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SPECTRAL PICARD–VESSIOT FIELDS FOR ALGEBRO-GEOMETRIC SCHRÖDINGER OPERATORS

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Abstract. — This work is a Galoisian study of the spectral problem $L\Psi = \lambda\Psi$, for an algebro-geometric second order differential operators L, with coefficients in a differential field, whose field of constants C is algebraically closed and of characteristic zero. Our approach regards the spectral parameter λ as an algebraic variable over C, forcing the consideration of a new field of coefficients for $L - \lambda$, whose field of constants is the field $C(\Gamma)$ of the spectral curve Γ. Since $C(\Gamma)$ is no longer algebraically closed, the need arises of a new algebraic structure, generated by the solutions of the spectral problem over Γ, called "Spectral Picard–Vessiot field" of $L-\lambda$. An existence theorem is proved using differential algebra, allowing to recover classical Picard–Vessiot theory for each $\lambda = \lambda_0$. For rational spectral curves, the appropriate algebraic setting is established to solve $L\Psi = \lambda\Psi$ analytically and to use symbolic integration. We illustrate our results for Rosen-Morse solitons.

Résumé. — Ce travail est une étude Galoisienne du problème spectral $L\Psi = \lambda\Psi$, pour les opérateurs différentiels algébro-géométriques du second ordre L, avec des coefficients dans un corps différentiel, dont le corps de constantes C est algébriquement clos et de caractéristique zéro. Notre approche considère le paramètre spectral λ une variable algébrique sur C, ce qui amène à considérer un nouveau corps de coefficients pour $L - \lambda$, dont le corps de constantes est le champ $C(\Gamma)$ de la courbe spectrale Γ. Puisque $C(\Gamma)$ n'est plus algébriquement clos, le besoin se fait sentir d'une nouvelle structure algébrique, générée par les solutions du problème spectral sur Γ, appelée « Corps spectral de Picard–Vessiot » de $L-\lambda$. On prouve un théorème d'existence en utilisant l'algèbre différentielle, permettant de retrouver la théorie classique de Picard–Vessiot pour chaque $\lambda = \lambda_0$. Pour les courbes spectrales rationnelles, on établit le cadre algébrique approprié pour résoudre $L\Psi = \lambda\Psi$ de manière analytique et pour utiliser l'intégration symbolique. Nous illustrons nos résultats pour les solitons de Rosen-Morse.

Keywords: Picard–Vessiot extension, Liouvillian extension, algebro-geometric operator, spectral curve.
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1. Introduction

Algebro-geometric operators are deeply linked to the integrabillity of partial differential equations of solitonic type, a review on this subject can be found in [4, 19, 21, 37]. In this paper we describe the differential field structure generated by the solutions of the spectral problem

$$L\Psi = \lambda\Psi, \tag{1.1}$$

for an algebro-geometric second order operator L, with coefficients in a differential field (Σ, ∂), whose field of constants C is algebraically closed and of characteristic zero. The novelty of our approach is to regard the spectral parameter λ to be an algebraic variable over C. This forces the consideration of a carefully chosen new field of coefficients for $L - \lambda$, whose field of constants provides the natural constants of this spectral problem, the field of rational functions on a plane algebraic curve, the so called spectral curve.

The goal of this work is a Galoisian study of the algebro-geometric spectral problem (1.1). In [5], the Galoisian approach has also been followed from the point of view of D-module theory. One of our guiding ideas is the strong connection between the integrability (ie, solvability in closed form) of the direct and the inverse problems for the Schrödinger equation. By "direct problem" we understand, given the potential to obtain the eigenfunctions and the eigenvalues. The "inverse problem" would be to obtain the potential from some suitable spectral data. In fact, this was also the motivation for Drach in his 1919 papers [17, 18], about the integrability in closed form of the equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = (\varphi(x) + h)y \quad (I)$$

where h is the spectral parameter. So he wrote in [17]:

> *« Il est donc très important de connaître les cas où une simplification se presente dans l'intégration de (I), en laissant le paramètre h arbitraire. Nous avons réussi à déterminer la fonction φ dans tous les cas où l'intégrale y peut s'obtenir par quadratures... »*

Moreover he said that to study this problem it is possible to use the classical theory of Picard about linear equations (Picard–Vessiot theory). In other words, he considered the integrability of the direct problem in the sense of the Picard–Vessiot theory. One could use the classical Picard–Vessiot theory [49] for each choice of $\lambda = \lambda_0 \in C$ and obtain the minimal field

extension of the coefficient field of $L - \lambda_0$ that contains the solutions of $L\Psi = \lambda_0\Psi$. A first attempt to use classical Picard–Vessiot theory was made by Y.V. Brezhnev in [7] that, following the ideas of Drach [17, 18], exhibited formulas for the solutions $\Psi = \Psi(x, \lambda)$ by means of theta functions. But along his papers Drach indicates that the constants are functions of the spectral parameter h.

The alternative we present considers λ as an algebraic parameter over the coefficient field K, which is not a free parameter but verifies the equation of a plane algebraic curve, the spectral curve, defined by a polynomial $f(\lambda, \mu) = 0$. Our construction allows new achievements in the study of algebro-geometric operators, as we explain next.

Algebro-geometric operators [56] are characterized in this paper by having a nontrivial centralizer; see also [23]. Moreover, in the case of second order operators we can prove that the centralizer is the ring $C[L, A]$, for an appropriate minimal odd order operator A, which is isomorphic to the ring $C(\Gamma)$ of an affine plane algebraic curve Γ. The curve Γ is the celebrated *spectral curve* discovered by Burchnall and Chaundy in their visionary article [10], where a correspondence was established between algebraic curves and pairs of commuting ordinary differential operators; see also [44]. Thus for algebro-geometric operators the spectral parameter λ is not a free parameter since $f(\lambda, \mu) = 0$, and it also provides an algebraic relation between the operators L and A, $f(L, A) = 0$.

In Section 2 we explain the connection between algebro-geometric operators, non trivial centralizers and the stationary Korteveg de Vries (KdV) hierarchy of differential equations. More precisely, algebro-geometric second order operators in normal form (i.e. with 0 coefficient in ∂) are Schrödinger operators $L = -\partial^2 + u$, with potential u verifying one of the equations of the KdV hierarchy (see Appendix A). In fact we establish the following result.

Theorem A. — *Given $L_s = -\partial^2 + u_s$ the following statements are equivalent.*

(1) *L_s is algebro-geometric.*
(2) *There exists a unique monic operator A_{2s+1} of minimal order $2s+1$ such that $\mathcal{C}(L_s) = C[L_s, A_{2s+1}]$ and $A_{2s+1}^2 + R_{2s+1}(L_s) = 0$, with $R_{2s+1}(\lambda)$ in $C[\lambda]$ of degree $2s+1$.*
(3) *u_s is a KdV-potential of KdV level s (i.e. it satisfies one of the KdV_s equations of the KdV-hierarchy, see Appendix A).*

Hence, without loss of generality, we restrict to this case. The spectral curve is then defined by $f_s(\lambda,\mu) = \mu^2 + R_{2s+1}(\lambda) = 0$ and it is appropriate to consider the smallest differential field $K = C\langle u_s\rangle$ containing u_s and C, as the coefficient field of L_s, to study the Galoisian properties of $L_s - \lambda$. Furthermore, the operator $L_s - \lambda$ determines as coefficient field an extended field of the curve, more precisely the fraction field $K(\Gamma_s)$ of the domain $K[\Gamma_s] = K[\lambda,\mu]/(f_s)$. An important contribution of this paper is to establish its field of constants in Theorem 3.5.

Theorem B. — *The field of constants of $K(\Gamma_s)$ is $C(\Gamma_s)$.*

In Section 4, we construct the minimal field extension $\mathcal{E}$ of $K(\Gamma_s)$ containing the solutions of (1.1). We call this new differential field structure *spectral Picard–Vessiot* field over the curve Γ_s, since

$$C(\Gamma_s) \subset K(\Gamma_s) \subset \mathcal{E}\ .$$

The field of constants $C(\Gamma_s)$ attached to the spectral coupled problem

$$L_s\Psi = \lambda\Psi\,, \quad A_{2s+1}\Psi = \mu\Psi\,, \tag{1.2}$$

is no longer an algebraically closed field, forcing an adapted new Picard–Vessiot theory where the structure of the spectral curve plays an essential role.

The differential algebra theory developed for the algebraic integration of the Risch differential equation in [9], by M. Bronstein and other authors, was essential to prove the main results of this paper. The Subresultant Theorem B.3, in Appendix B, guarantees the existence of an "intrinsic" right (common) factor $\partial-\phi_s$ of $L_s-\lambda$ (and $A_s-\mu$) in $K(\Gamma_s)[\partial]$, linked to the uniqueness of A_{2s+1}. The common solution of (1.2), is the transcendental element Ψ_s, which is the hyperexponential defined by $\partial\Psi_s = \phi_s\Psi_s$ over $K(\Gamma_s)$.

The main result of this paper is the following existence theorem for spectral Picard–Vessiot extensions. It is based on Theorem 4.11, where we prove that $\mathcal{E}$ is a transcendental Liouvillian extension $K(\Gamma_s)\langle\Psi_s\rangle$ of $K(\Gamma_s)$, whose field of constants is the field of the curve $C(\Gamma_s)$.

Theorem C. — *Let L_s be an algebro-geometric Schrödinger operator with spectral curve Γ_s. Let us consider $\partial - \phi_s$, the intrinsic right factor of $L_s - \lambda$ in $K(\Gamma_s)[\partial]$, and a nonzero solution Ψ_s defined by the differential relation $\partial(\Psi_s) = \phi_s\Psi_s$. Then $K(\Gamma_s)\langle\Psi_s\rangle$ is a spectral Picard–Vessiot field over the curve Γ_s of the equation $(L_s - \lambda)(\Psi) = 0$.*

Traditionally the spectral curve was assumed to be non-singular [7, 21, 31]. Nevertheless Burchnall and Chaundy in their 1931 paper [11] realized

that spectral curves with singularities need a special treatment. They studied the case of cuspidal curves, which are in fact singular rational curves, defined by $\mu^2 - \lambda^{2s+1} = 0$ for a particular type of Schrödinger operators.

We show how the study of (1.1) by means of classical Picard–Vessiot theory, for $\lambda = \lambda_0 \in C$, is recovered. It is important to note that the spectral curve may be singular, to consider the spectral problem (1.2) at each point $P_0 = (\lambda_0, \mu_0)$ of Γ_s, that is

$$L_s\Psi = \lambda_0\Psi, \quad A_{2s+1}\Psi = \mu_0\Psi. \tag{1.3}$$

We give the classical Picard–Vessiot extension $\mathcal{E}_{P_0}$ of K, the coefficient field of $L - \lambda_0$, in Section 5, Theorems 5.3 and 5.4, distinguishing the treatment of singular and non-singular points of Γ. The differential algebra developed to solve the Risch differential equation in [9] is again the key to prove these results. For all nonsingular points, but a finite number, this is the Liouvillian extension $K\langle y_0\rangle$ by a transcendental element y_0, which is the common solution of (1.3). In fact $\partial y_0 = \phi_0 y_0$, where the common factor $\partial - \phi_0 = \partial - \phi_s(P_0)$ of $L - \lambda_0$ and $A - \mu_0$ is obtained by the specialization to P_0 of the common factor of problem (1.2). Moreover, for $u = u(x) \in \mathcal{C}^\infty(\mathbb{R})$ it is the well known Baker–Akhiezer function $y_0 = \Psi(P_0, x, x_0)$ in [21]. Furthermore, at singular points we obtain a description of the sequence of differential field extensions of K to obtain $\mathcal{E}_{P_0}$ (see Theorem 5.4).

In Section 6 we restrict to the case of rational spectral curves, where the field of the curve is the field of rational functions $C(\tau)$ in an algebraic parameter τ. Considering a rational parametrization of Γ, say $\aleph(\tau) = (\chi_1(\tau), \chi_2(\tau))$, the spectral problem (1.1) in one-parameter form is

$$L\Psi = \chi_1(\tau)\Psi\ . \tag{1.4}$$

The chosen parametrization establishes an isomorphism between $K(\Gamma)$ and $K(\tau)$, which is now the coefficient field of $L - \chi_1(\tau)$.

More precisely, the isomorphism establised by the parametrization provides a right factor $\partial - \widetilde{\phi}_s$ of $L_s - \chi_1(\tau)$. In Section 6, we show that the spectral Picard–Vessiot field of $L_s - \lambda$ is isomorphic to a Liouvillian extension $K(\tau)\langle\Upsilon_s\rangle$ of the coefficient field $K(\tau)$ of $L - \chi_1(\tau)$, by a transcendental element Υ_s, see Theorem 6.3. We prove the next result.

Theorem D. — *Let L_s be an algebro-geometric Schrödinger operator with rational spectral curve Γ_s. The Liouvillian extension $K(\tau)\langle\Upsilon_s\rangle$ of $K(\tau)$, by a nonzero solution Υ_s of $(\partial - \widetilde{\phi}_s)\Upsilon = 0$ is isomorphic to a spectral Picard–Vessiot field over the curve Γ_s of the equation $(L_s - \lambda)(\Psi) = 0$.*

We can finally show the advantages of constructing solutions of the spectral problem (1.1) using a global rational parametrization of the spectral curve in $\mathbb{C}(\tau)^2$, instead of a local parametrization by Puiseux series in $\mathbb{C}\langle\langle\tau\rangle\rangle^2$. The coefficient field $K(\tau)$, where τ is now a free parameter, allows to say much more about the hyperexponential Υ_s. Whenever u_s is transcendental over $C(\tau)$ the algebraic integration algorithms in [9] would allow to compute Υ_s, see Remark 6.4. We illustrate this fact by means of a family of Rosen-Morse potentials in Example 6.5. Moreover, we established the appropriate algebraic setting to solve the spectral problem (1.1) analytically for rational curves, possibly with singularities. Whenever the potential $u = u(x)$ is an analytic potential in some complex domain, we describe the analytic character of the common solution of problem (1.4) in Theorem 6.6.

To finish, recall that for an algebro-geometric potential u_s, the spectral parameter λ of problem (1) is not a free parameter. On the contrary, the original spectral problem (1.1) in one-parameter form (1.4), where τ is a free parameter, falls into the recently developed parametrized Picard–Vessiot theory (see for example [2, 12, 25, 39]), and this provides new lines of research on the parametric behavior of the solutions of (1.1).

Organization of the paper

Section 2 presents the relation between algebro-geometric potentials, centralizers and the KdV hierarchy (proving Theorem A). Section 3 establishes the field of constants of the field $K(\Gamma)$, with Γ the spectral curve (Theorem B is part of Theorem 3.5). *Spectral Picard–Vessiot fields*, are defined and explicitly constructed in Section 4 (Theorems 4.11 and C). We recover the classical Picard–Vessiot extensions at each value $\lambda = \lambda_0$; this is done in Section 5, by Theorems 5.3 and 5.4. For rational spectral curves we prove Theorem D in Section 6. The existence of a free parameter τ, allows to use symbolic integration algorithms. We illustrate the obtained results by a family of Rosen-Morse potentials in Example 6.5. In this situation, $\lambda = \lambda(\tau)$, and Theorem 6.6 establishes the analyticity of the common solution in a domain of $\mathbb{C}^2$, around almost every point (x, τ). Some concluding remarks are contained in Section 7.

Notation

For concepts in differential algebra and differential Galois theory we refer the reader to [15, 40, 41, 49]. Let us consider algebraic variables λ and

μ with respect to ∂. Thus $\partial\lambda = 0$ and $\partial\mu = 0$ and we can extend the derivation ∂ of K to the polynomial ring $K[\lambda, \mu]$. Hence $(K[\lambda, \mu], \partial)$ is a differential ring whose ring of constants is $C[\lambda, \mu]$. Given a differential commutative ring R with non trivial derivation ∂, let us denote by $R[\partial]$ the ring of differential operators with coefficients in R and commutation rule $[\partial, a] = \partial a - a\partial = \partial(a)$, $a \in R$, where ∂a denotes the product in the noncommutative ring $R[\partial]$ and $a' = \partial(a)$ is the image of a by the derivation map.

2. Algebro-geometric operators of second order

Let us consider a differential operator L with coefficients in a differential field (Σ, ∂), whose field of constants C is algebraically closed and of characteristic zero. There are several characterizations of algebro-geometric operator, see for instance [56], and also [23]. We state next what we use as the base characterization of algebro-geometric operators for this work, the Burchnall and Chaundy Theorem [10], adapted from [56, Section 2]. We consider the nontrivial case of operators $L \notin C[\partial]$.

THEOREM 2.1. — *Let L be an order n differential operator in $\Sigma[\partial]\backslash C[\partial]$. The following are equivalent:*

(1) *There exists an operator P in $\Sigma[\partial]$ of order m, relatively prime with n, and a polynomial $f(\lambda, \mu) = \mu^n + R_m(\lambda)$ in $C[\lambda, \mu]$, with R_m of degree m, such that $f(L, P) = 0$.*
(2) *There exists an operator P in $\Sigma[\partial]$ of order m, relatively prime with n, such that $[L, P] = 0$.*

When one of these two equivalent conditions is satisfied, we will call L an algebro-geometric operator.

In the case of second order differential operators we would like to highlight the structure of the centralizer $\mathcal{C}(L)$ of L in the ring of differential operators $\Sigma[\partial]$.

THEOREM 2.2. — *Let L be a second order differential operator in $\Sigma[\partial]$. The following are equivalent:*

(1) *L is an algebro-geometric operator.*
(2) *The centralizer of L is nontrivial, $\mathcal{C}(L) \neq C[L]$. More precisely, there exists a unique monic operator A_{2s+1} of minimal order $2s+1$ such that $\mathcal{C}(L) = C[L, A_{2s+1}]$ and $A_{2s+1}^2 + R_{2s+1}(L) = 0$, with $R_{2s+1}(\lambda)$ in $C[\lambda]$ of degree $2s+1$.*

Proof. — Whenever the centralizer is nontrivial, $\mathcal{C}(L) \neq C[L]$, by [22, Theorem 1.2], there exists an operator X_{2s+1} of minimal order $2s+1$ in the centralizer such that $\mathcal{C}(L)$ equals the free $C[L]$-module of rank 2 with basis $\{1, X_{2s+1}\}$, that is

$$\mathcal{C}(L) = \{p_0(L) + p_1(L)X_{2s+1} \mid p_0, p_1 \in C[L]\} = C[L]\langle 1, X_{2s+1}\rangle. \tag{2.1}$$

Thus $\mathcal{C}(L)$ equals the C-algebra $C[L, X_{2s+1}]$ generated by L and X_{2s+1}.

In addition, any operator of the form $P = X_{2s+1} + p_0(L)$, with p_0 of degree $d \leqslant s$ is a generator of $\mathcal{C}(L) = C[L, P]$. By [22, Theorem 1.13] there exist $a, b \in C[\lambda]$ such that $X_{2s+1}^2 = a(L)X_{2s+1} + b(L)$. Since $P^2 = (X_{2s+1} + p_0(L))^2$ and $\{1, X_{2s+1}\}$ is a basis as $C(L)$ module of the centralizer, we obtain that for $p_0 = -a/2$ then $P = X_{2s+1} - (1/2)a(L)$ satisfies $P^2 + R_{2s+1}(L) = 0$ where $R_{2s+1}(\lambda) = -b(\lambda) - (1/4)a(\lambda)^2$ must be a polynomial of degree $2s+1$. Observe that P is unique up to multiples αP, $\alpha \in C$, so we choose the monic A_{2s+1}. □

We introduce next the terminology that will be useful to explain our contribution to the extensively studied commuting pair L, A_{2s+1}.

Definition 2.3. — *Given an algebro-geometric operator L of second order, we call the monic operator A_{2s+1} of minimal order $2s+1$ in $\mathcal{C}(L)$ such that $\mathcal{C}(L) = C[L, A_{2s+1}]$ and $A_{2s+1}^2 + R_{2s+1}(L) = 0$, with $R_{2s+1}(\lambda)$ in $C[\lambda]$ of degree $2s+1$,* the partner *of L, and say that L is an* algebro-geometric operator of level s *and denote it by L_s.*

Let us assume now that the algebro-geometric operator L_s of level s is in normal form, see [56, (2)]. Thus L_s is a Schrödinger operator

$$L_s = -\partial^2 + u_s, \text{ with } u_s \in \Sigma.$$

We prove in Appendix A that u_s verifies one of the equations of the celebrated stationary KdV hierarchy. This is probably a well known result, but we could not find a proof of it, so we include the proof there for completion.

Proof of Theorem A. — By Theorem 2.2 and Appendix A the equivalences follow. □

Given a Schrödinger operator L_s, to decide if it is algebro-geometric one has to look for a nontrivial commuting operator. To look for A_{2s+1} one possibility is forcing the commutator of L_s with an arbitrary operator of order $2n+1$ to be zero, starting with $n = 1$. The determination of the level s is intrinsically related to the determination of the vector of integration constants $\mathbf{c}^s \in C^s$, see Theorem A.1, and it is necessary for the effective computation of the partner A_{2s+1} of L_s. The algorithmic treatment of these results can be found in [42].

Observe that by Theorem A.1, A_{2s+1} is a differential operator with coefficients in the differential field $C\langle u_s\rangle$. Once we fix $u_s \in \Sigma$, we will work in the differential subfield $K = C\langle u_s\rangle$ of Σ.

3. The differential field of the spectral curve

Let us consider an algebro-geometric Schrödinger operator $L_s = -\partial^2 + u_s$ of level s and partner operator A_{2s+1}, as defined in Section 2. In other words, now we are fixing a KdV-potential u_s in Σ and considering commuting differential operators L_s and A_{2s+1} in $K[\partial]$ with $K = C\langle u_s\rangle$. By Theorem 2.2, there exists a polynomial

$$f_s(\lambda, \mu) = \mu^2 + R_{2s+1}(\lambda) \tag{3.1}$$

in $C[\lambda, \mu]$ such that $f_s(L_s, A_{2s+1}) = 0$. The constant coefficients polynomial f_s is called the *Burchnall–Chaundy (BC) polynomial* of the pair $\{L_s, A_{2s+1}\}$, since the first result of this sort appeared in [10]. We denote by Γ_s the affine algebraic curve in C^2 determined by $f_s(\lambda, \mu) = 0$, which is called the *spectral curve of the pair* $\{L_s, A_{2s+1}\}$.

Recall that the *rank* of a pair of differential operators is the greatest common divisor of their orders, [58]. Observe that all pairs $\{L_s, A_{2s+1}\}$ studied in this paper are rank 1 pairs. But also note that the same spectral curve could correspond to pairs of algebro-geometric operators with different rank. For instance the algebraic curve defined by $\mu^2 - \lambda^3 = 0$ is the spectral curve of the rank 1 pair

$$L_1 = -\partial^2 + \frac{2}{x^2} \quad \text{and} \quad A_3 = \partial^3 - \frac{3}{x^2}\partial + \frac{3}{x^3}\,,$$

in $C(x)[\partial]$ with $\partial = d/dx$, but it is also of the spectral curve of the famous pair of rank 2 operators posted by Dixmier see [48], which moreover is a "true rank" 2 pair (see also [48] for the definition of "true rank")

$$H^2 + 2x \quad \text{and } H^2 + \frac{3}{2}(xH + Hx)\,, \text{ with } H = \partial^2 + x^2.$$

The case of rank $r > 1$ corresponds to a vector bundle of rank r over the spectral curve. These bundles are related with the "inverse" spectral problem, [33, 34]. A difficult interesting problem is to give new "true rank" r pairs. Important contributions were made by Davletshina, Grinevich, Mironov, Mokhov, Oganesyan, Pogorelov Shamaev, and Zheglov (see [48], and the references therein).

Traditionally BC polynomials are computed as characteristic polynomials [21]. We would like to point out that once A_{2s+1} has been calculated,

by Previato's Theorem (see [47] or [42, Theorem 5.4]) one can compute f_s by means of the differential resultant. Moreover, by [58] the differential resultant of $L_s - \lambda$ and $A_{2s+1} - \mu$ is related with the rank r by

$$\partial \operatorname{Res}(L_s - \lambda, A_{2s+1} - \mu) = f_s^r. \tag{3.2}$$

In Appendix B we summarize the definition and main properties of this tool.

Observe that $f_s(\lambda, \mu)$ is an irreducible polynomial in $C[\lambda, \mu]$, since it has odd degree in λ and degree 2 in μ. Let us denote by (f_s) the prime ideal generated by f_s in $C[\lambda, \mu]$ or $K[\lambda, \mu]$, abusing the notation and distinguishing them by the context. Let us consider the monomial lexicographical order with $\mu > \lambda$ in $C[\lambda, \mu]$. Given $p \in C[\lambda, \mu]$, let us denote by p_N the normal form of p with respect to (f_s) (that is, p_N is the remainder of dividing p by f_s in $C[\lambda, \mu]$, see [14]). Observe that p_N is a polynomial in $C[\lambda, \mu]$ of degree one in μ. This reduction with respect to (f_s) will allow us to prove the next two results.

Theorem 3.1. — *Let $I = (L_s - \lambda, A_{2s+1} - \mu)$ be the ideal generated by $L_s - \lambda$ and $A_{2s+1} - \mu$ in $K[\lambda, \mu][\partial]$. Let (f_s) be the ideal generated by f_s in $C[\lambda, \mu]$. The differential elimination ideal $I \cap C[\lambda, \mu]$ verifies*

$$(f_s) = I \cap C[\lambda, \mu] = \{p \in C[\lambda, \mu] \mid p(L_s, A_{2s+1}) = 0\}. \tag{3.3}$$

Proof. — By Theorem B.2, $f_s \in I$. Thus we have the next chain of inclusions of ideals in $C[\lambda, \mu]$

$$(f_s) \subseteq I \cap C[\lambda, \mu] \subseteq J = \{p \in C[\lambda, \mu] \mid p(L_s, A_{2s+1}) = 0\}.$$

We will prove that $J \subset (f_s)$ and therefore the equality holds.

Given $p \in J$, we can write $p = hf_s + p_N$, for $h \in C[\lambda, \mu]$. If we assume that p_N is nonzero then $p_N = a(\lambda) + b(\lambda)\mu$, $a, b \in C[\lambda]$. Therefore $a(L_s) = -b(L_s)A_{2s+1}$ which is a contradiction since $a(L_s)$ has even order and $b(L_s)A_{2s+1}$ has odd order. This proves that p_N is identically zero and that $p \in (f_s)$. □

Let us denote by $C(\Gamma_s)$ and $K(\Gamma_s)$ the fraction fields of the domains

$$C[\Gamma_s] = \frac{C[\lambda, \mu]}{(f_s)} \text{ and } K[\Gamma_s] = \frac{K[\lambda, \mu]}{(f_s)} \tag{3.4}$$

respectively. Observe that $C(\Gamma_s)$ and $K(\Gamma_s)$ are usually interpreted as rational functions on the algebraic curve Γ_s defined by the polynomial f_s. The next result gives a description of the centralizer of L_s, alternative to

the famous one given by I. Schur [51], in terms of pseudodifferential operators. As a consequence the quotient field of the centralizer is a function field of one variable.

Proposition 3.2. — *The centralizer $\mathcal{C}_{\mathcal{D}}(L_s) = C[L_s, A_{2s+1}]$ of L_s in $\mathcal{D} = C\langle u_s\rangle[\partial]$ and the domain $C[\Gamma_s]$ are isomorphic commutative rings.*

Proof. — Given $p + (f_s)$ in $C[\Gamma_s]$ it has a representative given by the normal form $p_N = a(\lambda) + b(\lambda)\mu$ of p with respect to (f_s). By (2.1) we establish the isomorphism sending $p + (f_s)$ to $a(L_s) + b(L_s)A_{2s+1}$. □

In the history of this problem one can find different approaches that go from a local to a global treatment, "to work over the spectral curve". Note that the affine curve Γ_s could have singular points.

One could fix a point $P_0 = (\lambda_0, \mu_0)$ of Γ_s, in which case the differential operators

$$L_s - \lambda_0 \text{ and } A_{2s+1} - \mu_0 \text{ belong to } K[\partial]. \tag{3.5}$$

Hence one may even write $(\lambda_0, R_{2s+1}(\lambda_0)^{1/2})$, whenever $R_{2s+1}(\lambda_0) \neq 0$, as in [7, 21].

In the seminal works of Krichever [31, 32], the formal Baker–Akhiezer function is given using a local parametrization $(\tau^2, \mu(\tau))$. See also [43, 58]. Local parametrizations around a point P_0 are generally obtained as Puiseux series, see for instance [52, Section 2.5]. They always exist in the field of Puiseux series $C\langle\langle\tau\rangle\rangle$ and the differential operators

$$L_s - \tau^2 \text{ and } A_{2s+1} - \mu(\tau) \text{ belong to } K\langle\langle\tau\rangle\rangle[\partial]. \tag{3.6}$$

Their analytical behavior depends on the type of point of the curve (singular or regular). Those local expansions would allow a local parametric study of the spectral problem (3.6).

In the visionary works of Burchnall and Chaundy [10] and [11], the attention is driven towards the case of singular curves, for which their results regarding Abelian equations are no longer valid. In [11] they analyze cuspidal curves defined by $\mu^n - \lambda^m = 0$, n, m coprime, by means of the global parametrization $(\chi_1(\tau), \chi_2(\tau)) = (\tau^m, \tau^n)$.

We propose two new approaches "to work over the spectral curve". First, we consider λ and μ as generic variables and assume that

$$L_s - \lambda\,,\ A_{2s+1} - \mu \text{ belong to } K(\Gamma_s)[\partial]. \tag{3.7}$$

As operators in $K[\lambda, \mu][\partial]$, by Remark B.4 (1), they do not have a common factor, since $\partial\,\mathrm{Res}(L_s - \lambda, A_{2s+1} - \mu)$ is nonzero, it equals the BC polynomial

f_s. Considering them as operators in $K(\Gamma_s)[\partial]$, they have a common right factor, see Section 4.

Second, in the case of rational curves, see Section 6, we establish the coupling governed by a global parameterization $(\chi_1(\tau), \chi_2(\tau))$ in $C(\tau)^2$ and consider that differential operators

$$L_s - \chi_1(\tau)\,,\ A_{2s+1} - \chi_2(\tau) \text{ belong to } K(\tau)[\partial]. \tag{3.8}$$

Observe that $K(\tau)$ is a much smaller field than $K\langle\langle\tau\rangle\rangle$. The possibility of obtaining a global parametrization depends on the genus g of the curve Γ_s and ultimately of its singular locus, see for instance [3]. If the curve Γ is rational, $g = 0$, there are symbolic algorithms to obtain a global parametrization [52]. If Γ_s is an elliptic curve, $g = 1$, it can be parametrized by elliptic functions in $\mathbb{C}$, [53]. For $g \geqslant 2$ this is a difficult open problem, some contributions have been made in this direction, for instance by Y.V. Brezhnev in [6].

We can properly describe now the problem solved in this paper. We work with an algebro-geometric Schrödinger operator

$$L_s - \lambda \quad \text{in } K(\Gamma_s)[\partial] \tag{3.9}$$

to define, in Section 4, the minimal field extension of $K(\Gamma_s)$ that contains the solutions of $(L_s - \lambda)\Psi = 0$. In the remaning of this section we prove that $C(\Gamma_s)$ is the field of constants of $K(\Gamma_s)$.

We extend the derivation ∂ of K to the polynomial ring $K[\lambda, \mu]$ by

$$\partial\left(\sum a_{i,j}\lambda^i\mu^j\right) = \sum \partial(a_{i,j})\lambda^i\mu^j,\ \ a_{i,j} \in K \tag{3.10}$$

with ring of constants $C[\lambda, \mu]$. Let (f_s) be the ideal generated by f_s in $K[\lambda, \mu]$. Observe that for any $p \in K[\lambda, \mu]$ it holds that

$$\partial(pf_s) = \partial(p)f_s + p\partial(f_s) = \partial(p)f_s \tag{3.11}$$

since $f_s \in C[\lambda, \mu]$. This implies that (f_s) is a differential ideal in $(K[\lambda, \mu], \partial)$. Let us consider the domains in (3.4) and observe that $C[\Gamma_s] \hookrightarrow K[\Gamma_s]$. Secondly we consider the standard differential structure of the quotient ring $K[\Gamma_s]$ given by the following:

$$\widetilde{\partial}(q + (f_s)) = \partial(q) + (f_s),\ \ q \in K[\lambda, \mu]. \tag{3.12}$$

Observe that $\widetilde{\partial}$ is a derivation in $K[\Gamma_s]$ since (f_s) is a differential ideal. By abuse of notation we also denote by $\widetilde{\partial}$ its extension to the fraction field

$K(\Gamma_s)$. The next commutative diagram summarizes the situation:

$$\begin{tikzcd} K[\Gamma_s] \arrow[r, hook] & K(\Gamma_s) \\ C[\Gamma_s] \arrow[u, hook] \arrow[r, hook] & C(\Gamma_s) \arrow[u, hook] \end{tikzcd}$$

To work in the ring $K[\Gamma_s]$ we will consider special representatives of its elements, fixing the monomial lexicographical order with $\mu > \lambda$ in $K[\lambda, \mu]$. Given $p + (f_s) \in K[\Gamma_s]$, with representative $p \in K[\lambda, \mu]$, let us denote by p_N the normal form of p with respect to (f_s) (see [14]). We will call p_N the *normal form of p on Γ_s*. Observe that p_N is a polynomial in $K[\lambda, \mu]$ of degree one in μ. The following observations will be very important in what follows.

Remark 3.3. — Given q and h polynomials in $K[\lambda, \mu]$ of degree one in μ.

(1) If h is a factor of q then $q = \beta h$ for some nonzero β in $K[\lambda]$.
(2) $q = a + b\mu$ is irreducible in $K[\lambda, \mu]$ if and only if $\gcd(a, b) = 1$.
(3) Given q of degree one in μ, we can factor it as $q = \Lambda \widehat{q}$ where $\Lambda \in K[\lambda]$ and $\widehat{q}$ is irreducible of degree one in μ.

Proposition 3.4. — *Let q be a polynomial in $K[\lambda, \mu]$ of degree greater or equal than one in μ.*

(1) *Let $q + (f_s)$ be a nonzero element in $K[\Gamma_s]$. There exists a nonzero $T \in K[\lambda, \mu]$ and a nonzero $\Lambda \in K[\lambda]$ such that $Tq + (f_s) = \Lambda + (f_s)$.*
(2) *Every nonzero $\frac{p}{q}$ in $K(\Gamma_s)$ equals $\frac{r}{\Lambda}$ in $K(\Gamma_s)$ where $r \in K[\lambda, \mu]$ and $\Lambda \in K[\lambda]$.*

Proof. — The polynomials q and f_s can be seen as polynomials in μ with coefficients in the field $K(\lambda)$. By hypothesis, f_s does not divide q in $K[\lambda, \mu]$. This implies that f_s does not divide q in $K(\lambda)[\mu]$, since f_s is monic by (3.2); therefore they are coprime. There exist $A, B \in K(\lambda)[\mu]$ such that $Aq + Bf_s = 1$, see [14, Chapter 1, Section 5, Proposition 6]. We can write $A(\lambda, \mu) = g(\lambda, \mu)/a(\lambda)$ and $B(\lambda, \mu) = h(\lambda, \mu)/b(\lambda)$, with $g, h \in K[\lambda, \mu]$ and nonzero $a, b \in K[\lambda]$. Thus $bgq + ahf_s = ab$ in $K[\lambda, \mu]$. In $K[\Gamma_s]$ we have

$$(T + (f_s))(q + (f_s)) = \Lambda(\lambda) + (f_s), \text{ with } T = bg \text{ and } \Lambda = ab. \tag{3.13}$$

Given a nonzero $p + (f_s)$ in $K[\Gamma_s]$ from (3.13) we have $Tqp + (f_s) = \Lambda p + (f_s)$ thus $\frac{p}{q}$ equals $\frac{Tq}{\Lambda}$ in $K(\Gamma_s)$ and for $r = Tq$ statement (2) follows. □

We are now ready to prove Theorem B.

Theorem 3.5.

(1) *The ring of constants of* $(K[\Gamma_s], \widetilde{\partial})$ *is* $C[\Gamma_s]$.
(2) *(Theorem B) The field of constants of* $(K(\Gamma_s), \widetilde{\partial})$ *is* $C(\Gamma_s)$.

Proof. — Let us consider $p + (f_s)$ in $K[\Gamma_s]$ such that $\widetilde{\partial}(p + (f_s)) = 0$. Let $p_N = a + b\mu$ be its normal form on Γ_s, then $\partial(a) + \partial(b)\mu \in (f_s)$ in $K[\lambda, \mu]$. Then $\partial(a) = \partial(b) = 0$. Hence $p + (f_s)$ belongs to $C[\Gamma_s]$, which proves statement (1).

Let us also consider $v \in K(\Gamma_s)$ such that $\widetilde{\partial}(v) = 0$. By Proposition 3.4 (2), we have $v = \frac{p}{\Lambda_1}$ in $K(\Gamma_s)$, with $p \in K[\lambda, \mu]$ and $\Lambda_1 \in K[\lambda]$.

If $p = \Lambda_2 \in K[\lambda]$ then $\widetilde{\partial}(v) = \frac{H}{\Lambda_1^2}$ in $K(\Gamma_s)$ with $H = \partial(\Lambda_2)\Lambda_1 - \partial(\Lambda_1)\Lambda_2$. Thus $0 = \Lambda_1^2\widetilde{\partial}(v) = H + (f_s)$. Since H is a polynomial in $K[\lambda]$ then $H = 0$. Hence $\Lambda_2 = \gamma\Lambda_1$, with γ in the field of constants $C(\lambda)$ of $K(\lambda)$, and $v = \gamma$ in $K(\Gamma_s)$. Consequently $v \in C(\Gamma_s)$.

If $p \notin K[\lambda]$, by Remark 3.3 we can write $v = \frac{p_N}{\Lambda_1}$ in $K(\Gamma_s)$, where the normal form on Γ_s of p equals $p_N = a + b\mu$. Now $0 = \Lambda_1^2\widetilde{\partial}(v) = H + (f_s)$, with

$$H = \Lambda_1\partial(p_N) - \partial(\Lambda_1)p_N.$$

Since the degree in μ of H equals one then $H = 0$ in $K[\lambda, \mu]$. This implies that p_N divides $\partial(p_N)$ and by Remark 3.3, $\partial(p_N) = \beta p_N$, with $\beta \in K[\lambda]$. Thus a and b are solutions of the linear differential equation $\partial(\Psi) = \beta(\lambda)\Psi$ then $a = cb$, $c = c_1/c_2$ with $c_1, c_2 \in C[\lambda]$. Therefore $v = \frac{b(c_2+\mu)}{c_1\lambda_1}$ in $K(\Gamma_s)$ and

$$0 = \widetilde{\partial}(v) = (c_1 + \mu)\widetilde{\partial}(w) \Longleftrightarrow \widetilde{\partial}(w) = 0, \text{ with } w = \frac{b}{c_1\Lambda_1} \text{ in } K(\Gamma_s).$$

Thus $w \in C(\Gamma_s)$, which proves that $v \in C(\Gamma_s)$, and statement (2) is proved. □

4. Spectral Picard–Vessiot fields

We are ready now to introduce the main concept of this paper, the *spectral Picard–Vessiot (PV) field* of the equation

$$(L_s - \lambda)\Psi = 0, \tag{4.1}$$

where, as before, $L_s = -\partial^2 + u_s$ is an algebro-geometric Schrödinger operator of level s. We consider $(L_s - \lambda)(\Psi) = 0$ as an homogeneous linear differential equation of second order with coefficients in $(K(\Gamma_s), \widetilde{\partial})$. Since $\widetilde{\partial}$

extends the derivation ∂ of K, as can be deduced from (3.12), when there is no room for confusion we write ∂ instead of $\widetilde{\partial}$.

Let us recall the definition of Picard–Vessiot extension following Kaplansky (see [15, 29, 49] for instance).

Definition 4.1. — *Let $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0$ be a linear homogeneous differential equation with coefficients in the differential field Σ. We say that a differential field $\mathcal{E}$ containing Σ is a* Picard–Vessiot extension of Σ for the above equation, *if the following conditions are satisfied:*

(1) *$\mathcal{E} = \Sigma\langle u_1, \ldots, u_n\rangle$, the differential field extension of Σ generated by a fundamental set of solutions $\{u_1, \ldots, u_n\}$ of $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0$.*
(2) *$\mathcal{E}$ and Σ have the same field of constants.*

In this paper we use Picard–Vessiot extensions as in Definition 4.1, but, since L_s is an algebro-geometric operator we will be able to give a precise description of its Picard–Vessiot field in connection with its spectral curve. We will call this particular field structure *spectral Picard–Vessiot field over the curve*. Due to its importance for this paper we present next the definition.

Definition 4.2. — *A differential field extension $\mathcal{E}$ of $K(\Gamma_s)$ is called a* spectral Picard–Vessiot field over the curve Γ_s *of the equation $(L_s-\lambda)(\Psi) = 0$ if the following conditions are satisfied:*

(1) *$\mathcal{E} = K(\Gamma_s)\langle\Psi_1, \Psi_2\rangle$, the differential field extension of $K(\Gamma_s)$ generated by Ψ_1, Ψ_2, where $\{\Psi_1, \Psi_2\}$ is a fundamental set of solutions of $(L_s - \lambda)(\Psi) = 0$.*
(2) *$\mathcal{E}$ and $K(\Gamma_s)$ have the same field of constants $C(\Gamma_s)$.*

Afterwards, we will prove its existence highlighting the importance of its field of constants $C(\Gamma_s)$ and proving that it is a transcendental Liouvillian extension of $K(\Gamma_s)$.

For the convenience of the reader, we recall some definitions from differential algebra. Let K be a differential field and Σ a differenttial extension of K. An element t in Σ is a *primitive over K* if $Dt \in K$; the element t is an *hyperexponential over K* if $Dt/t \in K$; and t is called *Liouvillian element of* Σ if t is either algebraic, or a primitive or an hyperexponential over K. The field Σ is a *Liouvillian extension of K* if there are $t_1, \ldots, t_n$ in Σ such that $\Sigma = K(t_1, \ldots, t_n)$ and t_i is Liouvillian over $K(t_1, \ldots, t_{i-1})$ for $i = 1, \ldots, n$.

The next proposition shows that, as an element of $K(\Gamma_s)[\partial]$, the algebro-geometric operator $L_s - \lambda$ has an intrinsic order one (right) factor, that is linked to the (unique) partner A_{2s+1} of L_s.

Proposition 4.3. — *The monic greatest common (right) divisor of the differential operators $L_s - \lambda$ and $A_{2s+1} - \mu$ in $K(\Gamma_s)[\partial]$ is the order one operator*

$$\partial - \phi_s, \text{ where } \phi_s = \frac{\mu + \alpha(\lambda)}{\varphi(\lambda)},$$

for α and φ nonzero polynomials in $K[\lambda]$. Moreover ϕ_s is nonzero in $K(\Gamma_s)$ and we call $\partial - \phi_s$ the intrinsic right factor of $L_s - \lambda$.

Proof. — As defined in Appendix B, let us consider the differential resultant G_0 and first subresultant G_1 of $L_s - \lambda$ and $A_{2s+1} - \mu$ in (B.2) and (B.3) respectively. Since G_0 is zero in $K(\Gamma_s)$ by Theorem B.3 the greatest common factor of $L_s - \lambda$ and $A_{2s+1} - \mu$ is nontrivial. In addition G_1 is an order one differential operator

$$G_1 = (-\mu - \alpha(\lambda)) + \varphi(\lambda)\partial$$

where α and φ are nonzero polynomials in $K[\lambda]$ and by Theorem B.3 it is the greatest common right divisor.

We will also write ϕ_s to denote the element ϕ_s in $K(\Gamma_s)$. Observe that $\phi_s = 0$ in $K(\Gamma_s)$ if and only if $\mu + \alpha + (f_s) = 0$ in $K[\Gamma_s]$. But this is not possible since f_s, which has degree 2 in μ, is not a factor of $\mu + \alpha$ in $K[\lambda, \mu]$. This proves the last claim. □

Remark 4.4. — We proved Proposition 4.3 using the Differential Subresultant Theorem B.3. In addition the first subresultant $G_1 = \varphi_2\partial + \varphi_1$, see (B.3), of $L_s - \lambda$ and $A_{2s+1} - \mu$ can be used to compute the factor $\partial - \phi_s$, see [42] for an algorithmic approach.

We have the next factorization over the spectral curve

$$L_s - \lambda = (-\partial - \phi_s)(\partial - \phi_s), \text{ in } K(\Gamma_s)[\partial], \tag{4.2}$$

since, once this right factor is set, the only possibility as a left factor is $-\partial - \phi_s$. Let us define $\phi_+ := \phi_s$ before obtaining another factorization of $L_s - \lambda$. Observe that the BC-polynomial of $L_s - \lambda$ and $A_{2s+1} + \mu$ is also $f_s(\lambda, -\mu) = f_s(\lambda, \mu)$. Applying Proposition 4.3, we obtain another factorization of $L_s - \lambda$, namely

$$L_s - \lambda = (-\partial - \phi_-)(\partial - \phi_-), \text{ in } K(\Gamma_s)[\partial], \tag{4.3}$$

with

$$\phi_- = \frac{-\mu + \alpha(\lambda)}{\varphi(\lambda)}.$$

Both ϕ_+ and ϕ_- are solutions of the Riccati equation $\partial(\phi) + \phi^2 = u_s + \lambda$ with coefficients in $K(\Gamma_s)$.

Nonzero solutions Ψ_+ and Ψ_- of $(L_s - \lambda)(\Psi) = 0$ are defined respectively by the differential relations

$$\partial(\Psi_+) = \phi_+\Psi_+ \text{ and } \partial(\Psi_-) = \phi_-\Psi_- \tag{4.4}$$

and hence Ψ_+ and Ψ_- belong to the differential closure of the field $K(\Gamma_s)$, [30]. Therefore

$$\frac{\partial(\Psi_+)}{\Psi_+} = \phi_+ \text{ and } \frac{\partial(\Psi_-)}{\Psi_-} = \phi_-$$

belong to $K(\Gamma_s)$ and $K(\Gamma_s)\langle\Psi_+\rangle$ and $K(\Gamma_s)\langle\Psi_-\rangle$ are Liouvillian extensions of $K(\Gamma_s)$.

Lemma 4.5. — *Given Ψ_+ and Ψ_- as in* (4.4)*, it holds that:*

(a) *$\{\Psi_+, \Psi_-\}$ is a fundamental set of solutions of $(L_s - \lambda)(\Psi) = 0$.*
(b) *$\Psi_+\Psi_- \in K(\Gamma_s)$.*

Proof. — Trivially Ψ_+ and Ψ_- are nonzero solutions of $L_s - \lambda$. We will prove that their wronskian is nonzero. Observe that μ is a constant in $K(\Gamma_s)$, that is $\mu \in C(\Gamma_s)$ and furthermore it is nonzero, and the operator $L_s - \lambda$ is in normal form. Since $\partial(w(\Psi_+, \Psi_-)) = 0$, $w(\Psi_+, \Psi_-)$ belongs to $C(\Gamma_s)$. The following computation

$$\frac{w(\Psi_+, \Psi_-)}{\Psi_+\Psi_-} = \frac{\partial(\Psi_+)}{\Psi_+} - \frac{\partial(\Psi_-)}{\Psi_-} = \phi_+ - \phi_- = \frac{2\mu}{\varphi} \tag{4.5}$$

implies that $w(\Psi_+, \Psi_-) \neq 0$ in $C(\Gamma_s)$. This formula implies that

$$\Psi_+\Psi_- = \frac{\varphi w(\Psi_+, \Psi_-)}{2\mu} \in K(\Gamma_s), \tag{4.6}$$

which completes the proof. □

Remark 4.6. — Notice that Lemma 4.5 (b) establishes that the product of two solutions belongs to the field of coefficients of the Schrödinger operator. Thus, the Picard–Vessiot structures $K(\Gamma_s)\langle\Psi_\pm\rangle$ will also benefit from this fact, analogously to the classical case, whenever a particular value $\lambda = \lambda_0$ is chosen. In this case, since Hermite [27], Halphen [24], Drach [17, 18], the study of the second symmetric power of the operator is fundamental to analyze the Lamé equation (see Wittaker–Watsom [57, page 570 and the references therein]).

We would like to point out that the function of Lemma 4.5 (b) was also used in the important work of Gelfand and Dickii to study the asymptotic behavior of the resolvent, see [20].

By Lemma 4.5 (b) the next equality of differential filed extensions of $K(\Gamma_s)$ holds

$$K(\Gamma_s)\langle\Psi_+,\Psi_-\rangle = K(\Gamma_s)\langle\Psi_+\rangle = K(\Gamma_s)\langle\Psi_-\rangle. \tag{4.7}$$

Let us denote by $\Psi_s = \Psi_+$, which is defined by the differential relation $\partial(\Psi_s) = \phi_s\Psi_s$. We will prove that the Liouvillian extension

$$K(\Gamma_s) \subset K(\Gamma_s)\langle\Psi_s\rangle \tag{4.8}$$

is transcendental, showing that Ψ_s is a transcendental element over $K(\Gamma_s)$. In fact, this is intrinsically related to the determination of the subfield of constants of the field $K(\Gamma_s)\langle\Psi_s\rangle$. For this purpose we will use results from the book of M. Bronstein (see [9] and the references therein). We quote here some results from [9] for the convenience of the reader.

Definition 4.7 ([9, Definition 3.4.3]). — *Let (F,δ) be a differential field. We say that $\phi \in F$ is a* logarithmic derivative of a F-radical *if there exist a nonzero v in F and an integer $n \neq 0$ such that $n\phi = \delta v/v$.*

Theorem 4.8 ([9, Theorem 5.1.2]). — *If t is an hyperexponential over F and Dt/t is not a logarithmic derivative of a F-radical, then t is a monomial over F, the field of constants of $F(t)$ equals the field of constants of F; and $\mathcal{S} = F$. Conversely, if t is transcendental and hyperexponential over F, and the field of constants of $F(t)$ equals the field of constants of F then Dt/t is not a logarithmic derivative of a F-radical.*

Theorem 4.9 ([9, Corollary 3.3.1]). — *Let (F,D) be a differential field and let E be a separable algebraic extension of F. Let also $\mathbf{C}$ be the constant field of F with respect to the derivation D and let $\overline{\mathbf{C}}^E$ be the algebraic closure of $\mathbf{C}$ in E, i.e. the subfield of all the elements of E that are algebraic over $\mathbf{C}$. Then D can be extended uniquely to E, call it $\widetilde{D}$, and the field of constants of E with respect to $\widetilde{D}$ is also $\overline{C}^E$. In addition, if E is algebraically closed, then the field of constants of E with respect to $\widetilde{D}$ is an algebraic closure of $\mathbf{C}$.*

The next diagram corresponds to Theorem 4.9.

$$\begin{tikzcd}
\overline{\mathbf{C}}^E \arrow[r, hook] \arrow[d, dash] & (E,\widetilde{D}) \arrow[d, dash] \\
\mathbf{C} \arrow[r, hook] & (F,D)
\end{tikzcd}$$

Observe that if there exists a nonzero $v \in K(\Gamma_s)$ and a nonzero integer n such that

$$\frac{\widetilde{\partial}(v)}{nv} = \phi_s = \frac{\widetilde{\partial}(\Psi_s)}{\Psi_s}$$

then for $c = \Psi_s^n/v$ we have $\widetilde{\partial}(c) = 0$ and also $\Psi_s^n - cv = 0$. This ensures that Ψ_s is algebraic over a differential field that is generated by $K(\Gamma_s)$ and a possibly new constant c. We will prove that this is not the case for Ψ_s.

Lemma 4.10. — *There does not exists a nonzero $v \in K(\Gamma_s)$ such that $\phi_s = \frac{\tilde{\partial}(v)}{nv}$ for a nonzero integer n. That is, ϕ_s is not a logarithmic derivative of a $K(\Gamma_s)$-radical.*

Proof. — Let us assume that there exists $v \in K(\Gamma_s)$, $v \neq 0$, $\widetilde{\partial}(v) \neq 0$ such that $\phi_s = \frac{\tilde{\partial}(v)}{nv}$ for a nonzero integer n. By Proposition 3.4 (2), we can write $v = \frac{p}{\Lambda_1}$ in $K(\Gamma_s)$ for some nonzero $p \in K[\lambda, \mu]$ and $\Lambda_1 \in K[\lambda]$. Let p_N be the normal form of p on Γ_s.

Let us consider the polynomial in $K[\lambda, \mu]$

$$H = [\partial(p_N)\Lambda_1 - p_N\partial(\Lambda_1)]\,\varphi - np_N\Lambda_1(\mu + \alpha). \tag{4.9}$$

Recall that $\phi_s = \frac{\mu+\alpha}{\varphi}$ as in Theorem 4.3, hence $\alpha, \varphi \in K[\lambda]$ and

$$\frac{\widetilde{\partial}(v)}{nv} - \phi_s = \frac{H}{np_N\Lambda_1\varphi} = 0 \quad \text{in } K(\Gamma_s).$$

Now we apply Proposition 3.4 (1), for $q = np_N\Lambda_1\varphi$. Then there exists nonzero $T \in K[\lambda, \mu]$ and a nonzero $\Lambda_2 \in K[\lambda]$ such that

$$0 = \frac{\Lambda_2 H}{np_N\Lambda_1\varphi} = \frac{\Lambda_2 H}{q} = \frac{TH}{1} \quad \text{in } K(\Gamma_s).$$

Therefore in $K(\lambda, \mu)$

$$\frac{\Lambda_2 H}{np_N\Lambda_1\varphi} = Nf_s, \text{ for some } N \in K[\lambda, \mu].$$

Finally we obtain

$$\Lambda_2\,[[\partial(p_N)\Lambda_1 - p_N\partial(\Lambda_1)]\,\varphi - np_N\Lambda_1(\mu + \alpha)] = np_N\Lambda_1\varphi N f_s. \tag{4.10}$$

If $p_N \in K[\lambda]$ then the degree in μ of the LHS of (4.10) is 1 and of RHS of (4.10) is at least 2. Thus this is not possible. We have proved that v cannot be equal to γ in $K(\Gamma_s)$, with $\gamma \in K(\lambda)$. Hence it remains to check the case where p_N is not in $K[\lambda]$.

Let us assume that $p_N \notin K[\lambda]$. By (4.10), p_N is a factor of $\Lambda_1\Lambda_2\partial(p_N)\varphi$. Then by Remark 3.3(1), $\partial(p_N) = \Lambda_3 p_N$, with $\Lambda_3 \in K[\lambda]$. Hence equality (4.10) becomes

$$\Lambda_2\left[[\Lambda_3\Lambda_1 - \partial(\Lambda_1)]\,\varphi - n\Lambda_1(\mu+\alpha)\right] = n\lambda_1\varphi N f_s. \tag{4.11}$$

Observe that the degree in μ of the LHS of (4.11) is 1 and of RHS of (4.11) is at least 2. But this is a contradiction. Therefore we conclude that such v does not exist, which proves the result. □

It is an immediate consequence of the results in [9] that the previous lemma is equivalent to the next theorem.

Theorem 4.11. — *Let L_s be an algebro-geometric Schrödinger operator with spectral curve Γ_s. Let us consider the intrinsic right factor of $L_s - \lambda$ from Proposition 4.3, $\partial - \phi_s$, in $K(\Gamma_s)[\partial]$. A nonzero solution Ψ_s of $(L_s - \lambda)\Psi = 0$ defined by the differential relation $\partial(\Psi_s) = \phi_s\Psi_s$ is transcendental over $K(\Gamma_s)$ and the field of constants of $K(\Gamma_s)\langle\Psi_s\rangle$ equals the field of constants of $K(\Gamma_s)$.*

Proof. — By Lemma 4.10, applying Theorem 4.8 ([9, Theorem 5.1.2]) to the hyperexponential $t = \Psi_s$ and the differential field $(K(\Gamma_s), \widetilde{\partial})$, the result follows. □

We then proved the existence of the spectral Picard–Vessiot field over the curve Γ_s of the equation $(L_s - \lambda)(\Psi) = 0$.

Proof of Theorem C. — Theorem C is a direct consequence of Lemma 4.5, Theorem 4.11 and Definition 4.2. □

We illustrate Theorem C with the following commutative diagram, whose second row shows the fields of constants:

$$\begin{tikzcd}
K(\Gamma_s) \arrow[r, hook] & K(\Gamma_s)\langle\Psi_s\rangle \\
C(\Gamma_s) \arrow[u] \arrow[r, equal] & C(\Gamma_s). \arrow[u, hook]
\end{tikzcd}$$

In the next two sections we show applications of this new structure, the spectral PV field of the operator $L_s - \lambda$.

5. Classical Picard–Vessiot fields

Let $L_s = -\partial^2 + u_s$ be an algebro-geometric Schrödinger operator with spectral curve Γ_s and nonconstant potential u_s. For any fixed $P_0 = (\lambda_0, \mu_0)$

in Γ_s, we will describe the Picard–Vessiot (PV) extension over $K = C\langle u_s\rangle$ of $L_s - \lambda_0$.

We recall that, when a particular value of the spectral parameter is fixed, in the definition of the PV extension for $L_s - \lambda_0$, the field of constants used to be an algebraically closed field C of characteristic 0, see [49] and also [1]. Then, now we are looking for a differential field $\Sigma = K\langle y_1, y_2\rangle$, the differential field extension of K generated by y_1, y_2, where $\{y_1, y_2\}$ is a fundamental set of solutions of $(L_s - \lambda_0)(y) = 0$, whose field of constants is also C. We will call Σ *classical Picard–Vessiot Extension* in this case.

For a nonsingular point P_0 of Γ_s, the dimension of the space of common solutions of $L_s - \lambda_0$ and $A_{2s+1} - \mu_0$ is known to be one, [58, Theorem 5.8]. In the next proposition we prove that this holds even for singular points.

Proposition 5.1. — *For every $P_0 = (\lambda_0, \mu_0)$ in Γ_s the differential operators $L_s - \lambda_0$ and $A_{2s+1} - \mu_0$ have a greatest common right factor of order one $\partial - \phi_0$ in $K[\partial]$ with nonzero*

$$\phi_0 = \frac{\mu_0 + \alpha(\lambda_0)}{\varphi(\lambda_0)} \in K. \tag{5.1}$$

Proof. — By the Subresultant Theorem B.3, $L_s - \lambda_0$ and $A_{2s+1} - \mu_0$ have a common factor of order one in $K[\partial]$, namely $\varphi_1(P_0) + \varphi_2(P_0)\partial$ the specialization of (B.3) to $P_0 = (\lambda_0, \mu_0)$, hence $\varphi_2(\lambda_0) \neq 0$ □

For a fixed $P_0 = (\lambda_0, \mu_0)$ (singular or not) of Γ_s, we have the factorization

$$L_s - \lambda_0 = (-\partial - \phi_0)(\partial - \phi_0), \quad \text{in } K[\partial]. \tag{5.2}$$

Let us denote by y_0 a nonzero solution of $(\partial - \phi_0)(y) = 0$. Thus y_0 is a common solution of

$$\begin{cases} (L_s - \lambda_0)(y) = 0, \\ (A_{2s+1} - \mu_0)(y) = 0. \end{cases} \tag{5.3}$$

For u_s an analytic function in a suitable complex domain, y_0 is the stationary Baker–Akhiezer function

$$y_0 = \Psi(P_0, x, x_0) = \exp\left(\int_{x_0}^{x} \phi_s(P_0, x')\,\mathrm{d}x'\right), \quad P_0 \in \Gamma_s.$$

Traditionally the Baker–Akhiezer function is only defined for nonsingular affine points of Γ_s ([21, (1.41)]).

Proposition 5.2. — *For any P_0 in Γ_s, let us consider a nonzero solution y_0 in a differential closure of the differential field K of $(\partial - \phi_0)y = 0$. The field of constants of the differential field $K\langle y_0\rangle$ is C.*

Proof. — Being defined by $y_0'/y_0 = \phi_0 \in K$, we know that $K\langle y_0\rangle$ is a Liouvillian extension of K. If y_0 is algebraic, the result follows by Theorem 4.9 ([9, Corollary 3.3.1]). If y_0 is transcendental the result follows by [9, Theorems 5.1.1 and 5.1.2]. □

To describe the PV extension of $L_s - \lambda_0$, we must distinguish two different types of point $P_0 = (\lambda_0, \mu_0)$ in the curve, the ones with $\mu_0 \neq 0$ and those with $\mu_0 = 0$, that is the finite set

$$Z_s = \Gamma_s \cap (C \times \{0\}) = \{(\lambda, 0) \mid R_{2s+1}(\lambda) = 0\}. \tag{5.4}$$

Observe that Z_s contains all the affine singular points and ramification points of Γ_s.

Theorem 5.3. — *Let us fix $P_0 = (\lambda_0, \mu_0)$ in $\Gamma_s \backslash Z_s$. The PV extension of the equation $(L_s - \lambda_0)y = 0$, is the Liouvillian extension $K\langle y_0\rangle$ of K by a nonzero solution y_0 of $(\partial - \phi_0)y = 0$ as in* (5.2).

Proof. — Applying Proposition 5.1 to the point $P_0' = (\lambda_0, -\mu_0)$ gives a new factorization

$$L_s - \lambda_0 = (-\partial - \phi_{0-})(\partial - \phi_{0-}), \quad \text{in } K[\partial].$$

Then we have

$$\phi_{0+} = \phi_0 = \frac{\mu_0 + \alpha(\lambda_0)}{\varphi(\lambda_0)} \quad \text{and} \quad \phi_{0-} = \frac{-\mu_0 + \alpha(\lambda_0)}{\varphi(\lambda_0)}. \tag{5.5}$$

Hence we consider nonzero solutions y_+ and y_- of the differential equations $\partial(y) = \phi_{0+}y$ and $\partial(y) = \phi_{0-}y$ respectively, in a differential closure of K. The equality

$$\frac{w(y_+, y_-)}{y_+ y_-} = \phi_{0+} - \phi_{0-} = \frac{2}{\varphi(\lambda_0)}\mu_0 \neq 0$$

implies that $W_0 = w(y_+, y_-) \neq 0$ in C. Therefore $\{y_+, y_-\}$ is a fundamental set of solutions of $(L_s - \lambda_0)(y) = 0$. Moreover

$$y_+ y_- = \frac{\varphi(\lambda_0) W_0}{2\mu_0} \in K,$$

hence $K\langle y_+, y_-\rangle = K\langle y_+\rangle$. In addition, by Proposition 5.2, $K\langle y_+\rangle$ and K have the same field of constants C, which proves that $K\langle y_+\rangle$ is the PV field of $L_s - \lambda_0$. □

Observe that $(\partial - \phi_0)(y_0) = 0$ implies $(-\partial - \phi_0)(y_0^{-1}) = 0$. Thus a solution of the Risch differential equation

$$\partial(y) - \phi_0 y = y_0^{-1} \tag{5.6}$$

over the differential field $K\langle y_0\rangle$, see [9, Section 6], would be a solution of $(L_s - \lambda_0)(y) = 0$ because of the factorization (5.2).

THEOREM 5.4. — *Let us fix $(\lambda_0, 0)$ in Z_s. Given nonzero solutions y_0 of $(\partial - \phi_0)y = 0$ as in (5.2) and y_1 of (5.6), we have a chain of Liouvillian extensions*

$$K \subset K\langle y_0\rangle \subset K\langle y_0, y_1\rangle \tag{5.7}$$

with field of constants C, and then $K\langle y_0, y_1\rangle$ is the PV extension of $(L_s - \lambda_0)y = 0$.

Proof. — By Proposition 5.2 the field of constants of $K\langle y_0\rangle$ is C. Let us assume that $y_1 \notin K\langle y_0\rangle$. We have a fundamental system of solutions $\{y_0, y_1\}$ of $(L_s - \lambda_0)(y) = 0$ since

$$\frac{w(y_0, y_1)}{y_0 y_1} = \frac{\partial(y_0)}{y_0} - \frac{\partial(y_1)}{y_1} = \frac{-1}{y_0 y_1}.$$

Remains to prove that the field of constants of $K\langle y_0, y_1\rangle$ is C.

Let $\widehat{y}_1$ be a nonzero solution of $\partial - y_0^{-1}$. Without loss of generality, we can take $y_1 = y_0\widehat{y}_1$ as the solution of (5.6). Observe that $K\langle y_0, y_1\rangle = K\langle y_0, \widehat{y}_1\rangle$ is a Liouvillian extension of $K\langle y_0\rangle$ and $\widehat{y}_1$ is a primitive element over $K\langle y_0\rangle$. If $\partial(\widehat{y}_1)$ is a not the derivative of an element of $K\langle y_0\rangle$, by [9, Theorem 5.1.1] the field of constants of $K\langle y_0, \widehat{y}_1\rangle$ is C. Otherwise $\partial(\widehat{y}_1) = \partial(N_0)$, $N_0 \in K\langle y_0\rangle$. Derivating $y_1/y_0 = \widehat{y}_1$ we have $\partial(y_1)/y_1 = \partial(y_0)/y_0 + \partial(\widehat{y}_1)$, and hence $\partial(y_1)/y_1 \in K\langle y_0\rangle$.

Moreover if the hyperexponential y_1 is a logarithmic derivative of a $K\langle y_0\rangle$-radical then, by (5.6), $y_1 \in K\langle y_0\rangle$, which is a contradiction. Thus Theorem 4.11 ([9, Theorem 5.1.2]) implies that the field of constants of $K\langle y_0, y_1\rangle$ is the field of constants of $K\langle y_0\rangle$, that is C. □

Remark 5.5. — We would like to point out that the classical solutions of the Lamé equation correspond exactly to the case contemplated here when $\mu = 0$, see [57] and the references therein. Then their Picard–Vessiot extensions are given by (5.7). Their Galois groups are studied in [40].

For all but a finite number of points in Γ_s, we summarize the situation in the next commutative diagram:

$$\begin{tikzcd}
C(\Gamma_s) \arrow[r] & K(\Gamma_s) \arrow[r, "\text{Liouvillian}"] & K(\Gamma_s)\langle\Psi_s\rangle \\
C \arrow[u] \arrow[r] & K \arrow[u] \arrow[r, "\text{at } P_0\in\Gamma_s"] & K\langle y_0\rangle \arrow[u]
\end{tikzcd}$$

for $K = C\langle u_s\rangle$. In fact, in previous notations, this diagram holds for each $P_0 \notin Z_s$ in (5.4). In other words, for the non-branch points of Γ_s, we have

a good specialization process to obtain the classical Picard–Vessiot field from the spectral Picard–Vessiot field.

6. One-parameter spectral PV fields for rational curves

In this section we assume that the spectral curve Γ_s of the pair $\{L_s, A_{2s+1}\}$ is an affine rational curve, i.e. its closure in $\mathbb{P}^2$ is birrationally equivalent to $\mathbb{P}^1$ ([26, p. 78]). Hence an open subset $\widetilde{U}$ of Γ_s is isomorphic to a Zariski open subset U of the affine line $\mathbb{A}^1$. Then we have

$$\mathbb{C}(\Gamma_s) \overset{\rho_1}{\simeq} \mathbb{C}(\tau) \quad \text{with } \rho_1(\lambda) = \chi_1(\tau) \text{ and } \rho_1(\mu) = \chi_2(\tau) \tag{6.1}$$

for a complex parameter τ such that

$$\aleph : U \ni \tau \to (\chi_1(\tau), \chi_2(\tau)) \in \Gamma_s \tag{6.2}$$

is a regular isomorphism, and then $\mathbb{C}(\Gamma_s) \simeq \mathbb{C}(U)$.

Let us consider next the differential structure of $K(\tau) = \mathbb{C}\langle u_s\rangle(\tau)$. Recall that τ is an algebraic indeterminate over $\mathbb{C}\langle u_s\rangle$, which allows to extend the derivation $\partial = d/dx$ of $\mathbb{C}\langle u_s\rangle$ to $\mathbb{C}\langle u_s\rangle(\tau)$ since $\partial(\tau) = 0$. Then we extend the isomorphism ρ_1 to an isomorphism of differential fields $\mathbb{C}(\Gamma_s)\langle u_s\rangle \simeq \mathbb{C}(\tau)\langle u_s\rangle$. Since $\mathbb{C}(\tau)\langle u_s\rangle = \mathbb{C}\langle u_s\rangle(\tau)$, the composition of ρ_1 with the natural isomorphism $\mathbb{C}\langle u_s\rangle(\Gamma_s) \simeq \mathbb{C}(\Gamma_s)\langle u_s\rangle$ gives an isomorphism, let us call it also ρ_1,

$$K(\Gamma_s) = \mathbb{C}\langle u_s\rangle(\Gamma_s) \simeq \mathbb{C}\langle u_s\rangle(\tau) = K(\tau). \tag{6.3}$$

The next commutative diagram of differential fields illustrates the differential algebraic situation:

$$\begin{array}{ccc} K(\Gamma_s) & \xrightarrow{\rho_1} & K(\tau) \\ \uparrow & & \uparrow \\ \mathbb{C}(\Gamma_s) & \xrightarrow{\rho_1} & \mathbb{C}(\tau) \end{array} \tag{6.4}$$

Moreover, ρ_1 extends naturally to an isomorphism ϱ between the rings of differential operators $K(\Gamma_s)[\partial]$ and $K(\tau)[\partial]$.

Using Proposition 4.3, we obtain the right common factor $\varrho(\partial - \phi_s)$ of $L_s - \chi_1(\tau) = \varrho(L_s - \lambda)$ and $A_{2s+1} - \chi_2(\tau) = \varrho(A_{2s+1} - \mu)$ in $K(\tau)[\partial]$. Let us define $\widetilde{\phi}_s = \rho_1(\phi_s)$. Observe that $\widetilde{\phi}_s$ is a nonzero element of $K(\tau)$ since, by Proposition 4.3, ϕ_s is nonzero in $K(\Gamma_s)$. Furthermore, since the isomorphism respects the ring structure, from (4.2) we obtain

$$L_s - \chi_1(\tau) = (-\partial - \widetilde{\phi}_s)(\partial - \widetilde{\phi}_s), \text{ in } K(\tau)[\partial]. \tag{6.5}$$

Furthermore, we have factorizations

$$L_s - \chi_1(\tau) = (-\partial - \widetilde{\phi}_+)(\partial - \widetilde{\phi}_+) = (-\partial - \widetilde{\phi}_-)(\partial - \widetilde{\phi}_-)$$

where $\widetilde{\phi}_+ = \widetilde{\phi}_s$ and $\widetilde{\phi}_- = \rho(\phi_-)$ are distinct solutions of the same Riccati equation $\partial(\phi) + \phi^2 = u_s - \chi_1(\tau)$, since ρ_1 respects the differential field structure. In the differential closure $\widehat{K(\tau)}$ of $K(\tau)$ [30], we consider nonzero solutions Υ_+ and Υ_- respectively of the ordinary differential equations

$$\partial(\Upsilon) = \widetilde{\phi}_+\Upsilon \quad \text{and} \quad \partial(\Upsilon) = \widetilde{\phi}_-\Upsilon. \tag{6.6}$$

Lemma 6.1. — *Let Υ_+ and Υ_- as in* (6.6)*, it holds that:*

(1) *$\{\Upsilon_+, \Upsilon_-\}$ is a fundamental set of solutions of $(L_s - \chi_1(\tau))(\Upsilon) = 0$.*
(2) *$\Upsilon_+\Upsilon_- \in K(\tau)$.*

Proof. — The proof is analogous to the proof of Lemma 4.5, noting that

$$\frac{w(\Upsilon_+, \Upsilon_-)}{\Upsilon_+\Upsilon_-} = \widetilde{\phi}_+ - \widetilde{\phi}_- = \rho_1(\phi_+ - \phi_-) = \rho_1\left(\frac{2}{\varphi}\mu\right) \neq 0$$

since $\frac{2}{\varphi}\mu \neq 0$ and ρ_1 is an isomorphism. □

By Lemma 6.1, 2 we have $K(\tau)\langle\Upsilon_+, \Upsilon_-\rangle = K(\tau)\langle\Upsilon_+\rangle = K(\tau)\langle\Upsilon_-\rangle$. We denote $\Upsilon_s = \Upsilon_+$. Now we can apply the differential algebraic results in [9], to the hyperexponential Υ_s and the differential field $(K(\tau), \partial)$, regarding the integration problem of $\widetilde{\phi}_s = \partial\Upsilon_s/\Upsilon_s$ in $K(\tau)$.

Lemma 6.2. — *The element $\widetilde{\phi}_s$ of $K(\tau)$ is not a logarithmic derivative of a $K(\tau)$-radical.*

Proof. — Let us assume that there exists $w \in K(\tau)$, $w \neq 0$, $\partial(w) \neq 0$ such that $\widetilde{\phi}_s = \frac{\partial(w)}{nw}$ for a nonzero integer n. Since ρ_1 is an isomorphism, $w = \rho_1(v)$, with $v \in K(\Gamma_s)$, $v \neq 0$, $\widetilde{\partial}(v) \neq 0$. Then

$$\widetilde{\phi}_s = \rho_1(\phi_s) = \rho_1\left(\frac{\widetilde{\partial}(v)}{nv}\right) = \frac{\partial(w)}{nw}$$

implies $\phi_s = \frac{\tilde{\partial}(v)}{nv}$ contradicting Lemma 4.10. □

Theorem 6.3. — *Let L_s be an algebro-geometric Schrödinger operator with rational spectral curve Γ_s parametrized by $(\chi_1(\tau), \chi_2(\tau)) \in \mathbb{C}(\tau)^2$. Let $\partial - \widetilde{\phi}_s$ be the intrinsic right factor of $L_s - \chi_1(\tau)$ as in* (6.5)*. The Liouvillian extension $K(\tau)\langle\Upsilon_s\rangle$ of $K(\tau)$, by a nonzero solution $\Upsilon_s \in \widehat{K(\tau)}$ of $(\partial - \widetilde{\phi}_s)\Upsilon = 0$, is a transcendental extension with field of constants $\mathbb{C}(\tau)$.*

Proof. — By Lemma 6.2 and [9, Theorem 5.1.2] the result follows. □

In Theorem C of the introduction, we gave the field structure of the spectral Picard–Vessiot field for any spectral curve. In Theorem D of the introduction, we give its parametric presentation whenever the spectral curve is a rational curve. We are ready to prove Theorem D.

Proof of Theorem D. — The isomorphism ρ_1 in (6.4) extends naturally to an isomorphism between differential fields

$$\widehat{\rho}_1 : K(\Gamma_s)\langle\Psi_s\rangle \longrightarrow K(\tau)\langle\Upsilon_s\rangle$$

by sending a nonzero solution Ψ_s of $(\partial - \phi_s)\Psi = 0$, as in Theorem 4.11, to Υ_s. Hence diagram (6.4) extends to the following commutative diagram of differential fields, whose second row shows the fields of constants:

$$\begin{tikzcd}
K(\Gamma_s)\langle\Psi_s\rangle \arrow[r, "\hat{\rho}_1"] & K(\tau)\langle\Upsilon_s\rangle \\
\mathbb{C}(\Gamma_s) \arrow[u, hook] \arrow[r, "\rho_1"] & \mathbb{C}(\tau) \arrow[u, hook]
\end{tikzcd}$$

Hence we have proved the required statement. □

The global rational parametrization of the spectral curve allows us to be more specific about the integral representation of Υ_s, which was defined in (6.6) as a solution of $(\partial - \bar{\phi}_s)\Upsilon = 0$. Moreover, we have the following sequence of differential fields,

$$\mathbb{C}(\Gamma_s) \equiv \mathbb{C}(\tau) \subset \mathbb{C}\langle u_s\rangle(\tau) = K(\tau) \subset K(\tau)\langle\Upsilon_s\rangle. \tag{6.7}$$

with Υ_s a transcendental element over $\mathbb{C}\langle u_s, \tau\rangle$, by Theorem 6.3. We can finally show the advantages of constructing solutions of the spectral problem (1.1) using a global rational parametrization of the spectral curve in $\mathbb{C}(\tau)^2$, instead of a local parametrization by Puiseux series in $\mathbb{C}\langle\langle\tau\rangle\rangle^2$. We do so by means of a family of Rosen-Morse potentials in Example 6.5 to illustrate Remark 6.4, and establishing the appropriate algebraic setting to solve problem (1.1) analytically obtaining Theorem 6.6.

Remark 6.4. — Recall that $\partial - \widetilde{\phi}_s$ is an operator in $K(\tau)[\partial]$ where

$$K(\tau) = \mathbb{C}\langle u_s\rangle(\tau) = \mathbb{C}(\tau)\langle u_s\rangle.$$

If u_s is a monomial over the differential field $\mathbb{C}(\tau)$ one can address the integration of $(\partial - \bar{\phi}_s)\Upsilon = 0$ in the differential algebraic setting of [9], Chapter 5. Then the differential integration theorems and algorithms in [9] can be used to compute Υ_s in an elementary extension of $\mathbb{C}(\tau)$ if it exists.

Example 6.5. — Let us consider the family of Rosen-Morse potentials

$$u_s = \frac{-s(s+1)}{\cosh^2(x)}, \quad s \geqslant 1, \tag{6.8}$$

which belong to the differential field $K = \mathbb{C}\langle\cosh(x)\rangle = \mathbb{C}(e^x)$, with derivation $\partial = d/dx$ and field of constants $\mathbb{C}$. It is well known that the Schrödinger operators $-\partial^2 + u_s$ are algebro-geometric, [42, 55]. In [42], we gave algorithms to compute A_{2s+1}. By means of the differential resultant $\partial\,\mathrm{Res}(L_s - \lambda, A_{2s+1} - \mu)$ the defining polynomial f_s of Γ_s can be computed,

$$f_s(\lambda,\mu) = \mu^2 + \lambda \prod_{\kappa=1}^{s} (\lambda + \kappa^2)^2,$$

see for instance [21, Example 1.31]. By means of the first differential subresultant of $L_s - \lambda$ and $A_{2s+1} - \mu$ their right common factor $\partial - \phi_s$ is obtained. For $s = 1$

$$\phi_1 = \frac{\mu + \frac{1}{2}\partial(\varphi)}{\varphi} = \frac{(z^2+1)^3\mu + z^4 - z^2}{(z^2+1)((z^2+1)^2\lambda + z^4 + z^2 + 1)},$$

where $\varphi = \lambda + 1 - \frac{1}{\eta^2}$, with $\eta = \cosh(x)$ and $z = e^x$. All the curves Γ_s are rational, in particular they admit a polynomial global parametrization

$$\aleph_s(\tau) = (\chi_1(\tau), \chi_2(\tau)) = \left(-\tau^2, -\tau \prod_{\kappa=1}^{s} (\tau^2 - \kappa^2)\right), \tag{6.9}$$

with τ transcendental over $\mathbb{C}(e^x)$. Replacing $\lambda = \chi_1(\tau)$ and $\mu = \chi_2(\tau)$ in ϕ_s we obtain $\widetilde{\phi}_s(x,\tau)$, which is a rational function in $\mathbb{C}(\tau)(z)$. Since $z = e^x$ is transcendental over $\mathbb{C}(\tau)$, we are in the situation of Remark 6.4. Using the symbolic integration package of Maple 18 to obtain Υ_s as $\mathrm{int}(\widetilde{\phi}_s, x)$. For instance the primitive of $\widetilde{\phi}_1$ equals

$$\Upsilon_1 = \frac{(\tau - 1)\, z^2 + \tau + 1}{z^2 + 1} e^{x\tau},$$

replacing $\lambda = -\tau^2$ and $\mu = -\tau(\tau^2 - 1)$ in ϕ_1; observe that all functions are analytic outside the analytic set $E = \left\{\, z^4 + z^2 + 1 - (z^2+1)^2\tau^2 = 0 \,\right\} \subset \mathbb{C}^2$. By (6.7) the corresponding sequence of differential fields is:

$$\mathbb{C}(\Gamma_1) \equiv \mathbb{C}(\tau) \subset \mathbb{C}\langle e^x\rangle(\tau) \subset \mathbb{C}(e^x,\tau)\langle\Upsilon_1\rangle,$$

since $\mathbb{C}\left\langle \frac{-2}{\cosh^2(x)} \right\rangle(\tau) = \mathbb{C}(e^x,\tau)$. Moreover, Theorem 6.3 guaranties that the function Υ_1 is transcendental over $\mathbb{C}(e^x,\tau)$, and Theorem D that

$\mathbb{C}(e^x,\tau)\langle\Upsilon_1\rangle$ is the minimal field extension where the solutions of the spectral problem

$$\left(-\partial^2+\frac{-2}{\cosh^2(x)}\right)\Psi=-\tau^2\Psi \tag{6.10}$$

can be expressed as an element of a differential field with field of constants $\mathbb{C}(\tau)$. In fact, any solution of the above problem is of the form

$$\Psi(x,\tau)=c_1(\tau)\Upsilon_1(x,\tau)+c_2(\tau)\Upsilon_-(x,\tau),$$

with $c_i(\tau)\in\mathbb{C}(\tau)$ and $\Upsilon_-(x,\tau)=\frac{\varphi}{2\mu\Upsilon_1}=\frac{(-(z^2+1)^2\tau^2+z^4+z^2+1)(z^2+1)^2}{-2\tau(\tau^2-1)((\tau-1)z^2+\tau+1)}e^{-x\tau}$.

Next we will establish the appropriate algebraic setting to solve the spectral problem (1.1) analytically.

For an analytic potential $u_s(x)$ in a complex domain D in $\mathbb{C}$, we will prove that $(\partial-\bar{\phi}_s)\Upsilon=0$ has an analytic solution in an open neighborhood of each point in some open subset of $\mathbb{C}^2$. Since Γ_s is defined by $\mu^2+R_{2s+1}(\lambda)$, its singular locus $\mathrm{Sing}(\Gamma_s)$ is contained in the finite set $Z_s=\{(\lambda,0)\in\mathbb{C}^2\ :\ R_{2s+1}(\lambda)=0\}$, and we can assume that $U\cap Z_s=\emptyset$. As a consequence, at any fixed $\tau_0\in U$ we have

$$\chi_1(\tau)=\frac{g_1(\tau)}{h_1(\tau)}\,,\quad \chi_2(\tau)=\frac{g_2(\tau)}{h_2(\tau)} \tag{6.11}$$

for some polynomials g_i,h_i in $\mathbb{C}[\tau]$ and a neighborhood U_{τ_0} of τ_0 in U where $h_i(\tau)\neq 0$. Then, in U, we break down the spectral problem:

$$\left(-\partial^2+u_s(x)-\chi_1(\tau)\right)\Upsilon=0\,,\quad \left(A_{2s+1}-\chi_2(\tau)\right)\Upsilon=0 \tag{6.12}$$

into two steps. First we compute the right common factor of the operator $-\partial^2+u_s(x)-\chi_1(\tau)$ and $A_{2s+1}-\chi_s(\tau)$ with τ in U to obtain a one parameter family of common right first order factors, say $\partial-\widetilde{\phi}_s(x,\tau)$. Secondly, we address the resolubility of the first-order equation that generates the one dimensional space of common solutions of (6.12):

$$\left(\partial-\widetilde{\phi}_s(x,\tau)\right)\Upsilon=0\,,\quad \text{for } x\in D,\ \tau\in U. \tag{6.13}$$

Finally we can give a fundamental matrix for the linear differential equation $(-\partial^2+u_s(x)-\chi_1(\tau))\Upsilon=0$ that varies continuously in $\tau\in U$, see Theorem 6.6. We use concepts from Complex Analytical Geometry (see for example [36]).

THEOREM 6.6. — *Let $-\partial^2+u_s(x)$ be an algebro-geometric Schrödinger operator, with $u_s(x)$ an analytic potential in a complex open set D. Let us assume that the spectral curve Γ_s of $L_s-\lambda$ is a rational curve. Then, for each c in a domain $\Omega_c\subset D\times U\subset\mathbb{C}^2$, the equation*

$$(\partial-\widetilde{\phi}_s(x,\tau))\Upsilon=0 \tag{6.14}$$

has an analytic solution $\Upsilon(x,\tau)$ in a subdomain of Ω_c.

Proof. — Having a rational parametrization of the spectral curve Γ_s allows to write

$$\widetilde{\phi}_s(x,\tau)=\frac{N_1(x,\tau)}{N_2(x,\tau)} \tag{6.15}$$

with $N_1,N_2\in\mathbb{C}\{u_s(x)\}[\tau]$. For some open domain $D_1\subseteq D$ then (6.15) is a well defined analytic function in $(D_1\times U)\backslash E\subset\mathbb{C}^2$ where E is the analytic set $\{N_2(x,\tau)=0\}$.

For a fixed $c=(x_0,\tau_0)\in(D_1\times U)\backslash E$ there exists an open domain $\Omega_c\subset(D_1\times U)\backslash E$ where the differential equation

$$(\partial-\widetilde{\phi}_s(x,\tau))\Upsilon=0 \tag{6.16}$$

has an analytic solution $\Upsilon(x,\tau)$ in an open subdomain $\widetilde{\Omega}_c$ of Ω_c (see [28]). □

Remark 6.7. — Observe that Theorem 6.6 guaranties the existence of a sheaf structure of spectral Picard–Vessiot fields, $\mathbf{\Lambda}=\mathbf{\Lambda}(\Gamma_s)$, on a tubular neighborhood $W\subset\mathbb{C}^2$ of Γ_s, possibly outside of an analytic set (E in the proof of the previous theorem). For almost all $c\in W$, each fiber $\mathbf{\Lambda}_c$ is isomorphic to the differential field of germs at c of meromorphic functions on the spectral curve Γ_s over K extended with a solution Υ_c of (6.14). Moreover, when $W'\subset W$ is quasi-compact, the space of sections $\Gamma(W',\mathbf{\Lambda})$ has a field structure and then $\mathbf{\Lambda}$ is a torsion free sheaf over W. The relation between the sheaf $\mathbf{\Lambda}$, the K-points of Γ_s, and its singular locus over $\mathbb{C}$, is an intriguing question that we will study in future works.

7. Concluding remarks

We have generalized the concept of classical Picard–Vessiot extension of $L_s-\lambda_0$ over K, with algebraically closed constant field C, to define a spectral Picard–Vessiot field for $L_s-\lambda$ over $K(\Gamma_s)$ with constant field $C(\Gamma_s)$, which is not necessarily algebraically closed. These spectral Picard–Vessiot fields admit a specialization process at each $P_0=(\lambda_0,\mu_0)$ in Γ_s

that allowed us to give the classical Picard–Vessiot extensions Σ_{P_0}/K of $L_s - \lambda_0$ in Section 5. In addition, one could analyze the Differential Galois group $Gal(\Sigma_{P_0}/K)$ with the approaches given in [1, 8, 54].

Whenever the spectral curve Γ_s is hyperelliptic, a solution Ψ_s of problem (1.1), is expressed in [21, Theorem 1.20] in terms of Θ-functions, this is the well known Baker–Akhiezer function. Also in the hyperelliptic case, the work of Brezhnev ([8] and references there in) combines the expression of ϕ_s in terms of Θ-functions, associated to the spectral curve, with the definition of classical Picard–Vessiot extensions. It remains open to combine the new spectral Picard–Vessiot structure we have defined with the extensively developed Θ-function approach for the hyperelliptic case.

In Section 6 we studied the case of rational spectral curves transforming the original spectral problem (1.1) into a spectral problem

$$(-\partial^2 + u_s - \chi_1(\tau))\Upsilon = 0 \tag{7.1}$$

with a free parameter τ. In this case the spectral Picard–Vessiot field of (1.1) is isomorphic to a transcendental Liouvillian extension $K(\tau)(\Upsilon_s)$. In this context much more can be said about the hyperexponential Υ_s. In the coefficient field $K(\tau) = C(\tau)\langle u_s\rangle$, the differential integration theorems and algorithms of M. Bronstein in [9], and other references there in, can now be used to compute Υ_s. In addition we showed how, in the case of analytic potentials, the algebraic techniques developed in Section 4 (for all spectral curves) combined with a global rational parametrization of the curve, allowed us to solve the spectral problem (1.1) analytically in closed form when the spectral curve is a rational curve and has singular points, see Theorem 6.6 and Remark 6.7.

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Appendix A. KdV potentials

The purpose of this appendix is to define KdV potentials and prove that $-\partial^2 + u_s$ is algebro-geometric only for a KdV potential u_s. We present the KdV hierarchy and the family of differential operators of its Lax representation with the language of differential algebra [50]. The KdV-hierarchy was studied for the first time in the paper [20]. We follow the normalization in [21], see also [42, Section 3].

Let us consider a differential indeterminate u over C. We will call *formal Schrödinger operator* to the operator $L(u) = -\partial^2 + u$ with coefficients in the ring of differential polynomials $C\{u\} = C[u, u', u'', \ldots]$, where u' stands for $\partial(u)$ and $u^{(n)} = \partial^n(u)$, $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of positive integers including 0.

Let us consider the pseudo differential operator

$$\begin{aligned} &\mathcal{R} = -\frac{1}{4}\partial^2 + u + \frac{1}{2}u'\partial^{-1} \\ &\text{and its (formal) adjoint } \mathcal{R}^* = -\frac{1}{4}\partial^2 + u - \frac{1}{2}\partial^{-1}u', \end{aligned} \tag{A.1}$$

in the ring of pseudo-differential operators in ∂ with coefficients in $C\{u\}$ (see [22]) where ∂^{-1} is the inverse of ∂, $\partial^{-1}\partial = \partial\partial^{-1} = 1$. Recall that the (formal) adjoint operator of $T = \sum_{k=-d}^{m} a_k\partial^k$, $a_k \in C\{u\}$ is defined by $T^* = \sum_{k=-d}^{m}(-1)^k\partial^k a_k$, see [46, Theorem 5.31].

Observe that $\mathcal{R}^* = \partial^{-1}\mathcal{R}\partial$. The operator $\mathcal{R}^*$ is a recursion operator of the KdV equation (see [46, p. 319]). Applying the recursion operator $\mathcal{R}$, we define:

$$\mathrm{kdv}_0 := u', \;\; \mathrm{kdv}_n := \mathcal{R}(\mathrm{kdv}_{n-1}), \text{ for } n \geqslant 1. \tag{A.2}$$

Applying $\mathcal{R}^*$ we define:

$$v_0 := 1, \;\; v_n := \mathcal{R}^*(v_{n-1}), \text{ for } n \geqslant 1. \tag{A.3}$$

Hence for $n \in \mathbb{N}$ it holds $2\partial(v_{n+1}) = \mathrm{kdv}_n$. We will call the differential polynomials kdv_n the *KdV differential polynomials*. By [42, Lemma 3.1], the formulas for kdv_n and v_n give differential polynomials in $C\{u\}$.

As in [21], we define a family of differential operators in $C\{u\}[\partial]$ of odd order (see also [16, 45])

$$P_1(u) := \partial, \;\; P_{2n+1}(u) := v_n\partial - \frac{1}{2}\partial(v_n) + P_{2n-1}(u)L(u), \text{ for } n \geqslant 1. \tag{A.4}$$

The operators $P_{2n+1}(u)$ have the important property (see for instance [42, Lemma 3.2])

$$[P_{2n+1}(u), L(u)] = \mathrm{kdv}_n(u) \tag{A.5}$$

it is the multiplication operator by the kdv_n differential polynomial. This is the famous Lax representation of kdv_n, see [21, 45]. We will call the differential operators $P_{2n+1}(u)$ the *KdV differential operators*.

Note that after replacing the differential variable u by a potential $u_s \in \Sigma$ we obtain a Schrödinger operator $L_s = L(u_s)$ and differential operators $\{P^s_{2n+1} = P_{2n+1}(u_s)\}_{n\geqslant 0}$, whose coefficients belong to the differential field $K = C\langle u_s\rangle$ with derivation ∂ and field of constants C. The next theorem tells us how to construct the partner A_{2s+1} of an algebro-geometric L_s.

Theorem A.1. — *Let $L_s = -\partial^2 + u_s$ be an algebro-geometric Schrödinger operator of level s with partner A_{2s+1}. Then there exists a vector of constants $\mathbf{c}^s = (c^s_1, \dots, c^s_s) \in C^s$ such that*

$$A_{2s+1} = P_{2s+1}(u_s) + c^s_1 P_{2s-1}(u_s) + \cdots + c^s_s P_1(u_s).$$

Furthermore u_s verifies the equation $\mathrm{KdV}_s(u, \mathbf{c}^s) = 0$ of the KdV-hierarchy defined by the differential polynomial in $C\{u\}$

$$\mathrm{KdV}_s(u, c^s) = \mathrm{kdv}_s(u) + c^s_1\,\mathrm{kdv}_{s-1}(u) + \cdots + c^s_s\,\mathrm{kdv}_0(u). \tag{A.6}$$

Proof. — Let us assume that A_{2s+1} is monic. Since $\{P^s_{2i+1}\}_{i\leqslant s}$ and $\{L^i_s\}_{i\leqslant s}$ are families of operators in $K[\partial]$ of odd and even orders less than $2s+1$ respectively, we divide A_{2s+1} by those families and write

$$A_{2s+1} = \sum_{i=0}^{s} q_{2i+1} P^s_{2i+1} + \sum_{i=0}^{s} q_{2i} L^i_s \tag{A.7}$$

with $q_{2s+1} = 1$ and $q_{2i+1}, q_{2i} \in K$. Let us compute next the commutator of the right hand side of (A.7) with L_s. Observe that $[a, L_s] = \partial^2(a) + 2\partial(a)\partial$, for $a \in K$ and

$$[q_{2i+1}P^s_{2i+1}, L_s] = -\partial^2(q_{2i+1})P^s_{2i+1} - 2\partial(q_{2i+1})\partial P^s_{2i+1} + q_{2i+1}\,\mathrm{kdv}_i(u_s)$$

and

$$[q_{2i}L^i_s, L_s] = [q_{2i}, L_s]L^i_s = (\partial^2(q_{2i}) + 2\partial(q_{2i})\partial)L^i_s.$$

The only term of order $2i+2$ is the leading term of ∂P^s_{2i+1} and the only term of order $2i+1$ is the leading term of ∂L^i_s. Since $[A_{2s+1}, L_s] = 0$ then $\partial(q_{2i}) = 0$ and $\partial(q_{2i+1}) = 0$. Therefore $[q_{2i}L^i_s, L_s] = 0$ and $q_{2i+1} = c^s_i \in C$, $i = 0, \dots, n$ implies that

$$0 = \sum_{i=0}^{s} c^s_i\,\mathrm{kdv}_i(u_s) \quad \text{and} \quad A_{2s+1} = \sum_{i=0}^{s} c^s_i P^s_i,$$

which proves the result. □

Let us review the converse of the previous theorem, which is [42, Theorem 5.3], see [21]: For a vector $\mathbf{c}_n = (c_1, \dots, c_n)$ of algebraic indeterminates it holds by (A.5) and (A.6) that,

$$\left[L(u)\,,\ \sum_{i=0}^{n} c_i P_i(u)\right] = \mathrm{KdV}_n(u, \mathbf{c}_n), \quad \text{in } C\{u\}[\partial]. \tag{A.8}$$

Let as assume we are given $L_s = -\partial^2 + u_s$ such that u_s is a solution of an equation of the KdV-hierarchy (A.6), which means the following. There exists a vector of constants $\mathbf{c}^n = (c_1^n, \dots, c_s^n) \in C^n$ such that after replacing u by u_s and $\mathbf{c}_n$ by $\mathbf{c}^n$ then

$$[L_s, \sum_{i=0}^{n} c_i^n P_i(u_s)] = \mathrm{KdV}_n(u_s, \mathbf{c}^n) = 0 \ \text{ in } K[\partial]. \tag{A.9}$$

If s is minimal with the previous property, then there exists a unique vector of constants $\mathbf{c}^s$ such that $\mathrm{KdV}_s(u_, \mathbf{c}^s) = 0$ and, by [42, Proposition 4.2], $\mathcal{C}(L_s) = C[L_s, A_{2s+1}]$. See for instance [42, Section 4] for more details on the behaviour of the integration constants of the KdV-hierarchy.

Definition A.2. — *We say that u_s in Σ is a* KdV-potential of KdV level s *if it verifies the* KdV_s *equation* (A.6) *of the KdV-hierarchy for a vector of constants $\mathbf{c}^s \in C^s$, that is $\mathrm{KdV}_s(u_s, \mathbf{c}^s) = 0$, and s is minimal with this property.*

Appendix B. Differential Subresultant Theorem

We summarize next the definition and some important properties of differential resultants and subresultants to be used in this article. We use mainly the presentation given in [13], see also [35] and the recent report [38].

Let us consider differential operators P and Q of orders n and m respectively with coefficients in a differential integral domain $(\mathbb{D}, \partial)$, whose quotient field $\mathbb{K}$ is equipped with the same derivation ∂. The ring $\mathbb{K}[\partial]$ is a left Euclidean integral domain and therefore every left ideal is left principal. If $\mathrm{ord}(P) \geqslant \mathrm{ord}(Q)$ then $P = qQ + r$ with $\mathrm{ord}(r) < \mathrm{ord}(Q)$, $q, r \in \mathbb{K}[\partial]$. Let us denote by $\gcd(P, Q)$ the greatest common (left) divisor of P and Q.

As in the commutative case of polynomials in one variable, $G = gcd(P, Q)$ can be expressed in a unique way as $G = AP + BQ$, where the orders of A and B satisfy the natural restrictions. The search for A and B is equivalent to the resolution of a linear system defined by a linear map

$$\begin{array}{rccc} S_k : & \mathbb{K}^{n+m-2k} & \longrightarrow & \mathbb{K}^{n+m-k} \\ & (a_{n-k-1}, \dots, a_0, b_{m-k-1}, \dots, b_0) & \longmapsto & \text{coefficients of } AP + BQ\,, \end{array}$$

when searching for a gcd G_k of order k, $k = 0, 1, \ldots, N := \min\{n, m\} - 1$. The matrix $S_k(P, Q)$ of this linear map is the coefficient matrix of the extended system of differential operator

$$\Xi_k = \{\partial^{m-1-k}P, \ldots \partial P, P, \partial^{n-1-k}Q, \ldots, \partial Q, Q\}.$$

Observe that $S_k(P, Q)$ is a matrix with $n + m - 2k$ rows and $n + m - k$ columns, with entries in $\mathbb{D}$. Let G_k be the *determinant polynomial* of $S_k(P, Q)$ in $\mathbb{D}[\partial]$ (see [35] or the proof of Theorem B.2 for the construction).

Definition B.1. — *The* subresultant sequence *of P and Q is the sequence $\{G_k\}_{k=0}^N$ of differential operators in $\mathbb{D}[\partial]$. The* differential resultant of P and Q *is the zero order operator*

$$\partial \operatorname{Res}(P, Q) := G_0 = \det(S_0(P, Q)). \tag{B.1}$$

Theorem B.2 ([38, Theorem 3.10]). — *Let $P, Q \in \mathbb{D}[\partial]$. The resultant $\partial \operatorname{Res}(P, Q)$ of P and Q belongs to the ideal (P, Q) generated by P and Q in $\mathbb{D}[\partial]$.*

The next theorem gives us a method to look for $\gcd(P, Q)$ and is essential for the main results of this paper.

Theorem B.3 ([13, Theorem 4]. Differential Subresultant Theorem). *Given differential operators P and Q in $\mathbb{D}[\partial]$, $\gcd(P, Q)$ is a differential operator of order r if and only if:*

(1) *G_k is the zero operator for $k = 0, 1, , \ldots, r - 1$ and,*
(2) *G_r is nonzero.*

Then $\gcd(P, Q) = G_r$.

Remark B.4.

(1) The $\gcd(P, Q)$ is nontrivial (it is not in $\mathbb{K}$) if and only if $G_0 = \partial \operatorname{Res}(P, Q) = 0$.
(2) Given $G_r = \gcd(P, Q)$ then $P = \overline{P} G_r$ and $Q = \overline{Q} G_r$, $\overline{P}, \overline{Q} \in \mathbb{K}[\partial]$.

In this paper, we will use the case of differential operators $P = \partial^2 + a_0$ and $Q = \partial^{2s+1} + \cdots + b_1\partial + b_0$, $s \geqslant 1$ in $K[\partial]$, for a differential field (K, ∂). Observe that $P - \lambda$ and $Q - \mu$ are differential operators with coefficients in the differential domain $\mathbb{D} = K[\lambda, \mu]$. The differential resultant

$$G_0 = \partial \operatorname{Res}(P - \lambda, Q - \mu) = -\mu^2 - \lambda^{2s+1} + \cdots \tag{B.2}$$

is the determinant of the matrix $S_0(P - \lambda, Q - \mu)$, whose coefficients are in $K[\lambda, \mu]$ and it belongs to the elimination ideal

$$(P - \lambda, Q - \mu) \cap K[\lambda, \mu].$$

The first subresultant G_1 is a differential operator of order 1, the determinant polynomial of the matrix $S_1(P-\lambda, Q-\mu)$, whose rows are the coefficients of

$$\Xi_1 = \{\partial^{2s}(P-\lambda), \partial(P-\lambda), P-\lambda, Q-\mu\}$$

and whose columns are indexed by $\partial^{2s+1}, \cdots, \partial, 1$. The first subresultant equals

$$G_1 = \varphi_1 + \varphi_2\partial, \tag{B.3}$$

with $\varphi_1 = \det(S_1^0)$ and $\varphi_2 = \det(S_1^1)$, where S_1^0 and S_1^1 are the submatrices of S_1 obtained by removing columns indexed by ∂ and 1 respectively. Observe that

$$\det(S_1^0) = -\mu - \alpha(\lambda) \quad \text{and} \quad \det(S_1^1) = \varphi_2(\lambda), \tag{B.4}$$

for polynmials $\alpha, \varphi_2 \in K[\lambda]$.

Example B.5. — Given $P = \partial^2 + a_0$ and $Q = \partial^3 + b_2\partial^2 + b_1\partial + b_0$ in $K[\partial]$. The differential resultant $\partial\,\mathrm{Res}(P-\lambda, Q-\mu)$ is the determinant of the matrix $S_0(P-\lambda, Q-\mu)$, whose rows are the coefficients of the polynomials in $\Xi_0 = \{\partial^2(P-\lambda), \partial(P-\lambda), P-\lambda, \partial(Q-\mu), Q-\mu\}$ and whose columns are indexed by $\partial^4, \ldots, \partial, 1$. Take now the coefficient matrix

$$S_1(P-\lambda, Q-\mu) = \begin{pmatrix} 1 & 0 & a_0-\lambda & a_0' \\ 0 & 1 & 0 & a_0-\lambda \\ 1 & b_2 & b_1 & b_0-\mu \end{pmatrix}$$

of $\Xi_1 = \{\partial P, P-\lambda, Q-\mu\}$. The first subresultant $G_1 = \det(S_1^0) + \det(S_1^1)\partial$ where

$$S_1^0 = \begin{pmatrix} 1 & 0 & a_0' \\ 0 & 1 & a_0-\lambda \\ 1 & b_2 & b_0-\mu \end{pmatrix}, \quad S_1^1 = \begin{pmatrix} 1 & 0 & a_0-\lambda \\ 0 & 1 & 0 \\ 1 & b_2 & b_1 \end{pmatrix}.$$

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