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Young-Jun CHOI & Georg SCHUMACHER

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EXTENSION OF THE CURVATURE FORM OF THE RELATIVE CANONICAL LINE BUNDLE ON FAMILIES OF CALABI–YAU MANIFOLDS AND APPLICATIONS

by Young-Jun CHOI & Georg SCHUMACHER (*)

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ABSTRACT. — Given a proper surjective holomorphic map of Kähler manifolds, whose general fibers are Calabi–Yau manifolds, the volume forms for the Ricci-flat metrics induce a hermitian metric on the relative canonical bundle over the regular locus of the family. We show that the curvature form extends as a closed positive current. Consequently the Weil–Petersson metric extends as a positive current. In the projective case, the Weil–Petersson form is known to be the curvature of a certain determinant line bundle, equipped with a Quillen metric. As an application we get that after blowing up the singular locus, the determinant line bundle extends, and the Quillen metric extends as singular hermitian metric, whose curvature is a positive current.

RÉSUMÉ. — Étant donnée une application holomorphe surjective entre variétés de Kähler, dont les fibres générales sont des variétés de Calabi–Yau, les formes de volume pour les métriques Ricci-plates induisent une métrique hermitienne sur le faisceau canonique relatif sur le lieu régulier de la famille. Nous montrons que la forme de courbure s'étend comme un courant positif fermé. Par conséquent, la métrique de Weil–Petersson s'étend comme un courant positif. On sait que dans le cas projectif la forme de Weil–Petersson est la courbure d'un certain faisceau de lignes déterminant, équipé d'une métrique de Quillen. Comme application, nous obtenons qu'après éclatement du lieu singulier, le faisceau de lignes déterminant s'étend et la métrique de Quillen s'étend comme métrique hermitienne singulière, dont la courbure est un courant positif.

1. Introduction

When studying moduli spaces, the extension of the Weil–Petersson metric into the boundary of the moduli space, more precisely in the situation

of a degenerating family, is of interest. This could be done for families
of canonically polarized manifolds or stable holomorphic vector bundles.
The approach in both cases is to use a fiber integral formula for the Weil–
Petersson metric in terms of a certain positive $(n+1, n+1)$-form on the
total space (where n denotes the dimension of the fibers). Such a form has
to be extended as a positive current, and a push forward is taken. This
approach is closely related to the existence of Kähler–Einstein and Her-
mite–Einstein metrics resp. In the former case, uniform C^0-estimates for
the solution of the resp. Monge–Ampère equations in a degenerating family
ultimately provide the extension property.

Here, we will treat the case of Calabi–Yau manifolds, i.e. polarized Käh-
ler manifolds with vanishing first real Chern class, equipped with Ricci-flat
metrics according to Yau's theorem [11]. Our result will be based upon the
Ohsawa–Takegoshi extension theorem for canonical forms on submanifolds
in bounded pseudoconvex domains. The statement of the main theorem
will not depend on the type of the singularities of the degeneration.

Let $f : X \to S$ be a proper surjective holomorphic mapping between
complex manifolds, where X is Kähler . We denote by $W \subset S$ the analytic
set of singular values of f, and let $X' := f^{-1}(S')$ with $S' = S \setminus W$. Suppose
that for every $s \in S'$, the fiber $X_s = f^{-1}(s)$ is a Calabi–Yau manifold, i.e. a
compact Kähler manifold, whose first real Chern class $c_1(X_s)$ vanishes. Let
ω be a Kähler form on X. Then Yau's theorem implies that there exists a
unique Ricci-flat metric ω_s^{KE} in the class of $\omega|_{X_s}$ on each smooth fiber X_s,
$s \in S'$. One can see that the family of Ricci-flat metrics induces a fiberwise
Ricci-flat d-closed smooth real $(1, 1)$-form ρ on X' such that

$$\rho|_{X_s} = \omega_s^{KE}.$$

It can be normalized by the condition

$$(1.1) \qquad f_*(\rho^{n+1}) = \int_{X'/S'} \rho^{n+1} = 0, \text{ where } n = \dim X_s$$

in which case ρ is unique [7, Proposition 3.6]. By adding the pull-back of
a Kähler form on S' to ρ, (like the fiber integral $\int_{X'/S'} \omega^{n+1}$, or the Weil–
Petersson form, if the family is effectively parametrized) we can achieve
that $f_*(\rho^{n+1})$ is positive definite. Conversely, if $f_*(\rho^{n+1}) > 0$, then $\rho -
f^*(f_*(\rho^{n+1})/\mathrm{Vol}(X_s))$ satisfies the former normalization condition. A form
ρ satisfying the latter condition is also called *fiberwise Ricci-flat metric*. A
fiberwise Ricci-flat metric ρ is not uniquely determined, but it always exists
globally on X' under our assumptions. The corresponding *relative volume
form* on $X' \to S'$ is independent of the normalization, however, later we

will use (1.1). It induces a hermitian metric $h^\rho_{X'/S'}$ on the relative canonical line bundle (see Section 2). Now the Weil–Petersson metric $\omega^{WP}_{S'}$ and the curvature $\Theta_{h^\rho_{X'/S'}}(K_{X'/S'})$ of $h^\rho_{X'/S'}$ defined on S' satisfy the following equation according to [5]. (Note that the volume of the smooth fibers is constant.)

$$(1.2) \qquad \frac{1}{\mathrm{Vol}(X_s)} f^* \omega^{WP} = \Theta_{h^\rho_{X'/S'}}(K_{X'/S'}).$$

This implies that the curvature $\Theta_{h^\rho_{X'/S'}}(K_{X'/S'})$ of the relative canonical line bundle is semi-positive on X'. The main goal of this paper is the extension of $\Theta_{h^\rho_{X'/S'}}(K_{X'/S'})$.

THEOREM 1.1. — *Let $f : X \to S$ be a proper surjective holomorphic mapping of Kähler manifolds. Suppose that every regular fiber X_s is a Calabi–Yau manifold for $s \in S'$. Then the curvature $\Theta_{h^\rho_{X'/S'}}(K_{X'/S'})$ of the relative canonical line bundle extends to X as a d-closed positive $(1,1)$-current Θ.*

On S'

$$\omega^{WP} = f_* \left(f^*(\omega^{WP}) \wedge \omega^n \right) = \int_{X'/S'} f^*(\omega^{WP}) \wedge \omega^n,$$

holds, where ω can be replaced by any Kähler form that induces the same polarization on the fibers.

With wedge products taken in the sense of Bedford and Taylor, the Theorem together with (1.2) will imply that

$$\int_{X/S} \Theta \wedge \omega^n$$

defines an extension of ω^{WP} as a positive, real current.

COROLLARY 1.2. — *The Weil–Petersson form extends to all of S as a positive, closed current.*

The proof of the Theorem uses is inspired by a result of Păun from [8], applying Demailly's approximation theorem of plurisubharmonic functions by logarithms of absolute values of holomorphic functions [6].

In [7, Section 10], for projective families of Ricci-flat manifolds (more generally extremal Kähler manifolds), we showed that a certain determinant line bundle λ on the base of the family could be equipped with a Quillen metric h^Q such that the curvature form is up to a numerical constant equal to the Weil–Petersson form. Using the extension theorem from [9, 10],

we can see that after a blow-up (λ, h^Q) can be extended into the singular locus of such a projective family.

Acknowlegements

2. Preliminaries

2.1. Construction of a background metric

Let $f : X \to S$ be a proper surjective holomorphic mapping between Kähler manifolds. Let $d = \dim S$, and $\dim X = n + d$. Let $S' \subset S$ be the set of regular values and $X' = f^{-1}(S')$ as above. Then the restriction $f' : X' \to S'$ is a surjective holomorphic submersion.

Let ω be a Kähler form on X, or more generally a smooth, d-closed, real $(1,1)$ form, which is positive definite along the smooth fibers of f. We will use the notations $\omega^n = \omega \wedge \ldots \wedge \omega/n!$ and analogous.

Since the relative canonical bundle $K_{X/S}$ is defined as $K_X \otimes f^* K_S^{-1}$ a hermitian metric $h^\omega_{X/S}$ can be defined as follows:

Given any local coordinate system $(z^1, \ldots, z^{n+d})$ on $U \subset X$ we denote the corresponding Euclidean volume form by $dV_X = dV_X(z)$, and for local coordinates $(s^1, \ldots, s^d)$ on the image of U under f in S we use the Euclidean volume form $dV_S = dV_S(s)$. Let

$$(2.1) \qquad \omega^n \wedge f^* dV_S = \chi \cdot dV_X,$$

with a certain non-negative, differentiable function χ. At singular points of f the function χ will vanish, and we write

$$(2.2) \qquad \chi = e^{\psi_U},$$

where ψ_U takes the value $-\infty$ at singular points of f.

On the regular locus of f in X the function χ is the absolute value squared of a holomorphic function so that $e^{-\psi_U}$ defines a singular hermitian metric $h^\omega_{X/S}$ on $K_{X/S}$.

The corresponding curvature form on X' is given by

$$(2.3) \qquad \sqrt{-1}\Theta_{h^\omega_{X/S}}(K_{X/S}) = dd^c\psi_U|X' = dd^c\log(\omega^n \wedge f^*dV_S).$$

So far, the curvature is a positive current on X, and differentiable on the regular locus of f in X.

2.2. Existence and curvature of a fiberwise Ricci-flat metric

Let the restriction of f now denote a smooth family of Calabi–Yau manifolds $f' : X' \to S'$, and let ω to be a Kähler form on X. Let $\omega_s = \omega|_{X_s}$. For $s \in S'$ we take the Kähler class $[\omega_s]$ as polarization, and apply Yau's theorem.

Since $[\mathrm{Ric}(\omega_s)] = 0$, for any $s \in S'$, by the dd^c-lemma, there exists a unique function $\eta_s \in C^\infty(X_s)$ such that

$$(2.4) \qquad dd^c\eta_s = \mathrm{Ric}(\omega_s)$$

$$(2.5) \qquad \int_{X_s} e^{\eta_s}\omega_s^n = \int_{X_s} \omega_s^n = \mathrm{Vol}(X_s).$$

Then the implicit function theorem says that the function η, which is defined by $\eta(x) = \eta_s(x)$ for $s = f(x)$, is a smooth function on X'.

There exists a d-closed, smooth, real $(1,1)$-form ρ such that $\rho|_{X_s}$ is the Ricci-flat metric on the polarized manifold $(X_s, [\omega_s])$. This follows, since X' is assumed to be Kähler (cf. [5]): Take ω_s as initial metric for the solution of the Monge–Ampère equation in Yau's solution of the Calabi problem, and set $\rho_s = \omega_s + dd^c\varphi_s$ on X_s, $s \in S'$. (Here we use the assumption that ω is globally defined.) Then the form

$$\rho = \omega + dd^c\varphi \text{ on } X'$$

restricts to ρ_s on each fiber X_s. It is possible to normalize ρ in the sense of (1.1) or to make it a fiberwise Ricci-flat metric.

Using (2.4) and (2.5) we obtain

$$(2.6) \qquad e^{\eta_s}\omega_s^n = \rho_s^n$$

for each $s \in S'$. Multiplying this equation by the Euclidean volume form dV_S, we get

$$e^\eta\omega^n \wedge f^*(dV_S) = \rho^n \wedge f^*(dV_S)$$

on X'. Hence

$$dd^c\log(e^\eta\omega^n \wedge f^*(dV_S)) = dd^c\log(\rho^n \wedge f^*(dV_S))$$

holds over S'. Now we apply (2.3) to the left-hand side of the above equation. Concerning the right-hand side, the same argument provides us with the curvature of the relative canonical bundle with respect to ρ:

$$(2.7) \quad dd^c(\eta+\psi_U) = dd^c\eta + \sqrt{-1}\Theta_{h^\omega_{X'/S'}}(K_{X'/S'}) = \sqrt{-1}\Theta_{h^\rho_{X'/S'}}(K_{X'/S'}).$$

2.3. Deformation theory of polarized Calabi–Yau manifolds, and the Weil–Petersson metric

For general facts about the Weil–Petersson metric, in particular (1.2), we refer to [5]. Let $f : X \to S$ be a (smooth) polarized, holomorphic family of Calabi–Yau manifolds, and ρ be a form as above such that the restriction to a fiber X_s is the Ricci-flat form on the polarized fiber with the normalization

$$(2.8) \qquad\qquad \int_{X/S} \rho^{n+1} = 0.$$

Previously, according to [7, Theorem 7.8, Remark 7.3], the following fiber integral formula for the Weil–Petersson metric was known:

$$(2.9) \qquad\qquad \omega^{WP} = \int_{X/S} \Theta_{h^\rho_{K_{X/S}}} \wedge \omega^n,$$

which clearly follows from (1.2). Such fiber integrals fit into the framework of the Riemann–Roch–Hirzebruch–Grothendieck formalism (cf. Section 4).

3. Extension of the curvature form of the relative canonical line bundle

In this section, we will derive an upper bound of a local potential of the curvature $\Theta' = \Theta_{h^\rho_{X'/S'}}(K_{X'/S'})$, which is induced by ρ on the relative canonical line bundle and prove Theorem 1.1. Since this curvature is known to be positive, the boundedness from above of a local potential near singular fibers will yield a local extension of the curvature current Θ' into singular fibers of the total space. The null-extensions (cf. [6, (1.19)]) of these currents will fit together and yield a global positive current.

Let x_0 be a point which belongs to a singular fiber X_{s_0}, i.e., $s_0 \in W$. Let $(U, z^1, \ldots, z^{n+d})$ be a local coordinate at x_0 in X where U is biholomorphic to a unit ball in $\mathbb{C}^{n+d}$ and $(s^1, \ldots, s^d)$ be a local coordinate at s_0 which is defined in a neighborhood containing $f(U)$ in S. Take a neighborhood

$V \subset U$ of x_0 and $r > 0$ such that $B_r(x) \subset U$ for any $x \in V$, where $B_r(x)$ is a geodesic ball centered at x with radius r with respect to ω. Let ψ_U be the local potential of $\Theta_{h^\omega_{X/S}}(K_{X/S})$ which is defined in Section 2.1. Define a function $\theta : U \cap X' \to \mathbb{R}$ by

$$(3.1) \qquad \qquad \theta := \psi_U + \eta.$$

Then θ is a local potential of $\Theta_{h^\rho_{X'/S'}}(K_{X'/S'})$ in $U \cap X'$ by (2.7). Since θ is plurisubharmonic, it is enough for the extension of $\Theta_{h^\rho_{X'/S'}}(K_{X'/S'})$ to show that θ is bounded from above in $V \cap X'$.

Fix a point $x \in V \cap X'$ and denote by $s = f(x)$. Then X_s is a smooth fiber and $U_s := U \cap X_s$ is a Stein manifold of U. We claim that the restriction of θ to U_s

$$\theta_s := \theta|_{U_s} = \psi_s + \eta_s$$

is uniformly bounded from above in V_s. We now adopt an argument of M. Păun from [8] to apply an approximation argument of Demailly. The idea is to approximate the function θ_s by the logarithm of absolute values of holomorphic functions.

THEOREM 3.1. — Let $\mathcal{H}_s^{(m)}$ be the Hilbert space defined as follows:

$$\mathcal{H}_s^{(m)} := \left\{ \phi \in \mathcal{O}(U_s) : \|\phi\|_{m,\,s}^2 := \int_{U_s} |\phi|^2 \, e^{-m\theta_s} (\rho_s)^n < \infty \right\}.$$

Then we have

$$\theta_s(x) = \lim_{m \to \infty} \sup \left\{ \frac{1}{m} \log |\phi(x)|^2 : \phi \in \mathcal{H}_s^{(m)} \;\; s.t. \;\; \|\phi\|_{m,s}^2 \leqslant 1 \right\}$$

for every $x \in U_s$.

Proof. — The proof is essentially same with the proof of [6, Theorem 14.2]. We may assume that U is biholomorphic to a bounded strongly pseudoconvex domain in C^{n+d}, so U_s is a Stein manifold. This implies that $\mathcal{H}_s^{(m)}$ is a nontrivial seperable Hilbert space. Hence there exists an orthonormal basis $\{\sigma_k^s\}_{k \in \mathbb{N}}$ for $\mathcal{H}_s^{(m)}$. Now we define the Bergman kernel φ_m^s by

$$\varphi_m^s(x) = \frac{1}{m} \log \sum_{k=1}^{\infty} |\sigma_k^s(x)|^2.$$

It follows immediately that the Bergman kernel satisfies the extremal property:

$$\varphi_m^s(x) = \sup \left\{ \frac{1}{m} \log |\phi(x)|^2 : \phi \in \mathcal{H}_s^{(m)} \;\; s.t. \;\; \|\phi\|_{m,\,s}^2 \leqslant 1 \right\}.$$

Take a local coordinate neighborhood $B_\varepsilon := \{\zeta \in U_s : |x - \zeta| < \varepsilon\}$ in U_s. Denote by $d\lambda$ the Lebesgue measure in B_ε. For any $\phi \in \mathcal{H}_s^{(m)}$ with $\|\phi\|_{m,s}^2 \leqslant 1$, for $0 < r < \varepsilon$, the mean value inequality implies

$$|\phi(x)|^2 \leqslant \frac{n!}{\pi^n r^{2n}} \int_{B_r} |\phi(\zeta)|^2 \, d\lambda(\zeta)$$

$$\leqslant \frac{n!}{\pi^n r^{2n}} \sup_{\zeta \in B_r} \left(e^{m\theta_s(\zeta)} \right) \int_{B_r} |\phi(\zeta)|^2 \, e^{-m\theta_s} d\lambda(\zeta).$$

Let $C_s := \sup_{B_r} d\lambda/(\rho_s)^n$. Then it follows that

$$|\phi(x)|^2 \leqslant \frac{n! C_s}{\pi^n r^{2n}} \sup_{\zeta \in B_r} \left(e^{m\theta_s(\zeta)} \right) \int_{B_r} |\phi(\zeta)|^2 \, e^{-m\theta_s} (\rho_s)^n$$

$$\leqslant \frac{n! C_s}{\pi^n r^{2n}} \sup_{\zeta \in B_r} \left(e^{m\theta_s(\zeta)} \right).$$

Hence we have

$$\frac{1}{m} \log |\phi(x)|^2 \leqslant \sup_{\zeta \in B_r} \theta_s(\zeta) + \frac{1}{m} \left(\log \frac{n!}{\pi^n r^{2n}} + \log C_s \right).$$

Since θ_s is upper-semicontinuous, taking $r = 1/m$ it follows that

$$\theta_s(x) \geqslant \lim_{m \to \infty} \varphi_m^s(x).$$

Conversely, first note that

$$\sigma = \frac{dz^1 \wedge \cdots \wedge dz^{n+d}}{f^*(ds^1 \wedge \cdots \wedge ds^d)}$$

is a holomorphic section of $K_{X'/S'}$ over $U \cap X'$. We remark that as a relative canonical form σ can be written as follows: Assume that on some open subset of U' the functions $(z^1, \ldots, z^n)$ define local coordinates for U_s, $s \in S'$. Then

$$(3.2) \qquad \sigma = \left(\frac{\partial(f^1, \ldots, f^d)}{\partial(z^{n+1}, \ldots, z^{n+d})} \right)^{-1} \cdot dz^1 \wedge \ldots \wedge dz^n.$$

In particular, K_{U_s} is trivial. There exist n bounded holomorphic functions on U_s which form a local coordinate near x because U_s is embedded in U. Therefore, for any $a \in \mathbb{C}$, Ohsawa–Takegoshi extension theorem (e.g. Proposition 3.2) says that there exists $\phi \in \mathcal{O}(U_s)$ such that $\phi(x) = a$ and

$$\int_{U_s} |\phi|^2 \, e^{-m\theta_s} (\rho_s)^n \leqslant C_s' \, |a|^2 \, e^{-m\theta_s(x)},$$

where the constant C'_s does not depend on m. If we choose $a \in \mathbb{C}$ such that the right hand side is 1, then

$$\int_{U_s} |\phi|^2 e^{-m\theta_s} (\rho_s)^n \leqslant 1, \text{ i.e. } \phi \in \mathcal{H}_s^{(m)}.$$

It follows that

$$\frac{1}{m} \log |\phi(x)|^2 = \frac{1}{m} \log |a|^2 \geqslant \theta_s(x) - \frac{1}{m} \log C'_s,$$

hence

$$\varphi_m^s(x) \geqslant \theta_s(x) + \frac{1}{m} \log C'_s.$$

This completes the proof of Theorem 3.1. $\hfill\square$

PROPOSITION 3.2. — Let (W, ω) be a Stein manifold with a Kähler metric such that K_W is trivial. Suppose that there exists bounded holomorphic functions $q^1, \ldots, q^n$ which form a local coordinate near a given point $x \in W$. Then for $a \in \mathbb{C}$, there exist a global holomorphic function $\phi \in \mathcal{O}(W)$ with the following properties:

(i) $\phi(x) = a$

(ii) for any plurisubharmonic function Ψ on W

$$\int_W |\phi|^2 e^{-\Psi} dV_{W,\omega} \leqslant C_n \frac{1}{|\Lambda^n(dq)(x)|_\omega^2} \cdot |a|^2 e^{-\Psi(x)},$$

where the constant C_n depends only upon n, and not on Ψ.

Proof. — Let $C := e \sup_W \sqrt{\sum_{j=1}^n |q^j|^2}$. We consider $q = \frac{1}{C}(q^1, \ldots, q^n)$ as a holomorphic section of the trivial vector bundle $E = W \times \mathbb{C}^n$. We denote by $|q|$ the Euclidean norm of the section s. Then we have

$$|q|^2 \leqslant \frac{1}{e^2}$$

holds. Note that the curvature Θ_E of E vanishes.

The (trivial) anti-canonicalline bundle $-K_W$ is equipped with the hermitian metric $h_L = e^{-\Psi_L}$ with $\Psi_L = \Psi$. Then

$$\sqrt{-1}\Theta_{-K_X} + n\sqrt{-1}\partial\bar{\partial} \log |q|^2 \geqslant 0$$

in terms of currents.

Now we apply [6, Theorem 13.6]: There exists a global holomorphic section ϕ of $K_X - K_X = \mathcal{O}(W)$ with $\phi(x) = a$ and

$$\int_W \frac{|\phi|^2 e^{-\Psi}}{|q|^{2n}(-\log |q|)^2} dV_{W,\omega} \leqslant C_n \frac{|a|^2}{|\Lambda^n(dq)(x)|_\omega^2} e^{-\Psi(x)}.$$

Since $|q| \leqslant (1/e)$, it follows that

$$\int_W |\phi|^2 e^{-\Psi} dV_{W,\,\omega} \leqslant C_n \frac{|a|^2}{|\Lambda^n(dq)(x)|_\omega^2} e^{-\Psi(x)}. \qquad \square$$

Proof of Theorem 1.1. — Let $\phi \in \mathcal{H}_s^{(m)}$ be a holomorphic function on U_s with $\|\phi\|_{m,\,s}^2 \leqslant 1$. Then the Hölder inequality implies that

$$(3.3) \qquad \int_{U_s} |\phi|^{2/m} e^{-\theta_s} \rho_s^n \leqslant \left(\int_{U_s} |\phi|^2 e^{-m\theta_s} \rho_s^n \right)^{\frac{1}{m}} \left(\int_{U_s} \rho_s^n \right)^{\frac{m-1}{m}}$$

$$\leqslant (\mathrm{Vol}(X_s))^{\frac{m-1}{m}} \leqslant C$$

where C can be taken as $\max(1, \mathrm{Vol}(X_s))$, and the volume of X_s is given by (2.5) being taken with respect to the Kähler class $\omega|_{X_s}$. Hence this constant is independent of m and s.

On the other hand, the equation (3.1) implies that in local coordinates, it can be written as

$$e^{-\theta_s} \rho_s^n = e^{-\psi_U} e^{-\eta_s} \rho_s^n = e^{-\psi_U} \omega_s^n = \frac{dV_X}{\omega^n \wedge f^* dV_S} \cdot \omega_s^n = \frac{dV_X}{f^* dV_S},$$

where dV_X and dV_S are Euclidean volume form in U and $p(U)$ with respect to the coordinates $(z^1, \ldots, z^{n+d})$ and $(s^1, \ldots, s^d)$, respectively. Combining with (3.3), it follows that

$$\int_{U_s} |\phi|^{2/m} \frac{dV_X}{f^* dV_S} \leqslant C.$$

Applying the $L^{2/m}$ version of the Ohsawa–Takegoshi extension theorem [1, Proposition 0.2] to $\phi \in \mathcal{O}(U_s)$, there exists a holomorphic function $F \in \mathcal{O}(U)$ with the following properties (see Remark 3.3):

(1) The restriction of F to U_s is equal to ϕ.
(2) There exists a numerical constant $C_0 > 0$ independent of m and s such that

$$\int_U |F|^{2/m} dV_X \leqslant C_0 \int_{U_s} |\phi|^{2/m} \frac{dV_X}{f^*(dV_S)}.$$

Note that the measure $dV_X/f^*(dV_S)$ coincides with $dV_{U_s}/|\Lambda^r(ds)|^2$ by [6, Remark 13.7]. It is essential that the constant C_0 does not depend on s. In fact, the Ohsawa–Takegoshi theorem states that the constant C_0 only depends on the sup norm of the holomorphic functions that define the smooth subvariety, interpreted as section of a certain vector bundle. In our case the smooth subvariety is U_s and the section is given by the holomorphic family. Hence the sup norms of the sections are uniformly bounded in terms of the s-variable.

Now the mean value inequality can be applied to the global holomorphic function F on U: The mean value theorem implies that there exists a constant which depends only on ω and r such that

$$|F(x)|^{2/m} \leqslant C_r \int_{B_r(x)} |F(z)|^{2/m} \, dV_X.$$

All together, it follows that $|F(x)|^{2/m}$ is bounded from above by a constant which is independent of m and s. Now we have an estimate for the restriction $F|_{U_s} = \phi$. Therefore the weight function θ_s is bounded from above on $V \cap X_s$ (by Theorem 3.1 above).

This argument is carried out for all θ_s, and hence the function θ is uniformly bounded from above on $V \cap X'$ where $V \subset\subset U$. This implies that the curvature $\Theta_{h^\rho_{X'/S'}}(K_{X'/S'})$ extends as a d-closed positive real $(1,1)$-current on the total space X. It completes the proof. □

Remark 3.3. — [1, Proposition 0.2] only deals with the extension from a variety of codimension 1 in the unit ball. However, the proof is easily generalized to the case of the extension from a higher codimensional variety in a bounded pseudoconvex domain using [6, Corollary 13.9]. More precisely, we have the following:

Let $\Omega \in \mathbb{C}^{n+d}$ be a bounded pseudoconvex domain and $h : \Omega \to \mathbb{C}^d$ be a holomorphic function such that $\sup_\Omega |h| \leqslant 1$; moreover, we assume that $\Lambda^r ds \neq 0$ on $V := (h = 0)$. We denote by φ a plurisubharmonic function, such that its restriction to V is well-defined (i.e, $\varphi|_V \neq -\infty$). For any holomorphic function $f : V \to \mathbb{C}$ with the property that

$$\int_V |f|^{2/m} e^{-\varphi} \frac{d\lambda_V}{|\Lambda^r ds|^2}$$

there exists a function $F \in \mathcal{O}(\Omega)$ such that:

(i) $F|_V = f$ i.e., the function F is an extension of f;
(ii) The next $L^{2/m}$ bound holds

$$\int_\Omega |F|^{2/m} e^{-\varphi} d\lambda \leqslant C_0 \int_V |f|^{2/m} e^{-\varphi} \frac{d\lambda_V}{|\Lambda^r ds|^2},$$

where C_0 is an absolute constant.

The proof of Corollary 1.2 was given in Section 1.

4. Application to the projective case

4.1. Relative Riemann–Roch Theorem for hermitian vector bundles over Kähler manifolds

This part is based upon the theorem of Bismut, Gilet and Soulé, which generalizes the Riemann–Roch–Hirzebruch–Grothendieck theorem from cohomology classes to distinguished representatives. Given a proper, smooth Kähler morphism $f : Y \to S$ with relative Kähler form $\omega_{Y/S}$ and a hermitian vector bundle (F, h) on Y, there exists a Quillen metric h^Q on the determinant line bundle

$$\lambda = \lambda(F) := \det f_!(F)$$

taken in the derived category satisfying

THEOREM 4.1 ([2, 3, 4]). — *The Chern form of the determinant line bundle $\lambda(F)$ on the base S is equal to the component in degree two of the following fiber integral.*

$$(4.1) \qquad c_1(\lambda(F), h^Q) = - \left[\int_{Y/S} td(Y/S, \omega_{Y/S}) ch(F, h) \right]^{(2)}$$

Here ch and td resp. stand for the Chern and Todd character forms resp.

Based upon the universal properties of the construction, it was generalized from hermitian vector bundles (F, h^F) to elements of the Grothendieck group, i.e. formal differences of such objects. It proved to be consistent to define $ch(G - H, h^G, h^H) = ch(G, h^G) - ch(H, h^H)$.

4.2. Application to degenerating families of Calabi–Yau manifolds

We consider the situation of Theorem 1.1, and assume that there exists a hermitian, holomorphic line bundle (L, h_0) on X such that $\omega = c_1(L, h_0)$. Following [7, Section 10] we define

$$F = \left(K_{X'/S'} - K_{X'/S'}^{-1} \right) \otimes \left(L - L^{-1} \right)^{\otimes n}$$

on X. Note that the virtual bundles $K_{X'/S'} - K_{X'/S'}^{-1}$ and $L - L^{-1}$ are of rank zero so that the Chern character form in question is

$$c_1\left(F, h^F \right) = 2c_1\left(K_{X'/S'} - K_{X'/S'}^{-1}, h^\rho \right) \wedge 2^n c_1\left(L, h^L \right)^n + \dots,$$

where terms of degree larger than $2n + 2$ are omitted. At this point we use the normalization (1.1) for ρ.

Now by (4.1) and (2.9) the curvature of the determinant line bundle is

PROPOSITION 4.2.
$$c_1\left(\lambda(F), h^Q\right) = 2^{n+1}\omega^{WP}.$$

We apply [9, Theorem 2] (cf. [10]), and get the following fact for a map $f : X \to S$ as in this section.

THEOREM 4.3. — After a blow up $\widetilde{S} \to S$ of the set $S\backslash S'$ of singular values of f, the determinant line bundle $(\lambda(F), h^Q)$ extends to $\widetilde{S}$ as a holomorphic line bundle equipped with a singular hermitian metric, whose curvature current is positive.

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Young-Jun CHOI
Pusan National University
Department of mathematics
2, Busandaehak-ro, 63beon-gil, Geumjeong-gu,
Busan, 46241 (Republic of Korea)
youngjun.choi@pusan.ac.kr

Georg SCHUMACHER
Philipps-Universität Marburg
Fachbereich Mathematik und Informatik
Lahnberge, Hans-Meerwein-Straße
Marburg, D-35032, (Germany)
schumac@mathematik.uni-marburg.de