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STABILITY OF THE TANGENT BUNDLES OF COMPLETE INTERSECTIONS AND EFFECTIVE RESTRICTION

by Jie LIU

ABSTRACT. — For $n \geqslant 3$ and $r \geqslant 1$, let M be an $(n+r)$-dimensional irreducible Hermitian symmetric space of compact type and let $\mathcal{O}_M(1)$ be the ample generator of pic(M). Let $Y = H_1 \cap \cdots \cap H_r$ be a smooth complete intersection of dimension n, where $H_i \in |\mathcal{O}_M(d_i)|$ with $d_i \geqslant 2$. We prove a vanishing theorem for twisted holomorphic forms on Y. As an application, we show that the tangent bundle T_Y of Y is stable. Moreover, if X is a smooth hypersurface of degree d in Y such that the restriction pic(Y) → pic(X) is surjective, we establish some effective results for d to guarantee the stability of the restriction $T_Y|_X$. In particular, if Y is a general hypersurface in $\mathbb{P}^{n+1}$ and X is a general smooth divisor in Y, we show that $T_Y|_X$ is stable except for some well-known examples. We also address the cases where the Picard group increases by restriction.

RÉSUMÉ. — Soient M un espace hermitien symétrique irréductible de type compact et de dimension $(n+r)$ avec $n \geqslant 3$ et $r \geqslant 1$, $\mathcal{O}_M(1)$ le générateur ample de pic(M). Soit $Y = H_1 \cap \cdots \cap H_r$ une intersection complète lisse de dimension n où $H_i \in |\mathcal{O}_M(d_i)|$ avec $d_i \geqslant 2$. Nous montrons un théorème d'annulation pour le faisceau tordu des germes de p-formes holomorphes $\Omega_Y^p(\ell)$. Comme application, nous montrons que le fibré tangent T_Y de Y est stable. De plus, si X est une hypersurface lisse de degré d dans Y telle que la restriction pic(Y) → pic(X) soit surjective, nous obtenons des estimations effectives liées à la stabilité de la restriction $T_Y|_X$. En particulier, si Y est une hypersurface générale dans $\mathbb{P}^{n+1}$ et X est un diviseur général, nous montrons que $T_Y|_X$ est stable sauf certains exemples bien connus. Nous considérons aussi le cas où le nombre de Picard augmente par restriction.

1. Introduction

It has been one of main problems in Kähler geometry to study which Fano manifolds with $b_2 = 1$ admit a Kähler–Einstein metric. The celebrated

Yau–Tian–Donaldson conjecture asserts that a Fano manifold admits a Kähler–Einstein metric if and only if it is K-polystable. This conjecture has been solved recently (see [5, 6, 33] and the references therein). A weaker and more algebraic question related to the existence of Kähler–Einstein metrics asks whether the tangent bundle is (semi-)stable with respect to the anti-canonical divisor. Let us recall the definition of stability. Let (Z, H) be an n-dimensional polarized projective manifold, and let $\mathcal{E}$ be a non-zero torsion-free coherent sheaf over Z. The *slope* of $\mathcal{E}$ with respect to H is defined to be

$$\mu(\mathcal{E}) := \frac{c_1(\mathcal{E}) \cdot H^{n-1}}{\mathrm{rk}(\mathcal{E})}.$$

DEFINITION 1.1. — *Let (Z, H) be a polarized projective manifold, and let $\mathcal{E}$ be a non-zero torsion-free coherent sheaf over Z. The sheaf $\mathcal{E}$ is called H-stable (resp. H-semi-stable) if for any non-zero coherent subsheaf $\mathcal{F} \subset \mathcal{E}$ with $0 < \mathrm{rk}(\mathcal{F}) < \mathrm{rk}(\mathcal{E})$, we have*

$$\mu(\mathcal{F}) < \mu(\mathcal{E}) \qquad (\text{resp.} \quad \mu(\mathcal{F}) \leqslant \mu(\mathcal{E})).$$

By the works of Ramanan, Umemura, Azad–Biswas, Reid, Peternell–Wiśniewski and Hwang, the stability of tangent bundles is known for homogeneous spaces [1, 27, 34], Fano manifolds with index one [29], Fano manifolds of dimension at most six [13, 26], general complete intersections in $\mathbb{P}^N$ [26] and prime Fano manifolds with large index [14]. If X is a projective manifold with $\mathrm{Pic}(X) \cong \mathbb{Z}\mathcal{O}_X(1)$, then the stability of T_X is very much related to the cohomology vanishings of type

$$H^q(X, \Omega_X^p(\ell)),$$

where $\Omega_X^p(\ell) = \Omega_X^p \otimes \mathcal{O}_X(\ell)$. If M is an irreducible Hermitian symmetric space of compact type, then M is a Fano manifold with Picard number one. In particular, making use of the work of Kostant [18], Snow developed an algorithm in [30] and [31] to determine whether a given cohomology group $H^q(M, \Omega_M^p(\ell))$ vanishes. Based on the algorithm of Snow, Biswas, Chaput and Mourougane proved in [2] the following vanishing theorem.

THEOREM 1.2 ([2, Theorem D]). — *Let M be an n-dimensional irreducible Hermitian symmetric space of compact type which is not isomorphic to a projective space. Let $\mathcal{O}_M(1)$ be the ample generator of $\mathrm{Pic}(M)$, and let ℓ and p be two positive integers such that $H^q(M, \Omega_M^p(\ell)) \neq 0$ for some $q \geqslant 0$. Then we have*

$$\ell + q \geqslant p\frac{r_M}{n},$$

where r_M is the index of M, i.e., $\mathcal{O}_M(-K_M) \cong \mathcal{O}_M(r_M)$.

As an application of this vanishing theorem, we prove the following similar vanishing theorem for smooth complete intersections in M.

THEOREM 1.3. — For $n \geqslant 2$ and $r \geqslant 1$, let M be an $(n+r)$-dimensional irreducible Hermitian symmetric space of compact type, and let $\mathcal{O}_M(1)$ be the ample generator of $\mathrm{Pic}(M)$. Let $Y = H_1 \cap \cdots \cap H_r$ be a smooth complete intersection of dimension n where $H_i \in |\mathcal{O}_M(d_i)|$ with $d_i \geqslant 2$. Denote $\mathcal{O}_M(1)|_Y$ by $\mathcal{O}_Y(1)$. Then for any $q \geqslant 0$, $p \geqslant 1$, $\ell \in \mathbb{Z}$ such that $q + p \leqslant n - 1$, the following statements hold.

   (1) If $H^q(Y, \Omega_Y^p) \neq 0$, then $p = q$.
   (2) If $H^q(Y, \Omega_Y^p(\ell)) \neq 0$ for some $\ell \neq 0$, then

$$\ell + q > p\frac{r_Y}{n},$$

   where $r_Y = r_M - d_1 - \cdots - d_r$.

In Theorem 1.3, if $n \geqslant 3$, the natural map $\mathrm{Pic}(M) \to \mathrm{Pic}(Y)$ is an isomorphism by Lefschetz hyperplane theorem [12, Corollary 3.3]. In particular, we have $\mathrm{Pic}(Y) \cong \mathbb{Z}\mathcal{O}_Y(1) = \mathbb{Z}\mathcal{O}_M(1)|_Y$. As the first application of our vanishing theorem, we generalize a result of Peternell–Wiśniewski (see [26, Theorem 1.5]) in the following theorem.

THEOREM 1.4. — For $n \geqslant 3$ and $r \geqslant 1$, let M be an $(n+r)$-dimensional irreducible Hermitian symmetric space of compact type, and let $\mathcal{O}_M(1)$ be the ample generator of $\mathrm{Pic}(M)$. If $Y = H_1 \cap \cdots \cap H_r$ is a smooth complete intersection of dimension n where $H_i \in |\mathcal{O}_M(d_i)|$ with $d_i \geqslant 2$, then the tangent bundle T_Y is stable.

Let (Z, H) be a polarized projective manifold, and let $Y \in |dH|$ be a general smooth hypersurface of degree d. Let $\mathcal{E}$ be a torsion-free coherent sheaf over Z. Then it is easy to see that, if $\mathcal{E}$ is an H-unstable sheaf, then $\mathcal{E}|_Y$ is $H|_Y$-unstable. Equivalently, $\mathcal{E}$ is semi-stable if $\mathcal{E}|_Y$ is $H|_Y$-semi-stable. Nevertheless, the converse is false in general as shown by the following easy counterexample.

Example 1.5. — The tangent bundle $T_{\mathbb{P}^n}$ of $\mathbb{P}^n$ is stable with $\mu(T_{\mathbb{P}^n}) = (n+1)/n$. However, if Y is a hyperplane, then the restriction $T_{\mathbb{P}^n}|_Y$ is unstable since T_Y is a subbundle of $T_{\mathbb{P}^n}|_Y$ with $\mu(T_Y) = n/(n-1)$.

By a result of Mehta–Ramanathan, the restriction of a (semi-)stable sheaf is (semi-)stable for sufficiently large d. In general, we have the following important effective restriction theorem (cf. [10, 19, 22]).

THEOREM 1.6 ([19, Theorem 5.2 and Corollary 5.4]). — *Let (Z, H) be a polarized projective manifold of dimension n. Let $\mathcal{E}$ be a torsion-free H-(semi-)stable sheaf of rank $p \geqslant 2$. Let $Y \in |dH|$ be a general smooth hypersurface. If*

$$d > \frac{p-1}{p}\Delta(\mathcal{E})H^{n-2} + \frac{1}{p(p-1)H^n},$$

then $\mathcal{E}|_Y$ is $H|_Y$-(semi-)stable. Here $\Delta(\mathcal{E}) = 2pc_2(\mathcal{E}) - (p-1)c_1^2(\mathcal{E})$ is the discriminant of $\mathcal{E}$.

In [2], an optimal effective theorem is established for tangent bundles of irreducible Hermitian symmetric spaces of compact type.

THEOREM 1.7 ([2, Theorem A and B]). — *For $n \geqslant 3$, let M be an n-dimensional irreducible Hermitian symmetric space of compact type, and let Y be a smooth hypersurface of M. Then the restriction $T_M|_Y$ is stable unless Y is a linear section and M is isomorphic to either $\mathbb{P}^n$ or a smooth quadric hypersurface Q^n.*

In [2], this theorem was stated for the cotangent bundle Ω_M^1 of M, which is equivalent to Theorem 1.7, since a vector bundle E is H-stable over a polarized projective manifold (Z, H) if and only if its dual bundle E^* is H-stable.

As the second application of Theorem 1.3, we reduce the effective restriction problem of tangent bundles to the existence of certain twisted vector fields (cf. Proposition 5.3), and then we can derive the following result.

THEOREM 1.8. — *For $n \geqslant 3$, let M be an $(n + r)$-dimensional irreducible Hermitian symmetric space of compact type, and let $\mathcal{O}_M(1)$ be the ample generator of $\mathrm{Pic}(M)$. Let $Y = H_1 \cap \cdots \cap H_r$ be a smooth complete intersection of dimension n where $H_i \in |\mathcal{O}_M(d_i)|$ with $2 \leqslant d_1 \leqslant \cdots \leqslant d_r$. Let $X \in |\mathcal{O}_Y(d)|$ be a smooth hypersurface of degree d. Assume moreover that the composite of restrictions*

$$\mathrm{Pic}(M) \longrightarrow \mathrm{Pic}(Y) \longrightarrow \mathrm{Pic}(X)$$

is surjective. Then the restriction $T_Y|_X$ is stable if one of the following conditions holds.

   (1) *Y is a Fano manifold and M is neither isomorphic to the projective space $\mathbb{P}^{n+r}$ nor any smooth quadric hypersurface Q^{n+r}.*
   (2) *Y is a Fano manifold, $d \geqslant d_1$ and M is isomorphic to the projective space $\mathbb{P}^{n+r}$.*
   (3) *Y is a Fano manifold, $d \geqslant 2$ and M is isomorphic to a smooth quadric hypersurface Q^{n+r}.*

(4) X is general and $d > d_r - r_Y/n$, where $r_Y = r_M - d_1 - \cdots - d_r$.

In the case where Y is a general smooth hypersurface in $\mathbb{P}^{n+1}$, using the strong Lefschetz property of the Milnor algebra of Y, we can prove an extension theorem for twisted vector fields on X (see Theorem 4.4), and an optimal answer to the effective restriction problem can be given in this setting.

THEOREM 1.9. — For $n \geqslant 3$, let Y be a general smooth hypersurface in the projective space $\mathbb{P}^{n+1}$. Let $X \in |\mathcal{O}_Y(d)|$ be a general smooth hypersurface of degree d in Y. Assume furthermore that the restriction homomorphism $\mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ is surjective, then $T_Y|_X$ is stable unless $d = 1$, and Y is isomorphic to either $\mathbb{P}^n$ or Q^n.

In each exceptional case, the tangent bundle of X will destabilize $T_Y|_X$, so our result above is sharp. The stability of restrictions of tangent bundles with an increase of Picard group was also considered in [2]. According to Lefschetz's hyperplane theorem, the map $\mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ is always surjective if $n \geqslant 4$. In fact, Lefschetz proved an even more general version, the so-called Noether–Lefschetz theorem, in [21]: a very general complete intersection surface X in $\mathbb{P}^N$ contains only curves that are themselves complete intersections unless X is an intersection of two quadric threefolds in $\mathbb{P}^4$, or a quadric surface in $\mathbb{P}^3$, or a cubic surface in $\mathbb{P}^3$ (see also [11, 16]). In these exceptional cases, the possibilities of the pair (Y, X) are as follows:

(1) The manifold Y is the projective space $\mathbb{P}^3$ and X is a quadric surface or a cubic surface.
(2) The manifold $Y \subset \mathbb{P}^4$ is a quadric threefold and X is a linear section or a quadric section.
(3) The manifold $Y \subset \mathbb{P}^4$ is a cubic threefold and X is a linear section of Y.

When Y is a quadric threefold or the projective space $\mathbb{P}^3$, according to [2, Theorem B], the restriction $T_Y|_X$ is semi-stable with respect to $-K_X$ unless Y and X are both projective spaces; and $T_Y|_X$ is stable with respect to $-K_X$ if X is not a linear section. In the following result, we address the stability of the restriction $T_Y|_X$ in the case where Y is a cubic threefold and X is a linear section.

THEOREM 1.10. — Let $Y \subset \mathbb{P}^4$ be a general cubic threefold, and let $X \in |\mathcal{O}_Y(1)|$ be a general smooth linear section. Then the restriction $T_Y|_X$ is stable with respect to $\mathcal{O}_X(1)$.

This paper is organized as follows. In Section 2, we introduce the basic notions concerning Hermitian symmetric spaces and the Lefschetz properties

of Artinian algebras. In Section 3, we collect the results about the coho-
mologies of $(n-1)$-forms of Hermitian symmetric spaces and we introduce
the notion of special cohomologies. In Section 4, we investigate the twisted
vector fields over complete intersections in Hermitian symmetric spaces and
we prove some extension results in various settings. In Section 5, we ad-
dress the stability of tangent bundles of complete intersections in Hermitian
symmetric spaces and study the effective restriction problem. In particu-
lar, we prove Theorem 1.3, Theorem 1.4, Theorem 1.8 and Theorem 1.9.
In Section 6, we consider the case where the Picard number increases by
restriction and prove Theorem 1.10.

Convention

For an n-dimensional projective variety Y, we denote by Ω_Y^1 the sheaf of
Kähler differentials and denote by Ω_Y^p the sheaf $\wedge^p \Omega_Y^1$. For a line bundle
$\mathcal{O}_Y(1)$ and a coherent sheaf $\mathcal{F}$ on Y, we will denote $\mathcal{F} \otimes \mathcal{O}_Y(\ell)$ by $\mathcal{F}(\ell)$,
and the number $h^i(Y, \mathcal{F})$ is the dimension of $H^i(Y, \mathcal{F})$ over $\mathbb{C}$. Moreover,
the dual sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_Y)$ of $\mathcal{F}$ is denoted by $\mathcal{F}^*$. If Y is smooth, the
canonical divisor, denoted by K_Y, is a Weil divisor associated to Ω_Y^n. For a
submanifold X of a polarized manifold $(Y, \mathcal{O}_Y(1))$, we denote by $\mathcal{O}_X(1)$ the
restriction $\mathcal{O}_Y(1)|_X$. If Y is a projective manifold with Picard number one,
we consider always the polarization given by the ample generator $\mathcal{O}_Y(1)$ of
$\mathrm{Pic}(Y)$ unless otherwise stated.

2. Hermitian symmetric spaces and Lefschetz properties

In this section, we collect some basic materials about Hermitian sym-
metric spaces of compact type and the Lefschetz properties of Artinian
algebras. We refer to [3] and [24] for further details.

2.1. Hermitian symmetric spaces

Let (M, g) be a Riemannian manifold. A non-trivial isometry σ of (M, g)
is said to be an *involution* if $\sigma^2 = \mathrm{id}$. A Riemannian manifold (M, g) is
said to be *Riemannian symmetric* if at each point $x \in M$ there exists an
involution σ_x such that x is an isolated fixed point of σ_x.

DEFINITION 2.1. — *Let (M, g) be a Riemannian symmetric manifold. Then (M, g) is said to be a Hermitian symmetric manifold if (M, g) is a Hermitian manifold and the involution σ_x at each point $x \in M$ can be chosen to be a holomorphic isometry.*

A Hermitian symmetric space M is called *irreducible* if it cannot be written as the non-trivial product of two Hermitian symmetric spaces. It is well-known that the irreducible Hermitian symmetric spaces of compact type are Fano manifolds of Picard number one. We will denote by $\mathcal{O}_M(1)$ the ample generator of $\mathrm{Pic}(M)$. In this case, the index of M is defined to the unique positive integer r_M such that $\mathcal{O}_M(-K_M) \cong \mathcal{O}_M(r_M)$.

The Hermitian symmetric spaces are homogeneous under their isometry groups. According to Cartan, there are exactly six types of irreducible Hermitian symmetric spaces of compact type: Grassmannians (type A_n), quadric hypersurfaces (type B_n or D_n), Lagrangian Grassmannians (type C_n), spinor Grassmannians (type D_n) and two exceptional cases (type E_6 and E_7).

2.2. Lefschetz properties of Artinian algebras

Let $R = \mathbb{C}[x_1, \ldots, x_r]$ be the graded polynomial ring in r variables over $\mathbb{C}$. Let

$$A = R/I = \bigoplus_{i=0}^{n} A_i$$

be a graded Artinian algebra. Then, by definition, A is finite-dimensional over $\mathbb{C}$.

DEFINITION 2.2. — *Let A be a graded Artinian algebra.*

(1) *We say that A has the maximal rank property (MRP) if for any d, the homomorphism induced by multiplication by f*

$$\times f \colon A_i \longrightarrow A_{i+d}$$

has maximal rank for all i (i.e., is injective or surjective), whenever f is a general form of degree d.

(2) *We say that A has the strong Lefschetz property (SLP) if for any d, the homomorphism induced by multiplication by ℓ^d*

$$\times \ell^d \colon A_i \longrightarrow A_{i+d}$$

has maximal rank for all i (i.e., is injective or surjective), whenever ℓ is a general linear form.

Remark 2.3. — The strong Lefschetz properties have been extensively investigated in the literature (see [24] and the references therein), while the maximal rank property has only been introduced in [23] by Migliore and Miró-Roig. Moreover, it is easy to see that SLP implies MRP by semi-continuity.

Both concepts are motivated by the following theorem which was proved by Stanley in [32] using algebraic topology, by Watanabe in [37] using representation theory, by Reid, Roberts and Roitman in [28] with algebraic methods.

THEOREM 2.4 ([24, Theorem 1.1]). — *Let $R = \mathbb{C}[x_1, \ldots, x_r]$, and let I be the Artinian complete intersection $\langle x_1^{d_1}, \ldots, x_r^{d_r} \rangle$, where d_i's are positive integers. Then R/I has the SLP.*

Let $\mathbb{P}^{n+1}$ be the $(n+1)$-dimensional complex projective space, and let $Y \subset \mathbb{P}^{n+1}$ be a hypersurface defined by a homogeneous polynomial h of degree d. We denote by

$$J(Y) = \langle \partial h/\partial x_0, \ldots, \partial h/\partial x_{n+1} \rangle$$

the *Jacobian ideal* of Y, where $[x_0 : \ldots : x_{n+1}]$ are the coordinates of $\mathbb{P}^{n+1}$. Then the *Milnor algebra* of Y is defined to be the graded $\mathbb{C}$-algebra

$$M(Y) := \mathbb{C}[x_0, \ldots, x_{n+1}]/J(Y).$$

Remark 2.5 ([7, p. 109]). — One observes that the Hilbert series of the Milnor algebra $M(Y)$ of a general degree d hypersurface Y in $\mathbb{P}^{n+1}$ is

$$H(M(Y))(t) = (1 + t + t^2 + \cdots + t^{d-2})^{n+2},$$

and $\rho = (d-2)(n+2)$ is the top degree of $M(Y)$. The celebrated Macaulay's theorem (cf. [35, Théorème 18.19]) says that the multiplication map

$$\mu_{i,j} : M(Y)_i \times M(Y)_j \longrightarrow M(Y)_{i+j}$$

is non-degenerated for $i + j \leqslant \rho$. Using the perfect pairing

$$M(Y)_i \times M(Y)_{\rho-i} \longrightarrow M(Y)_\rho \cong \mathbb{C},$$

we see that the dimension of $M(Y)_i$ is symmetric. Recall that an element $f \in M(Y)$ of degree j is called *faithful* if the multiplication $\times f : M(Y)_i \to M(Y)_{i+j}$ has maximal rank for all i. Since the dimension of $M(Y)_i$ is strictly increasing over the interval $[0, \rho/2]$, an element f of degree j is faithful if and only if it induces injections $M(Y)_i \to M(Y)_{i+j}$ for $i \leqslant (\rho - j)/2$, equivalently it induces surjections $M(Y)_i \to M(Y)_{i+j}$ for $i \geqslant (\rho - j)/2$.

The proof of Theorem 1.9 relies on the nonexistence of certain twisted vector fields over X. To prove this, we reduce the problem to the nonexistence of certain twisted vector fields over Y by proving an extension result (cf. Theorem 4.4). The main ingredient of the proof of Theorem 4.4 is the SLP of the Milnor algebra $M(Y)$ which is well-known to experts. Recall that the Fermat hypersurface of degree d in $\mathbb{P}^{n+1}$ is defined by the equation $x_0^d + \cdots + x_{n+1}^d = 0$.

PROPOSITION 2.6. — *Let $Y \subset \mathbb{P}^{n+1}$ be a general hypersurface of degree d. Then the Milnor algebra $M(Y)$ of Y has the SLP. In particular, $M(Y)$ has the MRP.*

Proof. — Thanks to Theorem 2.4, the Milnor algebra of the Fermat hypersurface of degree d in $\mathbb{P}^{n+1}$ has the SLP. Then we conclude by semi-continuity. $\square$

3. Twisted $(n-1)$-forms and special cohomologies

In this section, we collect some vanishing results about the cohomologies of twisted $(n-1)$-forms of n-dimensional irreducible Hermitian symmetric spaces of compact type. Moreover, we introduce the notion of special cohomologies and we prove that all irreducible Hermitian symmetric spaces of compact type have special cohomologies. This notion is very useful in studying the twisted vector fields over complete intersections in Hermitian symmetric spaces in the next section. We start with a result due to Snow.

DEFINITION 3.1 ([2, Definition 2.4]). — *Let $\ell, n \in \mathbb{N}$ be two fixed positive integers. An n-tuple of integers $\mathbf{a}_n = (a_i)_{1 \leqslant i \leqslant n}$ is called an ℓ-admissible C_n-sequence if $|a_i| = i$ and $a_i + a_j \neq 2\ell$ for all $i \leqslant j$. Its weight is defined to be $p(\mathbf{a}_n) = \sum_{a_i > 0} a_i$ and its ℓ-cohomological degree is defined to be*

$$q(\mathbf{a}_n) = \#\{\, (i,j) \mid i \leqslant j \text{ and } a_i + a_j > 2\ell \,\}.$$

PROPOSITION 3.2 ([31, Section 2.1]). — *Let $M = Sp(2n)/U(n)$ be a type C_n irreducible Hermitian symmetric space of compact type. If we have*

$$H^q(M, \Omega_M^p(\ell)) \neq 0,$$

then there exists an ℓ-admissible C_n-sequence such that its weight is p and its ℓ-cohomological degree is q.

Example 3.3. — Denote by M the 10-dimensional Lagrangian Grassmannian $Sp(8)/U(4)$. Then M is a Fano manifold with index 5. Moreover, if

ℓ is an integer such that $1 \leqslant \ell \leqslant 4$, then we have $H^q(M, \Omega_M^9(\ell)) = 0$ for any $q \geqslant 0$. In fact, if $H^q(M, \Omega_M^9(\ell)) \neq 0$, by Proposition 3.2, there exists an ℓ-admissible C_4-sequence $\mathbf{a}$ with ℓ-cohomological degree q and weight 9. This implies

$$\mathbf{a} = (-1, 2, 3, 4).$$

As $1 \leqslant \ell \leqslant 4$, one can easily see that $\mathbf{a}$ cannot be ℓ-admissible.

Before giving the statement in the general case, we recall the cohomologies of the twisted holomorphic p-forms of projective spaces and smooth quadric hypersurfaces.

THEOREM 3.4 ([4]). — *Let* n, p, q *and* ℓ *be integers, with* n *positive and* p *and* q *nonnegative. Then*

$$h^q(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^p(\ell)) = \begin{cases} \binom{n+\ell-p}{\ell}\binom{\ell-1}{p}, & \text{if } q = 0, 0 \leqslant p \leqslant n, \ell > p; \\ 1, & \text{if } \ell = 0, p = q; \\ \binom{p-\ell}{-\ell}\binom{-\ell-1}{n-p}, & \text{if } q = n, 0 \leqslant p \leqslant n, \ell < p - n; \\ 0, & \text{otherwise.} \end{cases}$$

As a consequence, $H^q(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^{n-1}(\ell)) \neq 0$ for some $\ell \in \mathbb{Z}$ if and only if $q = 0$ and $\ell \geqslant n$, or $q = n - 1$ and $\ell = 0$, or $q = n$ and $\ell \leqslant -2$.

THEOREM 3.5 ([30, Theorem 4.1]). — *Let* X *be an* n-*dimensional smooth quadric hypersurface.*

(1) *If* $-n + p \leqslant \ell \leqslant p$ *and* $\ell \neq 0$, $-n + 2p$, *then* $H^q(X, \Omega_X^p(\ell)) = 0$ *for all* q.
(2) $H^q(X, \Omega_X^p) \neq 0$ *if and only if* $q = p$.
(3) $H^q(X, \Omega_X^p(-n + 2p)) \neq 0$ *if and only if* $p + q = n$.
(4) *If* $\ell > p$, *then* $H^q(X, \Omega_X^p(\ell)) \neq 0$ *if and only if* $q = 0$.
(5) *If* $\ell < -n + p$, *then* $H^q(X, \Omega_X^p(\ell)) \neq 0$ *if and only if* $q = n$.

In particular, if X is a smooth quadric hypersurface of dimension n, then $H^q(X, \Omega_X^{n-1}(\ell)) \neq 0$ for some $\ell \in \mathbb{Z}$ if and only if $q = 0$ and $\ell \geqslant n$, or $q = 1$ and $\ell = n - 2$, or $q = n - 1$ and $\ell = 0$, or $q = n$ and $\ell \leqslant -2$. The following general result is essentially proved in [30] and [31].

PROPOSITION 3.6. — *Let* M *be an* n-*dimensional irreducible Hermitian symmetric space of compact type. Then* $H^q(M, \Omega_M^{n-1}(\ell)) \neq 0$ *if and only if one of the following conditions is satisfied.*

(1) $q = 0$ *and* $\ell \geqslant \min\{n, r_M\}$.

(2) $q = n - 1$ and $\ell = 0$.

(3) $q = n$ and $\ell \leqslant -2$.

(4) $M \cong Q^n$, $q = 1$ and $\ell = n - 2$.

Proof. — If $n \leqslant r_M$ or $n \leqslant 3$, then X is isomorphic to $\mathbb{P}^n$ or Q^n and we can conclude by Theorem 3.4 and Theorem 3.5. On the other hand, if $\ell \geqslant r_M$, by [31, Proposition 1.1], the cohomological degree of $\Omega_M^{n-1}(\ell)$ is 0. As a consequence, $H^q(M, \Omega_M^{n-1}(\ell)) \neq 0$ if and only if $q = 0$. Moreover, it is well-known that $H^q(M, \Omega_M^p) \neq 0$ if and only if $q = p$. So we shall assume that $n - 1 \geqslant r_M \geqslant \ell + 1$, $n \geqslant 4$ and $\ell \neq 0$. In particular, M is not of type B_n.

If $\ell \leqslant -2$, by Serre duality, we obtain that $H^q(M, \Omega_M^{n-1}(\ell)) \neq 0$ if and only if $H^{n-q}(M, \Omega_M^1(-\ell)) \neq 0$. Recall that the cohomological degree of the sheaf $\Omega_M^1(-\ell)$ is 0 if $-\ell \geqslant 2$ by [31, Proposition 1.1]. So $H^q(M, \Omega_M^{n-1}(\ell)) \neq 0$ if and only if $q = n$ if $\ell \leqslant -2$.

If $\ell = -1$, by Serre duality again, we see that $H^q(M, \Omega_M^{n-1}(-1)) \neq 0$ if and only if $H^{n-q}(M, \Omega_M^1(1)) \neq 0$. Thanks to [30, Theorem 2.3], we have $H^{n-q}(M, \Omega_M^1(1)) = 0$ for all $q \geqslant 0$ if M is not of type C_n. On the other hand, if M is of type C_n, the vanishing of $H^q(M, \Omega_M^1(1))$ follows from [31, Theorem 2.3].

If $1 \leqslant \ell \leqslant r_M - 1$, we can prove the result case by case. If M is of type E_6 or E_7, from [31, Table 4.4 and Table 4.5], we have $H^q(M, \Omega_M^{n-1}(\ell)) = 0$ for any $q \geqslant 0$. If M is of type A_n, as M is not isomorphic to $\mathbb{P}^n$ or Q^n, we get $H^q(M, \Omega_M^{n-1}(\ell)) = 0$ for all $q \geqslant 0$ by [30, Theorem 3.4(3)]. Here we remark that $Gr(2, 4)$ is isomorphic to Q^4. If M is of type C_n and $n \neq 4$, we have $H^q(M, \Omega_M^{n-1}(\ell)) = 0$ for all $q \geqslant 0$ by [31, Theorem 2.4(3)]. If M is of type C_4, then M is isomorphic to the 10-dimensional homogeneous space $Sp(8)/U(4)$, and we get $H^q(M, \Omega_M^9(\ell)) = 0$ for all $q \geqslant 0$ according to Example 3.3. If M is of type D_n, it follows from [31, Theorem 3.4(3)] that $H^q(M, \Omega_M^{n-1}(\ell)) = 0$ for all $q \geqslant 0$ if $n \geqslant 5$. If M is of type D_n and $n \leqslant 4$, then M is isomorphic to either $\mathbb{P}^n$ or Q^n, which is impossible by our assumption. $\qquad\square$

As a direct application, we get the following result which is useful to describe the twisted vector fields over complete intersections.

COROLLARY 3.7. — *Let M be an n-dimensional irreducible Hermitian symmetric space of compact type. If $n \geqslant 3$, then $H^{n-1}(M, \Omega_M^1(\ell)) \neq 0$ if and only if $\ell = -n + 2$ and M is isomorphic to a smooth quadric hypersurface Q^n.*

Proof. — As $H^{n-1}(M, \Omega^1_M(\ell)) \neq 0$ if and only if $H^1(M, \Omega^{n-1}_M(-\ell)) \neq 0$ by Serre duality, thus the result follows from Proposition 3.6 directly. $\square$

Moreover, one can easily derive the following result for smooth hypersurfaces in projective spaces by Bott's formula.

LEMMA 3.8. — *For $n \geqslant 3$, let $Y \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d. Then we have $H^{n-1}(Y, \Omega^1_Y(-r_Y + t)) = 0$ for $t > d$, where $r_Y = n + 2 - d$.*

Proof. — By Bott's formula (cf. Theorem 3.4) and the following exact sequence of sheaves

$$0 \longrightarrow \Omega^1_{\mathbb{P}^{n+1}}(-r_Y + t - d) \longrightarrow \Omega^1_{\mathbb{P}^{n+1}}(-r_Y + t) \longrightarrow \Omega^1_{\mathbb{P}^{n+1}}(-r_Y + t)|_Y \longrightarrow 0,$$

we see that $H^{n-1}(Y, \Omega^1_{\mathbb{P}^{n+1}}(-r_Y + t)|_Y) = 0$ for any $t \in \mathbb{Z}$. Thus, the following exact sequence of $\mathcal{O}_Y$-sheaves

$$0 \longrightarrow \mathcal{O}_Y(-r_Y + t - d) \longrightarrow \Omega^1_{\mathbb{P}^{n+1}}(-r_Y + t)|_Y \longrightarrow \Omega^1_Y(-r_Y + t) \longrightarrow 0$$

induces an injective map of groups

$$H^{n-1}(Y, \Omega^1_Y(-r_Y + t)) \longrightarrow H^n(Y, \mathcal{O}_Y(-r_Y + t - d)).$$

Then we can conclude by Kodaira's vanishing theorem. $\square$

DEFINITION 3.9. — *Let $(Z, \mathcal{O}_Z(1))$ be a polarized projective manifold of dimension $n \geqslant 4$. We say that $(Z, \mathcal{O}_Z(1))$ has special cohomologies if $H^q(Z, \Omega^1_Z(\ell)) = 0$ for all $\ell \in \mathbb{Z}$ and all $2 \leqslant q \leqslant n - 2$.*

We remark that our definition of special cohomologies is much weaker than that given in [26].

Example 3.10. — By [25, Corollary 2.3.1], an n-dimensional smooth complete intersection Y in a projective space has special cohomologies if $n \geqslant 4$. Moreover, if $\widetilde{Y}$ is a cyclic covering of Y, then Y has special cohomologies (see [26, Theorem 1.6]).

Example 3.11. — Let Y be a smooth weighted complete intersection of dimension n in a weighted projective space, and let $\mathcal{O}_Y(1)$ be the restriction to Y of the universal $\mathcal{O}(1)$-sheaf from the weighted projective space. Then $(Y, \mathcal{O}_Y(1))$ has special cohomologies by [9, Satz 8.11].

PROPOSITION 3.12. — *Let $(M, \mathcal{O}_M(1))$ be an n-dimensional irreducible Hermitian symmetric space of compact type. If $n \geqslant 4$, then $(M, \mathcal{O}_M(1))$ has special cohomologies.*

Proof. — By Serre duality, it suffices to consider $H^{n-q}(M, \Omega^{n-1}_M(-\ell))$. As $2 \leqslant q \leqslant n - 2$, we get $2 \leqslant n - q \leqslant n - 2$. Now it follows from Proposition 3.6. $\square$

4. Extension of twisted vector fields

This section is devoted to study global twisted vector fields over smooth complete intersections in an irreducible Hermitian symmetric space of compact type. The main aim is to show that the global twisted vector fields over complete intersections can be extended to be global twisted vector fields over the ambient space (cf. Theorem 4.2).

4.1. Twisted vector fields over complete intersections

Let $(Z, \mathcal{O}_Z(1))$ be a polarized manifold, and let $Y \subset Z$ be a subvariety. Then we have a natural restriction map

$$\rho_t \colon H^0(Z, T_Z(t)) \longrightarrow H^0(Y, T_Z(t)|_Y)$$

for any $t \in \mathbb{Z}$. The following proposition concerns on the surjectivity of ρ_t and its proof was communicated to me by the anonymous referee.

PROPOSITION 4.1. — *Let $(Z, \mathcal{O}_Z(1))$ be a polarized projective manifold. Let Y be the complete intersection $H_1 \cap \cdots \cap H_r$ where $H_i \in |\mathcal{O}_Z(d_i)|$. Assume moreover that there exists an integer $r_Z \in \mathbb{Z}$ such that $\mathcal{O}_Z(-K_Z) \cong \mathcal{O}_Z(r_Z)$. Given $t \in \mathbb{Z}$, if $(Z, \mathcal{O}_Z(1))$ has special cohomologies, $\dim(Y) \geqslant 2$ and $H^{\dim(Z)-1}(Z, \Omega^1_Z(-r_Z + d_i - t)) = 0$ for all $1 \leqslant i \leqslant r$, then the natural restriction*

$$\rho_t \colon H^0(Z, T_Z(t)) \longrightarrow H^0(Y, T_Z(t)|_Y)$$

is surjective.

Proof. — Denote by E the vector bundle $\mathcal{O}_Z(-d_1) \oplus \cdots \oplus \mathcal{O}_Z(-d_r)$ over Z. Consider the following Koszul resolution of $\mathcal{O}_Y$

$$0 \longrightarrow \wedge^r E \longrightarrow \wedge^{r-1} E \longrightarrow \cdots \longrightarrow E \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_Y \longrightarrow 0.$$

Tensoring it with $T_Z(t)$, we obtain the following exact sequence

$$(4.1) \quad 0 \longrightarrow T_Z(t) \otimes (\wedge^r E) \longrightarrow T_Z(t) \otimes (\wedge^{r-1} E)$$
$$\longrightarrow \cdots \longrightarrow T_Z(t) \otimes E \longrightarrow T_Z(t) \longrightarrow T_Z(t)|_Y \longrightarrow 0.$$

By definition, for any $1 \leqslant j \leqslant r$, the vector bundle $\wedge^j E$ splits into a direct sum of line bundles as $\bigoplus_i \mathcal{O}_Z(-d_{ij})$ with $d_{ij} \geqslant j$. Since $(Z, \mathcal{O}_Z(1))$ has special cohomologies and $r = \operatorname{codim}(Y) \leqslant n - 2$, by Serre duality, we have

$$H^j(Z, T_Z(t) \otimes (\wedge^j E)) \cong \bigoplus_i H^j(Z, T_Z(t - d_{ij})) = 0$$

for any $j \geqslant 2$. On the other hand, as $H^{\dim(Z)-1}(Z, \Omega_Z^1(-r_Z + d_i - t)) = 0$ by our assumption, by Serre duality again, we obtain

$$H^1(Z, T_Z(t) \otimes E) \cong \bigoplus_{i=1}^r H^{\dim(Z)-1}(Z, \Omega_Z^1(-r_Z + d_i - t) = 0.$$

Then the second quadrant spectral sequence associated to the complex (4.1) (see [20, Appendix B]) implies that the natural restriction

$$H^0(Z, T_Z(t)) \longrightarrow H^0(Y, T_Z(t)|_Y)$$

is surjective. $\qquad\square$

As an immediate application, we derive the following theorem which will play a key role in the proof of Theorem 1.8.

THEOREM 4.2. — *Let M be an $(n + r)$-dimensional irreducible Hermitian symmetric space of compact type which is not isomorphic to any smooth quadric hypersurface Q^{n+r}. Let $Y = H_1 \cap \cdots \cap H_r$ be a complete intersection of dimension n where $H_i \in |\mathcal{O}_M(d_i)|$. Assume moreover that $\dim(Y) = n \geqslant 2$. Then the natural restriction*

$$\rho_t \colon H^0(M, T_M(t)) \longrightarrow H^0(Y, T_M(t)|_Y).$$

is surjective for any $t \in \mathbb{Z}$.

Proof. — If $n + r \geqslant 4$, this follows from Corollary 3.7, Proposition 3.12 and Proposition 4.1. If $n + r = 3$, then Y is a hypersurface of degree d in M. Thanks to the following exact sequence

$$0 \longrightarrow T_M(t - d) \longrightarrow T_M(t) \longrightarrow T_M(t)|_Y \longrightarrow 0,$$

to prove the surjectivity of ρ_t, it is enough to show that

$$H^1(M, T_M(t - d)) = 0.$$

By Serre duality, it is equivalent to show that

$$H^2(M, \Omega_M^1(-r_M + d - t)) = 0.$$

Since M is not isomorphic to any smooth quadric hypersurface, we conclude by Corollary 3.7. $\qquad\square$

If M is a smooth quadric hypersurface, then we can also regard Y as a complete intersection of degree $(2, d_1, \ldots, d_r)$ in the projective space $\mathbb{P}^{n+r+1}$. In general, we have the following result.

THEOREM 4.3. — *Let $Y = H_1 \cap \cdots \cap H_r$ be a complete intersection where $H_i \in |\mathcal{O}_{\mathbb{P}^{n+r}}(d_i)|$. Assume moreover that H_1 is smooth, $\dim(Y) = n \geqslant 2$*

and $n + r \geqslant 4$. If $d_i \geqslant 2$ for all $1 \leqslant i \leqslant r$ and t is an integer such that $d_i - t > d_1$ for any $2 \leqslant i \leqslant r$, then the natural restriction

$$\rho_t \colon H^0(H_1, T_{H_1}(t)) \longrightarrow H^0(Y, T_{H_1}(t)|_Y)$$

is surjective.

Proof. — If $r = 1$, the statement is trivial. So we may assume that $r \geqslant 2$. If $n + r \geqslant 5$, by definition and [25, Corollary 2.3.1], the hypersurface H_1 has special cohomologies (cf. Example 3.10). Thanks to Proposition 4.1, it suffices to verify that we have

$$H^{\dim(Y_1)-1}(Y_1, \Omega^1_{Y_1}(-r_{Y_1} + d_i - t)) = 0$$

for all $2 \leqslant i \leqslant r$. Since $t \in \mathbb{Z}$ is an integer such that $d_i - t > d_1$ for any $2 \leqslant i \leqslant r$, it follows from Lemma 3.8 immediately.

If $n + r = 4$, then $Y = H_1 \cap H_2$. As $d_2 - t > d_1$ by assumption, thanks to Lemma 3.8, we have $H^2(H_1, \Omega^1_{H_1}(-r_{H_1} + d_2 - t)) = 0$. By Serre duality, we obtain $H^1(H_1, T_{H_1}(t - d_2)) = 0$. Then, it follows from the following exact sequence

$$0 \longrightarrow T_{H_1}(t - d_2) \longrightarrow T_{H_1}(t) \longrightarrow T_{H_1}(t)|_Y \longrightarrow 0$$

that the map $H^0(H_1, T_{H_1}(t)) \to H^0(Y, T_{H_1}(t)|_Y)$ is surjective.   □

4.2. Twisted vector fields over hypersurfaces in projective spaces

The global sections of $T_{\mathbb{P}^n}(t)$ can be expressed explicitly by homogeneous polynomials of degree $t + 1$. To see this, we consider the twisted Euler sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(t) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(t+1)^{\oplus(n+1)} \longrightarrow T_{\mathbb{P}^n}(t) \longrightarrow 0.$$

Using the fact $H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(t)) = 0$, we see that the restriction map

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(t+1)^{\oplus(n+1)}) \longrightarrow H^0(\mathbb{P}^n, T_{\mathbb{P}^n}(t))$$

is surjective, so a global section σ of $T_{\mathbb{P}^n}(t)$ is given by a vector field on the affine complex vector space $\mathbb{C}^{n+1}$

$$\sigma = f_0 \frac{\partial}{\partial x_0} + \cdots + f_n \frac{\partial}{\partial x_n},$$

where f_i's are homogeneous polynomials of degree $t + 1$. Let $Y \subset \mathbb{P}^n$ be a smooth hypersurface defined by a homogeneous polynomial h and let X be

a submanifold of Y. Then the restriction $\sigma|_X$ is a global section of $T_Y(t)|_X$ if and only if we have

$$\left(f_0 \frac{\partial h}{\partial x_0} + \cdots + f_n \frac{\partial h}{\partial x_n} \right)\Big|_X \equiv 0.$$

Furthermore, we have $\sigma|_X \equiv 0$ if and only if

$$(x_i f_j - x_j f_i)|_X \equiv 0, \quad 0 \leqslant i \leqslant j \leqslant n.$$

Let Y be a general complete intersection in an N-dimensional irreducible Hermitian symmetric space M of compact type such that $N \geqslant 3$, and let $X \in |\mathcal{O}_Y(d)|$ be a general hypersurface of Y. Thanks to [36], $H^0(M, T_M(t)) \neq 0$ for some $t < 0$ if and only if $M \cong \mathbb{P}^N$ and $t = -1$. According to Theorem 4.2 and Theorem 4.3, we get

$$H^0(Y, T_Y(t)) = H^0(X, T_Y(t)|_X) = H^0(M, T_M(t)) = 0$$

for any $t \leqslant -2$. In the following theorem, we generalize this result to show that if Y is a general hypersurface of $\mathbb{P}^{n+1}$, then the natural restriction

$$\alpha_t \colon H^0(Y, T_Y(t)) \longrightarrow H^0(X, T_Y(t)|_X)$$

is surjective for $t \leqslant t_0$ large enough depending only on n and the degrees of X and Y. This theorem is a key ingredient of the proof of Theorem 1.9.

THEOREM 4.4. — *Let* $Y \subset \mathbb{P}^{n+1}$ *be a general smooth hypersurface defined by the homogeneous polynomial* h *of degree* $d_h \geqslant 2$ *and let* $X \in |\mathcal{O}_Y(d)|$ *be a general smooth divisor. If* $n \geqslant 3$, *then the restriction map*

$$H^0(Y, T_Y(t)) \longrightarrow H^0(X, T_Y(t)|_X)$$

is surjective for any $t \leqslant (\rho + d)/2 - d_h$, *where* $\rho = (d_h - 2)(n + 2)$ *is the top degree of the Milnor algebra of* Y.

Proof. — Since the restriction $H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(d)) \to H^0(Y, \mathcal{O}_Y(d))$ is surjective, there exists a general homogeneous polynomial f of degree d such that $X = \{f = h = 0\}$. We denote by $M(Y)$ and $J(Y)$ the Milnor algebra and Jacobian ideal of Y, respectively. Since $H^0(X, T_Y(t)|_X)$ is a subset of $H^0(X, T_{\mathbb{P}^{n+1}}(t)|_X)$ and $H^0(\mathbb{P}^{n+1}, T_{\mathbb{P}^{n+1}}(t)) = 0$ for $t \leqslant -2$, by Theorem 4.2, we may assume $t \geqslant -1$.

Let $s \in H^0(X, T_Y(t)|_X)$ be a global section. By Theorem 4.2, the section s is the restriction of some global section $\sigma \in H^0(\mathbb{P}^{n+1}, T_{\mathbb{P}^{n+1}}(t))$. Therefore there exist some polynomials f_i of degree $t + 1$ such that

$$s = \sigma|_X = \left(f_0 \frac{\partial}{\partial x_0} + \cdots + f_{n+1} \frac{\partial}{\partial x_{n+1}} \right)\Big|_X$$

and

$$\left(f_0 \frac{\partial h}{\partial x_0} + \cdots + f_{n+1} \frac{\partial h}{\partial x_{n+1}} \right)\bigg|_X = 0.$$

As a consequence, there exist two homogeneous polynomials g and p (maybe zero) such that

$$f_0 \frac{\partial h}{\partial x_0} + \cdots + f_{n+1} \frac{\partial h}{\partial x_{n+1}} = gf + ph.$$

We claim that g is contained in the Jacobian ideal $J(Y)$ of Y. In fact, by Euler's homogeneous function theorem, it follows

$$\left(f_0 - \frac{1}{d_h} p x_0 \right) \frac{\partial h}{\partial x_0} + \cdots + \left(f_{n+1} - \frac{1}{d_h} p x_{n+1} \right) \frac{\partial h}{\partial x_{n+1}} = gf.$$

Thanks to Proposition 2.6, the Milnor algebra $M(Y)$ has maximal rank property, hence, by the genericity assumption of f, the multiplication map

$$(\times f) \colon M(Y)_{t+d_h-d} \longrightarrow M(Y)_{t+d_h}$$

has maximal rank. Moreover, by the assumption, we have

$$t + d_h - d \leqslant \frac{\rho - d}{2},$$

so the multiplication map $(\times f)$ is injective (cf. Remark 2.5). It follows that $g = 0$ in $M(Y)$, or equivalently, the polynomial g is contained in the Jacobian ideal of Y. Therefore there exist some homogeneous polynomials g_i's of degree $t - d + 1$ such that

$$g = g_0 \frac{\partial h}{\partial x_0} + \cdots + g_{n+1} \frac{\partial h}{\partial x_{n+1}}.$$

This yields

$$\left(f_0 \frac{\partial h}{\partial x_0} + \cdots + f_{n+1} \frac{\partial h}{\partial x_{n+1}} \right) - \left(g_0 f \frac{\partial h}{\partial x_0} + \cdots + g_{n+1} f \frac{\partial h}{\partial x_{n+1}} \right) = ph.$$

We denote by $\sigma' \in H^0(\mathbb{P}^{n+1}, T_{\mathbb{P}^{n+1}}(t))$ the global section defined by

$$g_0 f \frac{\partial}{\partial x_0} + \cdots + g_{n+1} f \frac{\partial}{\partial x_{n+1}}.$$

Then $(\sigma - \sigma')|_Y \in H^0(Y, T_Y(t))$. Moreover, as $\sigma'|_X \equiv 0$, we have

$$(\sigma - \sigma')|_X = \sigma|_X = s.$$

Hence the restriction map

$$H^0(Y, T_Y(t)) \longrightarrow H^0(X, T_Y(t)|_X)$$

is surjective. $\qquad\square$

5. Stability and effective restriction with invariant Picard group

This section is devoted to study the stability of the tangent bundles of smooth complete intersections in Hermitian symmetric spaces. As mentioned in the introduction, this problem was studied by Peternell and Wiśniewski in [26] in the projective spaces case. Moreover, we will also consider the effective restriction problem for tangent bundles.

5.1. Vanishing theorem and stability of tangent bundles

We start with an observation whose statement was suggested by the referee. It is very useful when we consider the cohomologies of smooth complete intersections in some projective manifolds with many cohomological vanishings.

LEMMA 5.1. — *For $n \geqslant 2$, let $(Z, \mathcal{O}_Z(1))$ be a polarized projective manifold of dimension $n + r$. Let $Y = H_1 \cap \cdots \cap H_r$ be a smooth complete intersection of dimension n where $H_i \in |\mathcal{O}_Z(d_i)|$. If there exist integers p, q and ℓ such that $p + q \leqslant n$ and $H^q(Y, \Omega_Y^p(\ell)) \neq 0$, then there exist integers $j_1, \ldots, j_r \in \mathbb{N}$ such that $0 \leqslant j_1 + \cdots + j_r \leqslant p - 1$ and such that one of the following statements holds.*

(1) There exists an integer $1 \leqslant s \leqslant r$ such that
$$H^{q+p}(Y, \mathcal{O}_Y(\ell - (p-k)d_s - j_{s+1}d_{s+1} - \cdots - j_r d_r)) \neq 0,$$
where $k := j_{s+1} + \cdots + j_r$.
(2) $H^{q+j_1+\cdots+j_r}(Y, \Omega_Z^{p-j_1-\cdots-j_r}(\ell - j_1 d_1 - \cdots - j_r d_r)|_Y) \neq 0$.

Proof. — Let us denote by Y_i $(1 \leqslant i \leqslant r)$ the scheme-theoretic complete intersection $H_1 \cap \cdots \cap H_i$. Then we have $Y = Y_r$. Moreover, for convenience, we will denote Z by Y_0. Since Y is smooth, each Y_i is smooth in a neighborhood of Y. In particular, the cotangent sheaf $\Omega_{Y_i}^1$ of Y_i is locally free in a neighborhood of Y. Let s be the minimal non-negative integer such that there exist integers $j_{s+1}, j_{s+2}, \ldots, j_r \in \mathbb{N}$ satisfying $0 \leqslant k \leqslant p - 1$ and
$$H^{q+k}(Y, \Omega_{Y_s}^{p-k}(\ell - j_{s+1}d_{s+1} - \cdots - j_r d_r)|_Y) \neq 0.$$
We note that the assumption $H^0(Y, \Omega_Y^p(\ell)) \neq 0$ implies the existence of s satisfying $s \leqslant r$. Moreover, we are in Case (2) if $s = 0$.

Now we assume that $s \geqslant 1$. Then, by the minimality of s, we see that for any $j \in \mathbb{N}$ such that $0 \leqslant j + k \leqslant p - 1$, we have

(5.1) $H^{q+j+k}(Y, \Omega_{Y_{s-1}}^{p-j-k}(\ell - jd_s - j_{s+1}d_{s+1} - \cdots - j_r d_r)|_Y) = 0.$

The restriction over Y of the conormal sequence of Y_s in Y_{s-1}

$$0 \longrightarrow \mathcal{O}_Y(-d_s) \longrightarrow \Omega^1_{Y_{s-1}}|_Y \longrightarrow \Omega^1_{Y_s}|_Y \longrightarrow 0$$

induces an exact sequence of vector bundles

$$0 \longrightarrow \Omega^{p-j-k-1}_{Y_s}(-d_s)|_Y \longrightarrow \Omega^{p-j-k}_{Y_{s-1}}|_Y \longrightarrow \Omega^{p-j-k}_{Y_s}|_Y \longrightarrow 0.$$

Tensoring it with $\mathcal{O}_Y(\ell - jd_s - j_{s+1}d_{s+1} - \cdots - j_r d_r)$, we obtain

$$0 \longrightarrow \Omega^{p-j-k-1}_{Y_s}(\ell - (j+1)d_s - j_{s+1}d_{s+1} - \cdots - j_r d_r)|_Y$$
$$\longrightarrow \Omega^{p-j-k}_{Y_{s-1}}(\ell - jd_s - j_{s+1}d_{s+1} - \cdots - j_r d_r)|_Y$$
$$\longrightarrow \Omega^{p-j-k}_{Y_s}(\ell - jd_s - j_{s+1}d_{s+1} - \cdots - j_r d_r)|_Y \longrightarrow 0.$$

Then (5.1) shows that for any $j \in \mathbb{N}$ such that $0 \leqslant j + k \leqslant p - 1$, the induced map

$$H^{q+j+k}(Y, \Omega^{p-j-k}_{Y_s}(\ell - jd_s - j_s d_s - \cdots - j_r d_r)|_Y)$$
$$\longrightarrow H^{q+j+k+1}(Y, \Omega^{p-j-k-1}_{Y_s}(\ell - (j+1)d_s - j_{s+1}d_{s+1} - \cdots - j_r d_r)|_Y)$$

is injective. Then by induction on j from 0 to $p - k - 1$, the assumption

$$H^{q+k}(Y, \Omega^{p-k}_{Y_s}(\ell - j_{s+1}d_{s+1} - \cdots - j_r d_r)|_Y) \neq 0.$$

implies

$$H^{q+p}(Y, \mathcal{O}_Y(\ell - (p-k)d_s - j_{s+1}d_{s+1} - \cdots - j_r d_r)|_Y) \neq 0.$$

Finally we conclude by setting $j_1 = \cdots = j_s = 0$. $\qquad\square$

Now we are in the position to prove Theorem 1.3. The idea is to relate the cohomologies of Y to the cohomologies of the ambient space M using Lemma 5.1 and the Koszul resolution.

Proof of Theorem 1.3. — As M is a Fano manifold, by Kobayashi–Ochiai's theorem (see [17]), we have $r_M \leqslant n + r + 1$ with equality if and only if $M \cong \mathbb{P}^{n+r}$. On the other hand, if $M \cong \mathbb{P}^{n+r}$, thanks to [25, Corollary 2.3.1], under our assumption, we have $H^q(Y, \Omega^p_Y(\ell)) \neq 0$ if and only if $q = p$ and $\ell = 0$. Hence the result holds if $r_M = n + r + 1$. From now on, we shall assume that $r_M \leqslant n + r$. As a consequence, we have $r_Y \leqslant n - 1$ since $d_i \geqslant 2$. Let us denote by E the vector bundle $\mathcal{O}_M(-d_1) \oplus \cdots \oplus \mathcal{O}_M(-d_r)$. Then we have the Koszul resolution of $\mathcal{O}_Y$

$$(5.2) \qquad 0 \longrightarrow \wedge^r E \longrightarrow \wedge^{r-1} E \longrightarrow \cdots \longrightarrow E \longrightarrow \mathcal{O}_M \longrightarrow \mathcal{O}_Y \longrightarrow 0.$$

Moreover, for any $1 \leqslant j \leqslant r$, the vector bundle $\wedge^j E$ splits into a direct sum of line bundles as $\bigoplus_i \mathcal{O}_M(-d_{ij})$ with $d_{ij} \geqslant 2j$.

Proof of (1). — As $H^q(Y, \Omega_Y^p) \neq 0$, thanks to Lemma 5.1, there exist integers $j_1, \ldots, j_r \in \mathbb{N}$ such that $0 \leqslant k := j_1 + \cdots + j_r \leqslant p - 1$ and such that we have either

$$(5.3) \qquad\qquad H^{q+k}(Y, \Omega_M^{p-k}(-j_1 d_1 - \cdots - j_r d_r)|_Y) \neq 0$$

or

$$(5.4) \quad H^{q+p}\left(Y, \mathcal{O}_Y\left(-\left(p - \sum_{l \geqslant s+1} j_l\right) d_s - j_{s+1} d_{s+1} + \cdots + j_r d_r\right)\right) \neq 0$$

for some $1 \leqslant s \leqslant r$. Note that we have

$$-(p - j_{s+1} - \cdots - j_r) d_s - j_{s+1} d_{s+1} - \cdots - j_r d_r < 0$$

as $p > j_1 + \cdots + j_r$. Therefore, as $1 \leqslant q + p \leqslant n - 1$, then Kodaira's vanishing theorem shows that (5.4) is impossible. Hence, (5.3) holds and the second quadrant spectral sequence associated to the complex (5.2) twisting with

$$\Omega_M^{p-k}(-j_1 d_1 - \cdots - j_r d_r)$$

implies that there exists an integer $0 \leqslant j \leqslant r$ such that

$$H^{q+k+j}(M, \Omega_M^{p-k}(-j_1 d_1 - \cdots - j_r d_r) \otimes \wedge^j E) \neq 0.$$

As $q + j + p \leqslant n + r - 1$ by assumption, then the Akizuki–Nakano vanishing theorem implies

$$-j_1 d_1 - \cdots - j_r d_r - d_{ij} \geqslant 0$$

for some d_{ij}. Since $d_{ij} > 0$ if $j > 0$, $d_l > 0$ and $j_l \geqslant 0$ ($1 \leqslant l \leqslant r$), we obtain

$$j = j_1 = \cdots = j_r = 0.$$

Equivalently, we have $H^q(M, \Omega_M^p) \neq 0$, and it follows that $p = q$.

Proof of (2). — If $H^q(Y, \Omega_Y^p(\ell)) \neq 0$, by Lemma 5.1 again, there exist integers $j_1, \ldots, j_r \in \mathbb{N}$ with $0 \leqslant k := j_1 + \cdots + j_r \leqslant p - 1$ such that we have either

$$(5.5) \qquad\qquad H^{q+k}(Y, \Omega_M^{p-k}(\ell - j_1 d_1 - \cdots - j_r d_r)|_Y) \neq 0$$

or

$$(5.6) \quad H^{q+p}\left(Y, \mathcal{O}_Y\left(\ell - \left(p - \sum_{l \geqslant s+1} j_l\right) d_s - j_{s+1} d_{s+1} - \cdots - j_r d_r\right)\right) \neq 0$$

for some $1 \leqslant s \leqslant r$.

1st Case. (5.5) *holds.* — In this case, the second quadrant spectral sequence associated to the complex (5.2) twisting with $\Omega_M^{p-k}(\ell - j_1 d_1 - \cdots - j_r d_r)$ implies that there exists an integer $0 \leqslant j \leqslant r$ such that

$$H^{q+k+j}(M, \Omega_M^{p-k}(\ell - j_1 d_1 - \cdots - j_r d_r) \otimes \wedge^j E) \neq 0.$$

In particular, there exists d_{ij} such that

$$H^{q+k+j}(M, \Omega_M^{p-k}(\ell - j_1 d_1 - \cdots - j_r d_r - d_{ij})) \neq 0.$$

As $q + j + p \leqslant n + r - 1$, Akizuki–Nakano vanishing theorem shows that we have

(5.7) $$\ell - j_1 d_1 - \cdots - j_r d_r - d_{ij} \geqslant 0$$

If the equality in (5.7) holds, then $H^{q+k+j}(M, \Omega_M^{p-k}) \neq 0$ and it follows that $q + k + j = p - k$. As a consequence,

$$\begin{aligned} n(\ell + q) &= n(j_1 d_1 + \cdots + j_r d_r + d_{ij} + p - 2k - j) \\ &= n(p + (d_1 - 2)j_1 + \cdots + (d_r - 2)j_r + d_{ij} - j) \end{aligned}$$

As $d_i \geqslant 2$, $d_{ij} \geqslant 2j$ and $r_Y \leqslant n - 1$, we obtain $n(\ell + q) \geqslant np > r_Y p$.

If the inequality (5.7) is strict, we can apply [2, Theorem D] to get

$$\frac{\ell - j_1 d_1 - \cdots - j_r d_r - d_{ij} + q + k + j}{p - k} \geqslant \frac{r_M}{n + r}.$$

It follows that

$$\ell + q \geqslant p \frac{r_M}{n+r} + \left(d_1 - 1 - \frac{r_M}{n+r}\right) j_1$$

$$+ \cdots + \left(d_r - 1 - \frac{r_M}{n+r}\right) j_r + d_{ij} - j.$$

As $d_{ij} \geqslant 2j$, $d_i \geqslant 2$ and $r_M \leqslant n + r$, we obtain

$$\ell + q \geqslant p \frac{r_M}{n+r}.$$

Since $r_Y \leqslant n - 2r$, it is easy to see $r_M/(n+r) > r_Y/n$, and we get the desired inequality.

2nd Case. (5.6) *holds.* — As $1 \leqslant q + p \leqslant n - 1$, by Kodaira's vanishing theorem, we get

$$\ell - \left(p - \sum_{l \geqslant s+1} j_l\right) d_s - j_{s+1} d_{s+1} - \cdots - j_r d_r \geqslant 0.$$

As a consequence, we obtain

$$\ell \geqslant d_{s+1}j_{s+1} + \cdots + d_r j_r + (p - j_{s+1} - \cdots - j_r)d_s$$
$$\geqslant d_{s+1}j_{s+1} + \cdots + d_r j_r + 2(p - j_{s+1} - \cdots - j_r)$$
$$\geqslant 2p + (d_{s+1} - 2)j_{s+1} + \cdots + (d_r - 2)j_r$$

The second inequality follows from the assumptions $p > k \geqslant j_{s+1} + \cdots + j_r$ and $d_s \geqslant 2$. Then we can easily get

$$\ell + q \geqslant \ell \geqslant 2p$$

as $d_i \geqslant 2$ for all i. In particular, we have $n(\ell + q) \geqslant 2np > pr_Y$. $\square$

Remark 5.2. — In view of our proof of Theorem 1.3 (1), we see that it holds even without the assumption $d_i \geqslant 2$. However, in the proof of Theorem 1.3 (2), the assumption $d_i \geqslant 2$ is necessary.

Theorem 1.4 is a direct consequence of Theorem 1.3.

Proof of Theorem 1.4. — To prove the stability of T_Y, it is equivalent to prove the stability of Ω_Y^1. Let $\mathcal{F} \subset \Omega_Y^1$ be a nonzero proper subsheaf of rank p $(1 \leqslant p \leqslant n - 1)$. By Lefschetz hyperplane theorem and our assumption, we have $\mathrm{Pic}(Y) \cong \mathbb{Z}\mathcal{O}_Y(1)$, where $\mathcal{O}_Y(1) = \mathcal{O}_M(1)|_Y$. Thus, we could denote by ℓ the unique integer such that $\det(\mathcal{F}) \cong \mathcal{O}_Y(-\ell)$. Then we have $H^0(Y, \Omega_Y^p(\ell)) \neq 0$ by assumption. Since $p \leqslant n - 1$, the Akizuki–Nakano vanishing theorem implies $\ell \geqslant 0$. As $p \geqslant 1$, Theorem 1.3 (1) implies $\ell > 0$, and hence $\mu(\mathcal{F}) < \mu(\Omega_Y^1)$ by Theorem 1.3 (2). In particular, it follows that Ω_Y^1 is stable. $\square$

5.2. Effective restriction of tangent bundles

In this subsection, we proceed to prove various effective restriction theorems for tangent bundles of complete intersections in irreducible Hermitian symmetric spaces of compact type. We use the standard cohomological arguments as in the proof of Theorem 1.4 to reduce the problem to the existence of twisted vector fields.

PROPOSITION 5.3. — *For $n \geqslant 3$ and $r \geqslant 1$, let M be an $(n + r)$-dimensional irreducible Hermitian symmetric space of compact type. Let $Y = H_1 \cap \cdots \cap H_r$ be a smooth complete intersection of dimension n with $H_i \in |\mathcal{O}_M(d_i)|$ with $d_i \geqslant 2$. Let $X \in |\mathcal{O}_Y(d)|$ be a smooth divisor. Assume that the composite of restrictions*

$$\mathrm{Pic}(M) \longrightarrow \mathrm{Pic}(Y) \longrightarrow \mathrm{Pic}(X)$$

is surjective. Moreover, if Y is isomorphic to some smooth quadric hypersurface Q^n, we assume in addition that $d \geqslant 2$. Then the vector bundle $T_Y|_X$ is stable if and only if

$$H^0(X, T_Y(t)|_X) = 0$$

for any integer $t \leqslant -r_Y/n$, where r_Y is the unique integer such that $\mathcal{O}_Y(-K_Y) \cong \mathcal{O}_Y(r_Y)$.

Proof. — The "only if" implication follows from the definition of stability. Now we assume $H^0(X, T_Y(t)|_X) = 0$ for any $t \leqslant -r_Y/n$. Note that $T_Y|_X$ is stable if and only if $\Omega^1_Y|_X$ is stable. Let $\mathcal{F}$ be a proper subsheaf of $\Omega^1_Y|_X$ of rank p. After replacing $\mathcal{F}$ by its saturation in $\Omega^1_Y|_X$, we may assume that $\mathcal{F}$ is saturated. We denote by ℓ the unique integer such that $\det(\mathcal{F}) = \mathcal{O}_X(-\ell)$. Then, by assumption, we get $H^0(X, \Omega^p_Y(\ell)|_X) \neq 0$. To prove the stability of $\Omega^1_Y|_X$, it suffices to show that the following inequality

$$\mu(\mathcal{F}) = \frac{-\ell}{p} \mathcal{O}_X(1)^{n-1} < \mu(\Omega^1_Y|_X) = \frac{-r_Y}{n} \mathcal{O}_X(1)^{n-1}$$

holds for all pairs of integers (ℓ, p) such that $H^0(X, \Omega^p_Y(\ell)|_X) \neq 0$ and $1 \leqslant p \leqslant n - 1$. We consider the following exact sequence

$$0 \longrightarrow \Omega^p_Y(\ell - d) \longrightarrow \Omega^p_Y(\ell) \longrightarrow \Omega^p_Y(\ell)|_X \longrightarrow 0.$$

As $1 \leqslant p \leqslant n - 1$ and $n \geqslant 3$, by Theorem 1.3, $H^0(Y, \Omega^p_Y(\ell)) \neq 0$ implies $n\ell > pr_Y$. Thus we may assume $H^0(Y, \Omega^p_Y(\ell)) = 0$. Then the fact $H^0(X, \Omega^p_Y(\ell)|_X) \neq 0$ implies $H^1(Y, \Omega^p_Y(\ell - d)) \neq 0$.

1st Case. $p \leqslant n - 2$. — As $p \geqslant 1$, if $\ell \neq d$, by Theorem 1.3 (2), we have

$$n\ell \geqslant n(\ell - d + 1) > pr_Y.$$

If $\ell = d$, then we have $p = 1$ by Theorem 1.3 (1). As a consequence, we get

$$(5.8) \qquad\qquad n\ell = nd > pr_Y = r_Y$$

unless $d = 1$ and $r_Y \geqslant n$. If $r_Y \geqslant n$, by Kobayashi–Ochiai theorem, Y is isomorphic to either $\mathbb{P}^n$ or Q^n. As $d_i \geqslant 2$, then Y is actually isomorphic to Q^n. However, by our assumption, if $Y \cong Q^n$, then we have $d \geqslant 2$. Hence the inequality (5.8) still holds in this case.

2nd Case. $p = n - 1$. — We denote by $\mathcal{Q}$ the quotient $\left(\Omega^1_Y|_X\right)/\mathcal{F}$. Since $\mathcal{F}$ is saturated, the quotient $\mathcal{Q}$ is a torsion-free coherent sheaf of rank one such that $\det(\mathcal{Q}) = \mathcal{Q}^{**} \cong \mathcal{O}_X(-r_Y + \ell)$. Since $\mathcal{Q}^*$ is a subsheaf of $T_Y|_X$, we get

$$H^0(X, T_Y(\ell - r_Y)|_X) \neq 0.$$

By our assumption, we get $\ell - r_Y > -r_Y/n$. As a consequence, we get

$$\frac{\ell}{p} = \frac{\ell}{n-1} > \frac{r_Y}{n}.$$

We have thus proved the proposition.   □

As an application of Proposition 5.3, we can derive Theorem 1.8 by the nonexistence of global twisted vector fields.

Proof of Theorem 1.8. — Let X be a projective manifold of dimension $N \geqslant 2$, and let L be an ample line bundle. Recall that $H^0(X, T_X \otimes L^{-1}) \neq 0$ if and only if $X \cong \mathbb{P}^N$ and $L \cong \mathcal{O}_{\mathbb{P}^N}(1)$ (cf. [36]). In particular, if M is not isomorphic to a projective space, then we have $H^0(M, T_M(t)) = 0$ for any integer $t < 0$.

Proof of (1). — Under our assumption, by Theorem 4.2, the natural restriction map

$$\rho_t \colon H^0(M, T_M(t)) \longrightarrow H^0(X, T_M(t)|_X)$$

is surjective for all $t \in \mathbb{Z}$. In particular, we have $H^0(X, T_M(t)|_X) = 0$ for all $t < 0$. This implies $H^0(X, T_Y(t)|_X) = 0$ for all $t < 0$ since $H^0(X, T_Y(t)|_X)$ is a subgroup of $H^0(X, T_M(t)|_X)$. As Y is Fano, we have $r_Y > 0$ and we conclude by Proposition 5.3.

Proof of (2). — If $d \geqslant d_1$, by Theorem 4.3, the natural restriction map

$$\rho_t \colon H^0(Y_1, T_{Y_1}(t)) \longrightarrow H^0(X, T_{Y_1}(t)|_X)$$

is surjective for all $t \leqslant -1$. In particular, we have $H^0(X, T_{Y_1}(t)|_X) = 0$ for all $t \leqslant -1$ if $d \geqslant d_1$. Again, since Y is Fano, we have $r_Y > 0$ and the result is again a consequence of Proposition 5.3.

Proof of (3). — If M is isomorphic to any smooth quadric hypersurface Q^{n+r} and $Y \subset M$ is a complete intersection of degree $(d_1, \ldots, d_r)$ such that $d_i \geqslant 2$ for all $1 \leqslant i \leqslant r$. Then Y is also a complete intersection in $\mathbb{P}^{n+r+1}$ of degree $(2, d_1, \ldots, d_r)$ and the result follows immediately from (2).

Proof of (4). — Since every smooth complete intersection in a quadric hypersurface is also a smooth complete intersection in some projective space, we may assume that M is not isomorphic to any smooth quadric hypersurface. Note that we have $d > d_1 - r_Y/n \geqslant 1$ as $d_1 \geqslant 2$ and $r_Y \leqslant n$. Thus, by Proposition 5.3, it suffices to show that

$$H^0(X, T_Y(t)|_X) = 0 \qquad \text{for} \quad t \leqslant -\frac{r_Y}{n}.$$

Since M is not isomorphic to any smooth quadric hypersurface, by Theorem 4.2, the natural restriction map

$$H^0(M, T_M(t)) \longrightarrow H^0(Y, T_M(t)|_Y) \longrightarrow H^0(X, T_M(t)|_X)$$

is surjective for all $t \in \mathbb{Z}$. Let $\sigma \in H^0(X, T_Y(t)|_X)$ be a global section. Then σ is also a global section of $T_M(t)|_X$. Therefore there exists a global twisted vector field $\widetilde{\sigma} \in H^0(M, T_M(t))$ such that $\widetilde{\sigma}|_X = \sigma$. Denote by Y_j $(1 \leqslant j \leqslant r)$ the scheme-theoretic complete intersection $H_1 \cap \cdots \cap H_j$. Then each Y_j is smooth in a neighborhood of Y, and $\widetilde{\sigma}|_X = \sigma$ is a global section of $T_{Y_j}(t)|_X$ for all $1 \leqslant j \leqslant r$. Consider the following exact sequence

$$0 \longrightarrow T_{Y_j}(t)|_Y \longrightarrow T_{Y_{j-1}}(t)|_Y \xrightarrow{\beta_j(t)} \mathcal{O}_Y(d_j + t) \longrightarrow 0.$$

Then $\sigma \in H^0(X, T_Y(t)|_X)$ implies that the image $\widehat{\beta}_j(t)(\widetilde{\sigma}|_Y)$ vanishes over X, where $\widehat{\beta}_j(t)$ is the induced map

$$H^0(Y, T_{Y_{j-1}}(t)|_Y) \longrightarrow H^0(Y, \mathcal{O}_Y(d_j + t)).$$

However, since X is general and $d > d_j + t$ for any $1 \leqslant j \leqslant r$ by our assumption, we get $\widehat{\beta}_j(t)(\widetilde{\sigma}|_Y) = 0$ for any $1 \leqslant j \leqslant r$. Therefore we have $\widetilde{\sigma}|_Y \in H^0(Y, T_{Y_j}(t)|_Y)$ for any $1 \leqslant j \leqslant r$, i.e., $\widetilde{\sigma}|_Y \in H^0(Y, T_Y(t))$. On the other hand, since T_Y is stable (cf. Theorem 1.4), we have

$$H^0(Y, T_Y(t)) = 0 \qquad \text{for} \quad t \leqslant -\frac{r_Y}{n}.$$

Hence, we obtain $\widetilde{\sigma}|_Y = 0$ and consequently $\sigma = 0$. $\qquad\square$

Though the statements (2), (3) and (4) in the theorem may be not optimal in general, they have the advantage to give a lower bound which is quite easy to compute. The next theorem deals with the case in which Y is a general smooth hypersurface of $\mathbb{P}^{n+1}$. The proof of this theorem can be completed by combining Theorem 4.4 and the method analogous to that used above.

Proof of Theorem 1.9. — If Y is isomorphic to either $\mathbb{P}^n$ or Q^n, it follows from [2, Theorem A]. So we shall assume that Y is a general smooth hypersurface defined by a homogeneous polynomial h of degree $d_h \geqslant 3$. By Proposition 5.3, it is enough to prove that $H^0(X, T_Y(t)|_X) = 0$ for

$t \leqslant -r_Y/n$. As $n \geqslant 3$, $d_h \geqslant 3$ and $r_Y = n + 2 - d_h$, we have

$$\left(\frac{\rho + d}{2} - d_h\right) - \left(-\frac{r_Y}{n}\right) \geqslant \frac{(d_h - 2)(n+2) + d}{2} - d_h - \left(\frac{d_h}{n} - \frac{n+2}{n}\right)$$
$$\geqslant \left(\frac{n}{2} - \frac{1}{n}\right)d_h - n - \frac{1}{2} + \frac{2}{n}$$
$$\geqslant \frac{3n}{2} - \frac{3}{n} - n - \frac{1}{2} + \frac{2}{n}$$
$$> 0.$$

This implies $-r_Y/n < (\rho + d)/2 - d_h$. According to Theorem 4.4, the map

$$H^0(Y, T_Y(t)) \longrightarrow H^0(X, T_Y(t)|_X)$$

is surjective for $t \leqslant -r_Y/n$. The theorem is now a direct consequence of the stability of T_Y. □

Remark 5.4. — In Theorem 1.9, if Y is an arbitrary smooth hypersurface, then the argument above does not work, since the strong Lefschetz property (SLP) of Milnor algebras of smooth hypersurfaces is still open.

6. Hyperplane of cubic threefolds

In this section, we consider the case where the map $\text{Pic}(Y) \to \text{Pic}(X)$ is not surjective. By Noether–Lefschetz theorem mentioned in the introduction, this happens if X is a quadric section of a quadric threefold Q^3, or X is a quadric surface in $\mathbb{P}^3$, or X is a cubic surface in $\mathbb{P}^3$. In these cases, X is always a del Pezzo surface, i.e., the anti-canonical divisor $-K_X$ is ample.

6.1. Projective one forms

We denote by $\pi \colon S_r \to \mathbb{P}^2$ the surface obtained by blowing-up $\mathbb{P}^2$ at $r(\leqslant 8)$ points $p_1, \ldots, p_r$ in general position and denote by E_j the exceptional divisor over p_j. Then S_r is a del Pezzo surface with degree $K_S^2 = 9 - r$. It is well-known that the cotangent bundle $\Omega^1_{S_r}$ is stable with respect to the anti-canonical polarization $-K_{S_r}$ for $r \geqslant 2$ (cf. [8]). There is a one-to-one correspondence between the saturated rank one subsheaves of $\Omega^1_{S_r}$ and the global sections of $\Omega^1_{\mathbb{P}^2}(a)$ which vanish only in codimension two. The global sections of $\Omega^1_{\mathbb{P}^2}(a)$ are usually called *projective one forms*. In the following we give a brief description of this correspondence and we refer the reader to [8] for further details.

On one hand, let L be a saturated rank one subsheaf of the cotangent bundle of $\Omega^1_{S_r}$. Then we have

$$c_1(L) = -aE_0 - \sum_{j=1}^{9-r} b_j E_j$$

for some integers $a, b_j \in \mathbb{Z}$, where E_0 is the pull-back of a line in $\mathbb{P}^2$. The global section of $\Omega^1_{\mathbb{P}^2}(a)$ associated to L is defined by the following map

$$H^0(S_r, \Omega^1_{S_r} \otimes L^{-1}) \longrightarrow H^0\left(S_r \setminus \cup E_i, \Omega^1_{S_r} \otimes L^{-1}|_{S_r \setminus \cup E_i}\right)$$
$$\cong H^0\left(\mathbb{P}^2 \setminus \cup\{p_i\}, \Omega^1_{\mathbb{P}^2}(a)|_{\mathbb{P}^2 \setminus \cup\{p_i\}}\right) = H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(a)).$$

On the other hand, let $\omega \in H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(a))$ be a global section vanishing only in codimension two. Then ω can be identified with a rational global section of $\Omega^1_{\mathbb{P}^2}$ with poles supported on a line T. Let E_0 be the pull-back of T by π. Then $\pi^*\omega$ is a global section of $\Omega^1_{S_r} \otimes \mathcal{O}_{S_r}(aE_0)$. Let $\mathrm{div}(\pi^*\omega)$ be the divisor defined by the zeros of $\pi^*\omega$. The saturated rank one subsheaf associated to ω is defined to the image of the induced morphism $\mathcal{O}_{S_r}(-aE_0 + \mathrm{div}(\pi^*\omega)) \to \Omega^1_{S_r}$.

Example 6.1. — We recall several examples given in [8].

(1) The form $\omega = x_0 \mathrm{d}x_1 - x_1 \mathrm{d}x_0 \in H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(2))$ defines a saturated subsheaf of $\Omega^1_{S_r}$ which is isomorphic to $\mathcal{O}_{S_r}(-2E_0 + 2E_j)$, where E_j is the exceptional divisor above $[0:0:1]$. Moreover, the only rank one saturated subsheaves of $\Omega^1_{S_r}$ with $a = 2$ are $\mathcal{O}_{S_r}(-2E_0 + 2E_j)$ and $\mathcal{O}_{S_r}(-2E_0)$.

(2) We choose four points $p_1 = [1:0:0]$, $p_2 = [0:1:0]$, $p_3 = [0:0:1]$ and $p_4 = [1:1:1]$ in $\mathbb{P}^2$. Then the form defined by

$$\omega = (x_1^2 x_2 - x_2^2 x_1)\mathrm{d}x_0 + (x_2^2 x_0 - x_0^2 x_2)\mathrm{d}x_1 + (x_0^2 x_1 - x_1^2 x_0)\mathrm{d}x_2$$

induces a saturated subsheaf L of $\Omega^1_{S_r}$ such that $c_1(L) = -4E_0 + 2\sum_{j=1}^{4} E_j$, and there does not exist a subsheaf L' of $\Omega^1_{S_r}$ such that

$$c_1(L') = -4E_0 + 2\sum_{j=1}^{4} E_j + E_i$$

for $5 \leqslant i \leqslant r$. In fact, let ω' be the corresponding projective one form of L'. Since L is a subsheaf of L', $\pi^*\omega'/\pi^*\omega$ can be identified to be a global section of $L' \otimes L^*$ whose zeros are supported on E_i. In particular, ω is actually proportional to ω'. Nevertheless, the zeros of ω are $p_1, \ldots, p_4$ and the points $[0:1:1]$, $[1:0:1]$ and $[1:1:0]$.

Since there are at most four points of these points which are in general position, the exceptional line E_i cannot be a zero of $\pi^*\omega'$.

In order to prove Theorem 1.10, we need the following lemma due to Fahlaoui.

LEMMA 6.2 ([8, Lemme 1]). — *Let L, L' be two saturated subsheaves of $\Omega^1_{S_r}$. If L is not isomorphic to L', then we have*

$$h^0(S_r, \mathcal{O}_{S_r}(K_{S_r}) \otimes L^* \otimes L'^*) \geqslant 1.$$

6.2. Subsheaves of cotangent bundles of cubic surfaces

A cubic surface $S \subset \mathbb{P}^3$ is a blow-up $\pi\colon S \to \mathbb{P}^2$ of six points p_j on $\mathbb{P}^2$ in general position. The exceptional divisor $\pi^{-1}(p_j)$ is denoted by E_j. Let K_S be the canonical divisor of S and E_0 the pull-back of a line in $\mathbb{P}^2$. Then we have

$$-K_S = 3E_0 - \sum_{j=1}^{6} E_j \sim H|_S,$$

where $H \in |\mathcal{O}_{\mathbb{P}^3}(1)|$ is a hyperplane in $\mathbb{P}^3$. Let us recall the following well-known classical result of cubic surfaces.

- There are exactly 27 lines lying over a cubic surface: the exceptional divisors E_j above the six blown up points p_j, the proper transforms of the fifteen lines in $\mathbb{P}^2$ which join two of the blown up points p_j, and the proper transforms of the six conics in $\mathbb{P}^2$ which contain all but one of the blown up points.

The following result gives an upper bound for the degree of the saturated subsheaves of Ω^1_S.

PROPOSITION 6.3. — *Let S be a cubic surface and let $L \subset \Omega^1_S$ be a saturated invertible subsheaf. Then we have*

$$c_1(L) \cdot (-K_S) \leqslant -3.$$

Proof. — Note that since $\mu(\Omega^1_S) = -3/2$ and Ω^1_S is stable, we get $c_1(L) \cdot (-K_S) \leqslant -2$. Suppose that we have $c_1(L) \cdot K_S = 2$ for some L. Then we have

$$c_1(L) = -aE_0 - \sum_{j=1}^{6} b_j E_j$$

for some $a, b_j \in \mathbb{Z}$ with $a \geqslant 2$. If $a = 2$, then L is isomorphic to $\mathcal{O}_S(-2E_0)$ or some $\mathcal{O}_S(-2E_0 + 2E_i)$. In the former case we have $c_1(L) \cdot K_S = 6$ and in

the latter case we have $c_1(L) \cdot K_S = 4$. So we only need to consider the case $a \geqslant 3$. For fixed i, applying Lemma 6.2 to L and $-2E_0 + 2E_i$, we obtain an effective divisors C_i such that

$$C_i \sim K_S - L - (-2E_0 + 2E_i) = (a-1)E_0 + (b_i - 1)E_i + \sum_{j \neq i}(b_j + 1)E_j.$$

Denote by $d = -K_S \cdot C_i = 3a + \sum_{j=1}^{6} b_j + 1$ the degree of C_i. By the assumption $c_1(L) \cdot K_S = 2$, we obtain $3a + \sum_{j=1}^{6} b_j = 2$, and hence $d = 3$. Moreover, as $a \geqslant 3$, we have $\sum_{j=1}^{6} b_j \leqslant -7$, and consequently there is at least one $b_j \leqslant -2$. The proof will be divided into three steps.

Step 1. We will show $b_j \geqslant -2$ for all $1 \leqslant j \leqslant 6$. — There exist some π-exceptional effective divisors $\sum_{j=1}^{6} c_{ij} E_j$ such that the effective divisors C_i' defined as

$$C_i' = C_i - \sum_{j=1}^{6} c_{ij} E_j \sim (a-1)E_0 + (b_i - c_{ii} - 1)E_i + \sum_{j \neq i}(b_j - c_{ij} + 1)E_j$$

do not contain π-exceptional components. We denote the integer $b_j - c_{ij}$ by b_{ij} and denote the degree $-K_S \cdot C_i'$ of C_i' by d_i', then we have

$$(6.1) \qquad\qquad b_{ij} \leqslant b_j \quad \text{and} \quad d_i' \leqslant d.$$

Since the exceptional divisor E_i is a line on S and $-K_S \sim H|_X$ for some hyperplane $H \subset \mathbb{P}^3$, we have $\mathrm{Bs} \, |-K_S - E_i| \subset E_i$. Moreover, since C_i' does not contain E_i, we obtain

$$(6.2) \quad (-K_S - E_i) \cdot C_i' \geqslant 0 \quad \text{and} \quad -b_{ii} + 1 = C_i' \cdot E_i \leqslant -K_S \cdot C_i' = d_i'.$$

Combining (6.1) and (6.2) gives

$$(6.3) \qquad\qquad -b_i \leqslant -b_{ii} \leqslant d_i' - 1 \leqslant d - 1 = 2.$$

Since i is arbitrary, we deduce that $b_i \geqslant -2$ for $i = 1, \dots, 6$.

Step 2. We show $b_j \leqslant -1$ for all $1 \leqslant j \leqslant 6$ and $\sum_{j=1}^{6} b_j \leqslant -8$. — Since there is at least one $b_j \leqslant -2$ and $b_i \geqslant -2$ for all i, without loss of generality we may assume $b_1 = -2$. As a consequence of inequality (6.3), we have

$$b_{11} = -2 \quad \text{and} \quad d_1' = d = 3.$$

It follows that $-K_S \cdot (C_i - C_i') = d - d_1' = 0$. As $C_i - C_i' \geqslant 0$ and $-K_S$ is ample, we obtain $C_1' = C_1$ and

$$(6.4) \qquad\qquad -K_S \cdot C_1 = E_1 \cdot C_1 = 3.$$

Since C_1 does not contain E_j, we must have $-b_j - 1 = C_1 \cdot E_j \geqslant 0$ for $j \geqslant 2$, which yields $b_j \leqslant -1$ for $j \geqslant 2$. As a consequence, we get

$$-12 \leqslant \sum_{j=1}^{6} b_j \leqslant -7 \ \text{ and } \ 3 \leqslant a \leqslant 4.$$

Let $C_{1\ell}$ be a component of C_1. Since $\mathrm{Bs}\,|-K_S - E_1| \subset E_1$ and C_1 does not contain E_1, we have $(-K_S - E_1) \cdot C_{1\ell} \geqslant 0$. Then the equality (6.4) implies $(-K_S - E_1) \cdot C_{1\ell} = 0$. Therefore, $C_{1\ell}$ is actually a plane curve and there exists a plane $H_\ell \subset \mathbb{P}^3$ such that $C_{1\ell} + E_1 \leqslant H_\ell|_S$. In particular, we have

$$-K_S \cdot C_{1\ell} = H_\ell|_S \cdot C_{1\ell} \leqslant 2.$$

On the other hand, as $-K_S \cdot C_1 = 3$, it follows easily that there exists at least one component of C_1, denoted by C_{11}, such that $-K_S \cdot C_{11} = 1$. In particular, C_{11} is a line on S. However, C_{11} is not π-exceptional, so the line C_{11} passes at least two π-exceptional divisors, and it follows that there exists some $j \ (\geqslant 2)$ such that

$$-2 \leqslant b_j = -1 - C_1 \cdot E_j \leqslant -1 - C_{11} \cdot E_j \leqslant -2.$$

Hence we obtain $\sum_{j=1}^{6} b_j \leqslant -8$.

Step 3. We exclude the case $c_1(L) \cdot K_S = 2$. — By our argument above, if $c_1(L) \cdot K_S = 2$, then we have

$$a \geqslant 3, \ \ -2 \leqslant b_j \leqslant -1 \ \text{ and } \ -12 \leqslant \sum_{j=1}^{6} b_j \leqslant -8.$$

Then the equality $3a + \sum_{j=1}^{6} b_j = 2$ shows $a = 4$ and $\sum_{j=1}^{6} b_j = -10$, and consequently L is a line bundle of the form

$$-4E_0 + 2E_1 + 2E_2 + 2E_3 + 2E_4 + E_5 + E_6.$$

Nevertheless, we have seen that such a line bundle cannot be a saturated subsheaf of Ω^1_S (cf. Example 6.1), a contradiction. □

6.3. Stability of restrictions of tangent bundles of general cubic threefolds

In this subsection, we will prove Theorem 1.10. First we consider the saturated subsheaves of $\Omega^1_Y|_X$ of rank two and we give an upper bound for the degree of $c_1(\mathcal{F})$ with respect to $-K_X$.

LEMMA 6.4. — *Let Y be a general smooth cubic threefold and let $X \in |\mathcal{O}_Y(1)|$ be a general smooth divisor. If $\mathcal{F} \subset \Omega_Y^1|_X$ is a saturated subsheaf of rank two, then we have*

$$c_1(\mathcal{F}) \cdot (-K_X) \leqslant -5.$$

Proof. — The natural inclusion $\mathcal{F} \subset \Omega_Y^1|_X$ implies

$$h^0(X, \Omega_Y^2|_X \otimes \det(\mathcal{F})^*) \geqslant 1.$$

Using the short exact sequence

$$0 \longrightarrow \Omega_X^1(-1) \otimes \det(\mathcal{F})^* \longrightarrow \Omega_Y^2|_X \otimes \det(\mathcal{F})^*$$
$$\longrightarrow \mathcal{O}_X(K_X) \otimes \det(\mathcal{F})^* \longrightarrow 0,$$

we obtain either

$$h^0(X, \Omega_X^1(-1) \otimes \det(\mathcal{F})^*) \geqslant 1$$
$$\text{or} \quad h^0(X, \mathcal{O}_X(K_X) \otimes \det(\mathcal{F})^*) \geqslant 1.$$

In the former case, the stability of Ω_X^1 implies

$$(c_1(\mathcal{F}) + c_1(\mathcal{O}_X(1))) \cdot (-K_X) < \frac{K_X \cdot (-K_X)}{2} = -\frac{3}{2}.$$

This yields

$$c_1(\mathcal{F}) \cdot (-K_X) < -c_1(\mathcal{O}_X(1)) \cdot (-K_X) - \frac{3}{2} = -\frac{9}{2} < -4.$$

In the latter case, we have $c_1(\mathcal{F}) \cdot (-K_X) \leqslant K_X \cdot (-K_X) = -3$ with equality if and only if $c_1(\mathcal{F}) = -K_X$, and the quotient $\mathcal{G} := (\Omega_Y^1|_X)/\mathcal{F}$ is a torsion-free sheaf of rank one.

If $c_1(\mathcal{F}) \cdot (-K_X) = -3$, then $\det(\mathcal{F}) \cong \mathcal{O}_X(K_X) \cong \mathcal{O}_X(-1)$ and consequently $\det(\mathcal{G}) = \mathcal{O}_X(-1)$. Since $\mathcal{G}^*$ is a subsheaf of $T_Y|_X$, we obtain

$$h^0(X, T_Y|_X \otimes \det(\mathcal{G})) = h^0(X, T_Y(-1)|_X) \geqslant 1.$$

Since $T_Y(-1)|_X$ is a subsheaf of $T_Y|_X$, we get $H^0(X, T_Y|_X) \neq 0$. Then, by Theorem 4.4, we obtain $H^0(Y, T_Y) \neq 0$. Nevertheless, it is well-known that there are no global holomorphic vector fields over a cubic threefold (cf. [15, Theorem 11.5.2]). This leads to a contradiction.

If $c_1(\mathcal{F}) \cdot (-K_X) = -4$, then $\det(\mathcal{F}) \cong \mathcal{O}_X(-1) \otimes \mathcal{O}_X(-T)$ for some line $T \subset X$. As a consequence, we have $\det(\mathcal{G}) = \mathcal{O}_X(-1) \otimes \mathcal{O}_X(T)$. Since $\mathcal{G}^*$ is a subsheaf of $T_Y|_X$, we get

$$h^0(X, T_Y(-C)|_X) > 0,$$

where C is a conic such that $\mathcal{O}_X(C) \cong \mathcal{O}_X(1) \otimes \mathcal{O}_X(-T)$. Note that the sheaf $T_Y(-C)|_X$ is a subsheaf of $T_Y|_X$, it follows that $H^0(X, T_Y|_X) \neq 0$. Similarly, Theorem 4.4 implies $H^0(Y, T_Y) \neq 0$, which is impossible. $\qquad\square$

Now we are in the position to prove the main theorem in this section.

Proof of Theorem 1.10. — It is enough to prove that $\Omega_Y^1|_X$ is stable with respect to $\mathcal{O}_X(1)$. Since $\mu(\Omega_Y^1|_X) = -2$, it suffices to prove that the following inequality holds for any proper saturated subsheaf $\mathcal{F}$ of $\Omega_Y^1|_X$.

$$\mu(\mathcal{F}) = \frac{c_1(\mathcal{F}) \cdot -K_X}{\mathrm{rk}(\mathcal{F})} < -2$$

1st Case. $\mathcal{F} \subset \Omega_Y^1|_X$ *is a saturated subsheaf of rank one.* — Since $\mathcal{F}$ is a reflexive sheaf of rank one and X is smooth, $\mathcal{F}$ is actually an invertible sheaf. Then the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-1) \otimes \mathcal{F}^* \longrightarrow \Omega_Y^1|_X \otimes \mathcal{F}^* \longrightarrow \Omega_X^1 \otimes \mathcal{F}^* \longrightarrow 0$$

implies that we have either $h^0(X, \mathcal{O}_X(-1) \otimes \mathcal{F}^*) \geqslant 1$ or $h^0(X, \Omega_X^1 \otimes \mathcal{F}^*) \geqslant 1$. In the former case, we have

$$\mu(\mathcal{F}) = c_1(\mathcal{F}) \cdot (-K_X) \leqslant c_1(\mathcal{O}_X(-1)) \cdot (-K_X) = -3 < -2.$$

In the latter case, let $\overline{\mathcal{F}}$ be the saturation of $\mathcal{F}$ in Ω_X^1, then Proposition 6.3 implies

$$\mu(\mathcal{F}) \leqslant \mu(\overline{\mathcal{F}}) = c_1(\overline{\mathcal{F}}) \cdot (-K_X) \leqslant -3.$$

2nd Case. $\mathcal{F} \subset \Omega_Y^1|_X$ *is a saturated subsheaf of rank two.* — In this case, by Lemma 6.4, we have

$$\mu(\mathcal{F}) = \frac{c_1(\mathcal{F}) \cdot (-K_X)}{2} \leqslant \frac{-5}{2} < -2.$$

This finishes the proof. $\qquad\square$

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argument to simplify the proofs of Theorem 1.3 and Proposition 4.1, and to drop the genericity assumption in the previous version of this paper.

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