Université Grenoble Alpes

ANNALES DE L'INSTITUT FOURIER

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Tome 71, nº 1 (2021), p. 89-96.

<http://aif.centre-mersenne.org/item/AIF_2021__71_1_89_0>

Ann. Inst. Fourier, Grenoble
71, 1 (2021) 89-96

CORRIGENDUM TO "DIFFERENTIATING THE STOCHASTIC ENTROPY IN NEGATIVELY CURVED SPACES UNDER CONFORMAL CHANGES"

Annales de l'institut Fourier, vol. 67 (2017), n°3, 1115–1183

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Abstract. — We correct an error in [5, Proposition 2.16], and explain the subsequent changes in the proof of the main results.

Résumé. — Nous corrigeons une erreur dans [5, Proposition 2.16] et nous détaillons les modifications à apporter aux démonstrations des principaux résultats.

There is an error in our paper [5, Proposition 2.16], in the expression of the entropy of a foliated diffusion due to our dismissing the non-symmetry of the diffusions. The main results on the regularities of the entropy (Proposition 3.2 and Theorem 5.1 (ii)) and the structure of their proofs are not affected. But in several places, the Martin kernels we used should be replaced by the Martin kernels of the formal adjoint operator and the details of the first step of the proof of Theorem 5.1 (ii) should be corrected.

1. Setting

In this section, we briefly recall the settings and the notations from [5]. We locate the error and introduce the new notations we use.

Let (M, g) be an m-dimensional closed Riemannian manifold with negative curvature, where g is C^3, $(\widetilde{M}, \widetilde{g})$ the universal cover and G the fundamental group of M that acts freely on $\widetilde{M}$ in such a way that M can be identified with a space of orbits of this action. We choose once for all a connected fundamental domain M_0 for the action of G on $\widetilde{M}$.

Keywords: formal adjoint, Martin kernel, stochastic entropy.
2020 *Mathematics Subject Classification:* 37D40, 58J65.

The unit tangent bundles of M and $\widetilde{M}$, denoted SM and $S\widetilde{M}$, can be identified with $M_0 \times \partial\widetilde{M}$ and $\widetilde{M} \times \partial\widetilde{M}$, respectively, where $\partial\widetilde{M}$ is the geometric boundary of $\widetilde{M}$. For each $\mathbf{v} = (x, \xi) \in S\widetilde{M}$, its *stable manifold* with respect to the geodesic flow on $S\widetilde{M}$, denoted $W^s(\mathbf{v})$, is the collection of initial vectors $\mathbf{w}$ of geodesics asymptotic to ξ and can be identified with $\widetilde{M}\times\{\xi\}$. The collection of $W^s(\mathbf{v})$ defines the *stable foliation* (lamination) $\mathcal{W}$ on SM, whose leaves are discrete quotients of $\widetilde{M}$ and are naturally endowed with the Riemannian metric induced from $\widetilde{g}$. A differential operator $\mathcal{L}$ on (the smooth functions on) SM with continuous coefficients is said to be *subordinate to the stable foliation* $\mathcal{W}$, if for every smooth function F on SM the value of $\mathcal{L}(F)$ at $v \in SM$ only depends on the restriction of F to the leaf of $\mathcal{W}$ that contains v. Given such an operator $\mathcal{L}$, a Borel measure $\mathbf{m}$ on SM is called *$\mathcal{L}$-harmonic* if it satisfies, for every smooth function f on SM,

$$\int \mathcal{L}(f)\, d\mathbf{m} = 0.$$

In this note, we are interested in the case $\mathcal{L} = \Delta + Y$, where Y is a section of the tangent bundle of $\mathcal{W}$ over SM of class $C_s^{k,\alpha}$ for some $k \geqslant 1$ and $\alpha \in [0,1)$ such that Y^*, the dual of Y in the cotangent bundle to $\mathcal{W}$ over SM satisfies $dY^* = 0$ leafwisely and such that the Pressure of the function $-\langle \overline{X}, Y\rangle$ is positive (the sign convention for the argument of the Pressure function in [2] is opposite to the usual one), where $\overline{X}$ is the geodesic spray and can be considered as a section of the tangent bundle of $\mathcal{W}$ over SM. Such a $\mathcal{L}$ has a unique $\mathcal{L}$-harmonic probability measure $\mathbf{m}$ ([2, Theorem A]).

Extend $\mathcal{L}$ to a G-equivariant operator on $S\widetilde{M} = \widetilde{M} \times \partial\widetilde{M}$ which we shall denote with the same symbol. The operator $\mathcal{L}$ defines a Markovian family of probabilities on $\widetilde{\Omega}_+$, the space of continuous paths of $\widetilde{\omega} : [0, +\infty) \to S\widetilde{M}$, equipped with the smallest σ-algebra $\mathcal{A}$ for which the projections $R_t :\ \widetilde{\omega} \mapsto \widetilde{\omega}(t)$ are measurable. Indeed, for $\mathbf{v} = (x, \xi) \in S\widetilde{M}$, let $\mathcal{L}_{\mathbf{v}}$ denote the laminated operator of $\mathcal{L}$ on $W^s(\mathbf{v})$. It can be regarded as an operator on $\widetilde{M}$ with corresponding heat kernel functions $p_{\mathbf{v}}(t, y, z)$, $t \in \mathbb{R}_+, y, z \in \widetilde{M}$. Define

$$\mathbf{p}(t, (x, \xi), d(y, \eta)) = p_{\mathbf{v}}(t, x, y) d\mathrm{Vol}(y) \delta_\xi(\eta),$$

where $\delta_\xi(\,\cdot\,)$ is the Dirac function at ξ. Then the diffusion process on $W^s(\mathbf{v})$ with infinitesimal operator $\mathcal{L}_{\mathbf{v}}$ is given by a Markovian family $\{\mathbb{P}_{\mathbf{w}}\}_{\mathbf{w}\in\tilde{M}\times\{\xi\}}$, where for every $t > 0$ and every Borel set $A \subset \widetilde{M} \times \partial\widetilde{M}$ we have

$$\mathbb{P}_{\mathbf{w}}\left(\{\widetilde{\omega} :\ \widetilde{\omega}(t) \in A\}\right) = \int_A \mathbf{p}(t, \mathbf{w}, d(y, \eta)).$$

The group G acts naturally and discretely on the space $\widetilde{\Omega}_+$ with quotient the space Ω_+ of continuous paths in SM, and this action commutes with the shift. Therefore, the measure $\widetilde{\mathbb{P}} = \int \mathbb{P}_{\mathbf{v}}\, d\widetilde{\mathbf{m}}(\mathbf{v})$ on $\widetilde{\Omega}_+$ is the extension of a shift invariant ergodic probability measure $\mathbb{P}$ on Ω_+. Elements in Ω_+ will be denoted by ω, which can be identified with its lift in $\widetilde{\Omega}_+$ starting from M_0. By [1], for each $\mathbf{v} \in S\widetilde{M}$, there exists the corresponding Green function $G_{\mathbf{v}}(\,\cdot\,,\cdot\,)$ on $\widetilde{M} \times \widetilde{M}$. We define the Green function $\mathbf{G}(\,\cdot\,,\cdot\,)$ on $S\widetilde{M} \times S\widetilde{M}$ as being

$$\mathbf{G}\left((y,\eta),(z,\zeta)\right) := G_{(y,\eta)}(y,z)\delta_\eta(\zeta), \text{ for } (y,\eta),(z,\zeta) \in S\widetilde{M},$$

where δ_η is the Dirac function at η. Moreover, for $x, y \in \widetilde{M}$ and $\eta \in \partial\widetilde{M}$, the following limit exists and defines the *Martin kernel* $k_{\mathbf{v}}(x,y,\eta)$:

$$k_{\mathbf{v}}(x,y,\eta) \;=\; \lim_{z\to\eta} \frac{G_{(x,\xi)}(x,z)}{G_{(x,\xi)}(y,z)}.$$

By [2], Corollary B7, $(x,\xi) \mapsto \nabla_y \log k_{(x,\xi)}(x,y,\xi)|_{y=x}$ is a Hölder continuous function.

Denote by $\widetilde{\mathbf{m}}$ the G-invariant measure which extends $\mathbf{m}$ on $\widetilde{M} \times \partial\widetilde{M}$. The *(stochastic) entropy* $h_{\mathcal{L}}$ of $\mathcal{L}$ (introduced by Kaimanovich ([3, 4])) is given by

$$h_{\mathcal{L}} := \lim_{t\to+\infty} -\frac{1}{t}\int_{M_0\times\partial\tilde{M}} \left(\log \mathbf{p}(t,(x,\xi),(y,\eta))\right) \mathbf{p}(t,(x,\xi),d(y,\eta))\, d\widetilde{\mathbf{m}}(x,\xi).$$

We have the following limit expressions for $h_{\mathcal{L}}$:

Proposition 1.1 ([5, Proposition 2.4]). — *For $\mathbb{P}$-a.e. paths $\omega \in \Omega_+$,*

$$\begin{aligned} h_{\mathcal{L}} &= \lim_{t\to+\infty} -\frac{1}{t}\log \mathbf{p}(t,\omega(0),\omega(t)) \\ &= \lim_{t\to+\infty} -\frac{1}{t}\log \mathbf{G}(\omega(0),\omega(t)). \end{aligned} \tag{1.1}$$

Our attempt in [5, Proposition 2.16], is to replace $\mathbf{G}$ in (1.1) by the Martin kernel and use it to find an integral expression. In its computation, we replaced at some point $\frac{G_{\mathbf{v}}(y,z)}{G_{\mathbf{v}}(y,x)}$, for y far away in $\widetilde{M}$, by $k_{\mathbf{v}}(x,z,\xi)$ whereas that ratio is close to the Martin kernel $k^*_{\mathbf{v}}(x,z,\xi)$ of the formal adjoint $\mathcal{L}^*_{\mathbf{v}}$, which we define as follows. For $\mathbf{v} \in S\widetilde{M}$, let $\mathcal{L}^*_{\mathbf{v}}$ be the *formal adjoint* of $\mathcal{L}_{\mathbf{v}}$, i.e., the operator such that for all $\varphi, \psi \in C^\infty_c(\widetilde{M})$,

$$\int_{\tilde{M}} \varphi\mathcal{L}^*_{\mathbf{v}}\psi\, d\mathrm{Vol}_{\tilde{g}} \;=\; \int_{\tilde{M}} (\mathcal{L}_{\mathbf{v}}\varphi)\psi\, d\mathrm{Vol}_{\tilde{g}}.$$

The operator $\mathcal{L}^*$ subordinate to $\mathcal{W}$ given by $\mathcal{L}^*\psi = \Delta\psi - \mathrm{Div}(\psi Y)$ admits $\mathcal{L}^*_{\mathbf{v}}$ as its laminated operator on $W^s(\mathbf{v})$. Observe that the operator $\mathcal{L}^*$

is also weakly coercive and that its Green function $\mathbf{G}^*(x,y)$ is given by $G^*_{\mathbf{v}}(x,y) = G_{\mathbf{v}}(y,x)$. Again, for $\mathbf{v} = (x,\xi) \in S\widetilde{M}$, $y \in \widetilde{M}$ and $\eta \in \partial\widetilde{M}$, the following limit exists and defines the *Martin kernel* $k^*_{\mathbf{v}}(x,y,\eta)$:

$$k^*_{\mathbf{v}}(x,y,\eta) \;=\; \lim_{z\to\eta} \frac{G^*_{(x,\xi)}(x,z)}{G^*_{(x,\xi)}(y,z)} \;=\; \lim_{z\to\eta} \frac{G_{(x,\xi)}(z,x)}{G_{(x,\xi)}(z,y)}. \tag{1.2}$$

By [2], Corollary B7, the function $(x,\xi) \mapsto \nabla_y \log k^*_{(x,\xi)}(x,y,\xi)|_{y=x}$ is Hölder continuous. Moreover, if we write $Y = \nabla \log f_{\mathbf{v}}$ for some C^2 function $f_{\mathbf{v}}$, we have $\mathcal{L}^*_{\mathbf{v}}\psi = f_{\mathbf{v}}\mathcal{L}_{\mathbf{v}}\left((f_{\mathbf{v}})^{-1}\psi\right)$. Thus, $\psi \mapsto \left((f_{\mathbf{v}})^{-1}\psi\right)$ is a one-to-one correspondence between $\mathcal{L}^*_{\mathbf{v}}$-harmonic functions and $\mathcal{L}_{\mathbf{v}}$-harmonic functions, so that, for all $\mathbf{v} = (x,\xi) \in S\widetilde{M}$, $y \in \widetilde{M}$, $\eta \in \partial\widetilde{M}$,

$$k^*_{\mathbf{v}}(x,y,\eta) = \frac{f_{\mathbf{v}}(y)}{f_{\mathbf{v}}(x)} k_{\mathbf{v}}(x,y,\eta). \tag{1.3}$$

In [2, Appendix B], Hamenstädt studied the operator $\mathcal{L}^*_{\mathbf{v}}$ and its Martin kernel $k^*_{\mathbf{v}}(x,z,\xi)$ and she showed that they enjoy many properties of the operator $\mathcal{L}$ and its Martin kernel $k_{\mathbf{v}}(x,z,\xi)$. In particular, our argument for Proposition 3.2 still holds after correcting $k_{\mathbf{v}}$ by $k^*_{\mathbf{v}}$. Finally, to get Theorem 5.1 (ii), we apply the above to operators which come from formally self-adjoint operators, it is not surprising that the final formula is the same as the one obtained when overlooking the lack of symmetry of the intermediate operator.

2. Stochastic entropy for laminated diffusions

With the above notations, the following replaces Proposition 2.16 in [5]:

Proposition 2.1. — *Let $\mathcal{L} = \Delta + Y$ be such that Y^*, the dual of Y in the cotangent bundle to the stable foliation over SM, satisfies $dY^* = 0$ leafwisely and* $\operatorname{pr}(-\langle \overline{X}, Y\rangle) > 0$. *Then,*

$$h_{\mathcal{L}} = -\int_{M_0\times\partial\tilde{M}} (\Delta + Y)_y \left(\log k^*_{(x,\xi)}(y,\xi)\right)\Big|_{y=x} d\widetilde{\mathbf{m}}(x,\xi) \tag{2.1}$$

$$= \int_{M_0\times\partial\tilde{M}} \left(\|\nabla_y \log k_{(x,\xi)}(y,\xi)|_{y=x}\|^2 - \operatorname{Div} Y - \|Y\|^2\right) d\widetilde{\mathbf{m}}(x,\xi). \tag{2.2}$$

Proof. — The proof of (2.1) copies the proof of Proposition 2.16 in [5], up to the point where one uses (1.2). For $\mathbb{P}$-a.e. path $\omega \in \Omega_+$, we still denote ω its projection to $\widetilde{M}$ and write $\mathbf{v} := \omega(0)$. When t goes to infinity, we see that

$$\limsup_{t\to+\infty} |\log G_{\mathbf{v}}(x,\omega(t)) - \log k^*_{\mathbf{v}}(\omega(t),\xi)| < +\infty.$$

Indeed, let z_t be the point on the geodesic ray $\gamma_{\omega(t),\xi}$ closest to x. Then, as $t \to +\infty$,

$$G_{\mathbf{v}}(x,\omega(t)) \asymp G_{\mathbf{v}}(z_t,\omega(t)) \asymp \frac{G_{\mathbf{v}}(y,\omega(t))}{G_{\mathbf{v}}(y,z_t)}$$

for all y on the geodesic going from $\omega(t)$ to ξ with $d(y,\omega(t)) \geqslant d(y,z_t)+1$, where $\asymp$ means up to some multiplicative constant independent of t. The first $\asymp$ comes from Harnack inequality using the fact that $\sup_t d(x,z_t)$ is finite $\mathbb{P}$-almost everywhere. (For $\mathbb{P}$-a.e. $\omega \in \Omega_+$, $\eta = \lim_{t\to+\infty} \omega(t)$ differs from ξ and $d(x,z_t)$, as $t \to +\infty$, tends to the distance between x and the geodesic asymptotic to ξ and η in opposite directions.) The second $\asymp$ comes from Ancona's inequality ([1]). Replace $G_{\mathbf{v}}(y,\omega(t))/G_{\mathbf{v}}(y,z_t)$ by its limit as $y \to \xi$, which is $k^*_{(z_t,\xi)}(\omega(t),\xi)$ by (1.2), which is itself $\asymp k^*_{(x,\xi)}(\omega(t),\xi)$ by Harnack inequality again.

Since the $\mathcal{L}$-diffusion has leafwise infinitesimal generator $\Delta + Y$ and is ergodic, we obtain

$$\begin{aligned} h_{\mathcal{L}} &= \lim_{t\to+\infty} -\frac{1}{t}\int_0^t \frac{\partial}{\partial s}(\log k^*_{(x,\xi)}(\omega(s),\xi))\,ds \\ &= -\lim_{t\to+\infty}\frac{1}{t}\int_0^t (\Delta+Y)_y\left(\log k^*_{(x,\xi)}(y,\xi)\right)\Big|_{y=\omega(s)} ds \\ &= -\int_{M_0\times\partial\tilde{M}} (\Delta+Y)_y\left(\log k^*_{(x,\xi)}(y,\xi)\right)\Big|_{y=x} d\widetilde{\mathbf{m}}(x,\xi). \end{aligned}$$

For proving (2.2), one starts by using (1.3) to write

$$h_{\mathcal{L}} \;=\; -\int_{M_0\times\partial\tilde{M}} (\Delta+Y)_y\left(\log k_{(x,\xi)}(y,\xi) + \log f_{(x,\xi)}(y)\right)\big|_{y=x} d\widetilde{\mathbf{m}}(x,\xi).$$

Since the Martin kernel $k_{(x,\xi)}(\,\cdot\,,\xi)$ satisfies $(\Delta+Y)_y(k_{(x,\xi)}(y,\xi))|_{y=x} = 0$,

$$(\Delta+Y)_y\left(\log k_{(x,\xi)}(y,\xi)\right)|_{y=x} = -\|\nabla_y\left(\log k_{(x,\xi)}(y,\xi)\right)|_{y=x}\|^2.$$

On the other hand, since $Y = \nabla_y \log f_{(x,\xi)}|_{y=x}$, we have

$$(\Delta+Y)_y\left(\log f_{(x,\xi)}(y)\right)|_{y=x} = \operatorname{Div} Y + \|Y\|^2.$$

Relation (2.2) follows. □

For the proof of Proposition 3.2 in [5], the places that are affected by Proposition 2.16 and should be corrected are as follows: the $u_1, \mathbf{Z}^1_t$ and σ_1 of page 1141 have to be replaced by $u_1^*, \mathbf{Z}_t^{1,*}$ and σ_1^* for k^*. For Lemma 3.11, the changes are only formal, replacing $k^0_{\mathbf{v}}(\omega(s),\xi)$ by $k^{0,*}_{\mathbf{v}}(\omega(s),\xi)$ on page 1160 and in the estimate of $\mathbf{I}_3$.

3. The entropy part regularity

Let $\lambda \in (-1,1) \mapsto g^\lambda = e^{2\varphi^\lambda} g$ be a C^3 curve in the space of C^3 negatively curved metrics with $g^0 = g$ and constant volume. Denote $\widetilde{g}^\lambda$ the G-invariant extension of g^λ to $\widetilde{M}$. Its geometric boundary for $\widetilde{M}$ can be identified with $\partial\widetilde{M}$ of $\widetilde{g}$. For each g^λ, let Δ^λ denote the laminated Laplacian for its stable foliation (and also the lift to $\{\widetilde{M} \times \{\xi\}\}_{\xi \in \partial\tilde{M}}$), $\mathbf{m}^\lambda$ the unique Δ^λ-harmonic probability measure with G-invariant lift $\widetilde{\mathbf{m}}^\lambda$ to $\widetilde{M} \times \partial\widetilde{M}$, h_λ the entropy of Δ^λ and $k^\lambda_{\mathbf{v}}$ the leafwise Martin kernels for Δ^λ. Note that the formal adjoint of Δ^λ is itself and hence has the same Martin kernel functions as Δ^λ. By [3] (see also (2.1)),

$$h_\lambda = -\int_{M_0 \times \partial\tilde{M}} \Delta^\lambda_y \left(\log k^\lambda_{(x,\xi)}(y,\xi)\right)\Big|_{y=x} d\widetilde{\mathbf{m}}^\lambda(x,\xi).$$

Let $\widehat{\mathcal{L}}^\lambda := \Delta + Z^\lambda$ with $Z^\lambda = (m-2)\nabla\varphi^\lambda \circ \phi$, where $\phi : SM \mapsto M$ is the projection. Then

$$\Delta^\lambda = e^{-2\varphi^\lambda \circ \phi} \widehat{\mathcal{L}}^\lambda. \tag{3.1}$$

Leafwisely, Z^λ is the dual of the closed form $(m-2)d\varphi^\lambda \circ \phi$. Moreover, the pressure of the function $-\langle \overline{X}, Z^0\rangle = 0$ is positive. Therefore, there exists $\delta > 0$ such that for $|\lambda| < \delta$, the pressure of the function $-\langle \overline{X}, Z^\lambda\rangle$ is still positive. For such λ, we denote $\widehat{\mathbf{m}}^\lambda$ for the G-invariant extension to $S\widetilde{M}$ of the harmonic probability measure corresponding to $\widehat{\mathcal{L}}^\lambda$ with respect to metric g. We denote $\widehat{k}^\lambda_{\mathbf{v}}$ the Martin kernels for $\widehat{\mathcal{L}}^\lambda$. Let $\widehat{\mathcal{L}}^{\lambda,*}$ be the formal adjoint of $\widehat{\mathcal{L}}^\lambda$ and its Martin kernel functions are denoted by $\widehat{k}^{\lambda,*}_{\mathbf{v}}$. Let $\widehat{h}_\lambda$ be the entropy for $\widehat{\mathcal{L}}^\lambda$. We have by Proposition 2.1 that

$$\widehat{h}_\lambda = -\int_{M_0 \times \partial\tilde{M}} \widehat{\mathcal{L}}^\lambda_y \left(\log \widehat{k}^{\lambda,*}_{(x,\xi)}(y,\xi)\right)\Big|_{y=x} d\widehat{\mathbf{m}}^\lambda(x,\xi).$$

The proof of Theorem 5.1 (ii) in [5] concerning the differential formula of h_λ in λ was divided into two parts, where the first part is to show

$$(\mathbf{I})_h := \lim_{\lambda \to 0} \frac{1}{\lambda}(h_\lambda - \widehat{h}_\lambda) = 0$$

and the other is to analyze the differential of $\widehat{h}_\lambda$ in λ. The proof of $(\mathbf{I})_h = 0$ is affected by the use of the wrong formula of [5, Proposition 2.16]. We correct it as follows.

Claim 3.1. — *We have* $(\mathbf{I})_h = 0$.

Proof. — For λ with $|\lambda| < \delta$, set $V_\lambda := \int_{M_0 \times \partial \tilde{M}} e^{-2\varphi^\lambda \circ \phi} \, d\widetilde{\mathbf{m}}^\lambda$. We show $\widehat{h}_\lambda = V_\lambda^{-1} h_\lambda$. Since the volume of g_λ is constant, $\lim_{\lambda \to 0} \frac{1}{\lambda}(V_\lambda - 1) = 0$, and the claim follows.

Using (3.1), we obtain that $\widehat{\mathcal{L}}^{\lambda,*}\alpha = 0$ iff $\Delta^\lambda \big(e^{-(m-2)\varphi^\lambda \circ \phi} \alpha\big) = 0$. Consequently, we have

$$\widehat{k}_{\mathbf{v}}^{\lambda,*} = e^{(m-2)\varphi^\lambda \circ \phi} k_{\mathbf{v}}^\lambda.$$

Using $d\widehat{\mathbf{m}}^\lambda = V_\lambda^{-1} e^{-2\varphi^\lambda \circ \phi} \, d\widetilde{\mathbf{m}}^\lambda$ and (3.1), we compute that

$$\begin{aligned} V_\lambda \widehat{h}_\lambda &= - \int_{M_0 \times \partial \tilde{M}} \Delta_y^\lambda \left(\log \widehat{k}_{(x,\xi)}^{\lambda,*}(y,\xi) \right) \Big|_{y=x} d\widetilde{\mathbf{m}}^\lambda(x,\xi) \\ &= - \int_{M_0 \times \partial \tilde{M}} \Delta^\lambda \left(\log(k_{(x,\xi)}^\lambda(y,\xi) \right) \Big|_{y=x} d\widetilde{\mathbf{m}}^\lambda(x,\xi) \\ &\quad - (m-2) \int_{M_0 \times \partial \tilde{M}} \Delta^\lambda(\varphi^\lambda \circ \phi) \, d\widetilde{\mathbf{m}}^\lambda \\ &= - \int_{M_0 \times \partial \tilde{M}} \Delta^\lambda \left(\log(k_{(x,\xi)}^\lambda(y,\xi) \right) \Big|_{y=x} d\widetilde{\mathbf{m}}^\lambda(x,\xi) = h_\lambda, \end{aligned}$$

where the third equality holds since $\mathbf{m}^\lambda$ is Δ^λ harmonic. □

For the computation of the derivative, we have to replace u_1 by u_1^*. Since in our case, $\widehat{\mathcal{L}}_0 = \Delta$ is formally self-adjoint, $u_1^* = u_1$ and our previous computation is unchanged.

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Manuscrit reçu le 31 janvier 2019,
accepté le 3 août 2020.

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