(p, r)$ to get

- $g^{1\bar{1}} = 1 + K(p)|z|^2 + \dots;$
- $\frac{\partial\phi}{\partial z} = \frac{\partial\phi}{\partial z}(p) + \omega_p\bar{z} + \dots = \bar{z} + \dots.$

Thus, the *osculating* constant coefficient operator is

$$D_m^p f = 2 \frac{\partial^2}{\partial z \partial \bar{z}} f + K(p) f.$$

Let \mathcal{P}_k denote homogeneous polynomials of degree k in z, \bar{z} . It is better to arrange the terms of the Taylor expansion of D_m at p into terms

$$D_m = L_{-2} + L_{-1} + L_0 + L_1 + \dots$$

where $L_j : \mathcal{P}_k \rightarrow \mathcal{P}_{k+j}$.

Thus, $L_{-2} = \frac{\partial^2}{\partial z \partial \bar{z}}, L_{-1} = 0, L_0 = K_{[2]} \frac{\partial^2}{\partial z \partial \bar{z}} - 2m[\frac{\partial f}{\partial \bar{z}}]\bar{z} + K(p)$ etc. Note that $L_{-1} = 0$ because $dg^{1\bar{1}}(p) = 0$ and $\partial\phi(p) = 0$, so neither the second or first derivative terms contribute at this order.

Also expand

$$f(z) = f_1(p)z + f_{\bar{1}}(p)\bar{z} + f_{11}(p)z^2 + f_{1\bar{1}}(p)|z|^2 + f_{\bar{1}\bar{1}}(\bar{z})^2 + \dots + f_{[k]} + \dots,$$

where $f_{[k]} \in \mathcal{P}_k$ is homogeneous of order k .

The following is the generalization of the Bers local expansion theorem to complex line bundles.

LEMMA 5.8. — *Let z_0 be a zero of $f_{m,j}(dz)^m$. The first non-zero homogeneous term $f_{[n]}$ of the Taylor expansion of an eigenfunction is a harmonic homogeneous polynomial. If the order of vanishing is n , $f_{[n]}(z) = a\Re z^n + ib\Im z^n$. In particular, at a non-degenerate zero, the first homogeneous term is $f_{m,j} = a\Re z + ib\Im z$.*

Proof. — It is evident that $L_{-2} = \frac{\partial^2}{\partial z \partial \bar{z}} : \mathcal{P}_k \rightarrow \mathcal{P}_{k-2}$. If $f_{[k]}$ is the term of lowest degree in the expansion of f then $\frac{\partial^2}{\partial z \partial \bar{z}} f_{[k]} = 0$, i.e., $f_{[k]}$ is a homogeneous harmonic polynomial. In real dimension 2 the only possibilities are linear combinations of the real and imaginary parts of z^k . By a well-known argument, the nodal set of the real and imaginary parts of f are topologically equivalent to those of the leading order homogeneous term. □

This completes the proof of the Proposition 5.7. □

6. Adapted Kaluza–Klein metrics

All of the Kaluza–Klein metrics are Riemannian metrics on principal S^1 bundles $P_h \rightarrow X$ associated to C^∞ complex Hermitian line bundles $L \rightarrow X$. Given P_h we recover L as an associated line bundle. Let (X, g) be any Riemannian surface. We denote the genus of X by g . Let $(L, h) \rightarrow X$ be any complex line bundle with Hermitian metric h . Associated to L is the $U(1)$ bundle P_h of orthonormal frames. Let $T = \frac{\partial}{\partial \theta}$ generate the $S^1 \cong U(1)$ action. We endow P_h with a connection α , that is, an S^1 invariant 1-form on P_h such that $\alpha(T) = 1$.

The connection defines a splitting

$$T_p P_h = H_p \oplus V_p$$

into horizontal and vertical spaces. The vertical space is given by orbits of the S^1 action. The horizontal space is defined by $H_p = \ker \alpha$ and is isomorphic under $d\pi_p$ to $T_z X$ where $\pi(p) = z$.

DEFINITION 6.1. — *The Kaluza–Klein metric on P_h is the S^1 -invariant metric G such that the horizontal space $H_p := \ker d\alpha$ is isometric to $T_{\pi(p)} X$, so that $V = \mathbb{R} \frac{\partial}{\partial \theta}$ is orthogonal to H and is invariant under the natural S^1 action and so that the fiber is a unit speed geodesic.*

A Kaluza–Klein metric on the principal S^1 bundle P_h is thus determined by the pair (g, α) where g is a metric on X and where α is a connection 1-form on F . In general, the metric and connection are chosen independently. In Section 2.6 we discuss the orthonormal frame bundle, where α is the Riemannian connection of g .

Given P_h and any character $\chi_m = e^{im\theta}$ of S^1 we obtain associated line bundles (resp, real rank 2 bundles) by

$$L^m = P_h \times_{\chi_m} \mathbb{C}.$$

For purposes of this paper it may be assumed that $m \geq 0$.

We often assume that X is equipped with a complex structure J and that L is a holomorphic line bundle. Let $D_h^* \subset L^*$ be the unit co-disc bundle with respect to h and let $P_h = \partial D_h^*$ be its boundary, an S^1 bundle $\pi : P_h \rightarrow X$.

Remark 6.2. — Not all S^1 invariant metrics on P_h are adapted Kaluza–Klein metrics. It would be interesting to consider more general S^1 -invariant metrics on SX or on other manifolds (of all dimensions), as well as invariant metrics under more general compact Lie groups.

6.1. Geometry and analysis of Kaluza–Klein metrics

We use the term Kaluza–Klein metric or Kaluza–Klein metric in the sense of Definition 6.1 to denote metrics G on the unit tangent bundles $\pi : P_h = SX \rightarrow X$ over surfaces for which the vertical and horizontal spaces are orthogonal and which is invariant under the free $S^1 = SO(2)$ action⁽²⁾. They are special cases of Riemannian submersions with totally geodesic fibers isometric to $\mathbb{R}/2\pi\mathbb{Z}$.

DEFINITION 6.3. — *If H is a compact Lie group, and if $\pi : M \rightarrow B$ is a principal H -bundle with fiber F , then one says that a connection θ is an H -connection for π if the H action preserves the horizontal spaces and preserves the connection 1-form. One says that a metric G is adapted to H if the fibers $\pi^{-1}(b)$ are totally geodesic and isometric to F and such that the horizontal distribution of θ is the orthogonal complement to the vertical.*

The following Lemma gives details on the equivalences of the various conditions and is implicitly contained in [3, Example 2.1] and is proved in [21, Theorem 3.5]. Hence we only sketch the proof.

LEMMA 6.4. — *Suppose that S^1 acts freely on M and that G is an S^1 invariant metric for which all orbits are geodesics isometric to $\mathbb{R}/2\pi\mathbb{Z}$. Then G is a Kaluza–Klein metric and $\pi : M \rightarrow M/S^1$ is a Riemannian submersion with totally geodesic fibers isometric to $\mathbb{R}/2\pi\mathbb{Z}$.*

Proof. — Under the freeness assumption, we have an S^1 bundle $\pi : M \rightarrow X := M/S^1$. Let $V = \mathbb{R} \frac{\partial}{\partial \theta}$ be the vertical space, i.e., the tangent space to the orbits. Let $H_x = V_x^\perp$. The metric G determines a quotient Riemannian metric on X using the isomorphisms $d\pi_x : H_x \rightarrow T_{\pi(x)}X$. By assumption, $|\frac{\partial}{\partial \theta}|_G = 1$ if we identify $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$. The only non-trivial statement is that the orbits are geodesics. Let $\alpha(t)$ denote the orbit of a point x under the S^1 action. Let $v(0)$ be a horizontal vector at $\alpha(0)$. Let $\sigma(t, s)$ be the parallel translation of $\alpha(t)$ along a curve in X with initial tangent vector $d\pi(v(0))$. Then $\frac{\partial}{\partial s}\sigma(t, 0)$ is a horizontal vector field along $\alpha(t)$. The arclength of the curve $t \rightarrow \sigma(s, t)$ is constant in s . The first variation formula implies that $\alpha(t)$ is geodesic. \square

These adapted metrics are special cases of invariant metrics on an S^1 -manifold M . For instance, in the case of the non-free action of S^1 on the standard S^2 by rotations around the x_3 -axis, $|\frac{\partial}{\partial \theta}|_{g_0}$ varies with the orbit,

⁽²⁾ A free action is one for which all isotropy groups are trivial.

and almost no orbits are geodesics. It would be interesting to consider generalizations to all S^1 invariant metrics.

Sections of K^m , i.e., differentials of type $(dz)^m$, lift to the dual line bundle $K^* = T^{(1,0)}$ as equivariant scalar functions $F : K^* \rightarrow \mathbb{C}$ transforming by $e^{im\theta}$ under the S^1 action of rotating a frame. Using the metric g we form the unit tangent bundle $\pi : SX \rightarrow X$. In this section, we review the relevant formulae for lifted operators and review the fact that the Bochner Laplacians are Fourier components of the horizontal Laplacian on SX .

6.2. Lifts to P_h

The natural inner product on $L^2(P_h, dV_G)$ is given by

$$\langle f, f \rangle = \int_{P_h} |f|^2 dV_G.$$

Sections s of L^m naturally lift to L^* and F_h by

$$\hat{s}(z, \lambda) := \lambda(s(z)).$$

It is straightforward to check that the lift of $s \in C(X, L^m)$ satisfies $\hat{s}(r_\theta x) = e^{im\theta} \hat{s}(x)$ and that

$$\int_{F_h} |\hat{s}(x)|^2 dV_G = \int_X \|s(z)\|_{h^m}^2 dV_g.$$

Indeed, if $x = r_\theta \frac{e_{L^*}(z)}{\|e_{L^*}(z)\|}$ then $\hat{s}(x) = e^{im\theta} \|e_L(z)\|_{h^m}^m$.

In the case of $L = K_X$, the lift has the form,

$$f(\widehat{(dz)^m}(Y)) = f(dz(Y))^m.$$

We define a orthonormal frame of T^*X by $\omega_1 = e^{-\phi} dz := \frac{dz}{|dz|_h}$ as above, and let $\|\frac{\partial}{\partial z}\|^{-1} \frac{\partial}{\partial z} = e^\phi \frac{\partial}{\partial z}$ be the dual frame. In local coordinates z, \bar{z} on X and in this local frame we define local coordinates (z, \bar{z}, θ) on SX corresponding to the point $e^{i\theta} e^\phi \frac{\partial}{\partial z}$.

Then $(dz)^m$ lifts to the function,

$$e_m(z, \bar{z}, \theta) = \widehat{(dz)^m} \left(e^{i\theta} e^\phi \frac{\partial}{\partial z} \right) = e^{im\theta} e^{m\phi(z)}.$$

Consequently, the eigendifferential $f_{m,j}(dz)^m$ lifts to

$$\phi_{m,j}(z, \bar{z}, \theta) = f_{m,j}(z) e^{im\theta} e^{m\phi(z)}.$$

In (1.2) we decomposed the lift into real and imaginary parts. We now relate them to the real and imaginary parts of $f_{m,j}$.

If we take the inner product of $u_{m,j}$ and $v_{m,j}$ just along the fiber and use orthogonality of $\cos m\theta, \sin m\theta$ and that $\int_0^{2\pi} (\cos^2 m\theta - \sin^2 m\theta) d\theta = 0$, and then integrate in $dA(z)$ we get

LEMMA 6.5. — $\langle u_{m,j}, v_{m,j} \rangle = 0$.

6.3. Eigenspace decompositions

The Kaluza–Klein Laplacian has the form

$$\Delta_G = \Delta_H + \frac{\partial^2}{\partial\theta^2}, \quad \text{where } \Delta_H = \xi_1^2 + \xi_2^2$$

is the horizontal Laplacian. The fact that the fiber Laplacian is $\frac{\partial^2}{\partial\theta^2}$ reflects the fact that S^1 orbits are geodesics isometric to $\mathbb{R}/2\pi\mathbb{Z}$.

The weight spaces are Δ_H -invariant, i.e., as an unbounded self-adjoint operator,

$$\Delta_H : \mathcal{H}_m \rightarrow \mathcal{H}_m.$$

Under the canonical identification

$$\mathcal{H}_m \cong L^2(X, L^m)$$

using the lifting map and $\Delta_H|_{\mathcal{H}_m} \cong D_m - m^2I$ under the lifting map.

We then consider joint eigenfunctions $\phi_{m,j}$ of the Kaluza–Klein Laplacian Δ_G and of $\frac{\partial}{\partial\theta}$. The commutation relations show that $[\Delta_G, \frac{\partial^2}{\partial\theta^2}] = 0$.

LEMMA 6.6. — *The Bochner Laplacian agrees with the horizontal Laplacian Δ_H . In the above local coordinates and frame,*

$$\nabla_m^* \widehat{\nabla_m(f(dz)^m)} = \Delta_H(\widehat{f(dz)^m}).$$

Note that except for the last identity, these statements are true for any isometric S^1 action, not just for adapted Kaluza–Klein metrics.

6.4. Equivariant decomposition

Since S^1 acts isometrically on (M, G) we may decompose into its weight spaces,

$$L^2(M, dV_G) = \bigoplus_{m \in \mathbb{Z}} \mathcal{H}_m,$$

where

$$\mathcal{H}_m = \{f : M \rightarrow \mathbb{C} : f(e^{i\theta} x) = e^{im\theta} f(x)\}.$$

The weight spaces are Δ_H -invariant, i.e., as an unbounded self-adjoint operator,

$$\Delta_H : \mathcal{H}_m \rightarrow \mathcal{H}_m.$$

The lifting map gives a canonical identification

$$\mathcal{H}_m \cong L^2(X, L^m).$$

7. Connectivity of nodal sets of Kaluza–Klein eigenfunctions

Given the preparations in Section 5, it is now a simple matter to prove Theorem 1.9. The following is an immediate consequence of Lemma 5.1:

LEMMA 7.1. — *If 0 is a regular value, then $\mathcal{N}_{u_{m,j}} \subset SX$ is a singular $2m$ -fold cover of X with blow-down singularities over points where $f_{m,j}(z)(dz)^m = 0$.*

Indeed, the $2m$ zeros of $\Im\omega_{m,j}(v) = 0$ in $S_z X$ give $2m$ points on the fiber $\pi^{-1}(z)$ in P_h . Since locally there exist $2m$ smooth determinations of the zeros, the nodal set is a covering map away from the singular points.

We have separated Lemma 7.1 from further geometric results on the map $\pi : \mathcal{N}_{u_{m,j}} \rightarrow X$ in the next section since it was stated separately from those results in Theorem 1.9.

Remark 7.2. — In the literature, $\pi : \mathcal{N}_{u_{m,j}} \rightarrow X$ is sometimes called a branched cover [1, Section 4], but as J. Y. Welschinger explained to us, the terminology is misleading since smooth branched covers are supposed to have $z^{1/m}$ singularities over the branch points, just as for holomorphic branched covers, while the inverse image of a zero of $f_{m,j}$ is an S^1 orbit and the singularity is blown up. In some sense, π is locally like the projection of a vertical helicoid onto the horizontal plane.

8. Nodal domains of real and imaginary parts

We now give a sketch of the proof of Theorem 1.5. By Proposition 1.8 (and (1.5)), $\mathcal{N}_{\Im\phi_{m,j}} \setminus (\mathcal{N}_{\Im\phi_{m,j}} \cap \Sigma) \rightarrow X \setminus \mathcal{Z}_{f_{m,j}}$ is a $2m$ -sheeted cover. Moreover, $P_h \setminus \Sigma \rightarrow X$ is an S^1 bundle and

$$i(P_h \setminus \Sigma) \setminus \mathcal{N}_{\Im\phi_{m,j}} \rightarrow X$$

is a fiber bundle whose fibers consist of the punctured fibers $\pi^{-1}(z) \setminus \mathcal{N}_{\Im\phi_{m,j}}$. The connected components of each punctured fiber consist of “arcs” along which $\Im\phi_{m,j}$ has a constant sign. We therefore express it as

$$(P_h \setminus \Sigma) \setminus \mathcal{N}_{\Im\phi_{m,j}} = \mathcal{P}_+ \cup \mathcal{P}_-$$

where $\text{sign } \Im\phi_{m,j} = \pm$ in \mathcal{P}_{\pm} . Each $\pi : \mathcal{P}_{\pm} \rightarrow X$ is a fiber bundle whose fiber consists of m arcs of the fibers of $\pi : P_h \rightarrow X$. Since the number of zeros in each regular fiber is $2m$, the number of connected components of \mathcal{P}_{\pm} is $\leq m$. When we take the closure of these sets (i.e., add in the singular fibers, on which $\Im\phi_{m,j} = 0$, the connected components of the closure are the nodal domains. It follows that there are $\leq 2m$ nodal domains. We now argue that the closure of \mathcal{P}_{\pm} is connected, so that there exist exactly 2 nodal domains.

We now use the local analysis in Section 5 of eigendifferentials of generic Bochner Laplacians around their zeros to determine how the sheets are connected at the singular fibers $\mathcal{C}_j = \pi^{-1}(z_j)$, corresponding to singular points (i.e., zeros) of $f_{m,j}(dz)^m$ i.e., we consider the maximal components $\mathcal{P}_{\pm,j}$ of

$$\mathcal{P}_{\pm} \setminus \bigcup_{j=1}^m \mathcal{C}_j = \bigcup_{j=1}^m \mathcal{P}_{\pm,j},$$

in which $\Im\phi_{m,j}$ has a single sign. When we union the left side with $\bigcup_{j=1}^m \mathcal{C}_j$ we glue together some of these domains along intervals of the singular fibers.

The gluing rule for the nodal domains is determined by the gluing rule for the nodal set, since the boundary of the each nodal domain is the nodal set. From the downstairs point of view, the gluing rule is the monodromy of the cover $\mathcal{N}_{u_{j,m,j}} \rightarrow X \setminus Z(\omega_{m,j})$. If we fix a singular point z_0 , then we get a monodromy representation

$$\rho : \pi_1(X \setminus Z(\omega_{m,j})) \rightarrow \text{Aut}(\pi^{-1}(z_0)),$$

determining how the sheets of the nodal set are changed as the point circles around z_0 .

By Proposition 5.7, the index of the singular points z_0 is $\frac{\pm 1}{m}$. In terms of the monodromy, this means precisely that each turn around a circle C enclosing z_0 lifts to an arc from one vector in the fiber to its nearest neighbor with the same sign of $\Re\phi_{m,j}$ (i.e., skipping the neighboring vector of the opposite sign).

It follows that both the + region and – region is connected in P_h . Hence there are just two nodal domains.

8.1. Counting the number of nodal domains

We now give a more detailed presentation.

Let D be an open disc. We first study connectivity of a certain graph that arise from a pair of partitions of D .

Let P and Q be partitions of D , i.e., P (resp. Q) is a collection of disjoint open-sets $\Omega_P(1), \dots, \Omega_P(n_P) \subseteq D$ (resp. $\Omega_Q(1), \dots, \Omega_Q(n_Q) \subseteq D$) such that

$$\overline{\bigcup_{k=1}^{n_P} \Omega_P(k)} = D \text{ (resp. } \overline{\bigcup_{k=1}^{n_Q} \Omega_Q(k)} = D).$$

Let $c_P : P \rightarrow \{0, 1\}$ and $c_Q : P \rightarrow \{0, 1\}$ be colorings of P and Q , and define the inversions of c_P and c_Q by $c'_P = 1 - c_P$ and $c'_Q = 1 - c_Q$.

We now define a graph $G_m(P, Q, c_P, c_Q)$ as follows:

The vertex set is

$$\left\{ \begin{array}{cccc} v_{1,1}, & v_{1,2}, & \cdots & v_{1,n_P}, \\ v_{2,1}, & v_{2,2}, & \cdots & v_{2,n_Q}, \\ v_{3,1}, & v_{3,2}, & \cdots & v_{3,n_P}, \\ v_{4,1}, & v_{4,2}, & \cdots & v_{4,n_Q}, \\ & \vdots & & \\ v_{4m,1}, & v_{4m,2}, & \cdots & v_{4m,n_Q} \end{array} \right\}$$

and edges are

$\{v_{4j,a}, v_{4j+1,b}\}$ such that $\Omega_Q(a) \cap \Omega_P(b) \neq \emptyset$, and $c'_Q(\Omega_Q(a)) = c_P(\Omega_P(b))$,
 $\{v_{4j+1,a}, v_{4j+2,b}\}$ such that $\Omega_P(a) \cap \Omega_Q(b) \neq \emptyset$, and $c_P(\Omega_P(a)) = c_Q(\Omega_Q(b))$,
 $\{v_{4j+2,a}, v_{4j+3,b}\}$ such that $\Omega_Q(a) \cap \Omega_P(b) \neq \emptyset$, and $c_Q(\Omega_Q(a)) = c'_P(\Omega_P(b))$,
 $\{v_{4j+3,a}, v_{4j+4,b}\}$ such that $\Omega_P(a) \cap \Omega_Q(b) \neq \emptyset$, and $c'_P(\Omega_P(a)) = c'_Q(\Omega_Q(b))$
for $j = 0, 1, \dots, m-1$ with the identification $v_{0,a} = v_{4m,a}$.

DEFINITION 8.1. — We say a pair of partitions (P, Q) generic, if

$$D - \left(\bigcup_{k=1}^{n_P} \Omega_P(k) \cup \bigcup_{k=1}^{n_Q} \Omega_Q(k) \right)$$

does not contain a closed curve.

LEMMA 8.2. — For a generic pair of partitions (P, Q) with any given colorings c_P and c_Q , any connected component of $G_m(P, Q, c_P, c_Q)$ contains at least one of the following $2m$ vertices:

$$v_{1,1}, v_{3,1}, \dots, v_{4m-3}, v_{4m-1}.$$

In particular, $G_m(P, Q, c_P, c_Q)$ has at most $2m$ connected components.

Proof. — We first consider the case $m = 1$. To claim $G_1(P, Q, c_P, c_Q)$ has only 2 connected components, it is sufficient to prove that if $c_P(\Omega_P(a_1)) = c_P(\Omega_P(a_2))$, then v_{1,a_1} and v_{1,a_2} are path-connected.

Because (P, Q) is a generic pair, one can find a chain of open-sets

$$\Omega_P(a_1) = \Omega_P(c_1), \Omega_Q(b_1), \Omega_P(c_2), \Omega_Q(b_2), \dots, \Omega_P(c_k) = \Omega_P(a_2)$$

such that two adjacent open-sets have non-trivial intersection.

Observe that if $\Omega_P(c) \cap \Omega_Q(b) \neq \emptyset$, then either

$$\{v_{1,c}, v_{2,b}\}, \text{ or } \{v_{1,c}, v_{4,b}\}$$

is an edge, and likewise either

$$\{v_{3,c}, v_{2,b}\}, \text{ or } \{v_{3,c}, v_{4,b}\}$$

is an edge.

Therefore the above chain of open-sets corresponds to a path connecting v_{1,a_1} with either v_{1,a_2} or v_{3,a_2} . However, from the assumption $c_P(\Omega_P(a_1)) = c_P(\Omega_P(a_2))$, and from the construction of $G_1(P, Q, c_P, c_Q)$, v_{1,a_1} cannot be connected to v_{3,a_2} , hence is connected to v_{1,a_2} .

Now for the rest, note that G_m is an m -covering of G_1 , and because $v_{1,1}$ and $v_{1,3}$ belongs to the different connected components of G_1 , any connected components of G_m must contain at least one vertex of the fiber of $v_{1,1}$ or $v_{1,3}$. □

For a large class of colorings, we can deduce a much stronger result.

LEMMA 8.3. — *Let (P, Q) be a generic pair of partitions. Assume that we are given with a pair of colorings c_P and c_Q :*

There exist four open sets $\Omega_P(a_1), \Omega_P(a_2), \Omega_Q(b_1), \Omega_Q(b_2)$ such that

$$\Omega_P(a_i) \cap \Omega_Q(b_j) \neq \emptyset$$

for $i = 1, 2$ and $j = 1, 2$, and that

$$c_P(\Omega_P(a_1)) + c_P(\Omega_P(a_2)) = c_Q(\Omega_Q(b_1)) + c_Q(\Omega_Q(b_2)) = 1.$$

Then the graph $G_m(P, Q, c_P, c_Q)$ has 2 connected components.

Proof. — Note that any connected component of G_m must contain either one of v_{4j,a_1} or one of v_{4j,a_2} with $j = 1, \dots, m$, because G_1 has only two connected components.

Without loss of generality, assume that

$$c_P(\Omega_P(a_1)) = c_Q(\Omega_Q(b_1)).$$

Then from the construction of the graph and from the assumption of the lemma

$$\begin{aligned} & \{v_{4j,a_1}, v_{4j+1,b_1}\}, & & \{v_{4j+1,b_1}, v_{4j+2,a_2}\}, \\ & \{v_{4j+2,a_2}, v_{4j+3,b_2}\}, & \text{and} & \{v_{4j+3,b_2}, v_{4j+4,a_1}\} \end{aligned}$$

are edges, hence v_{4j,a_1} and v_{4j+4,a_1} are connected. Likewise, v_{4j,a_2} and v_{4j+4,a_2} are connected. Therefore any connected component of G_m must contain either v_{4,a_1} or v_{4,a_2} . \square

8.2. The number of nodal domains of generic eigenfunctions

Let P be a principal S^1 bundle over a connected smooth compact Riemannian surface X with the covering map $\pi : P \rightarrow X$. Let m be a fixed integer, and assume that $\phi \in C^1(M)$ satisfies the following conditions:

CONDITION 8.4. — *For any small open $U \subset X$ such that $\pi^{-1}U \cong U \times S^1$, there exists a local coordinate (x, θ) of $\pi^{-1}U$ such that*

- (1) $\phi(x, \theta) = f(x)e^{im\theta}$,
- (2) the zero set of $\Re f$ (resp. $\Im f$) gives rise to a partition $P = P_U$ (resp. $Q = Q_U$) of U , and
- (3) (P_U, Q_U) is a generic pair of partitions of U .

In this section, we prove the following theorem.

THEOREM 8.5. — *Fix any point $x \in X$ such that $\phi(x, \theta) \neq 0$. Then any nodal domain of $\Re\phi$ has a nonempty intersection with $\pi^{-1}x$. In particular, the number of nodal domains of $\Re\phi$ is $\leq 2m$. Assume further that ϕ has a regular zero. Then the number of nodal domains of $\Re\phi$ is 2.*

We begin with few observations in terms of fixed U and a local coordinate (x, θ) of $\pi^{-1}U$.

PROPOSITION 8.6. — *If $\Re\phi$ is positive on two open sets $U_1 \subset \pi^{-1}U \cap \{\theta = \frac{k\pi}{2m}\}$ and $U_2 \subset \pi^{-1}U \cap \{\theta = \frac{(k+1)\pi}{2m}\}$ for some integer k , and if $\pi U_1 \cap \pi U_2 \neq \emptyset$, then U_1 and U_2 are contained in the same nodal domain of $\Re\phi$.*

Proof. — Let x_0 be a point in the intersection $\pi U_1 \cap \pi U_2$. Then from the equation

$$\Re\phi(x_0, \theta) = \Re f(x_0) \cos(m\theta) + \Im f(x_0) \sin(m\theta),$$

we see that $\Re\phi$ is positive along the curve

$$\left\{ (x_0, \theta) : \frac{k\pi}{2m} \leq \theta \leq \frac{(k+1)\pi}{2m} \right\},$$

which connects U_1 and U_2 . Therefore U_1 and U_2 are contained in the same nodal domain. \square

PROPOSITION 8.7. — Any nodal domain of $\Re\phi|_{\pi^{-1}U}$ must intersect $\pi^{-1}U \cap \{\theta = \frac{k\pi}{2m}\}$ nontrivially for some integer $k \in \mathbb{Z}$.

Proof. — Assume for contradiction that Ω is a nodal domain of $\Re\phi|_{\pi^{-1}U}$ that is contained in

$$\pi^{-1}U \cap \left\{ \frac{k\pi}{2m} < \theta < \frac{(k+1)\pi}{2m} \right\}.$$

From the equation

$$\Re\phi(x, \theta) = \Re f(x) \cos(m\theta) + \Im f(x) \sin(m\theta),$$

we see that for each fixed x , $\Re\phi(x, \theta)$ either vanishes identically or has at most one sign change along the curve

$$\left\{ (x, \theta) : \frac{k\pi}{2m} < \theta < \frac{(k+1)\pi}{2m} \right\}.$$

This implies that if $x \in \pi\Omega$, then

$$\Re\phi\left(x, \frac{k\pi}{2m}\right) = \Re\phi\left(x, \frac{(k+1)\pi}{2m}\right) = 0,$$

which contradicts the assumption that the zero set of $\Re f$ gives rise to a partition of U . \square

From these two propositions, we see that the nodal domains of $\Re\phi|_{\pi^{-1}U}$ can be understood from the nodal domains of the restrictions of $\Re\phi|_{\pi^{-1}U}$ to the $4m$ -hypersurfaces

$$\pi^{-1}U \cap \left\{ \theta = \frac{k\pi}{2m} \right\}, \quad k = 0, 1, 2, \dots, 4m - 1.$$

In particular, if we define c_{P_U} and c_{Q_U} in terms of the sign of $\Re f$ and $\Im f$, then the number of connected components of $G_m(P_U, Q_U, c_{P_U}, c_{Q_U})$ is equal to the number of nodal domains of $\Re\phi|_{\pi^{-1}U}$.

Proof of Theorem 8.5. — Let $x \in X$ be a point where $\phi(x, \theta) \neq 0$, and let U be a sufficiently small neighborhood of x . We may assume without loss of generality that the vertices

$$v_{1,1}, v_{3,1}, \dots, v_{4m-3}, v_{4m-1}$$

of $G_m(P_U, Q_U, c_{P_U}, c_{Q_U})$ correspond to the nodal domains of the restrictions of $\Re\phi|_{\pi^{-1}U}$ to the hypersurfaces

$$\pi^{-1}U \cap \left\{ \theta = \frac{k\pi}{2m} \right\}, \quad k = 1, 3, \dots, 4m - 3, 4m - 1,$$

that intersect the fiber $\pi^{-1}x$. Then Lemma 8.2 implies that any nodal domain of $\Re\phi|_{\pi^{-1}U}$ must intersect $\pi^{-1}x$.

Now assume that x' is another point in U . Then we may restate this as “any nodal domain of $\Re\phi|_{\pi^{-1}U}$ that intersect $\pi^{-1}x'$ must intersect $\pi^{-1}x$ ”, and equivalently, “any nodal domain of $\Re\phi$ that intersect $\pi^{-1}x'$ must intersect $\pi^{-1}x$ ”. Because we assumed that X is connected, by the freedom of choice of the pair of points x and x' , any nodal domain of $\Re\phi$ must intersect $\pi^{-1}x$. This proves the first part of the theorem.

For the latter part of the theorem, let p be a regular zero of ϕ , i.e.,

$$d\phi : T_pP \rightarrow \mathbb{C}$$

is a surjection. Choose a sufficiently small neighborhood $U \subset X$ of πp , and let f be the function that satisfies

$$\begin{aligned} \phi(x, \theta) &= f(x)e^{im\theta} \\ &= \Re f \cos(m\theta) + \Im f \sin(m\theta) + i(\Im f \cos(m\theta) - \Re f \sin(m\theta)) \\ &= \Re\phi + i\Im\phi. \end{aligned}$$

If $d\Re f$ and $d\Im f$ are linearly dependent, then a straightforward computation implies that $d\phi$ has rank ≤ 1 , so $d\Re f$ and $d\Im f$ are linearly independent.

This implies that πp is a regular zero of both $\Re f$ and $\Im f$. Also, linear independency implies that locally around πp , $\Re f = 0$ and $\Im f = 0$ define two curves intersecting transversally at πp . From this, we may find four open sets near p that are required for Lemma 8.3, and we infer that the number of nodal domains of $\Re\phi|_{\pi^{-1}U}$ is two.

Now because any nodal domain of $\Re\phi$ must intersect with $\pi^{-1}x$ for some $x \in U$, any nodal domain of $\Re\phi$ must contain one of the nodal domains of $\Re\phi|_{\pi^{-1}U}$, from which we conclude that $\Re\phi$ has only two nodal domains. \square

We are ready to prove our main theorem, Theorem 1.5.

Proof. — It is sufficient to verify the assumptions in Theorem 8.5 is satisfied. The first condition is trivial to verify. For the other conditions, note from the assumption that $P \rightarrow X$ is non-trivial, $\mathcal{Z}_{f_{m,j}}$ is non-empty, and Theorem 4.1 implies that it is discrete and consists only of regular zeros. \square

Remark 8.8. — If $\mathcal{Z}_{f_{m,j}}$ contains a closed curve that divides X into two connected components, then the number of nodal domain can be large. For instance, if $f_{m,j}$ vanishes on the boundary of small open disc $U \subset X$, and if it does not vanish on U , then $\Re\phi_{m,j}$ vanish identically on $\partial(\pi^{-1}U)$, and therein, $\Re\phi_{m,j}$ has $2m$ -distinct nodal domains. In particular, Theorem 8.5 fails even if $f_{m,j}$ has a regular zero elsewhere.

9. Surfaces of constant curvature

In this section, we illustrate the geometry of Kaluza–Klein metrics and the Kaluza–Klein eigenvalue problem on unit tangent bundles of surfaces of constant curvature.

9.1. Flat tori

Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. We use coordinates $z = x_1 + ix_2$. Its unit tangent bundle is $S\mathbb{T}^2 = \mathbb{T}^2 \times S^1$. The connection is flat and $\Delta_H = \Delta$ is simply the Laplacian of \mathbb{T}^2 . The Kaluza–Klein Laplacian is that $\Delta_G = \Delta + \frac{\partial^2}{\partial\theta^2}$ on $\mathbb{T}^2 \times S^1$. The Kaluza–Klein eigenfunctions are linear combinations of the product eigenfunctions,

$$\phi_{m,\vec{k}}(x_1, x_2, \theta) = e^{i\langle\vec{k},\vec{x}\rangle} e^{im\theta}, \quad \Delta_G \phi_{m,\vec{k}} = -(|\vec{k}|^2 + m^2)\phi_{m,\vec{k}}.$$

The multiplicity of the eigenvalue with fixed m is the number of ways of representing $|\vec{k}|^2$ as a sum of two squares. They correspond to eigendifferentials

$$f_{m,\vec{k}}(z)(dz)^m = e^{i\langle\vec{k},\vec{x}\rangle}(dz)^m.$$

In the notation (1.2),

$$\begin{cases} \Re\phi_{m,\vec{k}}(x_1, x_2, \theta) = u_{m,\vec{k}}(x_1, x_2, \theta) = \cos(\langle\vec{k},\vec{x}\rangle + m\theta), \\ \Im\phi_{m,\vec{k}}(x_1, x_2, \theta) = v_{m,\vec{k}}(x_1, x_2, \theta) = \sin(\langle\vec{k},\vec{x}\rangle + m\theta). \end{cases}$$

The nodal sets of the imaginary part are given by,

$$\mathcal{Z}_{v_{m,\vec{k}}} = \{(x_1, x_2, \theta) : \langle\vec{k},\vec{x}\rangle + m\theta \in \pi\mathbb{Z}\}.$$

$\mathcal{Z}_{v_{m,\vec{k}}}$ contains the set

$$\{(x_1, x_2, \theta) : \langle\vec{k},\vec{x}\rangle \in \pi\mathbb{Z}, \theta = \ell \frac{\pi}{m}, \ell = 1, \dots, m\}.$$

Note that $\phi_{m,\vec{k}}(x_1, x_2, \theta)$ has no zeros on $\mathbb{T}^2 \times S^1$ and $f_{m,\vec{k}}(z)(dz)^m$ has no zeros as an m -differential on \mathbb{T}^2 .

If we change the lattice to a general lattice $L \subset \mathbb{R}^2$, the eigenfunctions of \mathbb{T}^2 change to $e_{\vec{\lambda}}(\vec{x}) = e^{2\pi i\langle\vec{\lambda},\vec{x}\rangle}$ where $\vec{\lambda} \in \Lambda = L^*$, the dual lattice. For generic L , the eigenvalues have multiplicity 2 and the eigenspaces are spanned by the real and imaginary parts of $e_{\vec{\lambda}}$ or equivalently by $e_{\vec{\lambda}}$ and its complex conjugate $e_{-\vec{\lambda}}$. The same is true of the Kaluza–Klein eigenfunctions $\phi_{m,\vec{\lambda}} = e^{2\pi i\langle\vec{\lambda},\vec{x}\rangle} e^{im\theta}$. Again, $\phi_{m,\vec{\lambda}}$ has no zeros. Using the bifurcation of nodal sets of eigenfunctions under generic paths of metrics of [20], one can show that

CONJECTURE 9.1. — *for generic Kaluza–Klein metrics on $S\mathbb{T}^2$, the joint eigenfunctions $\phi_{m,j}$ have no zeros.*

We now give an explicit orthonormal eigenbasis of \mathbb{T}^3 such that all of them have exactly two nodal domains, hence proving Theorem 1.11.

To begin with, let $f_1(x) = \cos(2\pi x)$ and $f_0(x) = \sin(2\pi x)$. Then

$$\{f_{j_1}(m_1x_1)f_{j_2}(m_2x_2)f_{j_3}(m_3x_3) : j_k = 0 \text{ or } 1, m_k \in \mathbb{Z}_{\geq 0}\}$$

is an orthogonal eigenbasis of \mathbb{T}^3 . We consider four cases.

Case 1: $m_1m_2m_3 > 0$. — We first have

$$\begin{aligned} & \langle \{f_{j_1}(m_1x_1)f_{j_2}(m_2x_2)f_{j_3}(m_3x_3), f_{1-j_1}(m_1x_1)f_{1-j_2}(m_2x_2)f_{1-j_3}(m_3x_3)\} \rangle \\ &= \langle \{f_{j_1}(m_1x_1)f_{j_2}(m_2x_2)f_{j_3}(m_3x_3) \pm f_{1-j_1}(m_1x_1)f_{1-j_2}(m_2x_2)f_{1-j_3}(m_3x_3)\} \rangle \end{aligned}$$

Assume without loss of generality that $j_1 = 0$. Then

$$\begin{aligned} & \{f_{j_1}(m_1x_1)f_{j_2}(m_2x_2)f_{j_3}(m_3x_3) \pm f_{1-j_1}(m_1x_1)f_{1-j_2}(m_2x_2)f_{1-j_3}(m_3x_3)\} \\ &= \Re((f_{j_2}(m_2x_2)f_{j_3}(m_3x_3) \pm if_{1-j_2}(m_2x_2)f_{1-j_3}(m_3x_3))e^{2\pi im_1x_1}), \end{aligned}$$

has two nodal domains by Theorem 8.5, because

$$f_{j_2}(m_2x_2)f_{j_3}(m_3x_3) \pm if_{1-j_2}(m_2x_2)f_{1-j_3}(m_3x_3)$$

has a regular zero.

Case 2: exactly one m_k is zero, and the other two are different. — From the same reasoning, each eigenfunction in the new basis in the following has two nodal domains:

$$\begin{aligned} & \langle \{f_{j_1}(m_1x_1)f_{j_2}(m_2x_2), f_{j_1}(m_1x_1)f_{j_3}(m_2x_3) : j_k = 0 \text{ or } 1\} \rangle \\ &= \langle \{f_{j_1}(m_1x_1)f_{j_2}(m_2x_2) \pm f_{1-j_1}(m_1x_1)f_{j_3}(m_2x_3) : j_k = 0 \text{ or } 1\} \rangle, \end{aligned}$$

$$\begin{aligned} & \langle \{f_{j_2}(m_1x_2)f_{j_1}(m_2x_1), f_{j_2}(m_1x_2)f_{j_3}(m_2x_3) : j_k = 0 \text{ or } 1\} \rangle \\ &= \langle \{f_{j_2}(m_1x_2)f_{j_1}(m_2x_1) \pm f_{1-j_2}(m_1x_2)f_{j_3}(m_2x_3) : j_k = 0 \text{ or } 1\} \rangle, \end{aligned}$$

and

$$\begin{aligned} & \langle \{f_{j_3}(m_1x_3)f_{j_1}(m_2x_1), f_{j_3}(m_1x_3)f_{j_2}(m_2x_2) : j_k = 0 \text{ or } 1\} \rangle \\ &= \langle \{f_{j_3}(m_1x_3)f_{j_1}(m_2x_1) \pm f_{1-j_3}(m_1x_3)f_{j_2}(m_2x_2) : j_k = 0 \text{ or } 1\} \rangle. \end{aligned}$$

Case 3: exactly one m_k is zero, and the other two are equal. — Again by the same reasoning, each of the following

$$\begin{aligned} & f_0(mx_1)f_0(mx_2) \pm f_1(mx_1)f_0(mx_3), \quad f_0(mx_2)f_0(mx_3) \pm f_1(mx_2)f_0(mx_1), \\ & f_0(mx_3)f_0(mx_1) \pm f_1(mx_3)f_0(mx_2), \quad f_1(mx_1)f_1(mx_2) \pm f_1(mx_3)f_0(mx_1), \\ & f_1(mx_2)f_1(mx_3) \pm f_1(mx_1)f_0(mx_2), \quad f_1(mx_3)f_1(mx_1) \pm f_1(mx_2)f_0(mx_3) \end{aligned}$$

has two nodal domains, and these are the basis of

$$\langle \{f_{j_1}(mx_1)f_{j_2}(mx_2), f_{j_1}(mx_1)f_{j_3}(mx_3), f_{j_2}(mx_2)f_{j_3}(mx_3) : j_k = 0 \text{ or } 1\} \rangle.$$

Case 4: exactly one m_k is nonzero. — In this case, we consider orthogonal eigenfunctions

$$\begin{aligned} f_0(mx_1) + f_0(mx_2) - \frac{1}{2}f_0(mx_3), & \quad f_0(mx_1) + f_0(mx_3) - \frac{1}{2}f_0(mx_2), \\ f_0(mx_2) + f_0(mx_3) - \frac{1}{2}f_0(mx_1), & \quad f_1(mx_1) + f_1(mx_2) - \frac{1}{2}f_1(mx_3), \\ f_1(mx_1) + f_1(mx_3) - \frac{1}{2}f_1(mx_2), & \quad f_1(mx_2) + f_1(mx_3) - \frac{1}{2}f_1(mx_1), \end{aligned}$$

which span

$$\langle \{f_j(mx_1), f_j(mx_2), f_j(mx_3) : j = 0 \text{ or } 1\} \rangle.$$

Each of these has only two nodal domains from the following lemma.

LEMMA 9.2. — *Let m be a positive integer. Then*

$$\cos(mx_1) + \cos(mx_2) - \frac{1}{2}\cos(mx_3)$$

has only two nodal domains.

Proof. — Let $x_1 - x_2 = a$, $x_1 - x_3 = b$, and $x_2 + x_3 = c$. Then

$$\begin{aligned} e^{2\pi imx_1} + e^{2\pi imx_2} - \frac{1}{2}e^{2\pi imx_3} \\ = \left(e^{\pi ima} e^{\pi imb} + e^{-\pi ima} e^{\pi imb} - \frac{1}{2}e^{\pi ima} e^{-\pi imb} \right) e^{2\pi imc}, \end{aligned}$$

and from Theorem 8.5, it is sufficient to prove that

$$e^{\pi ima} e^{\pi imb} + e^{-\pi ima} e^{\pi imb} - \frac{1}{2}e^{\pi ima} e^{-\pi imb}$$

has a regular zero. Let $\pi m(a + b) = x$ and $\pi m(a - b) = y$, then this is equivalent to

$$\cos x + \frac{1}{2}\cos y + i \left(\sin x - \frac{3}{2}\sin y \right)$$

having a regular zero. Since $\cos x + \frac{1}{2}\cos y$ and $\sin x - \frac{3}{2}\sin y$ do not have singular points, it is sufficient to check if these two functions have a common zero, in other words, if

$$\cos x + \frac{1}{2}\cos y + i \left(\sin x - \frac{3}{2}\sin y \right) = 0$$

has a solution. Note that this is equivalent to

$$(9.1) \quad e^{ix} = -\frac{1}{2}\cos y + i\frac{3}{2}\sin y.$$

Because

$$\left| -\frac{1}{2} \cos y + i \frac{3}{2} \sin y \right| = \frac{1}{4} + 2 \sin^2 y,$$

for y such that $\frac{1}{4} + 2 \sin^2 y = 1$, there is x satisfying (9.1), and this completes the proof. \square

9.2. Kaluza–Klein metrics on S^3

Let (S^2, g_0) be the 2-sphere with its standard metric of curvature 1. Then its unit tangent $SS^2 = SO(3) = \mathbb{R}\mathbb{P}^3 = S^3 / \pm 1$ and the Kaluza–Klein metric is the standard metric of constant sectional curvature 1 on S^3 (divided by the antipodal group \mathbb{Z}_2). The Kaluza–Klein Laplacian is therefore the standard Laplacian Δ_{S^3} on \mathbb{Z}_2 -invariant functions.

Since S^3 is a group, $L^2(S^3) = \bigoplus_{N=0}^{\infty} V_N \otimes V_N$ where V_N is an irreducible representation of S^3 of dimension $N+1$. Alternatively, the eigenfunctions of S^3 are harmonic homogeneous polynomials on \mathbb{R}^3 . Moreover, $\Delta|_{V_N \otimes V_N} = N(N+2) = (N+1)^2 - 1$. The eigenfunctions of $\mathbb{R}\mathbb{P}^3$ are those where N is even.

We need explicit separation of variables expressions for equivariant spherical harmonics, and therefore need to introduce coordinate systems. We use “axis - angle” Hopf coordinates (α, θ, ϕ) on $S^3 \subset \mathbb{R}^4$ defined by

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = r \begin{pmatrix} \sin \alpha \cos \phi \\ \sin \alpha \sin \phi \\ \cos \alpha \cos \theta \\ \cos \alpha \sin \theta \end{pmatrix}.$$

Here $0 \leq \alpha \leq \pi/2, 0 \leq \theta, \phi \leq 2\pi$. This corresponds to writing

$$z_1 = e^{i\phi} \sin \alpha, \quad z_2 = e^{i\theta} \cos \alpha, \quad ((z_1, z_2) \in \mathbb{C}^2 \simeq \mathbb{R}^4).$$

There exist two commuting isometric S^1 actions generated by the Killing vector fields

$$X = \frac{\partial}{\partial \phi} + \frac{\partial}{\partial \theta}, \quad Y = \frac{\partial}{\partial \phi} - \frac{\partial}{\partial \theta}.$$

The metric is $(d\alpha)^2 + (\cos \alpha d\theta)^2 + (\sin \alpha d\phi)^2$. In these coordinates one has an orthogonal basis of eigenfunctions given by

$$\begin{aligned} \Phi_N^{m_+, m_-}(\alpha, \phi, \theta) &= C_N^{m_+, m_-} e^{i(m_+ + m_-)\phi} e^{i(m_+ - m_-)\theta} \\ &\cdot (1 - \cos 2\alpha)^{\frac{m_+ + m_-}{2}} (1 + \cos 2\alpha)^{\frac{m_+ - m_-}{2}} P_{\frac{N}{2} - m_+}^{m_+ + m_-, m_+ - m_-}(\cos 2\alpha), \end{aligned}$$

where $P_N^{(a,b)}$ is a Jacobi polynomial and where

$$|m_{\pm}| \leq \frac{N}{2}, \frac{N}{2} - m_{\pm} \in \mathbb{N}.$$

Here, weight m in our sense means that the eigenfunctions transform by $e^{im - (\phi - \theta)}$. $\Phi_N^{m_+, m_-}$ are also known as “Wigner D-functions” on $SU(2)$. Another expression is

$$\begin{aligned} T_N^{m_1, m_2} &= C_N^{m_1, m_2} (\cos \alpha e^{i\theta})^{m_1 + m_2} (\sin \alpha e^{i\phi})^{m_2 - m_1} P_{N/2 - m_2}^{(m_2 - m_1, m_2 + m_1)}(\cos(2\alpha)). \end{aligned}$$

Here $m_1 = m$ in our notation. These are manifestly joint eigenfunctions of Δ_{S^3} and of $\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}$.

LEMMA 9.3. — *The nodal sets of the equivariant eigenfunctions $\Phi_N^{m_+, m_-}$ (or equivalently $T_N^{m_1, m_2}$) have real dimension 2.*

Proof. — The only factors with zeros are the α -functions. These have roughly m discrete zeros in α . Hence, the complex nodal set is a union

$$\{(\theta, \phi, \alpha) : (\cos \alpha)^{m_1 + m_2} (\sin \alpha)^{m_2 - m_1} P_{N/2 - m_2}^{(m_2 - m_1, m_2 + m_1)}(\cos(2\alpha)) = 0\},$$

and thus has real dimension 2. □

As a result, these eigenfunctions do not satisfy the conditions of the generic Kaluza–Klein metrics to which our results apply, and their nodal sets are quite different.

As mentioned in the introduction, the numerical experiments of A. Barnett et al. [2] show that random spherical harmonics of degree N on S^3 also have different types of nodal sets than our generic eigenfunctions. Namely, the expected number of nodal domains has the asymptotics cN^3 for a certain $c > 0$. As proved in [15], the nodal sets of real/imaginary parts of random equivariant eigenfunctions with fixed m (a subspace isomorphic to V_N) have connected nodal sets. The difference is due to the fact that our random equivariant spherical harmonics are a thin subset of the random spherical harmonics of degree N on S^3 .

Remark 9.4. — In [15], we compute the expected genus of the single component of the nodal sets of real/imaginary parts of random equivariant spherical harmonics of degree (N, m) where $|m| \leq N, 2|(N - m)$. The expected Euler characteristic is of the form $m(N^2 - m^2) + N$ modulo lower order terms.

9.3. Hyperbolic surfaces \mathbb{H}^2

Although it differs from our prior discussion in the compact case, let us consider a finite area hyperbolic real Riemann surface of constant negative curvature -1 . Then $X = S_g^*M = \Gamma \backslash G$ where $G = \mathrm{PSL}(2, \mathbb{R})$. The total space X carries a Lorentz Cartan–Killing metric with indefinite Laplacian the Casimir operator Ω . It is well known that $\Omega = H^2 + V^2 - W^2$. We now change the sign of the third term to get the Kaluza–Klein Laplacian $\Delta_X = H^2 + V^2 + W^2$. The associated metric defines a Riemannian submersion $\pi : X \rightarrow M$ with fibers given by K -orbits. They are necessarily totally geodesic. It follows that the horizontal Laplacian $H^2 + V^2$ commutes with the vertical Laplacian W^2 . This is obvious because $0 = [\Omega, W^2] = [H^2 + V^2, W^2] = 0$.

The joint eigenfunctions of Ω, W are denoted by $\phi_{m,j}$. When $m = 0$ they are pullbacks of eigenfunctions of $M = \Gamma \backslash G/K$.

In particular the number of nodal domains of $\phi_{j,0}$ on X is the same as the number of nodal domains of ϕ_j on M . The former nodal sets are K -invariant and in the case of regular nodal components are 2-tori over circles.

The lift of weight m of an m -differential $f(dz)^m$ is given by

$$\Phi(x, y, \theta) = y^{m/2} f(x + iy) e^{-im\theta}.$$

Here, the Kähler potential is $\phi = \log y$, $d\phi = \frac{dy}{y}$, $\Delta\phi = y^2(\log y)'' = -1$. Also, $\|d\phi\|^2 = y^2 \|\frac{dy}{y}\|^2 = 1$ and $*d\phi = *(\phi_x dx + \phi_y dy) = (-\phi_x dy + \phi_y dx)y$. The Maass operator is

$$D_m = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - 2imy \frac{\partial}{\partial x}$$

and

$$D_m f_{m,j} = s(1-s) f_{m,j}.$$

Breaking up into real and imaginary parts gives the system,

$$\begin{cases} \Delta \Re f_{m,j} + 2my \frac{\partial}{\partial x} \Im f_{m,j} = s(1-s) \Re f_{m,j}, \\ \Delta \Im f_{m,j} - 2my \frac{\partial}{\partial x} \Re f_{m,j} = s(1-s) \Im f_{m,j}. \end{cases}$$

The raising/lowering operators are the Maass operators defined by

$$\begin{cases} K_k = (z - \bar{z}) \frac{\partial}{\partial z} + k = 2iy^{1-k} \frac{\partial}{\partial z} y^k, \\ L_k = (z - \bar{z}) \frac{\partial}{\partial z} - k = -2iy^{1+k} \frac{\partial}{\partial z} y^{-k} = \bar{K}_{-k}. \end{cases}$$

Then,

$$K_k : \mathfrak{H}_k \rightarrow \mathfrak{H}_{k+1}, \quad L_k : \mathfrak{H}_k \rightarrow \mathfrak{H}_{k-1},$$

and

$$D_{k+1}K_k = K_k D_k, D_k L_{k+1} = L_{k+1} D_k,$$

and

$$D_k = L_{k+1}K_k + k(k + 1) = K_{k-1}L_k + k(k - 1).$$

9.3.1. Automorphic forms on the full modular group

Now we consider the case $\Gamma = \text{PSL}_2(\mathbb{Z})$. Note that the quotient $\Gamma \backslash G$ is non-compact in this case. Nevertheless, it is known that $-\Delta_G$ has infinitely many discrete spectrum, where corresponding L^2 integrable eigenfunctions can be chosen so that they are in one-to-one correspondence with Maass–Hecke cusp forms or holomorphic Hecke cusp forms. We refer the readers to [10] for detailed background.

THEOREM 9.5. — *Let $X = \text{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}$, and let $\phi_{m,ir}$ be a weight m Maass–Hecke cusp form on*

$$\text{PSL}_2(\mathbb{Z}) \backslash \text{PSL}_2(\mathbb{R}).$$

Assume that the zeros of $\phi_{m,ir}$ are isolated. Then $\Re\phi_{m,ir}$ has only two nodal domains.

Proof. — The first statement of Condition 8.4 follows from the definition of Maass–Hecke cusp form, and the second statement follow from the fact that $\phi_{m,ir} : X \rightarrow \mathbb{C}$ can not be scaled to a real-valued function, and that $\phi_{m,ir}$ is analytic. The third statement follows from the assumption.

Now, because the first Hecke eigenvalue is 1, the first Fourier coefficient of $\phi_{m,ir}$ at the cusp does not vanish, meaning that $i\infty$ is a regular zero of $\phi_{m,ir}$. We conclude the proof by applying Theorem 8.5. □

Remark 9.6. — It is not hard to see that in the constant curvature case, the nodal set of $\phi_{2,ir}$ consists of the fibers over the critical point set $C_{\phi_{ir}}$ of ϕ_{ir} . At this time, it does not seem to be known whether $C_{\phi_{ir}}$ is necessarily a discrete set of points in the case of hyperbolic surfaces. This cannot be proved by a purely local calculation, since the critical point set of rotationally invariant Dirichlet/Neumann eigenfunctions on a compact rotationally invariant submanifold C_R of a hyperbolic cylinder $\mathbb{H}^2 / \langle \gamma_0 \rangle$ consists of a union of S^1 orbits. Here, γ_0 is a hyperbolic element and $\langle \gamma_0 \rangle$ is the cyclic group it generates. Thus, negative curvature does not rule out codimension 1 critical point sets. One can put any negatively curved S^1 invariant metric on C_R and obtain the same result, so it is not an effect of constant curvature. We conjecture that for compact hyperbolic surfaces without boundary, $C_{\phi_{ir}}$ is a finite set for every eigenfunction.

When we have holomorphicity of ϕ , we may remove the assumption that the zeros of ϕ being isolated. For instance, we have:

THEOREM 9.7. — *Let $X = \mathrm{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}$, and let $\phi_{m,0}$ be a Laplacian eigenfunction on $\mathrm{PSL}_2(\mathbb{Z}) \backslash \mathrm{PSL}_2(\mathbb{R})$ corresponding to a holomorphic Hecke cusp form F of weight m . Then $\Re\phi_{m,0}$ has only two nodal domains.*

Proof. — We first note that $\phi_{m,0}(z, \theta) = y^{m/2} F(z) e^{-im\theta}$, and F is holomorphic. Therefore Condition 8.4 is satisfied.

Because we assumed that F is a Hecke cusp form, the first Hecke eigenvalue is 1. Therefore $i\infty$ is a regular zero of $\phi_{m,0}$, and now the theorem follows from Theorem 8.5. \square

COROLLARY 9.8. — *There exist eigenfunctions on $\mathrm{PSL}_2(\mathbb{Z}) \backslash \mathrm{PSL}_2(\mathbb{R})$ that have only two nodal domains but with arbitrarily large eigenvalues.*

We remark here that Theorem 9.7 is false, without the assumption that F is a Hecke cusp form. To construct a counter example, let $\Delta(z)$ be the discriminant modular form given by

$$\begin{aligned} \Delta(z) &= \sum_{n=1}^{\infty} \tau(n) q^n \\ &= q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 - 16744q^7 + \dots, \end{aligned}$$

where $q = e^{2\pi iz}$. This is a weight 12 modular form on $\mathrm{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}$. Thus $\Delta(z)^2$ is a modular form of weight 24 on $\mathrm{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}$, and

$$\Phi = \Re(y^{12} \Delta(z)^2 e^{-24i\theta})$$

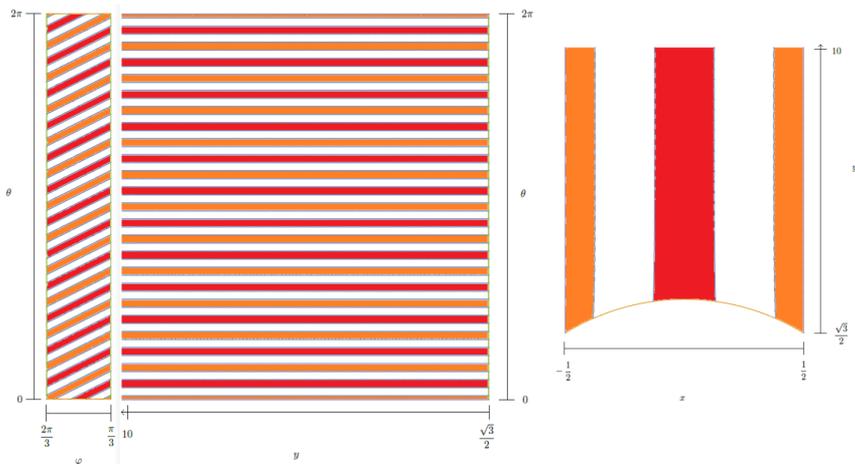
is a Laplacian eigenfunction on $\mathrm{PSL}_2(\mathbb{Z}) \backslash \mathrm{PSL}_2(\mathbb{R})$ of weight 24. To count the number of nodal domains of this eigenfunction, we let

$$\mathcal{F} = \left\{ x + iy : |x| \leq \frac{1}{2}, x^2 + y^2 \geq 1 \right\} \subset \mathbb{H}$$

be the fundamental domain of $\mathrm{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}$, and let $M_0 = \mathcal{F} \times \{\theta : 0 \leq \theta < 2\pi\}$.

We then consider the restrictions of Φ to the top $\theta = 2\pi$, side $x = -1/2$, and front $x^2 + y^2 = 1$ of the solid M_0 .

It can be shown that the nodal set of Φ on the side is that of $\cos(24\theta) = 0$, and on the front is that of $\cos(12(\varphi + 2\theta)) = 0$, where we define $\varphi = \arccos(x)$. We compute the nodal set of the restriction to the top numerically using Mathematica.



The nodal set of Φ on the front, the side, and the top of the solid M_0 .

Note that we may obtain $\mathrm{PSL}_2(\mathbb{Z}) \backslash \mathrm{PSL}_2(\mathbb{R})$ from M_0 by gluing the sides via $(x, y, \theta) = (x + 1, y, \theta)$ (corresponding to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$), the top and the bottom via $(x, y, \theta) = (x, y, \theta + 2\pi)$ (corresponding to $k(\theta) = k(\theta + 2\pi)$), and then the front with itself via $(\varphi, \theta) = (\pi - \varphi, \theta + \varphi)$ and $(\varphi, \theta) = (\varphi, \theta + 2\pi)$ (corresponding to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $k(\theta) = k(\theta + 2\pi)$).

From these, one can verify that Φ has exactly four nodal domains, where in the pictures above, two positive nodal domains are colored differently with red and orange.

BIBLIOGRAPHY

- [1] F. A. ARIAS & M. MALAKHALTSEV, “A generalization of the Gauss-Bonnet-Hopf-Poincaré formula for sections and branched sections of bundles”, *J. Geom. Phys.* **121** (2017), p. 108-122.
- [2] A. BARNETT, K. KONRAD & M. JIN, “Experimental Nazarov-Sodin constants, genus, and percolation on nodal domains for 2D and 3D random waves”, in preparation, 2017.
- [3] L. BÉRARD-BERGÉRY & J.-P. BOURGUIGNON, “Laplacians and Riemannian submersions with totally geodesic fibres”, *Ill. J. Math.* **26** (1982), no. 2, p. 181-200.
- [4] R. COURANT & D. HILBERT, *Methods of mathematical physics. Vol. I*, Interscience Publishers, 1953, xv+561 pages.
- [5] J.-P. DEMAILLY, *Analytic methods in algebraic geometry*, Surveys of Modern Mathematics, vol. 1, International Press; Higher Education Press, 2012, viii+231 pages.
- [6] A. ENCISO & D. PERALTA-SALAS, “Nondegeneracy of the eigenvalues of the Hodge Laplacian for generic metrics on 3-manifolds”, *Trans. Am. Math. Soc.* **364** (2012), no. 8, p. 4207-4224.
- [7] T. FUKUI & J. J. NUÑO-BALLESTEROS, “Isolated singularities of binary differential equations of degree n ”, *Publ. Mat., Barc.* **56** (2012), no. 1, p. 65-89.

- [8] A. GHOSH, A. REZNIKOV & P. SARNAK, “Nodal domains of Maass forms I”, *Geom. Funct. Anal.* **23** (2013), no. 5, p. 1515-1568.
- [9] ———, “Nodal domains of Maass forms, II”, *Am. J. Math.* **139** (2017), no. 5, p. 1395-1447.
- [10] H. IWANIEC & E. KOWALSKI, *Analytic number theory*, Colloquium Publications, vol. 53, American Mathematical Society, 2004, xii+615 pages.
- [11] S. U. JANG & J. JUNG, “Quantum unique ergodicity and the number of nodal domains of eigenfunctions”, *J. Am. Math. Soc.* **31** (2018), no. 2, p. 303-318.
- [12] J. JUNG & M. P. YOUNG, “Sign changes of the Eisenstein series on the critical line”, *Int. Math. Res. Not.* (2019), no. 3, p. 641-672.
- [13] J. JUNG & S. ZELDITCH, “Number of nodal domains and singular points of eigenfunctions of negatively curved surfaces with an isometric involution”, *J. Differ. Geom.* **102** (2016), no. 1, p. 37-66.
- [14] ———, “Number of nodal domains of eigenfunctions on non-positively curved surfaces with concave boundary”, *Math. Ann.* **364** (2016), no. 3-4, p. 813-840.
- [15] ———, “Topology of the nodal set of random equivariant spherical harmonics on S^3 ”, <https://arxiv.org/abs/1908.00979>, to appear in *Int. Math. Res. Not.*, 2019.
- [16] H. LEWY, “On the minimum number of domains in which the nodal lines of spherical harmonics divide the sphere”, *Commun. Partial Differ. Equations* **2** (1977), no. 12, p. 1233-1244.
- [17] M. MAGEE, “Arithmetic, zeros, and nodal domains on the sphere”, *Commun. Math. Phys.* **338** (2015), no. 3, p. 919-951.
- [18] A. STERN, “Bemerkungen über asymptotisches Verhalten von Eigenwerten und Eigenfunktionen. Math.- naturwiss. Diss.”, Göttingen, 30 S (1925)., 1925.
- [19] K. STREBEL, *Quadratic differentials*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 5, Springer, 1984, xii+184 pages.
- [20] K. UHLENBECK, “Generic properties of eigenfunctions”, *Am. J. Math.* **98** (1976), no. 4, p. 1059-1078.
- [21] J. VILMS, “Totally geodesic maps”, *J. Differ. Geom.* **4** (1970), p. 73-79.
- [22] S. ZELDITCH, “Logarithmic lower bound on the number of nodal domains”, *J. Spectr. Theory* **6** (2016), no. 4, p. 1047-1086.

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