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CYCLIC VECTORS AND INVARIANT SUBSPACES FOR THE BACKWARD SHIFT OPERATOR (1)

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1. Introduction.

Let **T** denote the unit circle and **D** the open unit disk in the complex plane. In [3] Beurling studied the closed invariant subspaces for the operator U which consists of multiplication by the coordinate function on the Hilbert space $H^2 = H^2(D)$. The operator U is called the *forward* (or *right*) *shift*, because the action of U is to transform a given function into one whose sequence of Taylor coefficients is shifted one unit to the right, that is, its action on sequences is

$$U : (a_0, a_1, a_2, \dots) \longrightarrow (0, a_0, a_1, \dots).$$

Strictly speaking, of course, the multiplication and the right shift operate on the distinct (isometric) Hilbert spaces H^2 and l^2. To keep the notation simple we shall, where no confusion is to be feared, blur this distinction, and sometimes speak of an element of H^2 as if it were an element of l^2 (the sequence of its Taylor coefficients). Likewise we denote both the shift operator on l^2 and multiplication by z on H^2 by the same symbol U. We shall also pass quite freely from the interpretation of elements of H^2 as holomorphic functions on **D** to that in which they are elements of $L^2(T)$. For a discussion of the basic concepts see Hoffman [12], Chapter 7.

Beurling's fundamental results can be summarized as follows :

i) A subspace of H^2 is closed and invariant for U if and only if it is of the form φH^2 for some inner function φ.

ii) For f in H^2 the necessary and sufficient condition that f be a cyclic vector for U (that is, the vectors $\{U^n f\}_{n=0}^{\infty}$ span H^2) is that f be outer, that is

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$$\log |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{it})| \, dt \, . \tag{1}$$

It is natural to ask what the situation is for the adjoint operator U^*. (This is the map which takes f to $\dfrac{f(z) - f(0)}{z}$ or, in terms of sequences, $(a_0, a_1, a_2, \ldots)$ to $(a_1, a_2, a_3, \ldots)$; it has therefore been called the *backward* (or *left*) *shift*). Evidently the closed invariant subspaces of H^2 under U^* are just the orthogonal complements of the spaces φH^2, and it may appear that the study of U^* presents no new problem. But, when we ask about the cyclic vectors of U^* an altogether different situation arises. Clearly, f fails to be cyclic for U^* if and only if it lies in a proper closed invariant subspace for U^*, that is, one of the spaces $(\varphi H^2)^\perp$ for some non-constant inner function φ. But, how are we to tell whether or not a given f lies in such a space ? For the forward shift the analogous question is equivalent to this : when does f have a non-constant inner factor ? And this is exactly the question answered by (1), that is, (1) is equivalent to the absence of an inner factor.

It appears that, for cyclic vectors of the backward shift, no necessary and sufficient condition as "nice" as (1) is available. We do obtain in Theorem 2.2.1 a necessary and sufficient condition for cyclicity in terms of a generalized notion of analytic continuation. Tumarkin has shown how to reformulate this condition in terms of approximation by rational functions (see § 4.1). Although Theorem 2.2.1 does not provide so "effective" a criterion as equation (1) above, it does enable us to identify various classes of cyclic and non-cyclic vectors for U^*, and generally speaking to determine whether a given f is cyclic whenever f is analytically continuable across some point of T. Moreover, it enables us to say a good deal about the structure of the set of all cyclic vectors and of the set of all non-cyclic vectors.

We can hardly claim that Theorem 2.2.1 is "deep" – on the contrary, the proof is almost a triviality. It is all the more surprising how many corollaries, some of them quite unexpected, follow from it !

In § 2 the main theorem relating cyclicity to the notion of pseudocontinuation along with certain corollaries are proved. Various classes of cyclic and non-cyclic vectors are exhibited and, in particular, it is shown that every function in H^2 with a lacunary Taylor series is cyclic. The relation between cyclicity and inner functions is investigated in § 3 along with the analytic continuability of the functions in an invariant subspace for U^*. The relation between some results of Tumarkin on rational approximation and cyclicity is explicated in § 4. Lastly, we indicate in § 5 other connections and possible extensions of our results.

Some of our results were announced earlier in [9]. We wish to acknowledge that a portion of these results were discovered independently by H. Helson and D.E. Sarason. In particular, we extend our thanks to Sarason for allowing us to incorporate one of his results in our paper (see Theorem 2.4.4 below). Helson kindly put at our disposal an unpublished manuscript dealing with cyclic vectors (which he calls "star-outer functions") ; however, we have not utilized this in writing the present paper, our point of view and methods being somewhat different, with analytic continuation playing the central role. Finally, we wish to thank G. Ts. Tumarkin, who communicated to us the results of § 4.1 and kindly consented to their inclusion in this paper.

2. The main theorem and some consequences.

2.1. *Preliminary definitions and notation.*

2.1.1. DEFINITION. – *Let Ω denote a Jordan domain in the extended complex plane (the Riemann sphere) whose boundary includes a rectifiable arc α, and suppose f is meromorphic on Ω and has a non-tangential limiting value $\varphi(t)$ at each point t of α, except for perhaps a set of (linear) measure zero. Then, f is said to have boundary values on α, and φ is the boundary function of f.* Where confusion cannot arise we shall write $f(t)$ for t in α to denote the boundary function.

By virtue of a theorem of Lusin and Privalov ([18], p. 212), f vanishes identically if it has non-tangential limiting value zero on a set of positive measure ; consequently, a meromorphic function which has boundary values almost everywhere on an arc is uniquely determined by its boundary function. (Actually, in this paper we shall work only with the functions of class $\mathcal{N}$, that is, of "bounded type" in Nevanlinna's sense, in which case the uniqueness theorem is somewhat more elementary, but is seems desirable for possible generalization to free the basic definitions from needless restrictions).

2.1.2. DEFINITION. – *For $i = 1, 2$ let Ω_i be disjoint Jordan domains in the extended complex plane whose boundaries each include the smooth arc α, and let f_i be meromorphic on Ω_i. Then f_1 and f_2 are pseudocontinuations of one another across α if they have boundary values on α which are equal almost everywhere. A pseudocontinuation, when it exists, is clearly unique, and an analytic continuation across α, if it exists, is a pseudocontinuation. We also express this relationship by saying : f_1 is pseudocontinuable across α into Ω_2 (and vice versa). This notion was introduced in* [22].

A useful remark concerning pseudocontinuation is

2.1.3. LEMMA. – *Suppose f is meromorphic on* $\mathbf{D}$, *and analytically continuable across all points of an open arc α on its boundary, with the exception of an isolated branch point on α. Then f is not pseudocontinuable across α into any contiguous domain.*

2.1.4. *The J-operator.* Let $\mathbf{D}_e$ denote the region $\{z \mid 1 < |z| \leqslant \infty\}$ of the extended complex plane. To each function f meromorphic on $\mathbf{D}$ we associate the function $F = Jf$ meromorphic on $\mathbf{D}_e$ by :

$$(Jf)(z) = F(z) = \overline{f(1/\overline{z})}\,.$$

If f is holomorphic on $\mathbf{D}$ with the Taylor series expansion $\sum_{n=0}^{\infty} a_n z^n$, then F is holomorphic on $\mathbf{D}_e$ and has the Taylor series expansion $\sum_{n=0}^{\infty} \overline{a}_n z^{-n}$. Note that if f has boundary values on $\mathbf{T}$, then so has F, and $F(t) = \overline{f(t)}$ for t in $\mathbf{T}$. Observe also that J, restricted to $H^2(\mathbf{D})$, maps it isometrically onto $H^2(\mathbf{D}_e)$. Lastly, every inner function φ has a pseudocontinuation throughout $\mathbf{D}_e$ as the meromorphic function $\widetilde{\varphi}$ defined by

$$\widetilde{\varphi}(z) = 1/(J\varphi)(z)\,.$$

Note that $\widetilde{\varphi}$ is the reciprocal of a bounded function, in fact of an inner (relative to $\mathbf{D}_e$) function.

2.1.5. LEMMA. – *Let u and v be in* $L^2[0, 2\pi]$ *and have the Fourier expansions* $\sum_{n=-\infty}^{\infty} a_n e^{int}$ *and* $\sum_{n=-\infty}^{\infty} b_n e^{int}$, *respectively. Then the product function uv is in* $L^1[0, 2\pi]$ *and has the Fourier expansion* $\sum_{n=-\infty}^{\infty} c_n e^{int}$, *where* $c_n = \sum_{k=-\infty}^{\infty} a_k b_{n-k}$.

Proof. – Apply the Parseval formula to u and $\bar{v}e^{int}$.

2.1.6. LEMMA. – *Let* R_1 *and* R_2 *be rectangular domains lying in the upper and lower half-planes, respectively, whose boundaries include the segment* $[a, b]$ *of the real axis, and let f_j be holomorphic in* R_j $(j = 1, 2)$. *Suppose that*

i) $\limsup_{y \to 0^+} \int_a^b |f_1(x + iy)| \, dx$ and $\limsup_{y \to 0^+} \int_a^b |f_2(x - iy)| \, dx$

are finite, and

ii) $\lim_{y \to 0^+} f_1(x + iy) = \lim_{y \to 0^+} f_2(x - iy)$ a.e.

Then f_1 and f_2 are ordinary analytic continuations of one another across each point of (a, b).

This is a standard exercise in the use of Cauchy's Theorem ; for more general continuation principles of this kind see Carleman [7], and Nyman [17].

The modification when $[a, b]$ is replaced by a circular arc (the case we shall actually require) is evident. Also, if i) is replaced by the analogous L^2 condition, the lemma remains true *a fortiori.*

2.1.7. *Notation.* – for u, v in $L^2(T)$, (u, v) denotes

$$\frac{1}{2\pi} \int_0^{2\pi} u(e^{it}) \overline{v(e^{it})} \, dt.$$

2.2.

We shall now state and prove our basic result concerning cyclic vectors. (Throughout the rest of this paper "cyclic" means "cyclic for U*" unless the contrary is stated).

2.2.1. THEOREM. – *A function* f *in* H^2 *is non-cyclic for* U^* *if and only if the following conditions hold :*

i) *there is a meromorphic function* $\tilde{f}$ *in* $\mathbf{D}_e$ *which is a pseudocontinuation of* f *across* $\mathbf{T}$, *and*

ii) *the function* $\tilde{f}$ *is of bounded (Nevanlinna) type in* $\mathbf{D}_e$.

Proof. – Suppose first that f has a Taylor series expansion $\sum_{n=0}^{\infty} a_n z^n$ and a pseudocontinuation $\tilde{f}$ of bounded type. We may write $\tilde{f} = \mathrm{G}/\mathrm{H}$, where G and H are in $\mathrm{H}^\infty(\mathbf{D}_e)$ and have Taylor series expansions $\mathrm{G}(z) = \sum_{n=0}^{\infty} b_n z^{-n}$ and $\mathrm{H}(z) = \sum_{n=0}^{\infty} c_n z^{-n}$. Moreover, there is no loss of generality in supposing $b_0 = 0$ since we can achieve this by replacing G and H by G/z and H/z, respectively. Now by hypothesis, $f(e^{it})\,\mathrm{H}(e^{it}) = \mathrm{G}(e^{it})$ a.e. Equating Fourier coefficients and applying Lemma 2.1.5 we obtain the equations

$$\begin{aligned} c_0 a_0 + c_1 a_1 + c_2 a_2 + \cdots &= b_0 = 0 \\ c_0 a_1 + c_1 a_2 + c_2 a_3 + \cdots &= 0\ ; \\ \vdots \qquad \vdots \qquad & \end{aligned} \tag{2}$$

hence the non-null l^2 vector $(c_0, c_1, c_2, \ldots)$ is orthogonal to $(a_0, a_1, a_2, \ldots)$ and all of its left shifts. Therefore f is not cyclic for U^*.

Conversely, suppose f is non-cyclic and let $(c_0, c_1, c_2, \ldots)$ be a non-null solution of (2) in l^2. Then by Lemma 2.1.5 the equations (2) are equivalent to the statement that the integrable function $f(e^{it})\,\mathrm{H}(e^{it})$ has a Fourier series expansion of the form $\sum_{n=1}^{\infty} b_n e^{-int}$, and is therefore the boundary function of some G in $\mathrm{H}^1(\mathbf{D}_e)$. Therefore, $\tilde{f} = \mathrm{G}/\mathrm{H}$ is a meromorphic pseudocontinuation of f across $\mathbf{T}$. Also it is of bounded type, since it is the quotient of an H^1 function by an H^2 function, and each of these is, in turn, the quotient of H^∞ functions. Thus Theorem 2.2.1 is proved.

2.2.2. *Remark.* – A curious consequence of the above proof is that if the left shifts of a sequence $(a_0, a_1, a_2, \ldots)$ in l^2 fail to span l^2, there exists a non-null sequence $(b_0, b_1, b_2, \ldots)$ orthogonal to all these left shifts that is not only in l^2 but which is actually the sequence of Taylor coefficients of an H^∞ function (we can even say, of an inner function). Of course this also follows from Beurling's theorem : every invariant subspace for U^* has the form $(\varphi H)^\perp$ for some inner function φ.

There exist non-cyclic vectors, for example, all the functions in $(\varphi H^2)^\perp$ for any inner function φ. Using Theorem 2.2.1 we can now construct cyclic vectors by many different paths.

2.2.3. THEOREM. – *If f is in H^2 and f is analytically continuable across all points of an arc α of T with the exception of an isolated branch point on α, then f is cyclic.*

Proof. – This follows at one from Theorem 2.2.1 and Lemma 2.1.3.

COROLLARY. – *The sequences* $\left\{\dfrac{1}{n+1}\right\}_{n=0}^{\infty}$ *and* $\{C_{1/2,n}\}_{n=0}^{\infty}$ *are cyclic vectors.*

Proof. – These sequences are the Taylor coefficients of the functions $-\log(1-z)$ and $(1+z)^{1/2}$, respectively, each of which has an isolated winding singularity at $z = 1$.

Of course, myriads of cyclic vectors can be obtained by choosing f to have an analytic continuation $\tilde{f}$ across some point of T, but such that $\tilde{f}$ fails to satisfy one of the conditions i) and ii) of Theorem 2.2.1. Thus, $f(z) = \exp(1/(z-2))$ is cyclic, for its analytic continuation is not meromorphic, whereas $f(z) = \sum_{k=1}^{\infty} n^{-3}\left(z - \left(1 + \frac{1}{n}\right)\right)^{-1}$ is cyclic because its analytic continuation, while meromorphic on D_e, is not of bounded type (it has too many poles).

In general, we know of no way to tell whether a function admits any pseudocontinuation, so if we are confronted with an f which is nowhere continuable in the ordinary sense (for instance $\sum_{n=1}^{\infty} n^{-1}z^{n^2}$

Theorem 2.2.1 does not provide an effective test for cyclicity. On the other hand, it is known (see [23]) that certain non-continuable functions, for example, $\sum_{n=0}^{\infty} a_n z^{2^n}$ with a_n such that the radius of convergence is one, are not even *pseudocontinuable* across any *subarc* of T into any contiguous domain, and hence, *a fortiori* these functions (if in H^2) are cyclic vectors. (We shall give a very simple proof of the latter below, see Theorem 2.5.1).

2.2.4. THEOREM. – *If* f *is holomorphic in* $|z| < R$ *for some* $R > 1$, *then* f *is either cyclic or a rational function (and hence non-cyclic).*

Proof. – That a rational function is non-cyclic is evident from Theorem 2.2.1. Suppose now that f is non-cyclic and holomorphic in $|z| < R$ with $R > 1$. Let $\tilde{f}$ denote its pseudocontinuation. Since in this case f must be a *bona fide* analytic continuation, f is extendible so as to be meromorphic on the Riemann sphere and hence is rational.

2.2.5. *Remarks.* – This Theorem is included in an earlier paper of Haplanov [11] (see also Kaz'min [13]), where the question of the totality of the set of remainders of a Taylor series expansion was studied.

2.2.6. We also mention (but leave as an exercise for the reader) that the span of the sequence $\{U^{*n} f\}_{n=0}^{\infty}$ is finite dimensional if and only if f is rational. This may be viewed as a reformulation of a classical theorem of Kronecker (cf. [4], p. 321).

2.2.7. The hypothesis of Theorem 2.2.4 can also be stated : the Taylor coefficients of f tend to zero exponentially. In this regard, it is of interest that no milder growth restriction on the Taylor coefficients suffices. This follows from the fact that, given any sequence of positive numbers $p_n \longrightarrow 0$, it is possible to choose positive numbers c_m so small that $a_n = O(e^{-np_n})$, where

$$\sum_{n=0}^{\infty} a_n z^n = \sum_{m=1}^{\infty} c_m (z - (1 + m^{-2}))^{-1} \qquad (3)$$

(cf. [20], p. 445, where such an estimation was carried out for another purpose). Yet, the function on the right hand side of (3) is non-cyclic as Theorem 2.2.1 shows.

One of the surprising consequences of Theorem 2.2.1 is that the sum of two non-cyclic vectors is non-cyclic. More generally,

2.2.8. THEOREM. – *Let f and g be non-cyclic and h cyclic. Then $f + g$ is non-cyclic and $f + h$ is cyclic. Moreover, fg and f/g are non-cyclic and fh and f/h are cyclic insofar as any of these is in* H^2.

Proof. – This is obvious by Theorem 2.2.1.

In particular, note that f is cyclic if and only if zf is cyclic.

2.2.9. COROLLARY. – *Every function f in* H^2 *is the sum of two cyclic vectors.*

Proof. – If f is cyclic, write $f = \frac{f}{2} + \frac{f}{2}$. If f is non-cyclic ; $f = (f - g) + g$ gives the desired representation where g is any cyclic vector.

2.2.10. COROLLARY. – *The set of non-cyclic vectors is a dense linear manifold in* H^2. *The set of cyclic vectors is dense in* H^2.

Proof. – Since the polynomials are non-cyclic, the first statement follows from the Theorem. For the second statement, observe that the set $\{f + p\}$, where f is a fixed cyclic vector and p an arbitrary polynomial, consists only of cyclic vectors and is dense in H^2.

It seems plausible that, although they form a dense linear manifold in H , the non-cyclic vectors are more "rare" than the cyclic. Corollary 2.2.9 points in this direction as does Theorem 2.3.3 where it is shown that they form a set of first category.

2.2.11. *Notation.* – Let $\mathfrak{N}(\mathbf{D})$ and $\mathfrak{N}(\mathbf{D}_e)$ denote the classes of meromorphic functions of bounded (Nevanlinna) type in **D** and $\mathbf{D}_e$, respectively. When no confusion will arise, we also use these letters to denote the boundary values (on **T**) of the functions in the corresponding class.

Our next theorem provides yet another method for constructing cyclic vectors which will be expanded on in the Corollary and remarks following Theorem 2.2.16.

2.2.12. THEOREM. – *Let u be a non-null function in* $L^2[-\pi, \pi]$ *such that* $\int_{-\pi}^{\pi} \log |u(t)|\, dt = -\infty$. *Then the sequence* $\{c_n\}_{n=0}^{\infty}$, *where*

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(t)\, e^{-int}\, dt\,, \tag{4}$$

is a cyclic vector in l^2.

Proof. – Suppose $\{c_n\}$ were non-cyclic. Let f denote the function in $H^2(D)$ with the Taylor series expansion $\sum_{n=0}^{\infty} c_n z^n$ and let $\tilde{f}$ be its pseudocontinuation. Moreover, let G be the function in $H^2(D_e)$ defined by $G(z) = \sum_{n=1}^{\infty} c_{-n} z^{-n}$. Then $Q = \tilde{f} + G$ is in $\mathcal{N}(D_e)$ and $Q(e^{it}) = u(t)$ a.e. Thus $\int_0^{2\pi} \log |Q(e^{it})|\, dt = -\infty$ and so Q must vanish identically on D_e, which implies $u \equiv 0$. This contradiction completes the proof.

2.2.13. COROLLARY. – *Let* A *denote a non-zero entire function of exponential type less than* π, *whose restriction to the real axis is in* $L^2(-\infty, \infty)$. *Then the sequence* $\{A(n)\}_{n=0}^{\infty}$ *is in* l^2 *and is cyclic.*

Proof. – That this sequence is in l^2 is well-known ([6], p. 101). By a theorem of Paley and Wiener, if a denotes the type of A, then $A(w) = \int_{-a}^{a} \alpha(t)\, e^{-itw}\, dt$ for some function α in $L^2[-a, a]$. Now defining

$$u(t) = \begin{cases} \dfrac{1}{2\pi}\, \alpha(t) & , \ |t| \leqslant a \\ 0 & , \ a < |t| \leqslant \pi \end{cases}$$

the desired result follows from the Theorem.

2.2.14. *Remarks.* – Clearly, the proof of Theorem 2.2.12 also establishes that the sequence $\{c_n\}_{n=0}^{\infty}$ defined by (4) (where we now assume only that u is in $L^2[-\pi, \pi]$) is cyclic, whenever we can ascertain for any reason that u is not the boundary function of any Q in $\mathcal{N}(D_e)$ (for instance, if u is equal to 1 on one interval and to 2 on another).

2.2.15. It is possible to obtain further results having the general character of the last Corollary. Without stating formal theorems, we shall merely outline the procedure. Suppose that A is an entire function of exponential type, with indicator diagram K, and *the image of* K *under the map* e^{-w} *does not separate the origin from* ∞. Then it is known (see [5], p. 7) that the function $f(z) = \sum_{n=0}^{\infty} A(n) z^n$ is analytically continuable along some radius to ∞, and the continued function admits, in the neighborhood of ∞, the Taylor series expansion $A(0) - \sum_{n=1}^{\infty} A(-n) z^{-n}$. If, in particular, $\sum_{n=0}^{\infty} |A(n)|^2 < \infty$, and the preceding series does not define a function of $\mathcal{N}(D_e)$, then the sequence $\{A(n)\}_{n=0}^{\infty}$ is cyclic. The same conclusion would follow if A were non-constant and the function $\sum_{n=1}^{\infty} A(-n) z^{-n}$ "too nice", for example, $\sum_{n=1}^{\infty} |A(-n)|^2 < \infty$, since then f would be analytically continuable across all of T.

On the other hand an example where a similar procedure leads to a class of *non-cyclic* vectors is contained in

2.2.16. THEOREM. – *Let* A *be an entire function such that* $|A(w)| \leqslant C_1 \exp(C_2 |w|^{1/2})$ *for suitable positive constants, and* $\sum_{n=0}^{\infty} |A(n)| < \infty$. *Then the sequence* $\{A(n)\}_{n=0}^{\infty}$ *is non-cyclic.*

Proof. – The hypotheses imply that the function

$$f(z) = \sum_{n=0}^{\infty} A(n) z^n$$

is continuable across all points of T except for $z = 1$, and the continued function belongs to $\mathcal{N}(\mathrm{D}_e)$. For details, as well as the construction of non-trivial functions A satisfying the hypotheses, see [24], pp. 329 ff.

2.3.

Our next Theorem asserts that the non-cyclic vectors form a set of first category in H^2. This result, communicated to us by G. Ts. Tumarkin, is certainly plausible in view of Theorem 2.2.8 (the non-cyclic vectors form a linear subspace) and a theorem of Banach (in a connected complete metrizable topological group, a proper subgroup which satisfies the condition of Baire must be of first category ([2], p. 22)). The difficulty in this approach lies in showing that the set of non-cyclic vectors satisfies the condition of Baire. It would be sufficient to show that it is a Borel set, and in Corollary 4.1.2 we shall show, in fact, using a Theorem of Tumarkin that it is an F_σ set. Here, however, we follow a more direct path.

2.3.1. Lemma. – *If f is a non-constant function in* H^2 *such that* $\int_0^{2\pi} \log |\operatorname{Re} f(e^{it})|\, dt = -\infty$, *then f is cyclic.*

Proof. – Let P denote the orthogonal projection of $\mathrm{L}^2(\mathrm{T})$ onto H^2. Then $\mathrm{P}(\operatorname{Re} f) = \frac{1}{2} f + \frac{1}{2}\bar{a}_0$, where $a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\, dt$. Hence by Theorem 2.2.8 it is enough to show that $\mathrm{P}(\operatorname{Re} f)$ is cyclic. Since f is non-constant, its real part cannot vanish a.e. on T, and the desired conclusion now follows from Theorem 2.2.12.

2.3.2. Lemma. – *For each real a the set*

$$\mathrm{E}_a = \left\{ f \in \mathrm{L}^2 : \frac{1}{2\pi}\int_0^{2\pi} \log |f(e^{it})|\, dt \geqslant a \right\}$$

is a closed, nowhere dense subset of $\mathrm{L}^2(\mathrm{T})$.

Proof. – Let $b = e^a$. Then E_a is precisely the set of functions f whose geometric mean is not less than b, that is, all f in L^2 for which

$$G(f) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{it})|\,dt\right) \geqslant b\ .$$

To show that E_a is closed, let $\{f_n\}_{n=0}^{\infty}$ be a sequence of functions in E_a converging in L^2 norm to a function f. Then we also have $\lim_{n\to\infty} \|f - f_n\|_p = 0$ for $0 < p < 2$, hence

$$\|f\|_p = \lim_{n\to\infty} \|f_n\|_p \geqslant \liminf_{n\to\infty} G(f_n) \geqslant b\ .$$

Since $G(g) = \lim_{p\to 0^+} \|g\|_p$ whenever g is in $L^q(T)$ for some $q > 0$, it follows that $G(f) \geqslant b$ and hence E_a is closed.

To show that E_a is nowhere dense, it is enough since E_a is closed to show that the complement is dense. This follows since if an arbitrary function g in $L^2(T)$ is redefined to be zero on a set of small measure, then for the perturbed function h we have $\|g - h\|$ small but $\int_0^{2\pi} \log|h(e^{it})|\,dt = -\infty$. Hence h is not in E_a and the proof is complete.

The next theorem is due to Tumarkin ; the proof presented here is different from his (see Corollary 4.1.2 for his proof).

2.3.3. THEOREM. – *The set of non-cyclic vectors is a set of first category in* H^2.

Proof. – Let H_0^2 denote the subspace of H^2 consisting of functions for which $\int_0^{2\pi} f(e^{it})\,dt = 0$. The shift operator maps H^2 isometrically onto H_0^2 and furthermore, it maps cyclic vectors onto cyclic vectors and non-cyclic vectors onto non-cyclic vectors (see the remark after Theorem 2.2.8). Thus it is sufficient to show that the set of non-cyclic vectors in H_0^2 is a set of the first category in H_0^2.

Let $\mathfrak{S}$ denote the set of real-valued functions u in $L^2(T)$ for which $\int_0^{2\pi} u(e^{it})\,dt = 0$; $\mathfrak{S}$ is complete in the L^2 metric. Further, the mapping $f \longrightarrow \sqrt{2}\,\mathrm{Re}\, f$ maps H_0^2 isometrically onto $\mathfrak{S}$. By Lemma 2.3.1 the image of the non-cyclic vectors in H_0^2 is contained in the set

$$\bigcup_{n=1}^{\infty} \left\{ u \in \mathfrak{S} : \int_0^{2\pi} \log |u(e^{it})|\, dt \geqslant -n \right\},$$

augmented by adjunction of the identically zero function, and by Lemma 2.3.2 this is a set of the first category.

2.4.

For our next result we require three lemmas. The first two are known and we state them without proof ; for a discussion of them see [16] and [19].

2.4.1. LEMMA. – *If φ is an inner function and f is in H^2, then the composed function $f \circ \varphi$ is in H^2 and*

$$\|f \circ \varphi\|_2 \leqslant \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}\, \|f\|_2 .$$

2.4.2. LEMMA. – *Let φ be an inner function, f a function in H^2 and $g = f \circ \varphi$. Then for almost all ζ in T both φ and g have radial limits, denoted by $\varphi(\zeta)$ and $g(\zeta)$, with $|\varphi(\zeta)| = 1$; moreover, f has the radial limit $f(\varphi(\zeta)) = g(\zeta)$ at $\varphi(\zeta)$. Thus for almost all e^{it} we have $g(e^{it}) = f(\varphi(e^{it}))$.*

2.4.3. LEMMA. – *If g is in H^2, φ is in H^∞, and P denotes the projection of $L^2(T)$ onto H^2 ; then $P(\overline{\varphi} g)$ is in K_g, where K_g is the closed subspace of H^2 spanned by the sequence $\{U^{*n} g\}_{n=0}^{\infty}$.*

Proof. – The subspace $(K_g)^\perp$ is invariant for U and hence has the form ψH^2 for some inner function ψ. We must show that $P(\overline{\varphi} g)$ is orthogonal to ψH^2 ; but for h in H^2 we have

$$(P(\overline{\varphi} g)\,, \psi h) = (\overline{\varphi} g\,, \psi h) = (g\,, \psi \varphi h) = 0$$

since g is orthogonal to ψH^2.

We can now state the theorem ; the "only if" statement is due to Sarason.

2.4.4. THEOREM. – *If φ is inner and f is in* H^2, *then f is cyclic if and only if $f \circ \varphi$ is cyclic.*

Proof. – Let f be non-cyclic with pseudocontinuation $\widetilde{f}$. Let $\widetilde{\varphi}$ denote the pseudocontinuation of φ (see 2.1.4) ; then $\widetilde{\varphi}$ maps $\mathbf{D}_e$ into itself. Let $g = f \circ \varphi$ and $\widetilde{g} = \widetilde{f} \circ \widetilde{\varphi}$. Then $\widetilde{g}$ is meromorphic and of bounded type in $\mathbf{D}_e$ (the property of being the quotient of two bounded analytic functions is preserved under composition). Lastly, g and $\widetilde{g}$ have the boundary values $(f \circ g)(e^{it})$ and $(\widetilde{f} \circ \widetilde{\varphi})(e^{it})$ in virtue of Lemma 2.4.2 and thus $\widetilde{g}$ is a pseudocontinuation of g. Therefore g is non-cyclic by Theorem 2.2.1.

Now assume that f is cyclic, let $g = f \circ \varphi$, and fix $n \geqslant 0$. There is a sequence of polynomials $\{p_k\}$ such that $\{p_k(U^*)f\}$ converges in norm to z^n. By Lemma 2.4.1 the sequence $\{(p_k(U^*)f) \circ \varphi\}$ converges to φ^n. Further, if $f = \Sigma\, a_n z^n$ then

$$(U^{*m} f) \circ \varphi = P(\bar{\varphi}^m g) - [a_0 \overline{\varphi(0)^m} + \cdots + a_{m-1} \overline{\varphi(0)}]\ .$$

Therefore $(p_k(U^*)f) \circ \varphi = P\, p_k(\bar{\varphi}) g + c_k$, where c_k are constants. If g were not cyclic them $K_g = (\psi H^2)^\perp$ for some inner function ψ. By Lemma 2.4.3, $P\, p_k(\overline{\varphi}) g$ is in K_g. Hence $P\, p_k(\overline{\varphi}) g + c_k$ is in $(z \psi H^2)^\perp$. It follows that φ^n is in this subspace. Hence

$$0 = (\varphi^n, z \psi g) = (\varphi^n \bar{z} \overline{\varphi}, g)$$

for all $g \in H^2$. Thus $z\psi/\varphi^n = z \psi \overline{\varphi}^n$ is in zH^2 ; in other words φ^n divides ψ. This must hold for all $n \geqslant 0$ which is impossible. Thus g is cyclic.

We obtain the following curious corollary :

2.4.5. COROLLARY. – *If φ is a finite Blaschke product and α is a complex number, $|\alpha| \geqslant 1$, then the rational function $\varphi - \alpha$ is not the square of any rational function ; in other words it has at least one zero or pole of odd order.*

Proof. – Let $f = (z - \alpha)^{1/2}$ (either branch) ; then f is cyclic and hence so is $f \circ \varphi$. But a rational function is non-cyclic ; thus neither branch of $(\varphi - \alpha)^{1/2}$ is rational.

Remark. – Similarly, $\varphi - \alpha$ cannot be a perfect cube, etc. We do not know a more direct proof of these assertions.

2.5.

We conclude with a few examples of *nowhere continuable* functions. Such functions can be cyclic or non-cyclic. An example of the latter is any nowhere continuable inner function φ. Recall that if φ is inner, $(J\varphi)^{-1}$ is a pseudocontinuation of class $\mathcal{N}(D_e)$ and hence φ is non-cyclic. Examples in the other direction are provided by the next two theorems. The first theorem is a special case of the second but the proof is much shorther and is based on a different idea.

2.5.1. THEOREM. – *Let* $f(z) = \Sigma\, a_k z^{2^k}$. *If* $a_k \neq 0$ *for infinitely many* k, *and* $\Sigma\, |a_k|^2 < \infty$, *then* f *is cyclic.*

Proof. – Suppose the contrary and let $\widetilde{f}$ denote the pseudocontinuation of f. Let λ_k denote a primitive 2^k-th root of unity. Then $f(z) - f(\lambda_k z) = p_k(z)$ is a polynomial. Therefore, $\widetilde{f}(z) - \widetilde{f}(\lambda_k z) = p_k(z)$, because this equation holds on T and hence throughout D_e. For suitable choice of k, the degree of p_k can be made arbitrarily great, and this contradicts the assumption that $\widetilde{f}(z)$ and hence $\widetilde{f}(z) - \widetilde{f}(\lambda_k z)$ has at worst a pole at infinity.

For the next theorem we need some lemmas which are all known. For completeness we include the proofs.

2.5.2. LEMMA. – *If* $\{n_k\}$ *is a lacunary sequence of positive integers, that is*

$$n_{k+1} \geqslant d n_k \quad (k = 1, 2, \ldots)$$

for some $d > 1$, *then there is a number* M *such that no positive integer* N *has more than* M *representations of the form* $N = n_j - n_k$.

Proof. – Choose M so that $d^{M-1}(d-1) \geqslant 1$. If N is given let i be the first integer such that $n_i > N$. It will be sufficient to show that if $N = n_j - n_k$ then $i \leqslant j < i + M$. The first inequality is obvious. For the second, if $j \geqslant i + M$ then

$$n_j - n_k \geqslant n_j - n_{j-1} \geqslant (d-1)\, n_{j-1} \geqslant (d-1)\, d^{M-1} n_i > N\,.$$

2.5.3. LEMMA. – *If* $b_n \geqslant 0$ *and* $\sum_{n=0}^{\infty} b_n < \infty$ *and if* $\{n_k\}$ *is lacunary, then*

$$\sum_{k=1}^{\infty} \sum_{j>k} b_{n_j - n_k} < \infty .$$

Proof. – It follows from the previous lemma that this series does not exceed $M \sum_{n=0}^{\infty} b_n$.

2.5.4. LEMMA. – *If* $b_n \geqslant 0$, $\sum_{n=0}^{\infty} b_n < \infty$, *and* $r_n = \sum_{k>n} b_k$, *then*

$$\sum_{n=0}^{\infty} b_n / r_n = \infty .$$

Proof. – $\sum_{n=0}^{\infty} b_n / r_n = \sum_{n=0}^{\infty} (r_{n-1} - r_n)/r_n = \sum_{n=0}^{\infty} ((r_{n-1}/r_n) - 1)$

which diverges since the product $\sum_{n=1}^{\infty} r_{n-1}/r_n$ diverges.

We now show that lacunary series are cyclic.

2.5.5. THEOREM. – *If* $\{n_k\}$ *is a lacunary sequence of positive integers, if* $a_k \neq 0$ *for infinitely many* k, *and if* $\Sigma\, |a_k|^2 < \infty$, *then* $f = \Sigma\, a_k z^{n_k}$ *is cyclic.*

Proof. – It is sufficient to show that z^n is in $K_f (n \geqslant 0)$. To do this it will be sufficient to show that every weak neighborhood of z^n meets K_f. We carry out the details for the case $n = 0$; the general case is similar. Finally, we may assume that $a_k \neq 0$ for all k, since in any case the set of indices k for which this holds forms a new lacunary sequence.

Let

$$f_k = (1/a_k)\, U^{*n_k} f = 1 + \sum_{j>k} (a_j/a_k)\, z^{n_j - n_k} = 1 + g_k .$$

It suffices to show that each weak neighborhood of 0 contains one of the g_k, and it suffices to consider weak neighborhoods of the form

$$N = \{h \in H^2 : |(h, h_i)| < 1, i = 1, \ldots, n\}$$

where $h_1, \ldots, h_n$ are given elements of H^2. It will be convenient to denote the Taylor coefficients of the functions h_i by putting a hat on the function, thus

$$h_i(z) = \sum_{n=0}^{\infty} \hat{h}_i(n) z^n .$$

We have

$$|(g_k, h_i)|^2 \leqslant \Big(\sum_{j>k} |(a_j/a_k)\, \hat{h}_i(n_j - n_k)|\Big)^2$$

$$\leqslant \sum_{j>k} |a_j/a_k|^2 \sum_{j>k} |\hat{h}_i(n_j - n_k)|^2 .$$

Assume that none of the g_k lie in N. Then

$$1 \leqslant \max_{1 \leqslant i \leqslant n} |(g_k, h_i)| \quad (k = 1, 2, \ldots) .$$

Hence

$$\frac{|a_k|^2}{\sum_{j>k} |a_j|^2} \leqslant \max_{1 \leqslant i \leqslant n} \sum_{j>k} |\hat{h}_i(n_j - n_k)|^2 \quad (k = 1, 2, \ldots) .$$

Summing on k we obtain a contradiction : the left side diverges by Lemma 2.5.4 while the right side converges by Lemma 2.5.3.

2.5.6. *Remarks.* – In the preceding proof, the fact that $\{n_k\}$ is a lacunary sequence is used only to establish Lemma 2.5.2. Thus, if $\{n_k\}$ is a sequence for which the conclusion of Lemma 2.5.2 is valid, if $a_k \neq 0$ for infinitely many k, and if $\sum_{k=0}^{\infty} |a_k|^2 < \infty$, then $f = \sum_{k=0}^{\infty} a_k z^{n_k}$ is cyclic.

2.5.7. As a corollary to the theorem we obtain the curious fact that no inner function has a lacunary Taylor series. This result is also a consequence of Szidon's theorem that if an L^∞ function has a lacunary Fourier series then the series must converge absolutely and so the function is continuous.

2.5.8. The preceding proof can be used to establish the cyclicity of a function with a lacunary Taylor series for a large class of weighted backward shifts (cf. § 5.5).

3. Cyclic vectors and inner functions.

3.1.

It is already clear that cyclicity is intimately related to inner functions. We prove here a few results which make this relationship more explicit. We begin with an alternate characterization of cyclic vectors that is essentially equivalent to Theorem 2.2.1. We offer two proofs deriving it both as a consequence of Theorem 2.2.1. and from the definition of cyclicity.

3.1.1. THEOREM. – *A necessary and sufficient condition that a function f in H^2 be non-cyclic is that there exist a pair of inner functions φ and ψ such that*

$$\frac{f}{\bar{f}} = \frac{\varphi}{\psi} \text{ almost everywhere on } \mathbf{T}\,. \tag{5}$$

First proof. – If (5) holds and $\mathrm{F} = \mathrm{J}f$, then $\mathrm{F}\varphi/\psi$, restricted to $\mathbf{T}$, is a boundary function for f and of class $\mathfrak{N}(\mathbf{D}_e)$; therefore f is non-cyclic by Theorem 2.2.1.

Conversely, suppose f is non-cyclic, and let $\tilde{f}$ denote its pseudo-continuation. It follows from the representation theory for the class $\mathfrak{N}(\mathbf{D}_e)$ (for the analogous theory of $\mathfrak{N}(\mathbf{D})$ see [18], pp. 75 ff.) that $\tilde{f} = \mathrm{G}/\Phi$ where G is in $\mathrm{H}^2(\mathbf{D}_e)$ and Φ is an inner function in $\mathbf{D}_e$. We can write $\mathrm{G} = \mathrm{J}g$ and $\Phi = \mathrm{J}\varphi$, where g is in $\mathrm{H}^2(\mathbf{D})$ and φ is an inner function in $\mathbf{D}$. Since $\mathrm{G} = \bar{g}$ and $\Phi = \bar{\varphi}$ on $\mathbf{T}$ (see § 2.1.4) we have $f = \tilde{f} = \bar{g}\varphi$ a.e. on $\mathbf{T}$. Since $|g| = |f|$ a.e. on $\mathbf{T}$, it follows that $g = f\psi_1/\psi_2$ where ψ_1 and ψ_2 are inner, and hence $f = \overline{f\psi_1}\,\psi_2\varphi$ a.e. on $\mathbf{T}$. But, this is just (5), with a slight change of notation.

Second proof. – If (5) holds and $n \geqslant 0$, then

$$(\mathrm{U}^{*n}f, z\varphi) = (f, z^{n+1}\varphi) = (1, z^{n+1}\varphi\bar{f}) = (1, z^{n+1}\psi f) = 0$$

showing that f and all its left shifts are orthogonal to $z\varphi$ and hence f is non-cyclic.

Conversely, assume f is non-cyclic and suppose φ is an inner function such that f is orthogonal to φH^2. Then, for g in H^2

$$0 = (f, \varphi g) = (f\overline{\varphi}, g) = (\overline{f}\varphi, \overline{g})$$

which implies that $\overline{f}\varphi = zh$ for some h in H^2. Since $|h| = |f|$ a.e. on T, $h = f\psi_1/\psi_2$ where ψ_1 and ψ_2 are inner, and so $\overline{f}\varphi = zf\psi_1\overline{\psi}_2$ a.e. on T, which can be written in the form (5) with a slight change of notation.

3.1.2. Corollary. – *A function is cyclic for* U^* *if and only if its outer factor is.*

3.1.3. *Remark.* – In view of Theorem 3.1.1 the cyclicity of a function in H^2 is completely determined by its argument. However, since the modulus of a function determines its outer factor, we see by the Corollary that the cyclicity is also determined by the modulus.

3.1.4. Definition. – *A holomorphic function in* D, *not identically zero, is normalized if the non-vanishing Taylor coefficient of lowest rank is real and positive.*

3.1.5. Theorem. – *A function f is non-cyclic for* U^* *if and only if there exists g in* H^2 *and an inner function φ such that*

$$f(e^{it}) = e^{-it}\,\overline{g(e^{it})}\,\varphi(e^{it}) \text{ a.e.} \tag{6}$$

Moreover, if we require that φ be normalized and relatively prime to the inner factor of g, then φ and g in (6) are uniquely determined. In this case the closed subspace generated by $\{U^{*n}f\}_{n=0}^{\infty}$ is precisely $(\varphi H^2)^{\perp}$.

3.1.6. *Remark.* – The inner function thus uniquely associated with each non-cyclic vector f, which we formally define as the *associated inner function* of f, plays a role in the study of the *left* shifts of f completely analogous to the role which the *inner factor* of f plays in the study of the *right* shifts.

Proof of Theorem 3.1.5. – That an element of the form (6) is non-cyclic is clear in view of the previous proof, as is the fact that every non-cyclic f satisfies a relation of the form (6). Suppose henceforth that f satisfies (6), with φ normalized and relatively prime to the inner factor of g. For the uniqueness, observe that

$$\bar{g}_1 \varphi_1 = \bar{g}_2 \varphi_2 \text{ a.e. on T}$$

with φ_i normalized and relatively prime to the normalized inner factor ψ_i of g_i $(i = 1, 2)$ we have, writing $g_i = h_i \psi_i$, that

$$\bar{h}_1 \varphi_1 \psi_2 = \bar{h}_2 \psi_1 \varphi_2 \text{ a.e. on T}$$

and since h_1 and h_2 are outer and have the same modulus a.e. on T they differ at most by a constant factor. Therefore $\varphi_1 \psi_2$ and $\psi_1 \varphi_2$ differ at most by a constant factor and, being normalized, they are identical. Therefore, φ_1 divides φ_2 and similarly φ_2 divides φ_1 so φ_1 and φ_2 are identical. It now follows that $\psi_1 = \psi_2$, $h_1 = h_2$, and hence $g_1 = g_2$.

Since clearly f and its left shifts are orthogonal to φH^2, it remains only to show that the closed span S of these is all of $(\varphi H^2)^\perp$. Now, in view of Beurling's theorem $S = (\psi H^2)^\perp$ for a certain normalized inner function ψ. Since f is orthogonal to ψH^2, it admits a.e. on T the representation $\bar{z} \bar{h} \psi$ where h is in H^2, and by cancelling redundant inner factors this may in turn be written $\bar{z} \bar{h}_1 \psi_1$, where h_1 is in H^2 and ψ_1 is a normalized inner function relatively prime to the inner factor of h_1. Observe that ψ_1 divides ψ. Now, by the uniqueness just established we conclude that $\psi_1 = \varphi$ and so φ divides ψ. On the other hand, since $\psi H^2 \supset \varphi H^2$, φ is in ψH^2 and ψ divides φ. Therefore $\varphi = \psi$ and the theorem is proved.

3.1.7. *Remarks.* – 1) Observe the symmetry in the relation of f to g : $f = \bar{z} \bar{g} \varphi$ if and only if $g = \bar{z} \bar{f} \varphi$.

2) If P denotes the orthogonal projection of $L^2(T)$ onto H^2, then $f = P(\bar{g}\varphi)$ is non-cyclic whenever g is in H^2 and φ is inner, since $f = \bar{g}\varphi + \bar{h}$ (with h in H^2) which shows that f has the pseudo-continuation $(Jg)\tilde{\varphi} + Jh$ in $\mathfrak{N}(D_e)$. Theorem 3.1.5 shows that all non-cyclic vectors arise in this way (even without projecting !).

The following Corollary has been obtained by a number of authors in special cases and in much greater generality by Sz.-Nagy and Foiaş, (cf. [26], pp. 248-9).

3.1.8. Corollary. – *If φ is an inner function, f is a function in $(\varphi H^2)^{\perp}$, and φ is analytically continuable across a point b of* $\mathbf{T}$, *then so is f.*

Proof. – Let $\tilde{f}$ be the pseudocontinuation of f to D_e. From (6) we see that $\tilde{f} = \tilde{\varphi}G/z$ where $G = Jg$ is in $H^2(D_e)$. Since $\tilde{\varphi}$ is by hypothesis holomorphic in a neighborhood of b (and hence bounded). Lemma 2.1.6 is applicable and tells us that $\tilde{f}$ is a (true) analytic continuation of f.

After proving the following lemma we obtain a somewhat more delicate corollary relating the behavior of $\tilde{\varphi}$ to that of $\tilde{f}$. A result in this direction had been obtained earlier by T. Kriete.

3.1.9. Lemma. – *Let f and φ be functions in* $H^1(\mathbf{D})$ *such that* i) *φ is inner,* ii) *f and φ have no common inner factor and* iii) *the quotient f/φ is bounded in a neighborhood of a point b in* $\mathbf{T}$. *Then φ is continuable across the point b.*

Proof. – We reduce the problem to the case of a bounded function on $\mathbf{D}$.

First of all let U be a disk centered at b such that $|f(z)/\varphi(z)| \leqslant M$ for z in $U \cap \mathbf{D}$ and let V be a disk centered at b properly contained in U. There exists an outer function h in $H^\infty(\mathbf{D})$ such that $|h(z)| \leqslant 1$ and $|f(z)\,h(z)| \leqslant 1$ for z in $\mathbf{D}$. Now factor $\varphi = \varphi_1\varphi_2$ such that φ_1 and φ_2 are inner functions ; $\varphi_2(z) \neq 0$ for z in $\mathbf{D} - V$ and the singular measure in the canonical representation for φ_2 is supported by $\bar{V} \cap \mathbf{T}$; and $\varphi_1(z) \neq 0$ for z in $V \cap \mathbf{D}$ and the singular measure in the canonical representation for φ_1 is supported by $\mathbf{T} - V$. It follows that $\varepsilon = \inf\{|\varphi_2(z)| : z \in \mathbf{D} - U\} > 0$.

Consider now the function $fh\varphi_1/\varphi$ defined on $\mathbf{D}$. For z in $U \cap \mathbf{D}$ we have

$$\left|\frac{f(z)\,h(z)\,\varphi_1(z)}{\varphi(z)}\right| \leqslant \left|\frac{f(z)}{\varphi(z)}\right| \leqslant M ,$$

and for z in $\mathbf{D} - U$ we have

$$\left|\frac{f(z)\,h(z)\,\varphi_1(z)}{\varphi(z)}\right| = \left|\frac{f(z)\,h(z)}{\varphi_2(z)}\right| \leqslant \frac{1}{\varepsilon} \cdot$$

Therefore $\dfrac{fh\varphi_1}{\varphi}$ is bounded on D and hence is in $H^\infty(D)$. It is now a simple consequence of the representation theory [18] that φ divides $fh\varphi_1$. This implies $\varphi_2 \equiv 1$ from which the result follows.

3.1.10. COROLLARY. – *If f is a non-cyclic vector with pseudocontinuation $\tilde{f}$ and f is analytically continuable across a point b of* T, *the same is true for every element of the closed span* S *of* $\{U^{*n} f\}_{n=0}^{\infty}$. *Moreover, if h is in* S, *then the pseudocontinuation of h can only have a pole where $\tilde{f}$ does and of a multiplicity not exceeding that of the corresponding pole of $\tilde{f}$.*

Proof. – Let φ denote the associated inner function of f. It is enough, in proving the first assertion, to establish that φ is continuable across b, in view of Corollary 3.1.8. This is the same as showing that $J\varphi = 1/\tilde{\varphi}$ is continuable (in the reverse direction) across b. From the theorem we have $zf = \bar{h}\varphi$, where φ and h have no common inner factor. Hence

$$z\tilde{f} = Jh/f\varphi \ .$$

Note that $J\varphi$ is an inner function in D_e, and that $J\varphi$ and Jh have no common inner factor in $H^2(D_e)$. The left side is bounded near b by hypothesis, and the result now follows from the analogue of the preceding lemma for D_e.

For the second assertion of the Corollary, observe that if $\tilde{f}$ has a pole of order m at the point $w \neq \infty$ in D_e, then $J\varphi$ has a zero of order precisely m at w (recall that Jg and $J\varphi$ have no common inner factor and hence no common zeros). If now h is in S, then $z\tilde{h} = Q/J\varphi$ for some Q in $H^2(D_e)$, and so $\tilde{h}$ can have at most a pole of order m at w.

Finally, if $\tilde{f}$ has a pole of order m at ∞, then $J\varphi$ must have a zero of order precisely $m + 1$ at ∞. Hence $\tilde{h} = Q/z\, J\varphi$ has at most a pole of order m at ∞. This completes the proof.

From the above analysis, one can describe fairly explicitly (in terms of their pseudocontinuations) the elements of $(\varphi H^2)^\perp$. Thus for example, when $\varphi(z) = \exp\left(\dfrac{z+1}{z-1}\right)$ the elements of $(\varphi H^2)^\perp$ have no singularities on the Riemann sphere except possibly at $z = 1$, hence they are of the form $F\left(\frac{z+1}{z-1}\right)$ where F is entire. It is easy to check moreover that F must be of exponential type, and it should not be very hard to describe precisely the F which arise in this way.

Similiar remarks apply to the explicit description of the closed U*-invariant space generated by a given non-cyclic vector. In this direction we have the following two corollaries.

3.1.11. Corollary. – *Let φ be inner and let f be in* H^2. *Then f is in $(\varphi H^2)^\perp$ if and only if f has a pseudocontinuation $\tilde{f}$ such that*

$$z\tilde{f}/\tilde{\varphi} \text{ is in } H^2(D_e)\ .$$

Proof. – If f is in $(\varphi H^2)^\perp$, then by the theorem there is a g in H^2 such that $f = \overline{z}\overline{g}\varphi$, that is, $\tilde{f} = \tilde{\varphi}G/z$, where $G = Jg$ is in $H^2(D_e)$.

Conversely, if $z\tilde{f}/\tilde{\varphi} = G$ is in $H^2(D_e)$, choose g in H^2 such that $G = Jg$. Then

$$zf/\varphi = z\tilde{f}/\tilde{\varphi} = G = \overline{g} \quad \text{(on T)}$$

or $f = \overline{z}\overline{g}\varphi$ and so f is in $(\varphi H^2)^\perp$.

3.1.12. Corollary. – *If f is in $(z\varphi H^2)^\perp$, then for each a in* D *the function h is in $(\varphi H^2)^\perp$, where*

$$h(z) = \frac{f(z) - f(a)}{z - a}\ .$$

Proof. – Clearly h has a pseudocontinuation $\tilde{h}$. We have

$$\frac{z\tilde{h}}{\tilde{\varphi}} = \frac{1}{z-a}\,\frac{z\tilde{f}}{\tilde{\varphi}} - \frac{z}{z-a}\,\frac{f(a)}{\tilde{\varphi}}\ .$$

The first term on the right is in $H^2(D_e)$ since $z\tilde{f}/\tilde{\varphi}$ is in $H^2(D_e)$ by Corollary 3.1.11 and the other factor is bounded. The second term on the right is bounded in D_e. The result now follows from the previous corollary.

In the same order of ideas we record the following which is a consequence of Lemma 2.4.3.

3.1.13. THEOREM. – *If* $\mathfrak{M}$ *is a closed subspace of* H^2 *invariant for* U^*, *and* f *in* $\mathfrak{M}$ *is divisible by the inner function* ψ, *then* f/ψ *is in* $\mathfrak{M}$.

3.2. The cyclic nature of U* invariant subspaces.

3.2.1. THEOREM. – *If* $\mathfrak{M}$ *is a closed subspace of* H^2, *invariant for* U^*, *then there is an element* f *in* $\mathfrak{M}$ *such that the sequence* $\{U^{*n} f\}_{n=0}^{\infty}$ *spans* $\mathfrak{M}$ *(in short : $\mathfrak{M}$ is a cyclic subspace).*

Proof. – By Beurling's theorem, $\mathfrak{M} = (\varphi H^2)^{\perp}$ for some inner function φ. If $f = U^*\varphi$, then f is in $\mathfrak{M}$ since for all g in H^2 we have $(f, \varphi g) = (\varphi, z\varphi g) = (1, zg) = 0$. Suppose now that h is in H^2 and $(U^{*n} f, h) = 0$ for $n \geq 0$. Then for $n \geq 1$ we have

$$0 = (h, U^{*n}\varphi) = (z^n h, \varphi)$$

showing that $h\bar{\varphi}$ is a boundary function of $H^2(D)$ and hence h is in $\varphi H^2 = \mathfrak{M}^{\perp}$. This proves the Theorem.

3.2.2. *Remark.* – If φ is inner and ψ divides φ, it is readily checked that $U^*\psi$ is in $(\varphi H^2)^{\perp}$. A question we have not been able to answer is the following :

Is $(\varphi H^2)^{\perp}$ *equal to the closed linear manifold spanned by the functions* $\{U^*\psi\}$, *where* ψ *ranges over the inner functions that divide* $\varphi(\psi \neq \varphi)$?

Here we must assume that φ is not a power of a single Blaschke factor. It is easy to give an affirmative answer in case φ is a Blaschke product, say with zeros at the points $\{z_n\}$ with multiplicities $\{m_n\}$, where we assume that there are at least two distinct zeros, that is, $n > 1$. Indeed, it can be shown that the functions

$$U^*\left(\frac{z - z_n}{1 - \bar{z}_n z}\right)^j \quad (j = 1, 2, \ldots, m_n) \tag{7}$$

are linearly independent and hence span the orthogonal complement of the subspace of functions with a zero of order at least m_n at z_n. Thus the functions (7) for $n = 1, 2, \ldots$ span $(\varphi H^2)^{\perp}$.[3]

[3] We have just received a preprint by Ahern and Clark entitled "On functions orthogonal to invariant subspaces". Their Corollary 8.1 gives an affirmative answer to the above question in general.

3.3. Reproducing kernels and overconvergence.

With the usual inner product, H^2 is a Hilbert space having the reproducing kernel (r.k.)

$$K_a(z) = K(z\,,a) = \frac{1}{1-\bar{a}z}$$

(K is also known as the Szegö kernel function of D). We assume familiarity with the most basic properties of reproducing kernels, see e.g. Aronszajn [1], Meschkowski [14]. The r.k. of φH^2 is $\overline{\varphi(a)}\,\varphi(z)\,K(z\,,a)$ and that of $(\varphi H^2)^\perp$ is

$$k_a(z) = k(z\,,a) = (1 - \overline{\varphi(a)}\,\varphi(z))\,K(z\,,a)\,.$$

Throughout this section we shall write $\mathfrak{M}_\varphi$ for $(\varphi H^2)^\perp$ and in the proof of the next lemma for notational convenience write Φ in place of $\tilde{\varphi}$, the pseudocontinuation of φ. The function Φ is analytic in D_e except for a pole at $1/\bar{b}$ (of appropriate multiplicity) corresponding to each zero b of φ in D. The pseudocontinuation of f shall, as usual, be denoted by $\tilde{f}$.

3.3.1. LEMMA. – *If λ is a point of D_e at which Φ is analytic, then the map $f \longrightarrow \tilde{f}(\lambda)$ is a bounded linear functional on $\mathfrak{M}_\varphi$.*

Proof. – Let f be in $\mathfrak{M}_\varphi$; then by Theorem 3.1.5 we have $f = \bar{g}\varphi$ for some g in H^2, $g(0) = 0$, and hence $\tilde{f} = G\Phi$, where $G = Jg$ is in $H^2(D_e)$. Let $A(\lambda)$ denote the norm of the (bounded) linear functional "evaluation at λ" in $H^2(D_e)$. Then $|\tilde{f}(\lambda)| = |\Phi(\lambda)|\,|G(\lambda)|$ and $|G(\lambda)| \leqslant A(\lambda)\,\|G\| = A(\lambda)\,\|f\|$, since $|G| = |f|$ on T. This proves the lemma with the bound $A(\lambda)\,|\Phi(\lambda)|$.

We next obtain a more useful estimate for the bound.

Let $a_1\,,a_2\,,\dots,a_n$ be distinct points of D, and $k_i = k_{a_i}$. Let λ be a point of D_e at which Φ is analytic and set

$$G_\lambda(z) = \frac{1-\overline{\Phi(\lambda)}\,\varphi(z)}{1-\bar{\lambda}z} = \frac{a}{\varphi(a)}\,\frac{\varphi(z)-\varphi(a)}{z-a}\quad (a = 1/\bar{\lambda})\,.$$

Note that G_λ is holomorphic in D. We now observe that G_λ is in $\mathfrak{M}_\varphi$; this is a consequence of Corollary 3.1.12, since φ is orthogonal to

$z\varphi H^2$. Therefore, $(G_\lambda, k_b) = G_\lambda(b)$ for $|b| < 1$; hence for complex numbers $c_1, c_2, \dots c_n$ we have

$$\left(G_\lambda, \sum_{i=1}^{n} c_i k_i\right) = \sum_{i=1}^{n} \bar{c}_i G_\lambda(a_i) \tag{8}$$

$$= \sum_{i=1}^{n} \bar{c}_i \frac{1 - \overline{\Phi(\lambda)}\,\varphi(a_i)}{1 - \bar{\lambda} a_i}$$

Let us write $f = \sum_{i=1}^{n} c_i k_i$ and

$$\tilde{f}(z) = \sum_{i=1}^{n} c_i \frac{1 - \overline{\varphi(a_i)}\,\Phi(z)}{1 - \bar{a}_i z} \quad \text{for } z \text{ in } \mathbf{D}_e$$

so that $\tilde{f}$ is the pseudocontinuation of f. From (8) we obtain, using the Cauchy-Schwartz inequality, that

$$|\tilde{f}(\lambda)| \leqslant \|G_\lambda\| \cdot \|f\| . \tag{9}$$

Since (9) holds for a dense set of f (namely, the finite linear combinations of the r.k.), and we know by Lemma 3.3.1 that the map $f \longrightarrow \tilde{f}(\lambda)$ defines a bounded linear functional on $\mathcal{M}_\varphi$, we conclude that (9) holds for all f in $\mathcal{M}_\varphi$.

3.3.2. DEFINITION. – *The singular set of an inner function φ is the union of the closure of the set of poles of $\tilde{\varphi}$ and the closed support of the singular measure on* **T** *which occurs in the canonical representation of φ.*

Thus the singular set is a compact subset of the closure of $\mathbf{D}_e$. Equivalently, it is the complement of the union of the domains of holomorphy of φ and $\tilde{\varphi}$. As we know by Corollary 3.1.8, if the singular set does not contain the point b in **T**, then all f in $\mathcal{M}_\varphi$ are analytically continuable across b. Moreover, from Lemma 3.3.1 it follows that f cannot have a pole in $\mathbf{D}_e$ except (possibly) at a point where $\tilde{\varphi}$ does.

In order to state the next theorem let us introduce the following notation. Let Ω denote the complement in the Riemann sphere of the singular set of φ. By the remarks of the last paragraph it is possible to extend f in $\mathcal{M}_\varphi$ to be a holomorphic function on Ω, which we will denote by f^*, as follows : for z in $\mathbf{D}$ define $f^*(z) = f(z)$, for z in $\mathbf{D}_e \cap \Omega$ define $f^*(z) = \widetilde{f}(z)$, and for z in $\mathbf{T} \cap \Omega$ let $f^*(z)$ be the value of the extended function whose existence is given by Corollary 3.1.8.

3.3.3. THEOREM. – *Let φ be an inner function and* E *be a compact subset of Ω. If $\{f_n\}_{n=1}^\infty$ is a Cauchy sequence which lies in $\mathcal{M}_\varphi$, then $\{f_n^*(z)\}_{n=1}^\infty$ converges uniformly on* E.

Proof. – It is clearly sufficient to show that the map $\lambda \longrightarrow f^*(\lambda)$ is a bounded linear functional on $\mathcal{M}_\varphi$ whose norm is less than some fixed bound for all λ in E. Now for f in $\mathcal{M}_\varphi$ of norm one, we have

$$|f^*(\lambda)|^2 = |f(\lambda)|^2 \leqslant k(\lambda, \lambda) = \frac{1 - |\varphi(\lambda)|^2}{1 - |\lambda|^2} \tag{10}$$

for λ in $\mathbf{D}$, while for λ in $\mathbf{D}_e$ we have by (9) that

$$|f^*(\lambda)|^2 = |\widetilde{f}(\lambda)|^2 \leqslant \| G_\lambda \| = \left|\frac{a}{\varphi(a)}\right|^2 \frac{1}{2\pi} \int_0^{2\pi} \left|\frac{\varphi(e^{it}) - \varphi(a)}{e^{it} - a}\right|^2 dt, \tag{11}$$

where $a = 1/\lambda$. It is clear from (10) and (11) that for f of norm one $|f^*(\lambda)|$ is bounded for λ in E by a bound that is independent of λ. For, if E contains points of $\mathbf{T}$, φ is analytically continuable across these points and the right hand sides of (10) and (11) remain uniformly bounded as λ approaches such points. This proves the Theorem.

3.3.4. *Remark.* - In the special case where φ is a Blaschke product, and the f_n are restricted to be finite linear combinations of r.k., this Theorem follows from general results in [21], when specialized to the space H^2.

4. Cyclic vectors and rational approximation.

This section is an elaboration of the connection between our subject and the work of G. Ts. Tumarkin on rational approximation. In § 4.1 a final characterization of cyclic vectors is stated in terms of rational approximation and is based on Theorem 2.2.1 and the work of Tumarkin. In § 4.2 we restate the result and give a direct proof. Using these same ideas we pursue a characterization of the invariant subspaces for U^* in terms of rational approximation in § 4.3.

4.1.

It will be convenient in what follows to introduce the following notation : if E is a finite sequence of numbers $z_1, z_2, \ldots, z_n$ in D_e, then S(E) shall denote $\sum_{i=1}^{n} (1 - |z_i|^{-1})$.

4.1.1. THEOREM. – (Tumarkin). *A necessary and sufficient condition that a function f in H^2 be non-cyclic is that there exist a sequence of rational functions $\{f_n\}_{n=1}^{\infty}$ with $\|f - f_n\|_2 \longrightarrow 0$, and such that* i) *the pole of f_n lie in D_e, and (if E_n denotes the poles of f_n counted with multiplicity)* ii) *the sequence $S(E_n)$ is uniformly bounded.*

Proof. – This result follows at once by combining Theorem 2.2.1 with general theorems of Tumarkin relating the boundary values of meromorphic functions of bounded type and the distribution of the poles of approximating rational functions. (See Theorem 1 of [28] ; see also [27] for a survey of Tumarkin's work in this general area).

The following corollary is a refinement of Theorem 2.3.3 ; it is due to Tumarkin.

4.1.2. COROLLARY. – *The set of non-cyclic vectors is an F_σ set of the first category in H^2.*

Proof. – For k a non-negative integer let R_k denote the set of rational functions f with poles in D_e and whose sequence of poles E satisfies $S(E) \leqslant k$. Let $\overline{R}_k$ denote the closure (in H^2) of R_k. By the Theorem, $\bigcup_{k=1}^{\infty} \overline{R_k}$ is the set of non-cyclic vectors. Moreover, $\overline{R_k}$ is nowhere dense, since it is closed and fails to interest the (dense) set of cyclic vectors.

4.2.

In this section we give a direct proof of Theorem 4.1.1 not requiring knowledge of Tumarkin's work. We present this proof both for completeness and for the light it sheds on an interesting aspect of the structure of the invariant subspaces for the backward shift operator.

Before beginning it will be convenient to introduce the following notation : if B is a (finite or infinite) Blaschke product, then $\sigma(\mathrm{B})$ shall denote the sum $\sum_n (1 - |z_n|)$, where $\{z_n\}$ is the sequence of zeros of B (multiple zeros counted multiply). For the empty product $\mathrm{B}(z) \equiv 1$, $\sigma(\mathrm{B})$ is defined as zero.

The following proposition, a slight reformulation of Theorem 4.1.1, is equivalent to it :

4.2.1. THEOREM. – *A necessary and sufficient condition that a function f in H^2 be non-cyclic is that there exist a sequence of Blaschke products $\{\mathrm{B}_n\}_{n=1}^{\infty}$ and a sequence of functions $\{f_n\}_{n=1}^{\infty}$ in H^2 such that*

i) $\sup_n \sigma(\mathrm{B}_n) \leqslant \mathrm{M} < \infty$,

ii) f_n *is in* $(\mathrm{B}_n \mathrm{H}^2)^{\perp}$, *and*

iii) $\|f - f_n\|_2 \longrightarrow 0$.

To show the equivalence of these two theorems it is enought to show that f can be approximated in the way described in Theorem 4.2.1 if and only if it can be approximated in the way described in Theorem 4.1.1.

Thus assume that $\{\mathrm{B}_n\}$ and $\{f_n\}$ are sequences of functions satisfying i), ii), iii) of Theorem 4.2.1. If $\{z_{n,1}, z_{n,2}, \ldots\}$ denotes the zeros (with multiplicities) of B_n, then on the basis of an earlier remark (in § 3.2.2), every function in $(\mathrm{B}_n \mathrm{H}^2)^{\perp}$ can be approximated in H^2 norm as closely as desired by rational functions whose poles (counting multiplicity) are a subset of $\{\bar{z}_{n,1}^{-1}, \bar{z}_{n,2}^{-1}, \ldots\}$. This together with i) and iii) implies the existence of a sequence of rational functions approximating f as described in Theorem 4.1.1.

Conversely, let f be the limit of a sequence of rational functions $\{f_n\}$ satisfying i) and ii) of Theorem 4.1.1. Then f_n is in $(B_n H^2)^\perp$, where B_n is the finite Blaschke product with zeros (counting multiplicities) at the conjugate reciprocal points to the poles of f_n. Also, $\sigma(B_n) \leqslant M$ follows from ii) of Theorem 4.1.1. Thus the functions f_n approximate f in the manner described in Theorem 4.2.1.

Thus, to establish Theorem 4.1.1 it is enough to demonstrate Theorem 4.2.1.

Proof(4) *of Theorem 4.2.1.* – If f is non-cyclic, then by Theorem 3.1.5 there exists an inner function φ and a function g in H^2 such that $g(0) = 0$ and $f(e^{it}) = \overline{g(e^{it})}\,\varphi(e^{it})$ a.e. If φ is a Blaschke product we are done (taking $B_n = \varphi$ and $f_n = f$ for $n = 1, 2, 3, \ldots$). In the general case, we use the fact that there exists a sequence of complex numbers $\lambda_n \longrightarrow 0$ such that each $B_n = \dfrac{\varphi - \lambda_n}{1 - \overline{\lambda}_n\varphi}$ is a Blaschke product. (This is a consequence of a Theorem due to Frostman ; for a simple proof see [12], pp. 175-6). The sequence $\{B_n\}_{n=1}^\infty$ converges to φ in the norm of H^∞. Define $f_n = P(\overline{g}B_n)$, where P denotes, as usual, the projection of L^2 onto H^2. Then f_n is in $(B_n H^2)^\perp$ and since

$$f - f_n = P(f - \overline{g}B_n) \ ,$$

we have

$$\|f - f_n\| \leqslant \|f - \overline{g}B_n\| = \|\overline{g}\varphi - \overline{g}B_n\| \leqslant \|g\|\,\|\varphi - B_n\|_\infty$$

which tends to 0 as $n \longrightarrow \infty$. Thus, it remains only to show that $\sup_n \sigma(B_n) < \infty$. Let $\{z_{n,j}\}_j$ denote the zeros of B_n. Let us suppose

(4) This proof is illuminated by the following result based on a lemma of Ahern and Clark (Lemma 4.1, Radial limits and invariant subspaces, preprint). To each non-null function φ in the unit ball of H^∞ assign the positive measure $\mu(\varphi)$ on $\mathbf{D}$ such that

$$d\mu = -\frac{1}{2\pi}\log|\varphi(e^{it})|\,dt + d\sigma + d\sigma_B$$

where σ is the singular measure on $\mathbf{T} \subset \overline{\mathbf{D}}$ associated with the singular inner factor of φ, B is the Blaschke factor of φ, and σ_B is the measure which assigns to each zero of B of multiplicity m the weight $m(1 - |z|)$. Then for a sequence $\{\varphi_n\}_{n=0}^\infty$ of non-null functions in the unit ball of H^∞, $\{\varphi_n\}$ converges uniformly on compact subsets of $\mathbf{D}$ to φ_0 if and only if $\mu(\varphi_n)$ converges in the weak*-topology to $\mu(\varphi_0)$. Added in proof. More general results of this nature, valid for all functions of bounded characteristic, were obtained several years ago by Tumarkin [29], [30].

first that $\varphi(0) \neq 0$. Since $B_n(0) \longrightarrow \varphi(0)$ we have for all sufficiently large n that

$$0 < \frac{1}{2} |\varphi(0)| \leqslant |B_n(0)| = \prod_j |z_{n,j}| ,$$

and by the elementary inequality

$$1 - x \leqslant -\log x \quad , \quad 0 < x \leqslant 1$$

we have

$$\sigma(B_n) = \sum_j (1 - |z_{n,j}|) \leqslant - \sum_j \log |z_{n,j}|$$

$$= \log \left(\prod_j |z_{n,j}|^{-1} \right) \leqslant \log (2/|\varphi(0)|)$$

for large n and we are done.

If $\varphi(0) = 0$, then write $\varphi = z^m \psi$, where ψ is an inner function and $\psi(0) \neq 0$. Approximate φ by the sequence $B_n^* = z^m B_n$, where $\{B_n\}$ are Blaschke products converging uniformly to ψ. Then the proof goes through in the same way. Thus the necessity part of Theorem 4.2.1 is established.

Assume now that f is in H^2 and $\{B_n\}_{n=1}^{\infty}$ and $\{f_n\}_{n=1}^{\infty}$ are sequences of functions satisfying i), ii) and iii). Write $B_n = B_n' B_n''$, where B_n' is a Blaschke product with zeros inside the closed disk of radius $1/2$, and B_n'' is a Blaschke product with zeros outside the closed disk of radius $1/2$(5). Since $\sigma(B_n') \leqslant \sigma(B_n)$ we see that the number of zeros of B_n' is bounded by $2M$. Choose now a subsequence such that $\{B_{n_k}'\}_{k=1}^{\infty}$ converges uniformly on $\mathbf{D}$ to a finite Blaschke product $\varphi_1 (\not\equiv 0)$ and $\{B_{n_k}''\}_{k=1}^{\infty}$ converges uniformly on compact subsets of $\mathbf{D}$ to a function φ_2 in H^{∞}. Then from the elementary inequality

$$-\log x \leqslant 2(1 - x) \; , \; \frac{1}{2} \leqslant x \leqslant 1$$

we obtain

$$\log (1/|B_{n_k}''(0)|) = - \log \left(\prod_j |z_{n_k,j}| \right)$$

$$= - \sum_j \log |z_{n_k,j}|$$

(5) As usual, an empty product is interpreted as 1.

$$\log(1/|B''_{n_k}(0)|) \leqslant 2 \sum_j (1 - |z_{n_k,j}|) \leqslant 2\sigma(B_{n_k}) ,$$

where $\{z_{n,j}\}_j$ denote the zeros of B''_n. Since $B''_{n_k}(0) \longrightarrow \varphi_2(0)$ we obtain that $\varphi_2(0) \neq 0$ and hence $\varphi_2 \not\equiv 0$. Thus $\varphi = \varphi_1 \varphi_2 \not\equiv 0$.

Now for h in H^2 the sequence $\{B_{n_k} h\}_{k=1}^{\infty}$ converges weakly to φh. Combining this with the fact that $\|f - f_n\|_2 \longrightarrow 0$ we obtain $0 = (f_n, B_n h) \longrightarrow (f, \varphi\ h)$. Hence f is in $(\varphi H)^{\perp}$ and is therefore non-cyclic. This completes the proof of Theorem 4.2.1.

4.3. We have also the following more general result.

4.3.1. Theorem. – *Let $\{B_n\}$ denote a sequence of Blaschke products such that $\sup_n \sigma(B_n) < \infty$. The set Q of f in H^2 which can be written as $f = \lim f_n$ (in the norm of H^2) where f_n is in $(B_n H^2)^{\perp}$ is a proper closed U^*-invariant subspace of H^2.*

Conversely, every proper closed U^*-invariant subspace of H^2 arises in this way by suitable choice of Blaschke products $\{B_n\}$ with $\sup_n \sigma(B_n) < \infty$.

4.3.2. *Remarks.* – Of course, the choice of the $\{B_n\}$ corresponding to a given subspace in not unique.

4.3.3. The theorem can be reformulated in terms of rational approximation with restricted poles, rather than Blaschke products and subspaces $(B_n H^2)^{\perp}$, as in the statement of Theorem 4.1.1. The present formulation was chosen because the proof of this version is technically more convenient.

Proof of Theorem 4.3.1.

1) Suppose we are given the B_n with $\sup_n B_n < \infty$. It is a fairly direct verification, which we omit, that Q is a closed linear subspace of H^2. Also, it is U^*-invariant, since $f = \lim f_n$ implies $U^* f = \lim U^* f_n$, and each $(B_n H^2)^{\perp}$ is U^*-invariant. Finally, it is proper, since by Theorem 4.2.1 it contains no cyclic vector.

2) Conversely, suppose we are given a closed proper U*-invariant subspace $S = (\varphi H^2)^\perp$, where φ is an inner function. As in the proof of the first half of Theorem 4.2.1 we can construct a sequence of Blaschke products B_n converging to φ in the norm of H^∞ and such that $\sup_n \sigma(B_n) < \infty$. It was shown there that every f in S can be written $f = \lim f_n$, where f_n is in $(B_n H^2)^\perp$. It therefore remains only to show that $f = \lim f_n$, where f_n is in $(B_n H^2)^\perp$ implies f is in S. However, this has been already done in the last paragraph of the proof of the second half of Theorem 4.2.1 (in our present situation $B_n h$ converges to φh in norm for all h in H^2).

4.4. *Remark.* – It is possible to reformulate Theorem 4.1.1 in a still different way which perhaps is of some interest. Observe that the eigenvalues of U* are precisely the points λ in **D**, and to each eigenvalue λ corresponds the unit eigenvector e_λ where

$$e_\lambda(z) = \frac{(1 - |\lambda|^2)^{1/2}}{1 - \lambda z}$$

(This was remarked by Beurling [3], p. 244).

Suppose f is a finite linear combination $\Sigma\, c(\lambda)\, e_\lambda$ of eigenvectors of U*, whereby all λ are assumed distinct. By the *rank* of such an f we mean $\Sigma(1 - |\lambda|)$, summed over those λ for which $c(\lambda) \neq 0$. Then the following is easily seen to be equivalent to Theorem 16 :

4.4.1. THEOREM. – *The necessary and sufficient condition that an element of* H^2 *be non-cyclic for* U* *is that it be the limit of a sequence of finite linear combinations of eigenfunctions of* U* *whose ranks are bounded.*

Proof. – We omit the simple verification.

A similar reformulation of Theorem 4.3.1 is of course also possible. It remains to be seen whether the non-cyclic vectors of any other interesting operators admit of an analogous characterization ("rank" being defined in an appropriate way).

5. Concluding remarks.

Here we gather miscellaneous comments and results, mostly without proof.

5.1. Cyclic vectors and Hankel matrices.

If $(a_0, a_1, a_2, \ldots)$ is a vector in l^2, then the matrix

$$A = (a_{i+j})_{i,j=0}^{\infty}$$

is called a Hankel matrix. A result of Nehari [15] states that A is a bounded operator on l^2 if and only if the $\{a_n\}$ are the Fourier coefficients of positive index of a bounded function. In any case A defines a closed, possibly unbounded, operator on l^2 with dense domain. The connection between A and the left shift U* is this : the successive column vectors of the matrix A are the successive left shifts of the vector $(a_0, a_1, a_2, \ldots)$; hence, A has dense range if and only if $(a_0, a_1, a_2, \ldots)$ is cyclic for U*. Alternatively, the successive row vectors of A are the successive left shifts of $(a_0, a_1, a_2, \ldots)$ and so A is one-to-one if and only if $(a_0, a_1, a_2, \ldots)$ is cyclic for U*.

5.2. Unitary functions.

A unitary function is a measurable function whose modulus is one almost everywhere. Recent results of Douglas and Rudin [8] tell us that the quotients of inner functions are uniformly dense in the unitary functions on T. Thus the problem posed by Theorem 3.1.1 of deciding when a unitary function is the quotient of two inner functions is an extremely delicate one.

5.3. Shifts of higher multiplicity.

Certain of the results of this paper can be carried over to shifts of higher multiplicity. Here we merely point out that it can be shown that any backward shift of countable multiplicity has a cyclic vector while the forward shift of multiplicity greater than one has no cyclic vector since the codimension of the range is greater than one (see [10], Problem 126).

Another possible generalization of our theory to shifts of higher multiplicity can be arrived at by considering the following definition of cyclic vector for U*. A function f in H^2 is cyclic for U* if and only if the subspace $\{P(\bar{\psi} f) \mid \psi \in H^\infty\}$ is dense in H^2.

5.4. Cyclic vectors in other H^p and l^p spaces.

There is no difficulty in extending the results of this paper to the spaces $H^p (1 < p < \infty)$ using the "near" duality of H^p with $H^q \left(\frac{1}{p} + \frac{1}{q} = 1\right)$. Some trouble occurs at the endpoints, however, since the dual spaces are more complicated. For example, $(H^1)^* = L^\infty/H^\infty$, and the Beurling theory for the forward shift is not available in this quotient space. The proof of Theorem 2.2.1, however, does not use the Beurling theory, and in fact, the proof, suitably modified works for H^1. There is no difficulty with the first half of the proof. In the second half, the numbers $\{c_n\}$ $(n \geqslant 0)$ are the Fourier coefficients of positive index of a bounded function (also, in equations (2) ordinary convergence must be replaced by Abel summability). It can now be shown that the function $G = fH$ is the projection of some L^1 function, and hence is in H^p for all $0 < p < 1$. This is enough to complete the proof.

A study similar to ours could also be made for the backward shift operator on H^∞ (endowed with the weak*-topology), or, in fact, on the algebra A of continuous functions on $\bar{D}$ which are analytic on D.

Further, viewing the backward shift as acting on l^2, it is natural to ask about the situation on l^p, $1 \leqslant p \leqslant \infty$. Considering the backward shift operator on the space l^∞ (endowed with the weak* topology), B. Nyman showed that if a vector is non-cyclic (he employs the term "mean-periodic") then the associated holomorphic function in D possesses a (*bona fide*) analytic continuation across every point of **T** with the exception of a closed set of measure zero, and the continued function is meromorphic in $\mathbf{D}_e$ ([17], p. 50, Theorem 10). (This is rather more difficult to prove than our Theorem 2.2.1 ; Nyman does not seem to obtain a complete characterization of the class of meromorphic functions in $\mathbf{D}_e$ that arise as analytic continuations of "mean-periodic" vectors. In particular it does not follow from his result that the sum of two non-cyclic vectors is non-cyclic, though this is easy to prove directly, since the intersection of two proper ideals in l^1 is a proper ideal).

5.5. Weighted backward shifts.

Suppose that $\{e_n\}$ is the standard orthonormal basis in l^2 and suppose $\{\alpha_n\}$ is a bounded sequence of positive numbers. The weighted shift A defined on l^2 by $\{\alpha_n\}$ is the operator defined $\mathrm{A}e_0 = 0$ and $\mathrm{A}e_{n+1} = \alpha_n e_n$ for $n \geqslant 0$. It is known (cf. [10] pp. 95-97) that if $\Sigma\, \alpha_n^2 < \infty$ and the sequence $\{\alpha_n\}$ is monotone decreasing, then a vector $(a_0, a_1, a_2, \ldots)$ is cyclic for A if and only if infinitely many of the a_n are different from zero. Dual to such a result is a result on the cyclicity of certain functions for the ordinary backward shift. In this instance one can show that if $\Sigma\, |a_k|^2 < \infty$, $a_k \neq 0$, and

$$\left(\sum_{k=\mathrm{N}+1}^{\infty} |a_k|^2 / |a_\mathrm{N}|^2 \right) \longrightarrow 0 \,,$$

then for any sequence of integers $\{n_k\}$, the function $f = \Sigma\, a_k z^{n_k}$ is cyclic for U*. From this point of view the results of Donoghue, Nikolskii and others (cf. [10] p. 97) can be considered as belonging to the subject of this paper.

The study of the cyclic vectors of a backward weighted shift would also seem to be of interest(6). Some of the techniques of this paper can be expected to be useful in such a study. In particular, the proof of Theorem 2.5.5 yields that functions with a lacunary Taylor series are cyclic for any monotone backward weighted shift, that is, an A with monotone decreasing weights.

(6) We have just received a preprinton this subject from Ralph Gellar entitled "Cyclic vectors and parts of the spectrum of a weighted shift".

Further, if instead of considering weighted shifts, one considers weighted sequence spaces (cf. [10] p. 48), then the space can, in many instances, be identified as a space of analytic functions on $\mathbf{D}$, while the dual can be identified with a space of analytic functions on $\mathbf{D}_e$. It would be interesting to know whether an analogue of Theorem 1 would hold in this case since the existence of radial limits could no longer be assumed.

5.6. Hyper-cyclic vectors.

It would be of interest in certain problems to know whether there exists a function f in H^2 such that φf is cyclic for every φ in H^∞ ($\varphi \neq 0$). This question seems to have intrinsic interest also, since its answer would presumably require identifying some further property possessed by cyclic vectors.

5.7.

Is the set of non-cyclic vectors equal to the range of some bounded operator on H^2 ?

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