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LIMIT SHAPE AND HEIGHT FLUCTUATIONS OF RANDOM PERFECT MATCHINGS ON SQUARE-HEXAGON LATTICES

by Cédric BOUTILLIER & Zhongyang LI (*)

Abstract. — We study perfect matchings on the contracting square-hexagon lattice, constructed row by row either from a row of the square grid or of the hexagonal lattice. Given $1 \times n$ periodic weights to edges, we consider the probabilities of dimers proportional to the product of edge weights. We show that the partition function equals a Schur function of the edge weights. We then prove the Law of Large Numbers (limit shape) and the Central Limit Theorem (convergence to the Gaussian free field) for the corresponding height functions. We also show that certain types of dimers near the turning corner converge in distribution to the eigenvalues of Gaussian Unitary Ensemble, and that in the scaling limit when each segment of the bottom boundary grows linearly with respect to the dimension of the graph, the frozen boundary is a cloud curve with multiple tangent points (depending on the period) along each horizontal boundary segment.

Résumé. — Nous étudions les couplages parfaits de graphes, construits en prenant, pour chaque ligne, une ligne soit du réseau carré, soit du réseau hexagonal. Étant donnés des poids sur les arêtes avec une période $1 \times n$, la fonction de partition est une fonction de Schur dépendant des poids. Nous obtenons dans la limite des grands systèmes une loi des grands nombres (forme limite) et un théorème central limite (convergence vers le champ libre) pour la fonction de hauteur associée. La distribution de certains dimères près du point de contact au bord converge vers celle des valeurs propres de l'ensemble unitaire gaussien. De plus, dans la limite d'échelle de systèmes pour lesquels chaque segment du bord croît linéairement avec la taille du graphe, le bord de la zone gelée est une *courbe nuage* avec des points de contact sur chaque segment du bord inférieur dont le nombre dépend de la période.

1. Introduction

A *perfect matching*, or a *dimer configuration*, is a subset of edges of a graph such that each vertex is incident to exactly one edge. We study the

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asymptotic behavior of periodically weighted random perfect matchings on a class of domains called the contracting square hexagon lattice. Each row of the lattice is either obtained from a row of a square grid or that of a hexagon lattice; see Figure 2.6 for an example. On such a graph we shall assign edge weights, satisfying the condition that the edge weights are invariant under horizontal translations, while changing row by row. We define a probability measure for dimer configurations on such a graph to be proportional to the product of edge weights.

When all the edge weights are 1, the underlying probability measure is the uniform measure. The uniform perfect matchings on a square grid or a hexagonal lattice have been studied extensively in the past few decades; see [15, 16, 42] for recent results about uniform perfect matchings on the hexagonal lattice, and [8] for recent results about uniform perfect matchings on the square grid. These results are obtained by applying and redeveloping the recent techniques developed to study the Schur processes; see [1, 2, 6, 7, 38, 40]. Dimer model on a more general graph, called the *railyard graph*, may also be studied by techniques of Schur processes; see [5].

Among the problems concerning the asymptotic behavior of perfect matchings on larger and larger graphs, two of them are of special interest: the Law of Large Numbers and the Central Limit Theorem. More precisely, when the underlying finite graphs on which the dimer configurations are defined become larger and larger whose rescaled version approximate a certain domain in the plane, the rescaled height functions (which is a random function defined on faces of the graph associated to each random perfect matching) are expected to converge to a deterministic function (limit shape); and the non-rescaled height function is expected to have Gaussian fluctuation. The limit shape behavior was first observed from the arctic circle phenomenon for dimer models on large Aztec diamond (which is a finite subgraph of the square grid with certain boundary conditions); see [21, 23]. In each component outside the inscribed the circle, with probability exponentially close to 1, all the present edges of the dimer configuration are along the same direction. This is called the frozen region. Inside the circle, the probability that an edge of any certain direction appears in the dimer configuration is non-degenerate and lies in the open interval $(0, 1)$; this is called the liquid region. The limit shape of non-uniform dimer models on square grids, with more general boundary conditions, was studied in [10], and the technique may be generalized to obtain a variational principle, limit shape, and equation of frozen boundary for dimer models on general periodic bipartite graphs; see [27]. The Aztec diamond with 2×2 period was studied in [9], with $2 \times n$ period was studied in [11].

A square-hexagon lattice may be constructed row by row from either a row of a square grid or a row of a hexagonal lattice. In this paper, we assign positive weights to edges of the square-hexagon lattice in such a way that the edge weights change row by row with a fixed finite period. We then consider a special finite subgraph of the square-hexagon lattice, called a contracting square-hexagon lattice. With the help of the branching formula for the Schur function, we then show that the partition function of dimer configurations on a contracting square hexagon lattice, can be computed by a Schur function depending on edge weights.

Note that Markov chains for sampling those random dimer configurations on finite square-hexagon lattices with certain boundary conditions (i.e. random tilings of tower graphs) were studied in [4].

We then study the limit shape of the dimer configurations when the mesh size of the graph goes to zero, and show that the height function converges a deterministic function with an explicit formula. We then find the equation of the frozen boundary, and show that the frozen boundary is again a cloud curve (similar results was obtained in [27] for the hexagonal lattice, and obtained in [8] for the square grid), whose number of tangent points to the bottom boundary depend not only on the number of segments with distinct boundary conditions on the bottom boundary, but also on the size of the period of edge weights. In particular, given our assignments of edge weights, the liquid region is a simply-connected domain, i.e. there are no "floating bubbles" in the liquid region. We then study the fluctuations of non-rescaled height function, and show that after a homeomorphism from the liquid region to the upper half plane, the law of non-rescaled height fluctuations are given in the limit by the Gaussian free field. This extends the framework in which such a result is available for non-flat boundary conditions. See [3, 12] for the first results of this type, and [13] for another method to show that a large class of models have this kind of fluctuations. We also study the distribution of present edges joining a row with odd index to a row with even index above it, and show that near the top boundary, these edges have the same distribution as the eigenvalues of a GUE random matrix, which was established in [39] for plane partitions and in [22] for the Aztec diamond. In [30, 31], the case when the periodic edge weights decay polynomially with respect to the size of the graph is investigated, the liquid region is proved to split to finitely many disconnected components, and the height fluctuation in each component of the liquid region is proved to be an independent Gaussian free field in the scaling limit.

The organization of the paper is as follows. In Section 2, we define the contracting square-hexagon lattice and prove the formula to compute the partition function of dimer configurations on such a lattice via Schur functions depending on edge weights. In Section 3, we prove an explicit formula for the limit of the rescaled height function. In Section 4, we prove an explicit formula for the density of the limit counting measure associated to the dimer configurations on each row of the contracting square-hexagon lattice, and define the frozen region to be the region whenever the density is 0 or 1. In Section 5, we prove an explicit formula for the frozen boundary (the boundary of the frozen region) and show that the frozen boundary a cloud curve. In Section 6, we show that the distribution of present edges joining an row with odd index to a row with even index above it near the top boundary is the same as that of eigenvalues of a GUE random matrix. In Section 7, we show that the fluctuation of the non-rescaled height function is a homeomorphism of the Gaussian free field in the upper half plane. In Section 8, we give simulations of the distribution of dimer models on the contracting square-hexagon lattice, and draw pictures of the limit shape.

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2. Combinatorics

In this section, we define the contracting square-hexagon lattice on which we shall study the perfect matching, or the dimer model. By an explicit bijection between perfect matchings on the contracting square-hexagon lattice and sequences of certain Young diagrams, we express the probability measure on perfect matchings, in which the probability of each configuration is proportional to product of edge weights, in terms of Schur functions. As a result, the partition function of dimer configurations on such contracting square-hexagon lattice can also be expressed in term of Schur functions.

This extends known results for the dimer model on the square grid [8] and hexagonal lattice [6, 42], where the underlying measure is uniform or a q-deformation of the uniform measure.

2.1. Square-hexagon Lattices

Consider a doubly-infinite binary sequence indexed by integers $\mathbb{Z} = \{\dots,-2,-1,0,1,2,\dots\}$.

$$\check{a} = (\dots, a_{-2}, a_{-1}, a_0, a_1, a_2, \dots) \in \{0,1\}^{\mathbb{Z}}.$$

The *whole-plane square-hexagon lattice* associated with the sequence $\check{a}$, is a bipartite plane graph $\mathrm{SH}(\check{a})$ defined as follows. Each vertex of $\mathrm{SH}(\check{a})$ is either black or white, and we identify the vertices with points on the plane. Its vertex set is a subset of $\frac{\mathbb{Z}}{2} \times \frac{\mathbb{Z}}{2}$. For $m \in \frac{\mathbb{Z}}{2}$, the vertices with ordinate m correspond to the $2m$th row of the graph. Vertices on even rows (for m integer) are colored in black. Vertices on odd rows (for m half integer) are colored in white.

- each black vertex on the $(2m)$th row is adjacent to two white vertices in the $(2m+1)$th row; and
- if $a_m = 1$, each white vertex on the $(2m-1)$th row is adjacent to exactly one black vertex in the $(2m)$th row; if $a_m = 0$, each white vertex on the $(2m-1)$th row is adjacent to two black vertices in the $(2m)$th row.

See Figure 2.1. Such a graph is also related to the rail-yard graph; see [5].

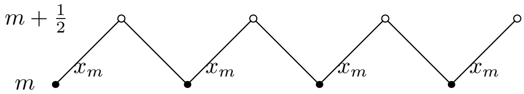
We shall assign edge weights to the whole-plane square-hexagon lattice $\mathrm{SH}(\check{a})$ as follows.

Assumption 2.1. — *For $m \geqslant 1$, we assign weight $x_m > 0$ to each NE-SW edge joining the $(2m)$th row to the $(2m+1)$th row of $\mathrm{SH}(\check{a})$. We assign weight $y_m > 0$ to each NE-SW edge joining the $(2m-1)$th row to the $(2m)$th row of $\mathrm{SH}(\check{a})$, if such an edge exists. We assign weight 1 to all the other edges.*

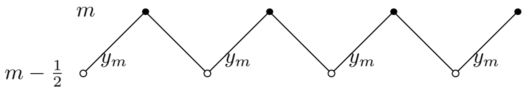
It is straightforward to check the following lemma describing the faces of a whole-plane square-hexagon lattice.

Lemma 2.2. — *Each face of $\mathrm{SH}(\check{a})$ is either a square (degree-4 face) or a hexagon (degree-6 face). Let $m \geqslant 1$ be a positive integer.*

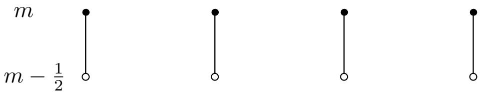
(1) *There exists a degree-6 face including both black vertices in the $(2m)$th row and black vertices in the $(2m+2)$th row if and only if $a_{m+1} = 1$.*



(a) Structure of $\mathrm{SH}(\check{a})$ between the $(2m)$th row and the $(2m+1)$th row



(b) Structure of $\mathrm{SH}(\check{a})$ between the $(2m-1)$th row and the $(2m)$th row when $a_m = 0$



(c) Structure of $\mathrm{SH}(\check{a})$ between the $(2m-1)$th row and the $(2m)$th row when $a_m = 1$

Figure 2.1. Graph structures of the square-hexagon lattice on the $(2m-1)$th, $(2m)$th, and $(2m+1)$th rows depend on the values of (a_m). Black vertices are along the $(2m)$th row, while white vertices are along the $(2m-1)$th and $(2m+1)$th row.

(2) *There exists a degree-4 face including both black vertices in the* $(2m)$*th row and black vertices in the* $(2m+2)$*th row if and only if* $a_{m+1} = 0$.

A *contracting square-hexagon lattice* is built from a whole-plane square-hexagon lattice as follows:

Definition 2.3. — *Let* $N \in \mathbb{N}$. *Let* $\Omega = (\Omega_1, \dots, \Omega_N)$ *be an* N*-tuple of positive integers, such that* $1 = \Omega_1 < \Omega_2 < \cdots < \Omega_N$. *Set* $m = \Omega_N - N$. *The contracting square-hexagon lattice* $\mathcal{R}(\Omega, \check{a})$ *is a subgraph of* $\mathrm{SH}(\check{a})$ *built of* $2N$ *or* $2N+1$ *rows. We shall now enumerate the rows of* $\mathcal{R}(\Omega, \check{a})$ *inductively, starting from the bottom as follows:*

- *The first row consists of vertices* (i, j) *with* $i = \Omega_1 - \frac{1}{2}, \dots, \Omega_N - \frac{1}{2}$ *and* $j = \frac{1}{2}$. *We call this row the boundary row of* $\mathcal{R}(\Omega, \check{a})$.
- *When* $k = 2s$, *for* $s = 1, \dots N$, *the* k*th row consists of vertices* (i, j) *with* $j = \frac{k}{2}$ *and incident to at least one vertex in the* $(2s-1)$*th row of the whole-plane square-hexagon lattice* $\mathrm{SH}(\check{a})$ *lying between the leftmost vertex and rightmost vertex of the* $(2s-1)$*th row of* $\mathcal{R}(\Omega, \check{a})$

- *When $k = 2s+1$, for $s = 1, \dots N$, the kth row consists of vertices (i,j) with $j = \frac{k}{2}$ and incident to two vertices in the $(2s)$th row of $\mathcal{R}(\Omega, \check{a})$.*

The transition from an odd row to the next even row in a contracting square-hexagon lattice can be of two kinds depending on whether vertices are connected to one or two vertices of the row above them. See Figures 2.4, 2.5, and 2.6 for examples of contracting square-hexagon lattices.

Definition 2.4. — *Let I_1 (resp. I_2) be the set of indices j such that vertices of the $(2j-1)$th row are connected to one vertex (resp. two vertices) of the $(2j)$th row. In terms of the sequence $\check{a}$,*

$$I_1 = \{k \in \{1, \dots, N\} \mid a_k = 1\}, \quad I_2 = \{k \in \{1, \dots, N\} \mid a_k = 0\}.$$

The sets I_1 and I_2 form a partition of $\{1, \dots, N\}$, and we have $|I_1| = N - |I_2|$.

2.2. Perfect Matching

Definition 2.5. — *A* dimer configuration, *or a* perfect matching *M of a contracting square-hexagon lattice $\mathcal{R}(\Omega, \check{a})$ is a set of edges $((i_1, j_1), (i_2, j_2))$, such that each vertex of $\mathcal{R}(\Omega, \check{a})$ belongs to a unique edge in M.*

The set of perfect matchings of $\mathcal{R}(\Omega, \check{a})$ is denoted by $\mathcal{M}(\Omega, \check{a})$.

Definition 2.6. — *Let $M \in \mathcal{M}(\Omega, \check{a})$ be a perfect matching of $\mathcal{R}(\Omega, \check{a})$. We call an edge $e = ((i_1, j_1), (i_2, j_2)) \in M$ a V-*edge *if* $\max\{j_1, j_2\} \in \mathbb{N}$ *(i.e. if its higher extremity is black) and we call it a* Λ-edge *otherwise. In other words, the edges going upwards starting from an odd row are V-edges and those ones starting from an even row are Λ-edges. We also call the corresponding vertices-(i_1, j_1) and (i_2, j_2) V-vertices and Λ-vertices accordingly.*

Lemma 2.7. — *Let $M \in \mathcal{M}(\Omega, \check{a})$ be a perfect matching of $\mathcal{R}(\Omega, \check{a})$. For each $1 \leqslant i \leqslant N$, the number of V-edges joining the $(2i-1)$th row and the $(2i)$th row is one more than the number of V edges joining the $(2i)$th row and the $(2i+1)$th row.*

Proof. — For $j \in \{2i, 2i+1\}$, let t_j be the number of vertices in the jth row of $\mathcal{R}(\Omega, \check{a})$. From the construction of $\mathcal{R}(\Omega, \check{a})$ in Definition 2.3, we have

$$t_{2i+1} = t_{2i} - 1. \tag{2.1}$$

Let s be the number of V-edges joining the $(2i-1)$th row and $(2i)$th row. Then there exists $(t_{2i}-s)$ Λ-edges joining the $(2i)$th row to the $(2i+1)$th row. Hence there are $(t_{2i+1}-t_{2i}+s)$ V-edges joining the $(2i+1)$th row and $(2i+2)$th row. Then the lemma follows from (2.1). ☐

Example 2.8.

(1) If for each $i \geqslant 1$, each vertex on the ith row of $\mathrm{SH}(\check{a})$ is adjacent to two vertices on the $(i+1)$th row of $\mathrm{SH}(\check{a})$, then the construction in Definition 2.3 gives us the rectangular Aztec diamond studied in [8].
(2) For each $i \geqslant 1$, each vertex on the $(2i-1)$th row of $\mathrm{SH}(\check{a})$ is adjacent to one vertex on the $(2i)$th row of $\mathrm{SH}(\check{a})$, then the construction in Definition 2.3 gives us the contracting hexagonal lattice studied in [7, 42].

2.3. Partitions and Young Diagrams

Following [8], we will use *signatures* to encode the perfect matchings of the contracting square-hexagons.

Definition 2.9. — *A* signature *of length N is a sequence of nonincreasing integers $\mu = (\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_N)$. Each μ_k is a* part *of the signature μ. The* length *N of the signature μ is denoted by $l(\mu)$. We say that μ is* non-negative *if $\mu_N \geqslant 0$. The* size *of a non-negative signature μ is*

$$|\mu| = \sum_{i=1}^{N} \mu_i.$$

$\mathbb{GT}_N$ denotes the set of signatures of length N, and $\mathbb{GT}_N^+$ is the subset of non-negative signatures.

To the boundary row $\Omega = (\Omega_1 < \cdots < \Omega_N)$ of a contracting square-hexagon lattice is naturally associated a non-negative signature ω of length N by:

$$\omega = (\Omega_N - N, \ldots, \Omega_1 - 1).$$

Non-negative signatures are the convenient objects to talk about integer partitions with a given number of zero parts. Most of the objects constructed from partitions are available for non-negative signatures, in particular Young diagrams, and interlacement relations which we recall now.

A graphic way to represent a non-negative signature μ is through its *Young diagram* Y_μ, a collection of $|\mu|$ boxes arranged on non-increasing rows aligned on the left: with μ_1 boxes on the first row, μ_2 boxes on the second row,... μ_N boxes on the Nth row. Some rows may be empty if the corresponding μ_k is equal to 0. The correspondence between non-negative signatures of length N and Young diagrams with N (possibly empty) rows is a bijection.

If all the parts of a non-negative signature μ are equal (say N parts equal to m), the Young diagram Y_μ has a rectangular shape. We then say that μ is *rectangular*, and note $\mu = N \times m$, and $Y_{N\times m}$ for its Young diagram.

Young diagrams included in $Y_{N\times m}$ are those corresponding to non-negative signatures of length N and parts bounded by m.

Definition 2.10. — *Let Y, W be two Young diagrams. We say that $Y \subset W$* differ by a horizontal strip *if the collection of boxes in $Z = W \setminus Y$ contains at most one box in every column. We say that they* differ by a vertical strip *if Z contains at most one box in every row.*

We say that two non-negative signatures λ and μ interlace, *and write $\lambda \prec \mu$ if $Y_\lambda \subset Y_\mu$ differ by a horizontal strip. We say they* co-interlace *and write $\lambda \prec' \mu$ if $Y_\lambda \subset Y_\mu$ differ by a vertical strip.*

Another way to graphically represent signatures is to use *Maya diagrams*, which usually represent a collection of white and black particles (here squares $\square$, $\blacksquare$) on the 1-dimensional lattice $\mathbb{Z}$. For our purposes, since we will work with non-negative signatures with Young diagram included in a rectangle of a given size, we will need finite version of Maya diagrams, defined below.

Definition 2.11. — *A* finite Maya diagram $\mathbf{m}$ *of length n is an element of $\{\square, \blacksquare\}^n$. The* origin *of the Maya diagram is a position between two successive elements of the sequence, such that the number of elements on the left (resp. on the right) of this position is equal to the number of $\blacksquare$ (resp. $\square$) particles.*

Non-negative signatures μ of length N with parts bounded by m corresponds bijectively to finite Maya diagrams m_μ of length $N+m$ and exactly N black particles, by the following coding of the non trivial part of the boundary of Y_μ seen as a lattice path of length $N+m$ connecting two opposite corners of $Y_{N\times m}$. A vertical step corresponds to a $\blacksquare$ and a horizontal step corresponds to a $\square$. See Figure 2.2.

This way, the signature of length N with all parts equal to 0 (resp. equal to m) corresponds to the Maya diagram where the N black particles are on the left (resp. on the right) of the m white particles.

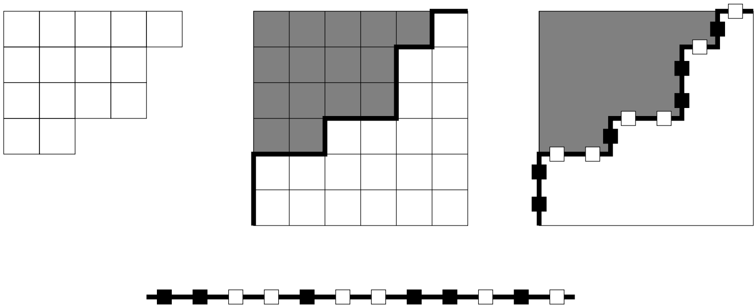


Figure 2.2. Top: the Young diagram of $(5,4,4,4,2,0,0)$, seen as a non-negative signature with 6 parts, all bounded by 6 (left); the same Young diagram included in the rectangle 6×6 and the path representing the boundary of the Young diagram from the lower left to the upper right corner (middle), the encoding of the steps of path with black and particles. Bottom, the actual finite Maya diagram of size $6+6$.

We shall associate to each perfect matching in $\mathcal{M}(\Omega, \check{a})$ a sequence of non-negative signatures, one for each row of the graph.

Construction 2.12. — Let $j \in \{1, \dots, 2N+1\}$. Assume that the jth row of $\mathcal{R}(\Omega, \check{a})$ has n_j V-vertices and m_j Λ-vertices. Then we first associate a finite Maya diagram of length $n_j + m_j$ with n_j black particles: every Λ-vertex (resp. V-vertex) is mapped to a white (resp. black) particle. This Maya diagram corresponds then to a Young diagram of a non-negative signature of length n_j and parts bounded by m_j, which in turn has a Young diagram Y_j fitting in a $n_j \times m_j$ rectangle.

Note that to perform this construction for the boundary row (i.e. $j = 1$), vertices with coordinates between Ω_1 and Ω_N, which are not present in the graph are considered a (virtual) Λ-vertices, and should be taken into account to compute n_1 and m_1.

The encoding of dimer configurations of finite contracting square-hexagon graphs with Maya diagrams allows then for a bijective correspondence with sequences of interlaces signatures. More precisely:

Theorem 2.13 ([8, Theorem 2.9], [5]). — *For given Ω, $\check{a}$, let ω be the signature associated to Ω. Then the construction 2.12 defines a bijection between the set of perfect matchings $\mathcal{M}(\Omega, \check{a})$ and the set $S(\omega, \check{a})$ of sequences of non-negative signatures*

$$\{(\mu^{(N)}, \nu^{(N)}, \dots, \mu^{(1)}, \nu^{(1)}, \mu^{(0)}\}$$

where the signatures satisfy the following properties:

- *All the parts of $\mu^{(0)}$ are equal to 0;*
- *The signature $\mu^{(N)}$ is equal to ω;*
- *The signatures satisfy the following (co)interlacement relations:*

$$\mu^{(N)} \prec' \nu^{(N)} \succ \mu^{(N-1)} \prec' \cdots \succ \mu^{(1)} \prec' \nu^{(1)} \succ \mu^{(0)}.$$

Moreover, if $a_m = 1$, then $\mu^{(N+1-k)} = \nu^{(N+1-k)}$.

Remark 2.14. — The interlacing relations have the following implications on the signatures and their Young diagrams:

- for all i, $l(\mu^{(i)}) = l(\nu^{(i)}) = i$;
- for all i, parts of $\mu^{(i)}$ (resp. $\nu^{(i)}$) are all bounded by $\Omega_N + i - t(i)$ (resp. $\Omega_N - t(i+1) + i + 1$).

where

$$t(i) = \#(I_1 \cap \{1, \dots, i-1\})$$

is the number of odd rows below with ordinate less than i, where vertices are connected to a single vertex of the row above them.

The following lemma relates the size of the signatures associated to rows of the graph with the number of NE-SW dimers connecting these rows:

Lemma 2.15. — *Let $1 \leqslant i \leqslant N$.*

(1) *If in the $(2i)$th row of $\mathcal{R}(\Omega, \check{a})$, the dimer configuration is given by the signature $\nu^{(N-i+1)}$; and in the $(2i+1)$th row, the dimer configuration is given by the signature $\mu^{(N-i)}$, then the number of present NE-SW edges joining the $(2i)$th row to the $(2i+1)$th row is $|\nu^{(N-i+1)}| - |\mu^{(N-i)}|$.*

(2) *Assume that each vertex in the $(2i-1)$th row of* $\mathrm{RH}(\check{a})$ *is adjacent to two vertices in the $(2i)$th row. If in the $(2i-1)$th row of $\mathcal{R}(\Omega, \check{a})$, the dimer configuration is given by the signature $\mu^{(N-i+1)}$; and in the $(2i)$th row, the dimer configuration is given by the signature $\nu^{(N-i+1)}$, then the number of present NE-SW edges joining the $(2i-1)$th row to the $(2i)$th row is $|\nu^{(N-i+1)}| - |\mu^{(N-i+1)}|$.*

Proof. — Edges present in a dimer configuration between rows $2i$ and $2i+1$ are Λ-edges, connecting Λ-vertices which in terms of Maya diagram are $\square$-particles.

Let $M_{\nu,i}$ (resp. $M_{\mu,i}$) be Maya diagrams corresponding to $\nu^{(N+1-i)}$ and $\mu^{(N-i)}$. Note that $M_{\nu,i}$ (resp. $M_{\mu,i}$) has exactly $N+1-i$ (resp. $N-i$) boxes to the left of its origin, and both $M_{\nu,i}$ and $M_{\mu,i}$ have the same number of boxes to the right or their origins. This follows from the fact that $M_{\nu,i}$ (resp. $M_{\mu,i}$) has $N+1-i$ (resp. $N-i$) black squares, while both $M_{\nu,i}$ and $M_{\mu,i}$ have the same number of white squares; see Definition 2.11. If we look at the $M_{\nu,i}$ and $M_{\mu,i}$ with their origin aligned, the presence of a NE-SW edge corresponds to a $\square$-particle in $M_{\nu,i}$ jumping to the right by one step in $M_{\mu,i}$ (whereas a NW-SE edge would correspond to a $\square$-particle staying at the same place in both $M_{\nu,i}$ and $M_{\mu,i}$; see Figure 2.3.

The number of NE-SW edges between these two rows is thus the total displacement of $\square$-particles. Since in a Maya diagram, moving a $\square$-particle to the right corresponds to removing a box in the Young diagram, thus decreasing the size of the partition by 1, it follows that the total displacement of the $\square$-particles is equal to $|\nu^{(N+1-i)}| - |\mu^{(N-i)}|$.

The second part is proved analogously. $\square$

A simple and direct consequence of Lemma 2.15 is the following:

Corollary 2.16. — *The total number of NE-SW edges in a perfect matching of $\mathcal{R}(\Omega, \check{a})$ is equal to $|\omega|$, the size of the non-negative signature corresponding to the boundary row Ω.*

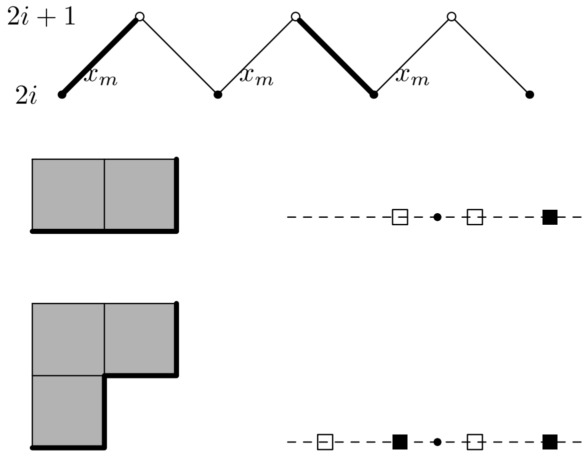


Figure 2.3. The top graph represents part of a dimer configuration of a square-hexagon lattice, more precisely, the configuration of edges between the $2i$th row and the $(2i+1)$th row. Present edges in the figure are Λ-edges. Endpoints of present edges in the figure are Λ-vertices; all the other vertices are V-vertices. The mid-left graph is the Young diagram associated to the $(2i+1)$th row; and the bottom-left graph is the Young diagram associated to the $2i$th row. The mid-right graph is the Maya diagram corresponding to the $(2i+1)$th row; and the bottom-right graph is the Maya diagram associated to the $2i$th row.

2.4. Schur Functions and Partition Function of Perfect Matchings

Recall that the *partition function* of the dimer model of a finite graph G with edge weights $(w_e)_{e\in E(G)}$ is given by

$$Z = \sum_{M\in\mathcal{M}} \prod_{e\in M} w_e,$$

where $\mathcal{M}$ is the set of all perfect matchings of G. The *Boltzmann dimer probability measure* on M induced by the weights w is thus defined by declaring that probability of a perfect matching is equal to

$$\frac{1}{Z}\prod_{e\in M} w_e.$$

In this section, we prove a formula which express the partition function of perfect matchings on a contracting square-hexagon lattice $\mathcal{R}(\Omega,\check{a})$ as a

Schur function depending on the boundary configuration Ω and the edge weights.

Definition 2.17. — *Let* $\lambda \in \mathbb{GT}_N$*. The* rational Schur function s_λ *associated to* λ *is the homogeneous symmetric function of degree* $|\lambda|$ *in* N *variables defined by:*

$$s_\lambda(u_1,\dots,u_N) = \frac{\det_{i,j=1,\dots,N}(u_i^{\lambda_j+N-j})}{\prod_{1\leqslant i<j\leqslant N}(u_i-u_j)}.$$

Let $\mu,\nu \in \mathbb{GT}_n^+$ be two non-negative signature of length n. It is well-known that Schur functions form a basis for the algebra of symmetric functions. Let $\beta = (\beta_1,\dots,\beta_n) \in \mathbb{C}^n$. We define as in [8] the coefficients $\mathrm{st}_\beta\left(\mu \to \nu\right)$ and $\mathrm{pr}_\beta\left(\lambda^{(i)} \to \mu^{(i-1)}\right)$ as follows:

$$\mathrm{pr}_\beta(\nu\to\lambda) = \begin{cases} \beta_1^{|\nu|-|\lambda|}\frac{s_\lambda(\beta_2,\dots,\beta_n)}{s_\nu(\beta_1,\dots,\beta_n)} & \text{if } \lambda\prec\nu \\ 0 & \text{otherwise,}\end{cases} \tag{2.2}$$

and

$$\mathrm{st}_\beta(\mu\to\lambda) = \begin{cases} \frac{1}{\prod_{j=1}^n(1+\beta_j)}\frac{s_\lambda(\beta_1,\dots,\beta_n)}{s_\mu(\beta_1,\dots,\beta_n)} & \text{if } \mu\prec'\lambda \\ 0 & \text{otherwise.}\end{cases} \tag{2.3}$$

By the branching formula for Schur polynomials, and the same argument as [8, Lemma 2.12] we have the following identities

$$\frac{s_\mu(u_1,\dots,u_n)}{s_\mu(\beta_1,\dots,\beta_n)}\prod_{j=1}^n\frac{(1+u_j)}{(1+\beta_j)} = \sum_{\lambda\in\mathbb{GT}_n}\mathrm{st}_\beta\left(\mu\to\lambda\right)\frac{s_\lambda(u_1,\dots,u_n)}{s_\lambda(\beta_1,\dots,\beta_n)}, \tag{2.4}$$

$$\frac{s_\nu(\beta_1,u_2,\dots,u_n)}{s_\nu(\beta_1,\beta_2\dots,\beta_n)} = \sum_{\lambda\in\mathbb{GT}_{n-1}}\mathrm{pr}_\beta\left(\nu\to\lambda\right)\frac{s_\lambda(u_2,\dots,u_n)}{s_\lambda(\beta_2,\dots,\beta_n)}. \tag{2.5}$$

from which we deduce that the following holds:

$$\sum_{\lambda\prec\nu}\mathrm{pr}_\beta(\nu\to\lambda) = 1, \qquad \sum_{\lambda\succ'\mu}\mathrm{st}_\beta(\mu\to\nu) = 1.$$

For $i \in \{1,2,\dots,i\}$, define

$$C_i = (x_i,x_{i+1},\dots,x_N) \in \mathbb{R}^{N-i+1}, \tag{2.6}$$

and for $i \in I_2$, define

$$B_i = y_iC_i = (y_ix_i, y_ix_{i+1}\dots,y_ix_N) \in \mathbb{R}^{N-i+1} \tag{2.7}$$

By homogeneity of the Schur functions, for each $i \in I_2$, and $\lambda \in \mathbb{GT}_{N-i}^+$, we have

$$s_\lambda(B_i) = (y_i)^{|\lambda|}\,s_\lambda(C_i). \tag{2.8}$$

For $i \in I_2$, define

$$\Gamma_i = \prod_{t=i+1}^{N} (1 + y_i x_t)\,. \tag{2.9}$$

Recall that $\mathcal{S}_\omega^N(\check{a})$, as defined in Theorem 2.13, is the set of all the sequences of partitions in bijection with the set $\mathcal{M}(\Omega, \check{a})$, which consists of all the perfect matchings on the contracting square-hexagon lattice with bottom boundary condition Ω and structures on rows given by $\check{a}$. We now define a probability measure on $\mathcal{S}_\omega^N(\check{a})$ as follows:

$$\begin{aligned}(2.10)\quad \mathbb{P}_\omega^N &\left(\mu^{(N)}, \nu^{(N)}, \ldots, \mu^{(1)}, \nu^{(1)}, \mu^{(0)}\right) \\ &= \mathbb{1}_{\{\mu^{(N)}=\omega\}} \prod_{j\in I_2} \mathrm{st}_{B_j}\left(\mu^{(N-j+1)} \to \nu^{(N-j+1)}\right) \\ &\qquad\qquad \times \prod_{i=1}^{N} \mathrm{pr}_{C_i}\left(\nu^{(N-i+1)} \to \mu^{(N-i)}\right).\end{aligned}$$

The following proposition connects this measure with the Boltzmann measure on dimers configurations of the associated contracting square-hexagon graph:

Proposition 2.18. — *The bijection described in Theorem 2.13 transports the probability measure* (2.10) *on* $\mathcal{S}_\omega^N(\check{a})$ *to a Boltzmann dimer measure on the perfect matchings of* $\mathcal{R}(\Omega, \check{a})$*, with the following weights*

- *each NE-SW edge joining the* $(2i)$*th row to the* $(2i+1)$*th row has weight* x_i*; and*
- *each NE-SW edge joining the* $(2i-1)$*th row to the* $2i$*th row has weight* y_i*, if such an edge exists;*
- *All the other edges have weight 1.*

Moreover, the dimer partition function on $\mathcal{R}(\Omega, \check{a})$ *for these weights is given by*

$$Z = \left[\prod_{i\in I_2} \Gamma_i\right] s_\omega(x_1, \ldots, x_N)$$

where ω *is the* N*-tuple corresponding to the boundary row of* $\mathcal{R}(\Omega, \check{a})$*, and* Γ_i *is defined as in* (2.9).

Proof. — By (2.10), (2.2), (2.3) and (2.8), we have

$$\begin{aligned}(2.11)\quad &\mathbb{P}_\omega^N\left(\mu^{(N)},\nu^{(N)},\dots,\mu^{(1)},\nu^{(1)},\mu^{(0)}\right)\\ &=1_{\{\mu^{(N)}=\omega\}}\frac{\prod_{i\in I_2}\left[(y_i)^{|\nu^{(N-i+1)}|-|\mu^{(N-i+1)}|}\right]\prod_{j=1}^N\left[(x_j)^{|\nu^{(N-j+1)}|-|\mu^{(N-j)}|}\right]}{\left[\prod_{i\in I_2}\Gamma_i\right]s_\omega(x_1,\dots,x_N)}.\end{aligned}$$

When $\mu^{(N)}=\omega$, the numerator of (2.11) is exactly by Lemma 2.15 the product of weights of present edges in the perfect matching corresponding to the sequence of non-negative signatures

$$\left(\mu^{(N)},\nu^{(N)},\dots,\mu^{(1)},\nu^{(1)},\mu^{(0)}\right)$$

Then the proposition follows. □

2.5. Examples

In this section, we provide a few examples of contracting square-hexagon lattices, compute the partition functions of dimer configurations on these graphs explicitly, and verify that these partition functions are equal to the formula given by Proposition 2.18.

2.5.1. Square Grid

Consider perfect matchings on a square grid with edge weights assigned as in the Figure 2.4.

Corollary 2.19. — *Let $\mathcal{R}(\Omega,\check{a})$ be a contracting square grid, with edge weights $x_1,x_2,\dots,$ on NE-SW edges; $a_i=0$ for all $i\geqslant 1$; and Ω is an N-tuple of integers. Then the partition function for perfect matchings on $\mathcal{R}(\Omega,\mathbf{0})$ is given by*

$$Z=\left[\prod_{i=1}^N\prod_{j=i}^N(1+y_ix_j)\right]s_\omega(x_1,\dots,x_N)$$

where ω is the N-tuple corresponding to the boundary row of $\mathcal{R}(\Omega,\mathbf{0})$, and Γ_i is defined as in (2.9).

Proof. — Note that when $\check{a}=\mathbf{0}$, the graph is a square grid. When Ω is an N-tuple of integers, we have $I_2=\{1,2,\dots,N\}$. Corollary 2.19 follows from Proposition 2.18. □

The case when all x_i and y_i are 1 is the one covered by [8].

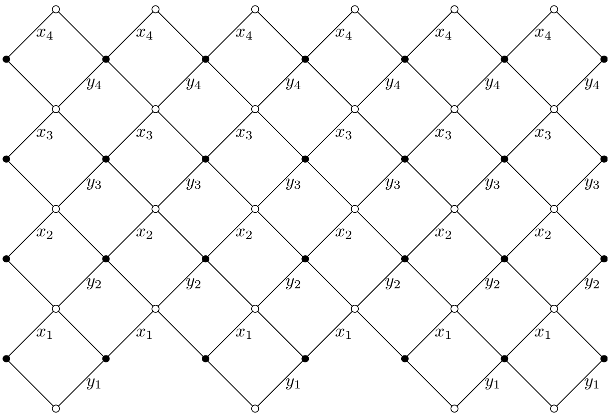


Figure 2.4. Rectangular Aztec diamond with $N = 4$, $m = 2$, $\Omega = (1, 3, 5, 6)$, and $a_i = 0$.

2.5.2. Hexagon Lattice

Corollary 2.20. — *Let $\mathcal{R}(\Omega, \check{a})$ be a contracting hexagonal lattice such that $a_i = 1$ for all $i \geqslant 1$ and Ω is an N-tuple of integers, with edge weights $x_1, x_2, \ldots,$ on NE-SW edges. Then the partition function for perfect matchings on $\mathcal{R}(\Omega, \mathbf{1})$ is given by*

$$Z = s_\omega(x_1, \ldots, x_N)$$

where ω is the N-tuple corresponding to the boundary row of $\mathcal{R}(N, \Omega, m)$, and Γ_i is defined as in (2.9).

Proof. — Note that when $\check{a} = \mathbf{1}$, the graph is a hexagon lattice. When Ω is an N-tuple of integers, we have $I_2 = \emptyset$. The Corollary 2.20 follows from Proposition 2.18. $\square$

The case when all the weights x_i are equal to 1 is the context of [42], although the results there were obtained through a q-deformation of the measure by setting $x_i = q^{-i}$ and taking the limit $q \to 1$.

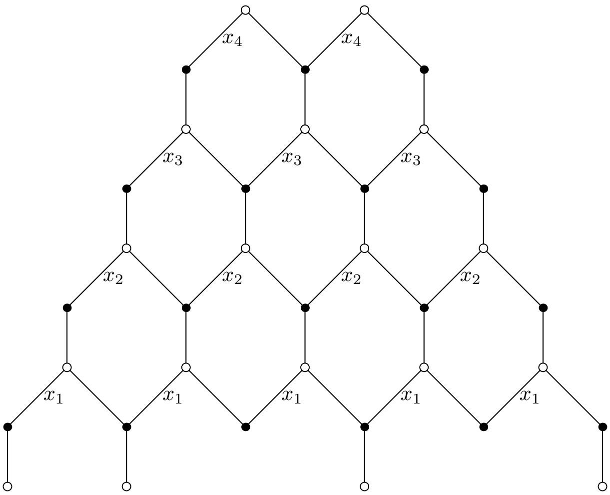


Figure 2.5. Contracting hexagon lattice with $N = 4$, $m = 2$, $\Omega = (1, 2, 4, 6)$, and $a_i = 1$.

2.5.3. A Square-Hexagon Lattice

Example 2.21. — The partition function of dimer configurations on a square-hexagon lattice as illustrated in Figure 2.6 is

$$Z = (1 + y_2x_2)(1 + y_2x_3) \times \left[x_1^3x_2 + x_1^3x_3 + x_1x_2^3 + x_1x_3^3 + x_2^3x_3 + x_2x_3^3 + x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2 + 2x_1x_2x_3(x_1 + x_2 + x_3)\right]. \tag{2.12}$$

Indeed, in the graph shown in Figure 2.6, we have $I_2 = \{2\}$ and the boundary signature is $\omega = (3, 1, 0)$. Then the partition function can be computed by applying Proposition 2.18. More precisely

$$Z = (1 + y_2x_2)(1 + y_2x_3)s_\omega(x_1, x_2, x_3).$$

Expanding $s_\omega(x_1, x_2, x_3)$, we obtain exactly (2.12).

For this case, as well as the previous cases, an alternative way to derive the partition function would be to apply Kasteleyn–Percus theory [24, 41] and write it as the determinant of a sign twisted, weighted, bipartite adjacency matrix of the graph, and get the same polynomials. But for this class of graphs, the machinery of symmetric functions gives a shorter derivation of the partition function.

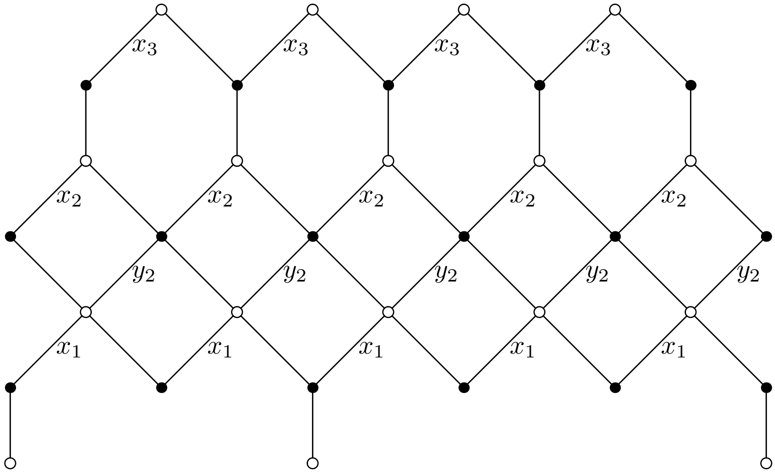


Figure 2.6. Contracting square-hexagon lattice with $N = 3$, $m = 3$, $\Omega = (1, 3, 6), (a_1, a_2, a_3) = (1, 0, 1)$.

2.6. Convergence of the free energy

We state now a result about the asymptotic behavior of the partition function Z of the dimer model on contracting square-hexagon graphs (defined in Proposition 2.18), subject to some regularity for the sequence of signatures describing the boundary of the graph. This partition function then grows exponentially with N^2, where N is the size of the graph, and the exponential growth rate

$$\lim_{N\to\infty} \frac{1}{N^2} \log Z$$

is called the *free energy*.

Let us introduce first some definition to state the hypotheses for the convergence result:

Let $\lambda \in \mathbb{GT}_N$ be a non-negative signature. We define the *counting measure* $m(\lambda)$ corresponding to λ as follows:

$$m(\lambda) = \frac{1}{N} \sum_{i=1}^{N} \delta\left(\frac{\lambda_i + N - i}{N}\right).$$

Let ρ be a probability measure on the set $\mathbb{GT}_N$ of all signatures. The push-forward of ρ with respect to the map $\lambda \mapsto m(\lambda)$ defines a random probability measure on $\mathbb{R}$ denoted by $m(\rho)$.

One natural setting for which one can prove that the free energy exists is when the corresponding sequence of signatures describing the boundary of our sequence of contracting square-hexagon graphs is *regular* [16], in the following sense:

Definition 2.22 ([16]). — *A sequence of signatures $\lambda(N) \in \mathbb{GT}_N$ is called* regular*, if there exists a piecewise continuous function $f(t)$ and a constant $C > 0$ such that*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N}\left|\frac{\lambda_j(N)}{N} - f\left(\frac{j}{N}\right)\right| = 0,$$
$$\text{and} \qquad \sup_{1\leqslant j\leqslant N}\left|\frac{\lambda_j(N)}{N} - f\left(\frac{j}{N}\right)\right| < C \qquad \text{for all } N \geqslant 1.$$

Since the renormalized logarithm of the Γ_i factors have a simple limit, the existence and the value of the free energy is determined by the existence of the logarithm of the renormalized Schur function.

For any positive integer $j \in \mathbb{N}$, let $\overline{j} = j \bmod n$.

Proposition 2.23 (Existence of the normalized free energy in the periodic case). — *Suppose that the following two conditions hold:*

- *$\{\lambda(N)\}_{N\in\mathbb{N}}$ is a regular sequence of signatures,*
- *as $N \to \infty$, $m(\lambda(N))$ converges weakly to a probability measure* $\mathbf{m}$ *on $\mathbb{R}$.*

Then:

(1) *For each N, $s_{\lambda(N)}(1,\dots,1) \geqslant 1$.*

(2) *For any $\beta = (\beta_1,\dots,\beta_n) \in \mathbb{R}^n$, and any sequence $\beta^{(N)} = \big((\beta_1^{(N)}, \dots, \beta_n^{(N)})\big)_N$ converging to β, the limit*

$$\lim_{N\to\infty}\frac{1}{N^2}\log\frac{s_{\lambda(N)}\left(\beta_{\overline{1}}^{(N)},\dots,\beta_{\overline{N}}^{(N)}\right)}{s_{\lambda(N)}(1,\dots,1)} \tag{2.13}$$

exists, and depends only on the limit β. In particular,

$$\lim_{N\to\infty}\frac{1}{N^2}\log\frac{s_{\lambda(N)}\left(\beta_{\overline{1}}^{(N)},\dots,\beta_{\overline{N}}^{(N)}\right)}{s_{\lambda(N)}(1,\dots,1)} = \lim_{N\to\infty}\frac{1}{N^2}\log\frac{s_{\lambda(N)}(\beta_{\overline{1}},\dots,\beta_{\overline{N}})}{s_{\lambda(N)}(1,\dots,1)}. \tag{2.14}$$

Proof. — By Corollary 2.20, $s_{\lambda(N)}(1,\dots,1)$ is the total number of dimer configurations on a contracting hexagonal lattice, in which the boundary configuration is given by $\lambda(N)$. Since there exists at least one dimer configuration on each such lattice, we obtain the first part.

The existence of the limit is a consequence of the successive application of two lemmas stated below: first Lemma 2.24 expressing the Schur function as a matrix integral, then Lemma 2.25 about limit of normalized logarithms of these integrals.

The convergence condition (2) implies that the empirical measures

$$\frac{1}{N}\sum_{i=1}^{N}\delta_{\beta^{(N)}_{\bar{i}}} \quad \text{and} \quad \frac{1}{N}\sum_{i=1}^{N}\delta_{\beta_{\bar{i}}}$$

converge to the same measure. Thus by Lemma 2.25, the renormalized logarithms of matrix integrals, and hence Schur functions, converge and have the same limit. □

Here are the two lemmas needed to conclude the proof of the previous proposition. The first one represents the Schur function as a matrix integral over the unitary group, the so-called Harish-Chandra–Itzykson–Zuber integral:

Lemma 2.24 ([19, 20]). — *Let $\lambda \in \mathbb{GT}^+_N$ be a non-negative signature, and let B be an $N \times N$ diagonal matrix given by*

$$B = \operatorname{diag}[\lambda_1 + N - 1, \ldots, \lambda_j + N - j, \ldots, \lambda_N + N - N]$$

Let $(a_1, a_2, \ldots, a_N) \in \mathbb{C}^N$, and let A be an $N \times N$ diagonal matrix given by

$$A = \operatorname{diag}[a_1, \ldots, a_N].$$

Then,

$$\frac{s_\lambda(e^{a_1}, \ldots, e^{a_N})}{s_\lambda(1, \ldots, 1)} = \prod_{1\leqslant i<j\leqslant N} \frac{a_i - a_j}{e^{a_i} - e^{a_j}} \int_{U(N)} e^{\operatorname{Tr}(U^* AUB)} \mathrm{d}U, \tag{2.15}$$

where $\mathrm{d}U$ is the Haar probability measure on the unitary group $U(N)$.

Note that Lemma 2.24 was originally proved when A is a Hermitian matrix with eigenvalues $a_1, a_2, \ldots, a_N$, hence $(a_1, a_2, \ldots, a_N) \in \mathbb{R}^N$. Since the right hand side of (2.15) depends only on the eigenvalues of A when A is a Hermitian matrix, the identity (2.15) is then true for $A = \operatorname{diag}[a_1, \ldots, a_N]$ with complex entries as well since both the left hand side and the right hand side in (2.15) are entire functions in $a_1, \ldots, a_N$.

The second lemma is about the convergence of the normalized logarithms of these integrals:

LEMMA 2.25 ([18, Theorem 1.1]). — *For an $N \times N$ Hermitian matrix A with eigenvalues $(a_1, \dots, a_N)$, we denote by*

$$\mathbf{m}_A^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{a_i}.$$

the spectral measure for A. Let $\{D_N\}_{N \in \mathbb{N}}$, $\{E_N\}_{N \in \mathbb{N}}$ be two sequences of diagonal, real-entry matrices, such that the three following conditions are satisfied:

- *there exists a compact subset $F \subset \mathbb{R}$ such that $\operatorname{supp} \mathbf{m}_{D_N}^N \subseteq F$ for all $N \in \mathbb{N}$;*
- *$\int x^2 \mathrm{d}\mathbf{m}_{E_N}^N$ is uniformly bounded with a bound independent of N;*
- *$\mathbf{m}_{E_N}^N$ and $\mathbf{m}_{D_N}^N$ converge weakly towards μ_E and μ_D, respectively.*

Then as $N \to \infty$,

$$\frac{1}{N^2} \log \int e^{N \operatorname{tr}(U D_N U^* E_N)} \mathrm{d}U,$$

has a limit depending only on μ_E and μ_D.

By Weyl's formula,

$$s_{\lambda(N)}(1, \dots, 1) = \prod_{1 \leqslant i < j \leqslant N} \frac{(\lambda_i(N) - i) - (\lambda_j(N) - j)}{j - i}.$$

From Proposition 2.23 we obtain that the existence of the free energy

$$\lim_{N \to \infty} \frac{1}{N^2} \log s_{\lambda(N)} \left(\beta_{\overline{1}}^{(N)}, \dots, \beta_{\overline{N}}^{(N)} \right)$$

under the assumption of Proposition 2.23 depends on the existence of

$$\lim_{N \to \infty} \frac{1}{N^2} \log \prod_{1 \leqslant i < j \leqslant N} \frac{(\lambda_i(N) - i) - (\lambda_j(N) - j)}{j - i},$$

which exists when the sequence of signatures is regular, and is given by the following integral

$$\iint_{0<x<y<1} \log\left(1 - \frac{f(y) - f(x)}{y - x}\right) \mathrm{d}x \mathrm{d}y.$$

3. Existence of Limit Shape

In this section, we study the convergence of the counting measure for the signature corresponding to the random dimer configuration on each row of the contracting square-hexagon lattice. We prove that the (random) moments of the counting measure converge to deterministic quantities, for which we give an explicit formula. This implies that the rescaled height function associated to the random perfect matching satisfies certain *law of large numbers*, and converges to a deterministic shape in the limit. This limit shape is also the solution of a variational problem, i.e., the unique deterministic function that maximizes the entropy; see [10].

We will need a more refined convergence, at order $\frac{1}{N}$ instead of $\frac{1}{N^2}$ for the free energy, and where a finite number of arguments of the Schur function (2.13) are allowed to vary.

If $(\lambda(N))$ is a regular sequence of signatures, then the sequence of counting measures $m(\lambda(N))$ converges weakly to a measure $\mathbf{m}$ with compact support. When the β_is are equal to 1, we have by Theorem 3.6 of [8] that there exists an explicit function $H_{\mathbf{m}}$, analytic in a neighborhood of 1, depending on the weak limit $\mathbf{m}$ such that

$$\lim_{N\to\infty} \frac{1}{N} \log\left(\frac{s_{\lambda(N)}(u_1,\dots,u_k,1,\dots,1)}{s_{\lambda(N)}(1,\dots,1)}\right) = H_{\mathbf{m}}(u_1)+\dots+H_{\mathbf{m}}(u_k), \tag{3.1}$$

and the convergence is uniform when $(u_1,\dots,u_k)$ is in a neighborhood of $(1,\dots,1)$. Precisely, $H_{\mathbf{m}}$ is constructed as follows: let $S_{\mathbf{m}}(z) = z + \sum_{k=1}^{\infty} M_k(\mathbf{m}) z^{k+1}$ be the moment generating function of the measure $\mathbf{m}$, where $M_k(\mathbf{m}) = \int x^k \mathrm{d}\mathbf{m}(x)$, and $S_{\mathbf{m}}^{(-1)}$ be its inverse for the composition. Let $R_{\mathbf{m}}(z)$ be the *Voiculescu R-transform* of $\mathbf{m}$ defined as

$$R_{\mathbf{m}}(z) = \frac{1}{S_{\mathbf{m}}^{(-1)}(z)} - \frac{1}{z}.$$

Then

$$H_{\mathbf{m}}(u) = \int_0^{\ln u} R_{\mathbf{m}}(t)\,\mathrm{d}t + \ln\left(\frac{\ln u}{u-1}\right). \tag{3.2}$$

In particular, $H_{\mathbf{m}}(1) = 0$, and

$$H'_{\mathbf{m}}(u) = \frac{1}{u S_{\mathbf{m}}^{(-1)}(\ln u)} - \frac{1}{u-1}.$$

Proposition 3.1. — *Assume that* $(\lambda(N) \in \mathbb{GT}_N,\ N = 1,2,\dots)$ *is a regular sequence of signatures, such that*

$$\lim_{N\to\infty} m(\lambda(N)) = \mathbf{m}.$$

Let $\mathbf{B}^{(N)} = \left(\beta_1^{(N)}, \beta_2^{(N)}, \dots, \beta_n^{(N)}\right) \in \mathbb{R}^n$, *such that one of the two following conditions holds:*

(1) *there exists a positive constant* $\alpha > 0$ *satisfying*

$$\lim_{N\to\infty} \max_{1\leqslant i\leqslant N} \left\{\left|\beta_i^{(N)} - 1\right|\right\} e^{N\alpha} = 0, \tag{3.3}$$

(2) *there exists a constant $C > 0$ such that*

$$|\lambda_1(N)| \leqslant C; \text{ and } \lim_{N\to\infty} \max_{1\leqslant i\leqslant N} \left\{\left|\beta_i^{(N)} - 1\right|\right\} N = 0 \tag{3.4}$$

for each $1 \leqslant i \leqslant n$.

Then for each fixed $k \in \mathbb{N}$, there is a small open complex neighborhood of $(1,\dots,1) \in \mathbb{C}^k$, such that for N large enough, $s_{\lambda(N)}\Big(u_1,\dots,u_k,\beta_{k+1}^{(N)},\dots,\beta_N^{(N)}\Big)$ is non-zero for $(u_1,\dots,u_k)$ in this neighborhood, and the following convergence occurs uniformly in this neighborhood:

$$\lim_{N\to\infty} \frac{1}{N} \log\left(\frac{s_{\lambda(N)}\left(u_1,\dots,u_k,\beta_{k+1}^{(N)},\dots,\beta_N^{(N)}\right)}{s_{\lambda(N)}\left(\beta_1^{(N)},\dots,\beta_N^{(N)}\right)}\right) = H_{\mathbf{m}}(u_1) + \cdots + H_{\mathbf{m}}(u_k).$$

Proof. — To simplify notation, we will simply write $s_\lambda(\mathbf{u},\mathbf{1})$ or $s_\lambda(\mathbf{u},\boldsymbol{\beta})$ for the following quantities $s_{\lambda(N)}(u_1,\dots,u_k,1,\dots,1)$ or $s_{\lambda(N)}\Big(u_1,\dots,u_k,\beta_{k+1}^{(N)},\dots,\beta_N^{(N)}\Big)$.

Note that

$$\log\left(\frac{s_\lambda(\mathbf{u},\boldsymbol{\beta})}{s_\lambda(\boldsymbol{\beta},\boldsymbol{\beta})}\right) = D_{N,1} + D_{N,2} + D_{N,3},$$

where

$$\begin{gathered} D_{N,1} = \log\left(\frac{s_\lambda(\mathbf{u},\boldsymbol{\beta})}{s_\lambda(\mathbf{u},\mathbf{1})}\right), \quad D_{N,2} = \log\left(\frac{s_\lambda(\mathbf{u},\mathbf{1})}{s_\lambda(\mathbf{1})}\right), \\ D_{N,3} = \log\left(\frac{s_\lambda(\mathbf{1},\mathbf{1})}{s_\lambda(\boldsymbol{\beta},\boldsymbol{\beta})}\right). \end{gathered} \tag{3.5}$$

By Theorem 3.6 of [8], we have

$$\lim_{N\to\infty} \frac{1}{N} D_{N,2} = H_{\mathbf{m}}(u_1) + \cdots + H_{\mathbf{m}}(u_k),$$

and the convergence is uniform in an open complex neighborhood of $(1,\dots,1)$. Let us first consider the term $D_{N,3}$, where there is no dependency in $\mathbf{u}$. Writing that the Schur function s_λ is a sum of $s_\lambda(\mathbf{1})$ homogeneous monomials of degree $|\lambda|$, we obtain that

$$|s_\lambda(\boldsymbol{\beta}) - s_\lambda(\mathbf{1})| \leqslant C|\lambda| \left(\sup |\beta_j^{(N)}|\right)^{|\lambda|} \max |\beta_j^{(N)} - 1|.$$

When Hypothesis (3.3) is satisfied, one has $|\lambda| = O(N^2)$ and $|\beta_i^{(N)} - 1| = O(e^{-N\alpha}) = o(N^{-2})$ Therefore, the ratio $s_\lambda(\boldsymbol{\beta})/s_\lambda(\mathbf{1})$ goes to 1 if $\beta_j^{(N)} = 1 + o(N^{-2})$ (uniformly in j) and $|\lambda| = O(N^{-2})$, which is the case under

Hypothesis (3.3). Therefore its logarithm converges to 0. The conclusion can be checked in a similar fashion for the second hypothesis (3.4).

The convergence for $D_{N,1}$ requires a better control of the $\beta_j^{(N)}$. Let us show that uniformly for $\mathbf{u} = (u_1, \ldots, u_k)$ in a small enough open neighborhood of 1, the ratio

$$\frac{s_\lambda(\mathbf{u}, \boldsymbol{\beta}) - s_\lambda(\mathbf{u}, \mathbf{1})}{s_\lambda(\mathbf{1}, \mathbf{1})}$$

converges to 0 as N goes to infinity under the hypothesis (3.3). The case (3.4) is similar.

We use the following formula for Schur functions:

$$s_\lambda(\mathbf{u}, \boldsymbol{\beta}) = \sum_{\mu \preceq \lambda} s_\mu(\mathbf{u}) s_{\lambda\setminus\mu}(\boldsymbol{\beta})$$

to write

$$s_\lambda(\mathbf{u}, \boldsymbol{\beta}) - s_\lambda(\mathbf{u}, \mathbf{1}) = \sum_{\mu} s_\mu(\mathbf{u}) \left(s_{\lambda\setminus\mu}(\boldsymbol{\beta}) - s_{\lambda\setminus\mu}(\mathbf{1}) \right)$$

Since s_μ has only k parameters, the only signatures μ contributing to the sum have at most k parts, which are at most equal to $\lambda(N)_1 \leqslant cN$ for some constant $c > 0$.

Fix such a signature μ. The skew Schur function $s_{\lambda\setminus\mu}$ is the sum of monomials indexed by skew semi-standard Young tableaux of shape $\lambda \setminus \mu$. There are precisely $s_{\lambda\setminus\mu}(\mathbf{1})$ of them. And each such monomial has degree $|\lambda \setminus \mu|$. Since all the $\beta_j^{(N)}$ are at distance at most $O(e^{-\alpha N})$ of 1, then the difference of each such monomial between its evaluation at $\mathbf{1}$ and at $\boldsymbol{\beta}$ can be bounded in absolute value, for some positive constant C uniform in N, by

$$C|\lambda \setminus \mu| \exp\left(C|\lambda \setminus \mu| e^{-\alpha N}\right) e^{-\alpha N} = O(N^2 e^{-\alpha N})$$

uniformly in N and μ. Moreover, $|s_\mu(\mathbf{u})| \leqslant s_\mu(|\mathbf{u}|)$. Putting these pieces together, we have

$$\left| \frac{s_\lambda(\mathbf{u}, \boldsymbol{\beta}) - s_\lambda(\mathbf{u}, \mathbf{1})}{s_\lambda(|\mathbf{u}|, \mathbf{1})} \right| \leqslant \sum_{\mu} r^{cN} s_\mu(\mathbf{1}) \frac{s_{\lambda\setminus\mu}(\mathbf{1})}{s_\lambda(|\mathbf{u}|, \mathbf{1})} CN^2 e^{-\alpha N} \leqslant CN^2 e^{-\alpha N}.$$

By [8, Theorem 3.6], the module of the ratio $s_\lambda(\mathbf{u}, 1)/s_\lambda(|\mathbf{u}|, \mathbf{1})$ is bounded from below uniformly in N by e^{-AN} for some $A > 0$, which tends to 0 as the radius r of the neighborhood around 1 goes to 0. Therefore, this radius r can be chosen close enough to 1 so that

$$\frac{s_\lambda(\mathbf{u}, \boldsymbol{\beta}) - s_\lambda(\mathbf{u}, \mathbf{1})}{s_\lambda(\mathbf{u}, \mathbf{1})}$$

tends to his quantity goes to 0. Adding 1 and taking the logarithm gives exactly the quantity $D_{N,1}$ which thus also tends to 0 as N goes to ∞. $\square$

3.1. The Schur generating function and moments of counting measures

Let

$$V(u_1,\dots,u_N)=\prod_{1\leqslant i<j\leqslant N}(u_i-u_j)$$

be the Vandermonde determinant with respect to variables $u_1,\dots,u_N$. Introduce the family $(\mathcal{D}_k)$ of differential operators acting on symmetric functions f with variables $u_1,\dots,u_N$ as follows:

$$\mathcal{D}_k f=\frac{1}{V}\left(\sum_{i=1}^{N}\left(u_i\frac{\partial}{\partial u_i}\right)^k\right)(V\cdot f). \tag{3.6}$$

For every $\lambda\in\mathbb{GT}_N$, the Schur function $s_\lambda(u_1,\dots,u_N)$ is an eigenfunction of $\mathcal{D}_k$, associated with the eigenvalue $\sum_i(\lambda_i+N-i)^k$, see [6, Proposition 4.3].

Therefore, one can use, in analogy with monomials for integer-valued random variables, a generating function which would give, by application of these differential operators, information about the moments of our random distribution of dimers on a given row of the graph. We thus adapt slightly the definition of *Schur generating functions*, introduced by Bufetov and Gorin [6] to fit our needs:

Definition 3.2. — *Let* $\mathbf{B}=(\beta_1,\dots,\beta_N)\in\mathbb{C}^N$. *Let* ρ *be a probability measure on* $\mathbb{GT}_N$. *The* Schur generating function $\mathcal{S}_{\rho,\mathbf{B}}(u_1,\dots,u_N)$ *with respect to parameters* $\mathbf{B}$ *is the symmetric Laurent series in* $(u_1,\dots,u_N)$ *given by*

$$\mathcal{S}_{\rho,\mathbf{B}}(u_1,\dots,u_N)=\sum_{\lambda\in\mathbb{GT}_N}\rho(\lambda)\frac{s_\lambda(u_1,\dots,u_N)}{s_\lambda(\beta_1,\dots,\beta_N)}.$$

When $\mathbf{B}=(1,\dots,1)$, this definition is the one found in Definition 4.4 of [6] and Definition 3.1 of [8].

The following result states that asymptotic behavior of the Schur generating function for random signatures implies the convergence for the associated random counting measures, in the same line as [8].

Lemma 3.3 ([6, Theorem 5.1]). — *Suppose that* $\mathbf{B}^{(N)}=\left(\beta_1^{(N)},\dots,\beta_N^{(N)}\right)$ *satisfies the condition described in* (3.3). *Let* $(\rho_N)_{N\geqslant1}$ *be a sequence of measures such that for each* N, ρ_N *is a probability measure on* $\mathbb{GT}_N$, *and for* *every j, the following convergence holds uniformly in a complex neighborhood of $(1,\dots,1)\in\mathbb{C}^j$*

$$\lim_{N\to\infty}\frac{1}{N}\log\mathcal{S}_{\rho_N,\mathbf{B}^{(N)}}\left(u_1,\dots,u_j,\beta^{(N)}_{j+1},\dots,\beta^{(N)}_{N}\right)\\ = Q(u_1)+\cdots+Q(u_j), \tag{3.7}$$

with Q an analytic function in a neighborhood of 1. Then the sequence of random measures $(m(\rho_N))_{N\geqslant 1}$ converges as $N\to\infty$ in probability in the sense of moments to a deterministic measure $\mathbf{m}$ on $\mathbb{R}$, whose moments are given by

$$\int_{\mathbb{R}} x^j\mathbf{m}(\mathrm{d}x)=\sum_{l=0}^{j}\frac{j!}{l!(l+1)!(j-l)!}\,\frac{\partial^l}{\partial u^l}\left(u^jQ'(u)^{j-l}\right)\bigg|_{u=1}.$$

Proof. — The proof is a direct adaption of the proof of Theorem 5.1 of [6]. The fact that Schur functions are eigenfunctions of the differential operators $\mathcal{D}_k$ allows one to rewrite the moments of the (random) moments of the counting measure associated to a random signature with distribution ρ on $\mathbb{GT}_N$ with Schur generating function $\mathcal{S}_{\rho,\mathbf{B}}$ as follows:

$$\mathbb{E}\left(\int_{\mathbb{R}}x^k m(\rho)(\mathrm{d}x)\right)^m\\ =\frac{1}{N^{m(k+1)}}(\mathcal{D}_k)^m\mathcal{S}_{\rho,\mathbf{B}}(u_1,\dots,u_N)|_{(u_1,\dots,u_N)=(\beta_1,\dots,\beta_N)}.$$

The key is then to notice that since the convergence is uniform and Q is analytic, then necessarily, $Q(1)=0$, as one can readily check from (3.7) for $j=1$. □

The Boltzmann probability measure from Proposition 2.18 on the set of perfect matchings of a contracting square-hexagon lattice $\mathcal{R}(\Omega,\check{a})$ induces a measure on the set of all possible configurations of $N-\lfloor\frac{k-1}{2}\rfloor$ V-edges in the k-th row, for $k=1,\dots 2N$, counting from the bottom. We can also think of it as a measure ρ^k on the signatures $\lambda\in\mathbb{GT}_{N-\lfloor\frac{k-1}{2}\rfloor}$.

Lemma 3.4. — *We have the following expressions for the measures ρ^k, depending on the parity of k:*

(1) *Assume that $k = 2t+1$, for $t = 0, 1, \dots, N-1$, then for $\lambda \in \mathbb{GT}_{N-t}$*

$$\rho^k(\lambda) = \sum_{\substack{\nu^{(a)} \in \mathbb{GT}_a, (N-t+1 \leqslant a \leqslant N);\\ \mu^{(b)} \in \mathbb{GT}_b, (N-t+1 \leqslant b \leqslant N-1)}} \prod_{i \in I_2 \cap \{1,2,\dots,t\}} \mathrm{st}_{B_i}(\mu^{(N-i+1)} \to \nu^{(N-i+1)}) \times \prod_{j=1}^{t} \left[\mathrm{pr}_{C_j}(\nu^{(N-j+1)} \to \mu^{(N-j)})\right]$$

where $\mu^{(N)} = \omega$, and $\mu^{(N-t)} = \lambda$.

(2) *Assume that $k = 2t+2$, for $t = 0, 1, \dots, N-1$, then for $\lambda \in \mathbb{GT}_{N-t}$*

(a) *If $t+1 \in I_2$, then*

$$\rho^k(\lambda) = \sum_{\mu^{(N-t)} \in \mathbb{GT}_{N-t}} \rho^{k-1}(\mu^{(N-t)})\, \mathrm{st}_{B_{t+1}}(\mu^{(N-t)} \to \lambda).$$

(b) *If $t+1 \notin I_2$, then $\rho^k(\lambda) = \rho^{k-1}(\lambda)$.*

Proof. — In Part (1), the expression for ρ^k is obtained by taking the formula for the probability of a configuration (2.10) and summing over all the intermediate signatures corresponding to rows below or above the kth one. The Markovian structure of the probability measure implies that when fixing the kth signature, the parts below and above are independent. But, when we fix the kth row to correspond to a signature λ, all the signatures above correspond to a contracting square hexagon lattice with a boundary row given by λ. Thus the sum over the signatures above level k is equal to 1. Part (2) follows from the definition of st and ρ^k. □

In order to study the limit shape, we make the following assumption of periodicity for the graph:

Assumption 3.5. — *The square-hexagonal lattice $\mathrm{SH}(\check{a})$ is periodic with period $2n$. More precisely, for any integer $i, j \in \mathbb{N}$, the jth row and the $(j+2ni)$th row in $\mathrm{SH}(\check{a})$ coincides and have the same edge weights.*

Under the periodic assumption 3.5, the sets I_1 and I_2 defined in Definition 2.4 are periodic, so $m \in I_2$ if and only if $\overline{m}(:= m \bmod n) \in I_2$. Moreover, all the independent edge weights are the x_i's for $i = 1, \dots, n$, for the NE-SW edges joining the $2i$th and $(2i+1)$th row (mod $2n$) and the y_m, for $m \in I_2 \cap \{1, 2, \dots, n\}$, for the NE-SW edges, joining the $(2i-1)$th and $(2i)$ rows. See Figure 2.6. Obviously $m \in I_2$ if and only if $\overline{m} \in I_2$.

Lemma 3.6. — *For any k between 0 and $2N-1$, define $t = \lfloor k/2 \rfloor$, and let*

$$X^{(N-t)} = (x_{\overline{t+1}}, \dots, x_{\bar{N}}), \quad \text{and} \quad Y^{(t)} = (x_{\bar{1}}, \dots, x_{\bar{t}}).$$

Then the generating Schur function $\mathcal{S}_{\rho^k,X^{(N-t)}}$ *is given by:*

$$\mathcal{S}_{\rho^k,X^{(N-t)}}(u_1,\dots,u_{N-t}) = \frac{s_\omega\left(u_1,\dots,u_{N-t},Y^{(t)}\right)}{s_\omega(X^{(N)})} \prod_{i\in\{1,\dots,t\}\cap I_2} \prod_{j=1}^{N-t} \left(\frac{1+y_{\bar{i}}u_j}{1+y_{\bar{i}}x_{\overline{t+j}}}\right).$$

Proof. — We prove the case when $k=2t+1$ is odd here; the case when k is even can be proved similarly.

Notice that for $1\leqslant i\leqslant N$, $\mathrm{pr}_{C_i}(\,\cdot\,)=\mathrm{pr}_{B_i}(\,\cdot\,)$, by the expression (2.2) of pr, and the expression of (2.7), (2.6) for B_i, C_i, respectively.

By Definition 3.2, we have

$$\mathcal{S}_{\rho^k,X^{(N-t)}}(u_1,\dots,u_{N-t}) = \sum_{\lambda\in\mathbb{GT}_{N-t}} \rho^k(\lambda)\frac{s_\lambda(u_1,\dots,u_{N-t})}{s_\lambda(x_{\overline{t+1}},\dots,x_{\bar{N}})}. \tag{3.8}$$

Plugging the expression of $\rho^k(\lambda)$ in Lemma 3.4(1) into (3.8), and applying (2.4), (2.5), and the previous remark about $\mathrm{pr}_{C_i}=\mathrm{pr}_{B_i}$ sequentially, we obtain the lemma. □

Proposition 3.7. — *Assume that the sequence of signatures* $(\omega(N))_N$ *corresponding the first row is regular, and* $\lim_{N\to\infty} m[\omega(N)]=\mathbf{m}_\omega$*. Assume that the edge weights* y_i *for* $i\in\{1,\dots,n\}\cap I_2$ *are independent of* N*, while the edge weights* $x_1^{(N)},x_2^{(N)},\dots,x_n^{(N)}$ *satisfy* (3.3) *or* (3.4)*. Let* (k_N) *be a sequence of nonnegative integers such that* $\lim_{N\to\infty}\frac{k_N}{2N}=\kappa\in[0,1]$*. Then, the sequence of random measures* $m(\rho^{k_N}(N))$ *converges as* $N\to\infty$ *in probability, in the sense of moments to a deterministic measure* $\mathbf{m}^\kappa$ *in* $\mathbb{R}$*, whose moments are given by*

$$\int_{\mathbb{R}} x^j\mathbf{m}^\kappa(\mathrm{d}x) = \frac{1}{2(j+1)\pi\mathbf{i}}\oint_1 \frac{\mathrm{d}z}{z}\left(zQ'(z)+\frac{z}{z-1}\right)^{j+1},$$

where

$$Q(u)=\frac{1}{1-\kappa}H_{\mathbf{m}_\omega}(u)+\frac{\kappa}{(1-\kappa)n}\sum_{l=\{1,2,\dots,n\}\cap I_2}\log\left(\frac{1+y_lu}{1+y_l}\right) \tag{3.9}$$

and the integration goes over a small positively oriented contour around 1.

Proof. — From Proposition 3.1 it follows that

$$\lim_{N\to\infty}\frac{1}{N}\log\left(\frac{s_{\omega(N)}\left(u_1,\dots,u_j,x_{\bar{1}}^{(N)},\dots,x_{\overline{N-j}}^{(N)}\right)}{s_{\omega(N)}\left(x_{\bar{1}}^{(N)},\dots,x_{\bar{N}}^{(N)}\right)}\right) = H_{\mathbf{m}_\omega}(u_1)+\dots+H_{\mathbf{m}_\omega}(u_j).$$

Let us look at what is happening in the kth row, for $k = k(N) = 2N(\kappa + o(1))$. Let ρ_N^k be the probability measure of perfect matchings along the kth row, given that the first row has a configuration $\omega(N)$. The kth row has $(N - t) = \left(N - \lfloor \frac{k-1}{2} \rfloor\right)$ V-squares. By Lemma 3.6, we have

$$
\begin{aligned}
(3.10)\quad & \lim_{N\to\infty} \frac{1}{(1-\kappa)N} \log\left(\mathcal{S}_{\rho_N^k}\left(u_1,\dots,u_j,x_{\overline{t+1}}^{(N)},\dots,x_{\overline{N-j}}^{(N)}\right)\right)\\
&= \lim_{N\to\infty} \frac{1}{(1-\kappa)N} \log\left(\frac{s_{\omega(N)}\left(u_1,\dots,u_j,x_{\overline{1}}^{(N)},\dots,x_{\overline{N-j}}^{(N)}\right)}{s_{\omega(N)}(x_{\overline{1}}^{(N)},\dots,x_{\overline{N}}^{(N)})}\right.\\
&\qquad\qquad \left.\times \prod_{l\in\{1,2,\dots,t/t+1\}\cap I_2} \prod_{i=1}^{j} \frac{1+y_{\overline{l}}u_i}{1+y_{\overline{l}}x_{\overline{N-j+i}}^{(N)}}\right)\\
&= \sum_{i=1}^{j}\left(\frac{1}{1-\kappa}H_{\mathbf{m}_\omega}(u_i) + \frac{\kappa}{(1-\kappa)n}\sum_{l=\{1,2,\dots,n\}\cap I_2} \log\left(\frac{1+y_l u_i}{1+y_l}\right)\right)\\
&= \sum_{i=1}^{j} Q(u_i).
\end{aligned}
$$

where in the last identity we use the assumption (3.3) or (3.4), and n is a fixed finite positive integer relating to the size of the fundamental domain.

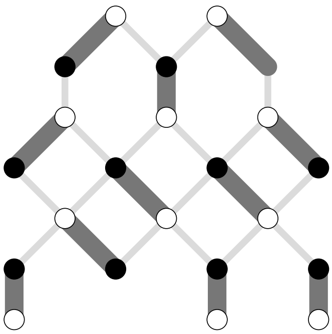
Then the proposition follows from Lemma 3.3 and [8, (3.14)]. The link between the expression of moments in Lemma 3.3 and this proposition is obtained by an explicit evaluation of the integral by residues. □

3.2. Height function

Let $\mathcal{R}(\Omega(N), \check{a})$ be a contracting square-hexagon lattice.

As for any bipartite planar graph, dimer configurations can be encoded by a height function with values on the faces of the graph. A convenient way to construct a height function (which will help also for the matter of discussing the scaling limit for these graphs), is to see a contracting square-hexagon lattice, as in fact a subgraph of the square lattice, by drawing the *vertical* edges of the hexagonal rows as diagonal NW-SE (the missing diagonal NE-SW diagonal on these rows correspond to the fusion of two unit squares to make hexagonal faces). The vertices of the contracting square-hexagon graph are thus on the sublattice $\frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z}$ with coordinates (i, j) satisfying $i + j \in \mathbb{Z}$.

The height function we will consider is then defined on the sublattice where $i+j \in \frac{1}{2}+\mathbb{Z}$, applying the same rule as for domino tilings [44]: the height increases (resp. decreases) by 1 when going counterclockwise around a white vertex (resp. black vertex), i.e., for j half-integer (resp. integer) as long as a dimer is not crossed. For definiteness, we fix the height to be 0 at the square face $(\frac{1}{2},0)$. When keeping $i=\frac{1}{2}$ and increasing j by one, we jump over a black vertex with two edges, with only one of them being a dimer. Then the height increases by 2. See Figure 3.1. Note that the height function under this definition is a constant multiple of the height function under the classical definition in [28].



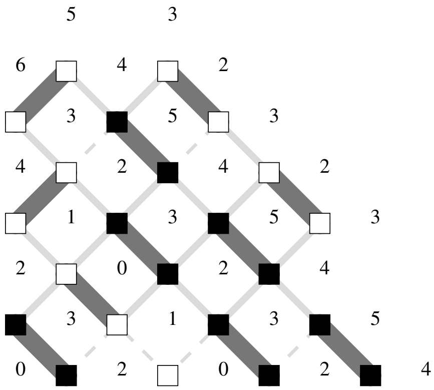


Figure 3.1. Two representations of a dimer configuration on a contracting square hexagon graph. Left: usual embedding, with the bipartite coloring of vertices. Right: as a subgraph of the square lattice, with the associated height function, and particles associated to Maya diagrams encoding each row.

The value of the height function can be related directly to the counting measures of the signatures associated to rows of the graph: the height at position (i,j) is given by the following formula:

$$h(i,j) = 2j-1+2\,\mathrm{Card}\{\blacksquare \text{ at the left of } i\} - 2\,\mathrm{Card}\{\square \text{ at the left of } i\}.$$

If λ is the signature of the $2j$th row (with ordinate j), the numbers $\lambda_k - k + N - 2j + 1$ are the positions of the $\blacksquare$, which is (up to a small shift), $N-2\lfloor j\rfloor$ times the atoms of the counting measure for this signature.

Therefore, if we define $N_j = N - \lfloor j \rfloor$, for j-half integer, the height along the $2j$th row, encoded by the signature $\mu^{(N_j)}$, is given by the following formula:

$$h(i,j) = 2j - 1 + 2N_j \int_{[0,\frac{i}{2N_j})} (2\mathrm{d}m(\mu^{(N_j)}) - \mathrm{d}x).$$

The convergence of the counting measures from Proposition 3.7 implies directly the following theorem about the convergence in probability of the rescaled height:

Theorem 3.8 (Law of large numbers for the height function). — *Consider $N \to \infty$ asymptotics such that all the dimensions of a contracting square-hexagon lattice $\mathcal{R}(\Omega(N), \check{a})$ linearly grow with N. Assume that*

- *the sequence of signatures $(\omega(N))$ corresponding to the first row is regular, and*
$$\lim_{N\to\infty} m[\omega(N)] = \mathbf{m}_\omega$$
in the weak sense; and
- *the edge weights are assigned as in Assumption 2.1 (see Figure 2.6 for an example) and satisfy the Assumption 3.5 of periodicity, such that for each $1 \leqslant i \leqslant n$ and $i \in I_2$, $y_i > 0$ are fixed and independent of N, while for $1 \leqslant i \leqslant n$ $x_i^{(N)} > 0$ satisfies (3.3) or (3.4).*

Let ρ_N^k be the measure on the configurations of the kth row, and let $\kappa \in (0,1)$, such that $j = [\kappa N] + \frac{1}{2}$, Then $m(\rho_N^k)$ converges to $\mathbf{m}^\kappa$ in probability as $N \to \infty$, and the moments of $\mathbf{m}^\kappa$ is given by Proposition 3.7.

Define

$$\mathbf{h}(\chi,\kappa) := 2\kappa - 2\chi + 4(1-\kappa)\int_0^{\frac{\chi}{1-\kappa}} \mathrm{d}\mathbf{m}^\kappa.$$

Then the random height function h_M associated to a random perfect matching M, satisfies the following law of large numbers: as N goes to infinity, its scaling by a factor N^{-1}

$$(\chi,\kappa) \longmapsto \frac{h_M([\chi N],[\kappa N]+\frac{1}{2})}{N}$$

converges uniformly, in probability, to the deterministic function $\mathbf{h}(\chi,\kappa)$, where χ,κ are new continuous parameters of the domain.

Remark 3.9. — The continuous variables (χ,κ) lie in a region R of the plane obtained from $\mathcal{R}(\Omega(N),\check{a})$ by translating each row from the second to the left such that the leftmost vertex of each row are on the same vertical line; then rescaled by $\frac{1}{N}$, and taking the limit as $N \to \infty$. When $\mathcal{R}(\Omega(N),\check{a})$ is a square grid, R is a rectangle; otherwise R is a trapezoid with two right angles on the left.

The convergence also occurs on the other sublattice, when j is integer, and i half integer, due to the fact that the discrete height function is Lipschitz.

4. Density of the Limit Measure

We work in this section with the same hypotheses as in Theorem 3.8. We obtain an explicit formula to compute the density of the limit of the random counting measure corresponding to the random signatures for dimer configurations on a row of the contracting square-hexagon lattice. This formula for the density of limit measure will be used in the next section to obtain the *frozen boundary*, i.e., the frontier between *frozen* regions where the density of the limit measure is 0 or 1, and a *temperate* region where the density lies strictly between 0 and 1.

Under the assumptions above, it is not hard to check that all the measures $\mathbf{m}^{\kappa}$, $\kappa \in (0,1)$ have compact support, and that they is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ with a density taking values in $[0,1]$.

Recall that the *Stieltjes transform* of a compactly supported measure $\mathbf{m}$ is defined by

$$\mathrm{St}_{\mathbf{m}}(t) = \int_{\mathbb{R}} \frac{\mathbf{m}(\mathrm{d}s)}{t-s}, \tag{4.1}$$

for $t \in \mathbb{C} \setminus \mathrm{Support}(\mathbf{m})$, which has an expansion as a series in t in a neighborhood of infinity whose coefficients are the moments of the measure $\mathbf{m}$: if the $M_j(\mathbf{m})$ is jth moment of $\mathbf{m}$, then for t large enough:

$$\mathrm{St}_{\mathbf{m}}(t) = \sum_{j=0}^{\infty} M_j(\mathbf{m}) t^{-j-1}.$$

By Proposition 3.7, we know in principle for any $\kappa \in (0,1)$, all the moments of the limiting measure $\mathbf{m}^{\kappa}$, expressed in terms of the function Q from Equation (3.9), so we can have an expression of the Stieltjes transform of $\mathbf{m}^{\kappa}$. However, Q depends on $H_{\mathbf{m}_\omega}$ which is itself expressed in terms of the Stieltjes transform $\mathbf{m}_\omega$. Indeed, for any measure $\mathbf{m}$, the function $H_{\mathbf{m}}$ is related to the Stieltjes transform of $\mathbf{m}$ by the following relation:

$$H'_{\mathbf{m}}(z) = \frac{\mathrm{St}_{\mathbf{m}}^{(-1)}(\log(z))}{z} - \frac{1}{z-1}.$$

See also Equation (3.2).

Introducing an additional variable t such that $\mathrm{St}_{\mathbf{m}}(t) = \log(z)$, one can then write for $\kappa \in (0,1)$:

$$\begin{aligned}F_\kappa(z,t) &:= zQ'(z) + \frac{z}{z-1} \\ &= \frac{z}{1-\kappa}\left(\frac{t}{z} - \frac{1}{z-1} + \frac{\kappa}{n}\sum_{i\in I_2\cap\{1,2,\dots,n\}}\frac{y_i}{1+y_iz}\right) + \frac{z}{z-1}.\end{aligned} \tag{4.2}$$

As a consequence, injecting the expression of the moments of the limiting measure into the definition of the Stieltjes transform, one gets an implicit equation to be solved: for any $x \in \mathbb{C}$, finding $(z,t) \in (\mathbb{C}\setminus\mathbb{R}_-)\times(\mathbb{C}\setminus \mathrm{Support}(\mathbf{m}_\omega))$ such that

$$\begin{cases} F_\kappa(z,t) = x \\ \mathrm{St}_{\mathbf{m}_\omega}(t) = \log(z), \end{cases} \tag{4.3}$$

allows one to express $\mathrm{St}_{\mathbf{m}^\kappa}$: let $x \mapsto z^\kappa(x)$ be the composite inverse of

$$u : z \longmapsto F_\kappa\left(z, \mathrm{St}_{\mathbf{m}_\omega}^{(-1)}(\log z)\right).$$

Note that $z^\kappa(x)$ is a uniformly convergent Laurent series in x when x is in a neighborhood of infinity, and

$$z^\kappa\left(F_\kappa\left(z, \mathrm{St}_{\mathbf{m}_\omega}^{(-1)}(\log z)\right)\right) = z.$$

See [8, Section 4.1].

The following identity holds when x is in a neighborhood of infinity

$$\mathrm{St}_{\mathbf{m}^\kappa}(x) = \log(z^\kappa(x)). \tag{4.4}$$

Indeed, by Proposition 3.7, the j-th moment of $\mathbf{m}^\kappa$ is given by

$$M_j(\mathbf{m}^\kappa) = \frac{1}{2(j+1)\pi\mathbf{i}}\oint_1 \frac{\mathrm{d}z}{z}[F_\kappa(z, \mathrm{St}_{\mathbf{m}_\omega}^{(-1)}(\log z))]^{j+1},$$

where the integral is along a small counterclockwise contour winding once around 1. Then the identity (4.4) follows from the same computation as in the proof of Lemma 4.1 in [8], by performing an integration by parts and a change of variable from z to u.

The first equation of the system (4.3) with F_κ from (4.2) is linear in t for given x and z, which gives with $c_i = \frac{1}{y_i}$ the value of t, as a function of

z, κ, and x:

$$t = t(z,\kappa,x) = x(1-\kappa) + \frac{\kappa z}{z-1} - \frac{\kappa z}{n} \sum_{i \in I_2 \cap \{1,2,\dots,n\}} \frac{1}{c_i + z} \tag{4.5}$$
$$= x(1-\kappa) + \kappa\frac{r}{n} + \kappa\left(\frac{1}{z-1} + \frac{1}{n}\sum_{i \in I_2 \cap \{1,\dots,n\}} \frac{c_i}{z+c_i}\right),$$

where

$$r = n - \mathrm{Card}(I_2) = \mathrm{Card}\{1,\dots,n\} \cap I_1. \tag{4.6}$$

For a given value $y \in \mathbb{R}$, and fixed x (and κ), we investigate properties of the complex numbers z such that $t(z,\kappa,x) = y$. In particular, we have the following:

Lemma 4.1. — *Let $c_i > 0$, for $i \in I_2 \cap \{1,2,\dots,n\}$. Let $\kappa \in (0,1)$, and $x, y \in \mathbb{R}$. Then the following equation in z*

$$t(z,\kappa,x) = y \tag{4.7}$$

has $m+1$ roots on the Riemann sphere $\mathbb{C} \cup \{\infty\}$, where m is the number of distinct values of c_i, and all these roots are real and simple.

Proof. — Let $0 < \gamma_1 < \dots < \gamma_m$ be all the possible distinct values for the c_i, and $n_1, \dots, n_m$ be their respective multiplicities among the c_i's. Define

$$H(z;x,y) = t(z,\kappa,x) - y = K + \kappa\left(\frac{1}{z-1} + \frac{1}{n}\sum_{j=1}^{m} \frac{n_j\gamma_j}{z+\gamma_j}\right) \tag{4.8}$$

where $K = x(1-\kappa) - y + \kappa\frac{r}{n}$, with r as in Equation (4.6). When writing $H(z;x,y)$ as a single rational fraction by bringing all the terms onto the same polynomial denominator (of degree $m+1$), the polynomial on the numerator has degree at most $m+1$. So there are at most $m+1$ roots in $\mathbb{C}$ (and exactly $m+1$ if we add roots at infinity). Notice that the denominator does not depend on x and y, but just on the γ_j's.

Moreover, each factor of the form $\frac{1}{z-b}$ with $b = -\gamma_j$ or 1 is a decreasing function of z on any interval where it is defined. As a consequence, on each of the intervals $(-\gamma_{j+1}, -\gamma_j)$, $j = 1, \dots, m-1$ and $(-\gamma_1, 1)$, H realizes a bijection with $\mathbb{R}$. In particular, the equation $H(z;x,y)$ has a unique solution in every such interval. It is also decreasing on $(\infty, -\gamma_m)$ and $(1, +\infty)$. Since the limits of $H(z;x,y)$ when z goes to $\pm\infty$ coincide, and

$$\lim_{z\to 1+} H(z;x,y) = +\infty, \qquad \lim_{z\to -\gamma_m^-} H(z;x,y) = -\infty,$$

the equation $H(z;x,y)=0$ has a unique solution in $(-\infty,-\gamma_m)\cup(1,+\infty)\cup\{\infty\}$, which is in fact infinite if and only if the limits of H at infinity is zero, that is, when $K=0$. This gives thus $m+1$ real roots (with possibly one at infinity). ☐

Remark 4.2. — Increasing the value of y translates downward the graph of the function $z\in\mathbb{R}\mapsto H(z;x,y)$. Since $H(z;x,y)$ is decreasing in any interval of definition, the roots present in the bounded intervals decrease. The one in $(-\infty,-\gamma_m)\cup(1,+\infty)\cup\{\infty\}$ moves also to the left, and if it started in $\mathbb{R}_-$, when it reaches $-\infty$, it jumps to the right part of $(1,+\infty)$ and then continues to decrease. In particular, it means that if $y<y'$, the respective roots $z_1<\cdots<z_{m+1}$ and $z'_1<\cdots<z'_{m+1}$ are interlaced:

- if $y<y'<x(1-\kappa)+\frac{\kappa r}{n}$,

$$z'_1<z_1<-\gamma_m<z'_2<z_2<-\gamma_{m-1}<\cdots<-\gamma_1<z'_{m+1}<z_{m+1}<1,$$

- if $y<x(1-\kappa)+\frac{\kappa r}{n}<y'$,

$$z_1<-\gamma_m<z'_1<z_2<-\gamma_{m-1}<\cdots<-\gamma_1<z'_m<z_{m+1}<1<z'_{m+1},$$

- if $x(1-\kappa)+\frac{\kappa r}{n}<y<y'$,

$$-\gamma_m<z'_1<z_1<-\gamma_{m-1}<z'_2<z_2<-\gamma_{m-1}<\cdots<1<z'_{m+1}<z_{m+1},$$

The limiting case when y or y' is equal to $x(1-\kappa)+\frac{\kappa r}{n}$ is obtained by sending the corresponding root in $(-\infty,-\gamma_m)\cup(1,+\infty)$ to ∞.

Rational fractions where zeros of the numerator and denominator interlace have interesting monotonicity properties, already used for example in [40], which are straightforwardly checked by induction using the decomposition of $R(z)$ into the sum of simple fractions:

Lemma 4.3. — *Let*

$$R(z)=\frac{(z-u_1)(z-u_2)\dots(z-u_s)}{(z-v_1)(z-v_2)\dots(z-v_s)},$$

where $\{u_i\}$ and $\{v_i\}$ are two sets of real numbers.

- *If $\{u_i\}$ and $\{v_i\}$ satisfy*

$$v_1<u_1<v_2<u_2<\cdots<v_s<u_s.$$

Then $R(z)$ is monotone increasing in each one of the following intervals

$$(-\infty,v_1),(v_1,v_2),\dots,(v_{s-1},v_s),(v_s,\infty).$$

- *If $\{u_i\}$ and $\{v_i\}$ satisfy*

$$u_1 < v_1 < u_2 < v_2 < \cdots < u_s < v_s.$$

Then $R(z)$ is monotone decreasing in each one of the following intervals

$$(-\infty, v_1), (v_1, v_2), \ldots, (v_{s-1}, v_s), (v_s, \infty).$$

The monotonicity on each interval of definition is still true if R has the form:

$$R(z) = \frac{(z-u_1)\ldots(z-u_{s-1})}{(z-v_1)\ldots(z-v_s)} \quad \text{with } v_1 < u_1 < \cdots < u_{s-1} < v_s$$

or

$$R(z) = \frac{(z-u_1)\ldots(z-u_{s+1})}{(z-v_1)\ldots(z-v_s)} \quad \text{with } u_1 < v_1 < \cdots < v_s < u_{s+1}.$$

This is helpful to determine the number of solutions of Equation (4.3), as shown in the following lemma:

Lemma 4.4. — *Suppose that $\mathbf{m}_\omega$ is a measure with a density with respect to the Lebesgue measure equal to the indicator of a union of intervals $\bigcup_{i=1}^s [a_i, b_i]$, with*

$$a_1 < b_1 < a_2 < b_2 < \cdots < a_s < b_s \quad \text{and} \quad \sum_{i=1}^{s} (b_i - a_i) = 1.$$

Then the system of equations (4.3) has at most one pair of complex (non real) conjugate solutions. Moreover,

- *if $b_i \neq x(1-\kappa) + \frac{\kappa r}{n}$, for all $1 \leqslant i \leqslant s$, then for each fixed $x \in \mathbb{R}$, (4.3) has at least $(m+1)s - 1$ distinct real roots;*
- *if $b_i = x(1-\kappa) + \frac{\kappa r}{n}$ for some i in $\{1, 2, \ldots, s\}$, then for each fixed $x \in \mathbb{R}$, (4.3) has at least $(m+1)s - 2$ distinct real roots.*

where m is the number of distinct c_is.

Proof. — The Stieltjes transform can be computed explicitly from the definition:

$$\mathrm{St}_{\mathbf{m}_\omega}(t) = \log \prod_{i=1}^{s} \frac{t-a_i}{t-b_j}.$$

We use the second expression from (4.5) to substitute $t(z, \kappa, x)$ in the second equation of (4.3), to get (after exponentiation)

$$z = \prod_{i=1}^{s} \frac{H(z; x, a_i)}{H(z; x, b_i)}. \tag{4.9}$$

Let us suppose that none of the a_i's or b_i's is equal to $x(1-\kappa)+\frac{\kappa r}{n}$. The rational fractions $\prod H(z;x,a_i)$ and $\prod H(z;x,b_i)$ have the same poles $m+1$ poles (of same order s) and according to Lemma 4.1 have $s(m+1)$ distinct real roots, which by Remark 4.2, interlace. Therefore, the ratio:

$$\prod_{i=1}^{s} \frac{H(z;x,a_i)}{H(z;x,b_i)}$$

is a rational fraction of the form described in the hypotheses of Lemma 4.3. Therefore, on each bounded interval between two consecutive poles, by monotonicity, the graph of the rational fraction will cross the first diagonal at least once and there are $(m+1)s-1$ such intervals.

If (no b_i, and exactly) one a_i is equal to $x(1-\kappa)+\frac{\kappa r}{n}$, the same argument is applicable. The only difference is that the rational fraction on the right hand side of Equation 4.9 has only $(s-1)(m+1)+m = s(m+1)-1$ zeros, but still $s(m+1)$ poles. Therefore we still get the same number $s(m+1)-1$ of intersection with the first diagonal, one on each finite interval between two consecutive poles.

If (no a_i and exactly) one b_i is equal to $x(1-\kappa)+\frac{\kappa r}{n}$, then this time the rational fraction has $s(m+1)-1$ finite real poles. Therefore, there is only $s(m+1)-2$ roots found by this approach between two successive poles. $\square$

Remark 4.5. — Note that when rewriting Equation 4.9 as a polynomial equation in z, it has degree

$$\begin{cases} s(m+1)+1 & \text{when no } b_i \text{ equals } x(1-\kappa)+\frac{\kappa r}{n}, \\ s(m+1) & \text{when a } b_i \text{ is equal to } x(1-\kappa)+\frac{\kappa r}{n}. \end{cases}$$

Indeed, in the last case, the leading coefficients of the numerator and denominator of the rational fraction are distinct, thus there is no cancellation of the monomials of higher degree when multiplying both sides by the denominator. In both case, it is exactly the number of real roots we found plus 2. Which means that Equation 4.9, and thus Equation 4.3 has at most a pair of complex conjugated roots.

When there are complex conjugated roots, the density of the counting measure is given by their the normalized argument as stated in the Theorem 4.8 below. In order to prove it, we need two additional lemmas, which are minor adaptations of the ones of [8] adapted to our context:

Lemma 4.6. — *Let $x_0 \in \mathbb{R}$ be such that Equation* (4.3) *has a pair of complex roots. Let $z_j(x_0)$ be the jth smallest real root of* (4.3). *Let $\mathcal{U}$ be a small complex neighborhood of x_0. Let $z_j(x)$ be the root of* (4.3) *approximating $z_j(x_0)$, when $x \to x_0$ (which is well defined if $\mathcal{U}$ is small enough).*

Then the derivative of $z_j(x)$ with respect to x at x_0 is non-negative. Moreover, it is equal to 0 if and only if $z_j(x_0) = 1$.

Proof. — Write

$$G(z,x) = \prod_{i=1}^{s} \frac{H(z;x,a_i)}{H(z;x,b_i)} = \frac{\prod_{i=1}^{s}(x(1-\kappa) + \frac{\kappa z}{z-1} - \frac{\kappa z}{n}\sum_{i=1}^{m}\frac{1}{z+c_i} - a_i)}{\prod_{i=1}^{s}(x(1-\kappa) + \frac{\kappa z}{z-1} - \frac{\kappa z}{n}\sum_{i=1}^{m}\frac{1}{z+c_i} - b_i)},$$

then follow the same argument as in the proof of Lemma 4.5 of [8] to show $G'_x(z(x_0),x_0) \leqslant 0$ and $G'_z(z(x_0),x_0) > 1$. Then the lemma follows. □

LEMMA 4.7. — *Let $\mathbf{m}_\omega$ be a measure as described in Lemma 4.4. Let*

$$\mathbf{Z}^\kappa(x) = \exp(\mathrm{St}_{\mathbf{m}^\kappa}(x)),$$

where $x \in \mathbb{C} \setminus \mathrm{Support}(\mathbf{m}^\kappa)$. Then $\mathbf{Z}^\kappa(x)$ is a solution of (4.3). Let x_0 be such that (4.3) has exactly one pair of non-real solutions. Then $\lim_{\epsilon\to 0+}\mathbf{Z}^\kappa(x_0+i\epsilon)$ coincides with the unique non-real root with negative imaginary part of (4.3) when $x = x_0$.

Proof. — The fact that $\mathbf{Z}^\kappa(x)$ solves (4.3) follows from Equation (4.4). The limit $\lim_{\epsilon\to 0+}\mathbf{Z}^\kappa(x_0+i\epsilon)$ has strictly negative imaginary part follows from the analysis of [8, p. 24], by applying Lemma 4.6. □

Here is the main theorem to be proved in this section.

THEOREM 4.8. — *The density of $\mathbf{m}^k$ is given by*

$$\mathrm{d}\mathbf{m}^\kappa(x) = \frac{1}{\pi}\,\mathrm{Arg}(\mathbf{z}_+^\kappa(x)),$$

where $\mathbf{z}_+^\kappa(x)$ is the unique complex root of the system of equations (4.3) which lies in the upper half plane. If such a complex root does not exist, the density is equal to 0 or 1.

Proof. — First we assume that $\mathbf{m}_\omega$ is a uniform measure with density one on a sequence of intervals, as described in Lemma 4.4. In this case, the theorem follows from Lemma 4.7, and from the classical fact about Stieltjes transform that if a measure μ has a continuous density f with respect to the Lebesgue measure then one can reconstruct f by the following identity (See e.g. [8, Lemma 4.2]):

$$f(x) = -\lim_{\epsilon\to 0+}\frac{1}{\pi}\Im(\mathrm{St}_\mu(x+i\epsilon)), \tag{4.10}$$

where $\Im$ denote the imaginary part of a complex number.

In the general case of a measure $\mathbf{m}_\omega$, there exists a sequence of measures $\{\mu_i\}$ converging weakly to $\mathbf{m}_\omega$, where each μ_i is a measure with the form as described in Lemma 4.4. Passing to the limit we obtain the theorem. □

5. Frozen Boundary

In this section, we study the *frozen boundary*, which is the curve separating the "liquid region" and the "frozen region" in the scaling limit of dimer models on a contracting square-hexagon lattice. We prove an explicit formula for the frozen boundary under the assumption that each segment of the boundary row of the square-hexagon lattice grows linearly with respect the dimension of the graph. We then prove that the frozen boundary is a curve of a certain type, more precisely, a cloud curve whose class depends on the size of the fundamental domain and the number of segments of the boundary row. Similar results for dimer configurations on the square grid or the hexagonal lattice with uniform measure or a q-deformation of the uniform measure was obtained in [8, 27].

We consider special sequences of contracting square-hexagon lattices $\mathcal{R}(\Omega, \check{a})$ with

$$(5.1)\quad \Omega = (A_1, A_1+1, \dots, B_1-1, B_1, A_2, A_2+1, \dots, B_2, \dots, A_s, A_s+1, \dots, B_s),$$

where

$$\sum_{i=1}^{s}(B_i - A_i + 1) = N.$$

In other words, Ω is an N-tuple of integers whose entries take values of all the integers in $\cup_{i=1}^{s}[A_i, B_i]$.

We shall consider $\Omega(N)$ changing with N, and discuss the asymptotics of the frozen boundary as $N \to \infty$.

Suppose that for each N, $\Omega(N)$ has corresponding $A_i(N)$, $B_i(N)$, for a fixed s. Assume also that $A_i(N), B_i(N), \Omega(N)_N - N$ have the following asymptotic growth:

$$(5.2)\quad \begin{gathered} A_i(N) = a_i N + o(N), \quad B_i(N) = b_i N + o(N), \\ \Omega(N)_N - N = \mu N + o(N) \end{gathered}$$

where $a_1 < b_1 < \cdots < a_s < b_s$ are new parameters such that

$$\sum_{i=1}^{s}(b_i - a_i) = 1.$$

Definition 5.1. — *Let $\mathcal{L}$ be the set of (χ, κ) inside $\mathcal{R}$ such that the density* $\mathrm{d}\mathbf{m}^{\kappa}\left(\frac{\chi}{1-\kappa}\right)$ *is not equal to 0 or 1. Then $\mathcal{L}$ is called the* liquid region. *Its boundary $\partial\mathcal{L}$ is called the* frozen boundary.

THEOREM 5.2. — *The frozen boundary $\partial\mathcal{L}$ of the limit of a contracting square-hexagon lattice satisfying* (5.1), (5.2) *is a rational algebraic curve C with an explicit parametrization $(\chi(t), \kappa(t))$ defined as follows:*

$$\chi(t) = t - \frac{J(t)}{J'(t)}, \quad \kappa(t) = \frac{1}{J'(t)},$$

where

$$J(t) = \Phi_s(t)\left[\frac{1}{\Phi_s(t)-1} - \frac{1}{n}\sum_{i\in I_2\cap\{1,2,\dots,n\}}\frac{1}{\Phi_s(t)+c_i}\right] = \frac{r}{n} + \frac{1}{\Phi_s(t)-1} + \frac{1}{n}\sum_{j=1}^{m}\frac{n_j\gamma_j}{\Phi_s(t)+\gamma_j} \tag{5.3}$$

and

$$\Phi_s(t) = \frac{(t-a_1)(t-a_2)\dots(t-a_s)}{(t-b_1)(t-b_2)\dots(t-b_s)}$$

Proof. — According to the discussions in Section 4, the frozen boundary is given by the condition that the following equation in the unknown z has a double root:

$$G\left(z, \frac{\chi}{1-\kappa}\right) = z; \tag{5.4}$$

where

$$G\left(z, \frac{\chi}{1-\kappa}\right) = \prod_{i=1}^{s}\frac{H(z;\frac{\chi}{1-\kappa}, a_i)}{H(z;\frac{\chi}{1-\kappa}, b_i)}, \tag{5.5}$$

and $H(z;x,y)$ is defined by Equation (4.8).

We can also rewrite the system of equations (4.3) as follows:

$$\begin{cases} \Phi_s(t) & = z; \\ (1-\kappa)F_\kappa(z) & = t - \kappa\underbrace{\left(\frac{r}{n} + \frac{1}{z-1} + \sum_{j=1}^{m}\frac{n_j\gamma_j}{z+\gamma_j}\right)}_{J(t)} = \chi. \end{cases}$$

We plug the expression of z from the first equation into the second equation, and note that the condition that the resulting equation has a double root is equivalent to the following system of equations

$$\begin{cases} \chi = t - \kappa J(t), \\ 1 = \kappa J'(t). \end{cases}$$

where $J(t)$ is defined by (5.3). Then the parametrization of the frozen boundary follows. □

The algebraic curve we obtain for the frozen boundary has special properties, that can be read from its dual curve, as described in the definition and the proposition below:

Definition 5.3 ([27]). — *A degree d real algebraic curve $C \subset \mathbb{R}P^2$ is* winding *if the following two conditions hold:*

(1) *it intersects every line $L \subset \mathbb{R}P^2$ in at least $d-2$ points counting multiplicity,*
(2) *there exists a point $p_0 \in \mathbb{R}P^2$ called center, such that every line through p_0 intersects C in d points.*

The dual curve of a winding curve is called a cloud curve*.*

Proposition 5.4. — *The frozen boundary C is a cloud curve of class $(m+1)s$, where s is the number of segments, and m is the number of distinct values of $c_i = \frac{1}{y_i}$ in one period. Moreover, the curve C is tangent to the following lines in the (χ, κ) coordinates:*

$$\mathcal{L} = \{\chi = a_i \mid i = 1, \dots, s\} \cup \{\chi + r\kappa - b_i = 0 \mid 1 \leqslant i \leqslant s\} \cup \{\kappa = 0\} \cup \{\kappa = 1\}.$$

So the proposition states that the dual curve is winding of degree $(m+1)s$, and passes through the points $(-\frac{1}{a_i}, 0)$ and $(-\frac{1}{b_i}, -\frac{r}{nb_i})$.

The result about the frozen boundary being a cloud curve extends the result of Kenyon and Okounkov [27] for the uniform measure of rhombus tilings of polygonal domains, and Bufetov and Knizel [8] for Aztec rectangles.

Proof. — We recall that the class of a curve is the degree of its dual curve. So we need to show that the dual curve $C^\vee$ has degree $(m+1)s$ and is winding.

We apply the classical formula to obtain from a parametrization $(x(t), y(t))$ of the curve C defining the frozen boundary, another one for its dual $C^\vee$, $(x^\vee(t), y^\vee(t))$:

$$x^\vee = \frac{y'}{yx' - xy'}, \quad y^\vee = -\frac{x'}{yx' - xy'}.$$

and obtain that the dual curve $C^\vee$ is given in the following parametric form:

$$C^\vee = \left\{ \left(-\frac{1}{t}, -\frac{J(t)}{t} \right) \middle| \, t \in \mathbb{C} \cup \{\infty\} \right\}. \tag{5.6}$$

from which we can read that its degree is $(m+1)s$. To show that $C^\vee$ is winding, we need to look at real intersections with straight lines.

First, from Equation (5.6), one sees that the first coordinate $x^\vee$ of the dual curve $C^\vee$ and the parameter t are linked by the simple relation $x^\vee t = -1$.

Using this relation to eliminate t from the expression of the second coordinate, we obtain that the points $(x^\vee, t)$ on the dual curve satisfy the following implicit equation:

$$y^\vee = x^\vee J\left(\frac{-1}{x^\vee}\right).$$

The points of intersection $(x^\vee(t), y^\vee(t))$ of the dual curve with a straight line of the form $y^\vee = cx^\vee + d$ have a parameter t satisfying:

$$(c - dt) = J(t) \tag{5.7}$$

but since J is the composition of two rational fractions Φ_s and an affine transformation of H, with interlacing poles and zeros, with degrees s and $m+1$ respectively, the exact same argument as in Lemma 4.6 (but with the role of s and $(m+1)$ exchanged) shows that Equation 5.7 has at least $(m+1)s-2$ distinct real solutions, yielding $(m+1)s-2$ points of intersections for the dual curve and the line $y^\vee = cx^\vee + d$. Moreover, if t_0 doesn't lie in a compact interval containing all the zeros of J, then any non vertical straight line passing through $(t_0, 0)$ will have $(m+1)s-1$ intersections with the graph of J. See Figure 5.1 for the graph of $J(t)$. This means that for $x_0^\vee$ in some closed interval, there are at least $(m+1)s-1$ real intersections of the dual curve with a line $y^\vee = cx^\vee + d$ passing through $(x_0^\vee, y_0^\vee)$, thus exactly $(m+1)s$ real intersections, since there cannot be a single complex one. Such points $(x_0^\vee, y_0^\vee)$ are candidates to be the center of the dual curve.

To consider the vertical lines $x^\vee = d$, we rewrite the equations in homogeneous coordinates $[x^\vee : y^\vee : z^\vee]$ and get that the line $x^\vee = \mathrm{d}z^\vee$ intersects the curve at the point $[0:1:0]$ with multiplicity $(m+1)s-1$, so again, by the same argument as above, $(m+1)s$ real intersections. The case of the line $z^\vee = 0$ is similar.

Recall that each point on the dual curve $C^\vee$ corresponds to a tangent line of C. The points $(x, y) = \left(-\frac{1}{a_i}, 0\right)$ belong to $C^\vee$. Indeed, they correspond to $t = a_i$ which is a zero of Φ_s, and thus of J, by (5.3). Similarly, $(x, y) = \left(-\frac{1}{b_i}, -\frac{r}{nb_i}\right)$ are also points of $C^\vee$, since they correspond to $t = b_i$ which is a pole of Φ_s, and thus $J(b_i) = \frac{r}{n}$, again by (5.3).

These families of points correspond to the families of tangent lines $(\{\chi = a_i\})_{1\leqslant i\leqslant s}$ and $(\{\chi + \frac{r}{n}\kappa - b_i = 0\})_{1\leqslant i\leqslant s}$. The former are vertical lines passing through $(a_i, 0)$, and the latter are lines passing through $(b_i, 0)$ with slope $-\frac{n}{r}$.

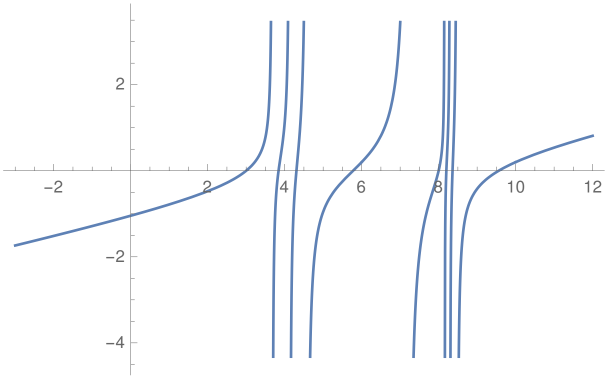


Figure 5.1. A plot of the graph of the function J, for the parameters $r = 1$, $n = 5$, $s = 2$, $m = 3$, $(n_1, n_2, n_3) = (2, 1, 1)$, $(\gamma_1, \gamma_2, \gamma_3) = (0.7, 0.4, 0.2)$ and $(a_1, a_2) = (3, 8)$, $(b_1, b_2) = (6, 10)$.

The point $(x^\vee, y^\vee) = (0, -1) \in C^\vee$ corresponds to the tangent line $\kappa = 1$ of C. By the parametrization of C given in Theorem 5.2, the roots of

$$\left(\Phi_s\left(-\frac{1}{x^\vee}\right) - 1\right)\prod_{i=1}^{m}\left(\Phi_s\left(-\frac{1}{x^\vee}\right) + c_i\right)$$

correspond to $(m+1)s-1$ points of tangency of C with the line $\kappa = 0$. □

6. Positions of V-edges in a row and eigenvalues of GUE random matrix

In this section, we prove that near a turning point of the frozen boundary, the present V-edges in a random perfect matching of the contracting square-hexagon lattice are distributed like the eigenvalues of a random Hermitian matrix from the Gaussian Unitary Ensemble (GUE).

A matrix of the GUE of size k is diagonalizable with real eigenvalues $\epsilon_1 \geqslant \epsilon_2 \geqslant \cdots \geqslant \epsilon_k$, whose distribution $\mathbb{P}_{\mathbb{GUE}_k}$ on $\mathbb{R}^k$ has a density with respect to the Lebesgue measure on $\mathbb{R}^k$ proportional to:

$$\prod_{1\leqslant i<j\leqslant k}(\epsilon_i - \epsilon_j)^2 \exp\left(-\sum_{i=1}^{k}\epsilon_i^2\right),$$

See [34, Theorem 3.3.1]. Here is the main theorem we prove in this section.

THEOREM 6.1. — *Let $\mathcal{R}(\Omega(N), \check{a})$ be a contracting square-hexagon lattice, such that*

- *$\Omega_N = (\Omega_1(N), \ldots, \Omega_N(N))$ is an N-tuple of integers denoting the location of vertices on the first row,*
- *the edge weights are assigned as in Assumptions 2.1 and 3.5,*
- *for every i, $x_i = 1 + o(N^{-2})$,*

Let $\lambda^k(N)$ be the signature corresponding to the dimer configuration incident to the $(N-k+1)$th row of white vertices in $\mathcal{R}(\Omega(N), \check{a})$, and for $1 \leqslant l \leqslant k$,

$$b_{kl}^N = \lambda_l^k(N) + N - l.$$

Let

$$\psi_1 = \int_{\mathbb{R}} x \mathrm{d}\mathbf{m}^1, \quad \psi_2 = \int_{\mathbb{R}} x^2 \mathrm{d}\mathbf{m}^1,$$

where $\mathbf{m}^1$ is the limit counting measure of signatures on the top of $\mathcal{R}(\Omega(N), \check{a})$. Let

$$\widetilde{b}_{kl}^{(N)} = \frac{\frac{b_{kl}^{(N)}}{\sqrt{N}} - \sqrt{N}\left(\psi_1 - \frac{1}{2} + \frac{1}{n}\sum_{i \in I_2 \cap \{1,\ldots,n\}} \frac{y_i}{1+y_i}\right)}{\psi_2 - \psi_1^2 - \frac{1}{12} + \frac{1}{n}\sum_{i \in I_2 \cap \{1,2,\ldots,n\}} \frac{y_i}{(1+y_i)^2}}, \quad 1 \leqslant l \leqslant k.$$

Then, for any fixed k, the distribution of $\left(\widetilde{b}_{kl}^{(N)}\right)_{l=1}^{k}$ converges weakly to $\mathbb{P}_{\mathbb{GUE}_k}$ as $N \to \infty$.

The convergence of the distribution of certain present edges near the boundary to the GUE minor process was proved in [22] in the case of uniform perfect matching on the Aztec diamond, and in [39] for plane partitions. In the case of the uniform perfect matching on a hexagon lattice, the result was proved in [16, 36]. In the case of q-distributed perfect matching on a hexagon lattice with $q = e^{\frac{-\gamma}{N}}$, the result was proved in [35].

Proof of Theorem 6.1. — In order to prove Theorem 6.1, we shall apply the following characterization of $\mathbb{P}_{\mathbb{GUE}_k}$, which follows directly from the definition. See e.g. [16, 36].

LEMMA 6.2. — *Let $(q_1, \ldots, q_k) \in \mathbb{R}^k$ be a random vector with distribution $\mathbb{P}$ and $Q = \mathrm{diag}[q_1, \ldots, q_k]$ the $k \times k$ diagonal matrix obtained by putting the q_i on the diagonal.*

Then $\mathbb{P}$ is $\mathbb{P}_{\mathbb{GUE}_k}$ if and only if for any matrix P,

$$\mathbb{E} \int_{U(k)} \exp[\mathrm{Tr}(PUQU^*)] \mathrm{d}U = \exp\left(\frac{1}{2} \mathrm{Tr}\, P^2\right).$$

It is in fact enough to check the case when P is diagonal, with real coefficients.

Let

$$P_k = \operatorname{diag}\left[v_1, \ldots, v_k\right], \quad Q_k^{(N)} = \operatorname{diag}\left[b_{k1}^{(N)}, \ldots, b_{kk}^{(N)}\right].$$

By Lemma 2.24, we have for any k:

$$\int_{U(k)} \exp\Big[\operatorname{Tr}\Big(P_k U Q_k^{(N)} U^*\Big)\Big] \mathrm{d}U = \left[\prod_{1\leqslant i<j\leqslant k} \frac{e^{v_i} - e^{v_j}}{v_i - v_j}\right] \frac{s_{\lambda^k(N)}(e^{v_1}, \ldots, e^{v_k})}{s_{\lambda^k(N)}(1, \ldots, 1)} \tag{6.1}$$

Let ρ_N^k be the distribution of dimer configurations restricted on the $(N+k-1)$th row of white vertices on $\mathcal{R}(\Omega(N), \check{a})$, and let

$$X_k^{(N)} = \left(x_{N-k+1}^{(N)}, \ldots, x_N^{(N)}\right).$$

Then

$$\begin{aligned}
&\sum_{\lambda_k(N)\in\mathbb{GT}_k^+} \rho_N^k(\lambda_k(N)) \int_{U(k)} \exp\left[\frac{1}{\sqrt{N}} \operatorname{Tr}\left(P_k U Q_k^{(N)} U^*\right)\right] \mathrm{d}U \qquad (6.2)\\
&= \prod_{1\leqslant i<j\leqslant k} \left[\sqrt{N} \frac{e^{v_i/\sqrt{N}} - e^{v_j/\sqrt{N}}}{v_i - v_j}\right] \\
&\qquad\qquad \times \sum_{\lambda_k(N)} \rho_N^k(\lambda_k(N)) \frac{s_{\lambda_k(N)}(e^{v_1/\sqrt{N}}, \ldots e^{v_k/\sqrt{N}})}{s_{\lambda_k(N)}(1, \ldots, 1)} \\
&= \prod_{1\leqslant i<j\leqslant k} \left[\sqrt{N} \frac{e^{v_i/\sqrt{N}} - e^{v_j/\sqrt{N}}}{v_i - v_j}\right] \times \mathcal{S}_{\rho_N^k, X_k^{(N)}}(e^{v_1/\sqrt{N}}, \ldots, e^{v_k/\sqrt{N}}) \\
&\qquad\qquad \times \frac{\sum_{\lambda_k(N)} \rho_N^k(\lambda_k(N)) \frac{s_{\lambda_k(N)}(e^{v_1/\sqrt{N}}, \ldots e^{v_k/\sqrt{N}})}{s_{\lambda_k(N)}(1, \ldots, 1)}}{\sum_{\lambda_k(N)} \rho_N^k(\lambda_k(N)) \frac{s_{\lambda_k(N)}(e^{v_1/\sqrt{N}}, \ldots e^{v_k/\sqrt{N}})}{s_{\lambda_k(N)}(x_1, \ldots, x_k)}},
\end{aligned}$$

the denominator of the last fraction being exactly $\mathcal{S}_{\rho_N^k, X_k^{(N)}}(e^{v_1/\sqrt{N}}, \ldots, e^{v_k/\sqrt{N}})$ by Definition 3.2. First, this fraction is converging to 1 as N goes to infinity. Indeed, all partitions $\lambda_k(N) \in \mathbb{GT}_k$ that contribute to the sum must have parts bounded by a constant times N by hypothesis. Since $|x_i - 1| = o(N^{-2})$, one has:

$$\begin{aligned}
&\left|\frac{s_{\lambda_k(N)}(x_1, \ldots, x_k) - s_{\lambda_k(N)}(1, \ldots, 1)}{s_{\lambda_k(N)}(1, \ldots, 1)}\right| \\
&\qquad\qquad \leqslant C o(N^{-2}) |\lambda_k(N)| C^{o(N^{-2})|\lambda_k(N)|} = o(1)
\end{aligned}$$

uniformly in $\lambda_k(N)$, which implies that the difference between the fraction and 1 is negligible as N goes to ∞.

We then use Lemma 3.6 to re-express the Schur generating function:

$$\begin{aligned}(6.3)\quad &\mathcal{S}_{\rho_N^k, X_k^{(N)}}(\zeta_1,\dots,\zeta_k)\\ &= \frac{s_\omega\left(x_{\overline{1}}^{(N)},\dots,x_{\overline{N-k}}^{(N)},\zeta_1,\dots,\zeta_k\right)}{s_\omega\left(x_{\overline{1}}^{(N)},\dots,x_{\overline{N}}^{(N)}\right)} \prod_{\substack{1\leqslant i\leqslant k\\ j\in I_2\cap\{1,2,\dots,N-k\}}} \left(\frac{1+y_{\overline{j}}\zeta_i}{1+y_{\overline{j}}x_{\overline{N-k+i}}}\right),\end{aligned}$$

where $\omega = (\omega_1 \geqslant \omega_2 \geqslant \dots \geqslant \omega_N) \in \mathbb{GT}_N^+$ is the signature corresponding to the first row.

The same estimate for Schur functions we used before to compare $s_{\lambda_k(N)}$ evaluated at $(1,\dots,1)$ and $(x_1^{(N)},\dots,x_k^{(N)})$ can be used this time for s_ω, using the fact that $|\omega| = O(N^2)$ by hypothesis. We have then that

$$\left|\frac{s_\omega(x_1^{(N)},\dots x_N^{(N)}) - s_\omega(1,\dots,1)}{s_\omega(1,\dots,1)}\right| \leqslant \epsilon(N)N^{-2}|\omega|C^{|\omega|\epsilon(N)N^{-2}} = o(1),$$

with $\lim_{N\to\infty}\epsilon(N) = 0$, so we can, up to a negligible correction, replace in the denominator $s_\omega(x_1^{(N)},\dots x_N^{(N)})$ by $s_\omega(1,\dots,1)$ in Equation (6.3).

This ratio of Schur functions appears also when using again Lemma 2.24 to rewrite the matrix integral over $U(N)$ this time. More precisely, let

$$\begin{aligned}R_N &= \operatorname{diag}\left[v_1,\dots,v_k,t_{\overline{1}}^{(N)},\dots,t_{\overline{N-k}}^{(N)}\right],\\ Q_N &= \operatorname{diag}\left[\omega_1+N-1,\omega_2+N-2\dots,\omega_N\right].\end{aligned}$$

where $t_i^{(N)} = \sqrt{N}\log\left[x_i^{(N)}\right]$ fo $1\leqslant i\leqslant n$. For $1\leqslant i\leqslant k$, let $\zeta_i = e^{\frac{v_i}{\sqrt{N}}}$. Then,

$$\begin{aligned}(6.4)\quad &\frac{s_\omega\left(x_{\overline{1}}^{(N)},\dots,x_{\overline{N-k}}^{(N)},\zeta_1,\dots,\zeta_k\right)}{s_\omega(1,\dots,1)}\\ &= \int_{U(N)} \exp\left[\frac{1}{\sqrt{N}}\operatorname{Tr}\left(R_N U Q_N U^*\right)\right]\mathrm{d}U \times \left[\prod_{1\leqslant i<j\leqslant k}\sqrt{N}\frac{e^{\frac{v_i}{\sqrt{N}}}-e^{\frac{v_j}{\sqrt{N}}}}{v_i-v_j}\right]^{-1}\\ &\quad\times\left[\prod_{\substack{1\leqslant i\leqslant k\\ 1\leqslant j\leqslant N-k}}\sqrt{N}\frac{e^{\frac{v_i}{\sqrt{N}}}-x_{\overline{j}}^{(N)}}{v_i-t_{\overline{j}}^{(N)}}\right]^{-1}\left[\prod_{1\leqslant i<j\leqslant N-k}\sqrt{N}\frac{x_{\overline{i}}^{(N)}-x_{\overline{j}}^{(N)}}{t_{\overline{i}}^{(N)}-t_{\overline{j}}^{(N)}}\right]^{-1}.\end{aligned}$$

Plugging the modified version of Equation (6.3) and (6.4) into (6.2), one gets:

$$\log \sum_{\lambda^k(N)\in\mathbb{GT}_k^+} \rho_N^k(\lambda^k(N)) \int_{U(k)} \exp\left[\frac{1}{\sqrt{N}} \operatorname{Tr}\left(P_k U Q_k^{(N)} U^*\right)\right] \mathrm{d}U$$

$$= \log \int_{U(N)} \exp\left[\frac{1}{\sqrt{N}} \operatorname{Tr}\left(R_N U Q_N U^*\right)\right] \mathrm{d}U - \sum_{\substack{1\leqslant i\leqslant k\\ 1\leqslant j\leqslant N-k}} \log\left[\frac{e^{v_i/\sqrt{N}} - e^{t_j/\sqrt{N}}}{\frac{v_i}{\sqrt{N}} - \frac{t_j}{\sqrt{N}}}\right]$$

$$- \sum_{1\leqslant i<j\leqslant N-k} \log\left[\frac{e^{t_i\sqrt{N}} - e^{t_j\sqrt{N}}}{\frac{t_i}{\sqrt{N}} - \frac{t_j}{\sqrt{N}}}\right]$$

$$+ \sum_{\substack{1\leqslant i\leqslant k\\ j\in I_2\cap\{1,\dots,N-k\}}} \log \frac{1+y_j e^{v_i\sqrt{N}}}{1+y_j e^{t_{N-k+i}/\sqrt{N}}} + O(N^{2-\alpha}).$$

Expand the sums using the two expansions $\log\frac{e^u-e^v}{u-v} = \frac{u+v}{2} + \frac{(u-v)^2}{24} + O(|u|^3+|v|^3)$, and $\log(\frac{1+ye^v}{1+y}) = \frac{y}{1+y}v + \frac{y}{2(1+y)^2}v^2 + O(|v|^3)$ as u and v tend to 0, to get that:

$$\sum_{\substack{1\leqslant i\leqslant k\\ 1\leqslant j\leqslant N-k}} \log\left[\frac{e^{v_i/\sqrt{N}} - e^{t_j/\sqrt{N}}}{\frac{v_i}{\sqrt{N}} - \frac{t_j}{\sqrt{N}}}\right] = o(1),$$

$$\sum_{1\leqslant i<j\leqslant N-k} \log\left[\frac{e^{t_i\sqrt{N}} - e^{t_j\sqrt{N}}}{\frac{t_i}{\sqrt{N}} - \frac{t_j}{\sqrt{N}}}\right] = (N-k)\sum_{i=1}^k\left(\frac{v_i}{2\sqrt{N}} + \frac{v_i^2}{24N} + O(N^{-3/2})\right),$$

$$\sum_{\substack{1\leqslant i\leqslant k\\ j\in I_2\cap\{1,\dots,N-k\}}} \log \frac{1+y_j e^{v_i\sqrt{N}}}{1+y_j e^{\frac{t_{N-k+i}}{\sqrt{N}}}}$$

$$= \sum_{j\in I_2\cap\{1,\dots N-k\}} \sum_{i=1}^k \frac{y_j}{1+y_j}\frac{v_i}{\sqrt{N}} + \frac{y_j}{(1+y_j)^2}\frac{v_i^2}{2N} + O(N^{-\frac{3}{2}}).$$

Then we use Lemma 6.3 below to get the asymptotic behavior of the integral over $U(N)$, to obtain that

$$\log \int_{U(N)} \exp\left[\frac{1}{\sqrt{N}} \operatorname{Tr}\left(R_N U Q_N U^*\right)\right] \mathrm{d}U$$

$$= \psi_1\left(\sum_{i=1}^k v_i\right) + \frac{\psi_2-\psi_1^2}{2}\left(\sum_{i=1}^k v_i^2\right) + o(1),$$

where ψ_1 and ψ_2 are respectively the first and second moments of the limiting measure m_ω, as N goes to infinity. Bringing all the pieces together, and defining

$$A = \psi_1 - \frac{1}{2} + \frac{1}{n} \sum_{j \in I_2 \cap \{1,\dots,n\}} \frac{y_j}{1+y_j}, \quad B = \psi_2 - \psi_1^2 - \frac{1}{12} + \frac{1}{n} \sum_{j \in I_2 \cap \{1,\dots,n\}} \frac{y_j}{1+y_j},$$

one finally obtain that

$$\int_{U(k)} \exp\left[\frac{1}{\sqrt{N}} \operatorname{Tr}\left(P_k U Q_k^{(N)} U^*\right)\right] \mathrm{d}U = \exp\left(\sqrt{N} A \sum_i v_i + \frac{1}{2} B \sum_i v_i^2 + o(1)\right).$$

Therefore, according to Lemma 6.2, the distribution of the diagonal coefficients of

$$\frac{1}{\sqrt{N} B}(Q_k^{(N)} - NA \, \mathrm{Id}_k)$$

which are exactly the $\widetilde{b}_{kl}^{(N)}$, converges to $\mathbb{P}_{\mathbb{GUE}_k}$. □

Lemma 6.3. — *Let $k \in \mathbb{N}$ be fixed. For any N, let $\omega(N) \in \mathbb{GT}_N^+$ and*

$$q_i^{(N)} = \omega_i(N) + N - i, \qquad Q_N = \operatorname{diag}\left[q_1^{(N)}, \dots, q_N^N\right],$$
$$P_N = \operatorname{diag}\left[v_1, \dots, v_k, \log\left(x_{\overline{1}}^{(N)}\right), \dots, \log\left(x_{\overline{N-k}}^{(N)}\right)\right].$$

Assume that

- $(\omega(N))_{N\in\mathbb{N}}$ *is a regular sequence of signatures,*
- *as $N \to \infty$, the counting measures $\mathbf{m}_{\omega(N)}$ converge weakly to $\mathbf{m}_\omega$,*
- *there exists a fixed positive integer n, such that for $1 \leqslant i \leqslant N-k$, $x_{\overline{i}}^{(N)} = x_{i \bmod n}^{(N)}$, and*

$$\forall \;\; 1 \leqslant j \leqslant n \;\; \lim_{N\to\infty} \left| x_j^{(N)} - 1 \right| N^2 = 0.$$

Then as $N \to \infty$, we have the following asymptotic expansion

$$\log \int_{U(N)} \exp\left[\frac{1}{\sqrt{N}} \operatorname{Tr}\left(P_N U Q_N U^*\right)\right] \mathrm{d}U = K_1 p_1(v_1,\dots,v_k) N^{1/2} + \frac{K_2}{2!} p_2(v_1,\dots,v_k) + o(1) \tag{6.5}$$

where K_d is $(d-1)!$ multiplying the dth free cumulant of the measure $\mathbf{m}_\omega$ (in particular we have $K_1 = \psi_1, K_2 = \psi_2 - \psi_1^2$) and the p_d is the dth power *sum:*

$$p_d(v_1,\dots,v_k)=\sum_{i=1}^{k} v_i^d.$$

Proof. — We write P_N as $P_N^0+P_N'$ where $P_N^0=\mathrm{diag}(v_1,\dots,v_k,0,\dots,0)$ and P_N' is diagonal, with the first k coefficients equal to 0 and the others are the x_is.

From [36, Corollary 6] (see also [17]), there is an asymptotic expansion (at any order) of the left hand side of (6.5)

$$\log\int_{U(N)}\exp\left[\frac{1}{\sqrt{N}}\operatorname{Tr}\left(P_N^0UQ_N\right)U^*\right]\mathrm{d}U\sim\sum_{d=1}^{\infty}\frac{K_d}{d!}p_d(v_1,\dots,v_k)N^{1-\frac{d}{2}},$$

the first two orders giving the right hand side of (6.5).

We now show that the additional perturbation coming from P_N' does not change the first two orders as long as the x_is are close enough to 1. For this, we rewrite the trace of $P_N'UQU^*$ as:

$$\operatorname{Tr}(P_N'UQ_NU^*)=\sum_{i=k+1}^{N}\sum_{j=1}^{N}\log(x_{i-k}^{(N)})q_j^{(N)}|U_{i,j}|^2.$$

Then $|\log(x_i^{(N)})|=o(N^{-2})$, $q_j^{(N)}=O(N)$ uniformly in i and j, and the sum $\sum_j|U_{i,j}|^2$ is equal to 1, as U is unitary. Therefore, this trace is $o(1)$, and

$$\exp\left(\frac{1}{\sqrt{N}}\operatorname{Tr}(P_N'UQ_NU^*)\right)=1+o(N^{-\frac{1}{2}}),$$

uniformly in U. Thus the correction induced by P_N' to the integral is of lower order than the two first terms of the asymptotic expansion above, thus yielding the result. □

7. Gaussian Free Field

In this section, we show that under a homeomorphism from the liquid region to the upper half plane, the (non-rescaled) height function of the dimer configurations on a contracting square-hexagon lattice converges to the Gaussian free field (GFF). The relationship between the dimer height function and GFF has been studied extensively in the past few decades. Here is an incomplete list of contributions to this question: the convergence of height functions to GFF in distribution for uniform perfect matchings on a simply-connected domain with Temperley boundary condition was proved in [25, 26]; for random perfect matchings on the isoradial double graph with an analogous Temperley boundary condition was proved in [29]; for uniform perfect matchings on a hexagonal lattice with sawtooth boundary was proved in [3, 42]; for uniform perfect matchings on a square grid with sawtooth boundary (rectangular Aztec diamond) was proved in [8]; for interacting particle systems with non-flat boundary was proved in [12, 13].

7.1. Mapping from the liquid region to the upper half plane

By Theorem 4.8 and Definition 5.1, (χ, κ) is in the liquid region if and only if the following system of equations

$$\begin{cases} F_\kappa(z,t) = \frac{\chi}{1-\kappa} \\ \mathrm{St}_{\mathbf{m}_\omega}(t) = \log(z) \end{cases}$$

has non-real roots, where

$$F_\kappa(z,t) = \frac{z}{1-\kappa}\left(\frac{t}{z} - \frac{1}{z-1} + \frac{\kappa}{n}\sum_{i\in I_2\cap\{1,2,\dots,n\}} \frac{1}{z+c_i}\right) + \frac{z}{z-1}.$$

By explicit computation, one obtains the following:

Lemma 7.1. — *Let $t \in \mathbb{H}$, where $\mathbb{H}$ is the upper half plane. Then $(z(t), t)$ is the solution of* (4.3) *if and only if the following equation holds*

$$\begin{aligned} &(7.1)\quad p^{\chi,\kappa}(t) \\ &:= \chi - \left(t + \kappa\left(\frac{1}{\exp(-\mathrm{St}_{\mathbf{m}_\omega}(t)) - 1} + \frac{1}{n}\sum_{i\in I_2\cap\{1,2,\dots,n\}} \frac{1}{c_i \exp(-\mathrm{St}_{\mathbf{m}_\omega}(t)) + 1}\right)\right) \\ &= 0 \end{aligned}$$

Note that there exists a unique solution $t \in \mathbb{H}$ of (7.1). Let $S(t) := \mathrm{St}_{\mathbf{m}_\omega}(t)$.

Proposition 7.2. — *Let $\mathcal{L}$ denote the liquid region, and $T_{\mathcal{L}}$ the mapping*

$$T_{\mathcal{L}} : \mathcal{L} \to \mathbb{H}$$

sending $(\chi, \kappa) \in \mathcal{L}$ to the corresponding unique root of (7.1) *in $\mathbb{H}$. Then, $T_{\mathcal{L}}$ is a homeomorphism with inverse $t \mapsto (\chi_{\mathcal{L}}(t), \kappa_{\mathcal{L}}(t))$ for all $t \in \mathbb{H}$, given by*

$$\chi_{\mathcal{L}}(t) = t - \frac{(t-\bar{t})\zeta(t)}{\zeta(t) - \zeta(\bar{t})} \tag{7.2}$$

$$\kappa_{\mathcal{L}}(t) = -\frac{t-\bar{t}}{\zeta(t) - \zeta(\bar{t})}, \tag{7.3}$$

where

$$\zeta(t) = -\frac{\exp(S(t))}{\exp(S(t))-1} + \frac{\exp(S(t))}{n} \sum_{i\in I_2\cap\{1,\dots,n\}} \frac{1}{\exp(S(t))+c_i}.$$

Proof. — The proof is an adaptation of Proposition 6.2 of [8]; see also Theorem 2.1 of [14].

It suffices to show all the following statements:

(1) $\mathcal{L}$ is nonempty.
(2) $\mathcal{L}$ is open.
(3) $T_{\mathcal{L}} : \mathcal{L} \to \mathbb{H}$ is continuous.
(4) $T_{\mathcal{L}} : \mathcal{L} \to \mathbb{H}$ is injective.
(5) $T_{\mathcal{L}} : \mathcal{L} \to T_{\mathcal{L}}(\mathcal{L})$ has continuous inverse for all $t \in T_{\mathcal{L}}(\mathcal{L})$.
(6) $T_{\mathcal{L}}(\mathcal{L}) = \mathbb{H}$.

We first prove (1). Explicit computations show that $(\chi_{\mathcal{L}}(t), \kappa_{\mathcal{L}}(t))$ satisfies (7.1). Since $\mathbf{m}_\omega$ is a measure on $\mathbb{R}$ with compact support, assume that $\mathrm{Support}(\mathbf{m}_\omega) \subset [a,b]$ where $a,b\in\mathbb{R}$. The Stieltjes transform satisfies

$$S(t) = \frac{1}{t} + \frac{\alpha}{t^2} + \frac{\beta}{t^3} + O(|t|^{-4}),$$

where α and β are the first two moments of the measure $\mathbf{m}_\omega$:

$$\alpha = \int_a^b x\mathbf{m}_\omega(\mathrm{d}x), \quad \beta = \int_a^b x^2\mathbf{m}_\omega(\mathrm{d}x).$$

After computations we have

$$\chi = \alpha - \frac{1}{2} + \frac{1}{n} \sum_{i=I_2\cap\{1,2,\dots,n\}} \frac{1}{1+c_i} + O(|t|^{-1}),$$

$$\kappa = 1 + \left(-\beta - \frac{1}{n} \sum_{i\in I_2\cap\{1,2,\dots,n\}} \frac{1}{(1+c_i)^2} + \frac{1}{12} + \alpha^2\right) \frac{1}{|t|^2} + O(|t|^{-3}).$$

Let λ be the Lebesgue measure on $\mathbb{R}$. Recall that $\mathbf{m}_\omega$ is a probability measure supported in the interval $[a,b]$ and $\mathbf{m}_\omega \leqslant \lambda$. Hence we have $b-a\geqslant 1$. Then we infer

$$a+\frac{1}{2} = \int_a^{a+1} x\mathrm{d}x \leqslant \alpha \leqslant \int_{b-1}^b x\mathrm{d}x = b-\frac{1}{2}.$$

Similarly,

$$\beta-\alpha^2 \geqslant \frac{1}{2}\int_0^1\int_0^1 (x-y)^2\mathrm{d}x\mathrm{d}y = \frac{1}{12}.$$

As a result, $(\chi,\kappa) \in \left(a+\frac{1}{2}, b-\frac{1}{2}\right) \times (0,1)$ whenever $|t|$ is sufficiently large. Then (1) follows.

The facts (2) and (3) follow from Rouché's theorem by the same arguments as in the proof of Proposition 6.2 in [8]. The facts (4)–(6) can also be obtained by the same arguments as in the proof of Proposition 6.2 in [8]. □

7.2. Gaussian Free Field

Let C_0^∞ be the space of smooth real-valued functions with compact support in the upper half plane $\mathbb{H}$. The *Gaussian free field* (GFF) Ξ on $\mathbb{H}$ with the zero boundary condition is a collection of Gaussian random variables $\{\Xi_f\}_{f\in C_0^\infty}$ indexed by functions in C_0^∞, such that the covariance of two Gaussian random variables Ξ_{f_1}, Ξ_{f_2} is given by

$$\mathrm{Cov}(\Xi_{f_1},\Xi_{f_2}) = \int_{\mathbb{H}}\int_{\mathbb{H}} f_1(z)f_2(w)G_{\mathbb{H}}(z,w)\mathrm{d}z\mathrm{d}\bar{z}\mathrm{d}w\mathrm{d}\overline{w}, \tag{7.4}$$

where

$$G_{\mathbb{H}}(z,w) := -\frac{1}{2\pi}\ln\left|\frac{z-w}{z-\overline{w}}\right|, \qquad z,w\in\mathbb{H}$$

is the Green's function of the Laplacian operator on $\mathbb{H}$ with the Dirichlet boundary condition. The Gaussian free field Ξ can also be considered as a random distribution on C_0^∞, such that for any $f\in C_0^\infty$, we have

$$\Xi(f) = \int_{\mathbb{H}} f(z)\Xi(z)\mathrm{d}z := \xi_f.$$

Here $\Xi(f)$ is the Gaussian random variable with respect to f, which has mean 0 and variance given by (7.4)) with f_1 and f_2 replaced by f. See [43] for more about the GFF.

Consider a contracting square-hexagon lattice $\mathcal{R}(\Omega,\check{a})$. Let ω be a signature corresponding to the boundary row.

By (2.10), we defined a probability measure on the set of signatures

$$\mathcal{S}^N = (\mu^{(N)},\nu^{(N)},\dots,\mu^{(1)},\nu^{(1)}).$$

By Proposition 2.18, this measure is exactly the probability measure on dimer configurations such that the probability of a perfect matching is proportional to the product of weights of present edges.

Define a function Δ^N on $\mathbb{R}_{\geqslant 0}\times\mathbb{R}_{\geqslant 0}\times\mathcal{S}\to\mathbb{N}$ as follows

$$\begin{aligned}\Delta^N &: (x,y,(\mu^{(N)},\nu^{(N)},\dots,\mu^{(1)},\nu^{(1)}))\\ &\qquad\longmapsto \sqrt{\pi}|\{1\leqslant s\leqslant N-\lfloor y\rfloor : \mu_s^{N-\lfloor y\rfloor}+(N-\lfloor y\rfloor)-s\geqslant x\}|\end{aligned}$$

This is, up to a deterministic shift, and an affine transformation, the height function defined in Section 3.2, extended to a piecewise constant function.

Let $\Delta^N_{\mathcal{M}}(x,y)$ be the push-forward of the measure P^N_ω on $\mathcal{S}^N$ with respect to Δ^N. For $z \in \mathbb{H}$, define

$$\boldsymbol{\Delta}^N_{\mathcal{M}}(z) := \Delta^N_M(N\chi_{\mathcal{L}}(z), N\kappa_{\mathcal{L}}(z)),$$

where $\chi_{\mathcal{L}}(z)$, $\kappa_{\mathcal{L}}(z)$ are defined by (7.2), (7.3), respectively.

Here is the main theorem we shall prove in the section.

Theorem 7.3. — *Let $\boldsymbol{\Delta}^N_{\mathcal{M}}(z)$ be a random function corresponding to the random perfect matching of the contracting square-hexagon lattice, as explained above, with the following hypothesis on the weights $\beta_i^{(N)}$:*

$$\lim_{N\to\infty} \sup_i |\beta_i^{(N)} - 1|N = 0.$$

Then $\boldsymbol{\Delta}^N_{\mathcal{M}}(z) - \mathbb{E}\boldsymbol{\Delta}^N_{\mathcal{M}}(z)$, seen as a random field, converges as N goes to ∞ to the Gaussian free field Ξ in $\mathbb{H}$ with Dirichlet boundary conditions, in the following sense: for $0 < \kappa \leqslant 1$, $j \in \mathbb{N}$, define:

$$M_j^\kappa = \int_{-\infty}^{+\infty} \chi^j(\boldsymbol{\Delta}^N_{\mathcal{M}}(N\chi, N\kappa) - \mathbb{E}\boldsymbol{\Delta}^N_{\mathcal{M}}(N\chi, N\kappa))\mathrm{d}\chi,$$

and

$$\mathbf{M}_j^\kappa = \int_{z\in\mathbb{H};\kappa_{\mathcal{L}}(z)=\kappa} [\chi_{\mathcal{L}}(z)]^j\, \Xi(z)\mathrm{d}\chi_{\mathcal{L}}(z).$$

Then:

$$M_j^\kappa \to \mathbf{M}_j^\kappa, \text{ as } N \to \infty.$$

7.3. Covariance

To derive this statement, we use the technology developed by Bufetov and Gorin, linking the asymptotic behavior of Schur generating functions to the fluctuations for the related tiling/interlacing signature model [6, 7].

Definition 7.4. — *A sequence of symmetric functions $\{H_N(u_1, \dots, u_N)\}_{N\geqslant 1}$ is* appropriate *(or* CLT-appropriate*) if there exist two collections of real numbers $\{\gamma_k\}_{k\geqslant 1}$, $\{\lambda_{k,l}\}_{k,l\geqslant 1}$, such that the following properties are satisfied:*

- *for any N, the function $\mathbf{u} \mapsto \log H_N(\mathbf{u})$ is holomorphic in an open complex neighborhood of $B^{(N)} = (\beta_1^N, \dots, \beta_N^{(N)})$, where $\mathbf{u} = (u_1, \dots, u_N)$, and*

$$\lim_{N\to\infty} \sup_{1\leqslant i\leqslant N} |\beta_i^{(N)} - 1|N = 0.$$

- *for any index i and $k \in \mathbb{N}$, we have*
$$\lim_{N\to\infty} \frac{\partial_i^k \log H_N(\mathbf{u})}{N}\bigg|_{\mathbf{u}=B^{(N)}} = \gamma_k.$$
- *For any distinct indices i, j and any $k, l \in \mathbb{N}$, we have*
$$\lim_{N\to\infty} \partial_i^k \partial_j^l \log H_N(\mathbf{u})\big|_{\mathbf{u}=B^{(N)}} = \lambda_{k,l}$$
- *For any $s \in \mathbb{N}$ and any indices $i_1, \dots, i_s$, such that there are at least three distinct numbers in $i_1, \dots, i_s$, we have*
$$\lim_{N\to\infty} \partial_{i_1}\partial_{i_2}\dots\partial_{i_s} \log H_N(\mathbf{u})\big|_{\mathbf{u}=B^{(N)}} = 0.$$
- *The power series*
$$\sum_{k=1}^{\infty} \frac{\gamma_k}{(k-1)!}(z-1)^{k-1}, \qquad \sum_{k=1,l=1}^{\infty} \frac{\lambda_{k,l}}{(k-1)!(l-1)!}(z-1)^{k-1}(w-1)^{l-1}$$
converge in an open neighborhood of $z = 1$ and $(z, w) = (1, 1)$, respectively.

The definition of CLT-apprioriateness is extended to measures using their Schur gererating functions as follows:

Definition 7.5. — *Let as before*
$$B^{(N)} = (\beta_1^N, \dots, \beta_N^{(N)}).$$
satisfying the hypothesis
$$\lim_{N\to\infty} \sup_{1\leqslant i\leqslant N} |\beta_i^{(N)} - 1|N = 0.$$
A sequence of measures $\rho = \{\rho_N\}_{N\geqslant 1}$ is appropriate *(or* CLT-appropriate*) if the sequence of Schur generating functions $\{\mathcal{S}_{\rho_N,B^{(N)}}(u_1, \dots, u_N)\}_{N\geqslant 1}$, defined as in Definition 3.2, is appropriate in the sense of Definition 7.4.*

For such a sequence we define functions
$$A_\rho(z) = \sum_{k=1}^{\infty} \frac{\gamma_k}{(k-1)!}(z-1)^{k-1}$$
$$B_\rho(z, w) = \sum_{k=1,l=1}^{\infty} \frac{\lambda_{k,l}}{(k-1)!(l-1)!}(z-1)^{k-1}(w-1)^{l-1}$$
$$C_\rho(z, w) = B_\rho(1+z, 1+w) + \frac{1}{(z-w)^2}.$$
Here $\{\gamma_k\}_{k\geqslant 1}$ and $\{\lambda_{k,l}\}_{k,l\geqslant 1}$ are obtained from the definition of CLT-appropriate symmetric functions with $H_N = \mathcal{S}_{\rho_N,B^{(N)}}$ as in Definition 7.4.

Lemma 7.6. — *Assume that the Schur generating function of a sequence of probability measures $\{\rho_N\}_{N\geqslant 1}$ on $\mathbb{GT}_N$ satisfies the conditions*

$$\lim_{N\to\infty}\frac{\partial_1\log\mathcal{S}_{\rho_N,B^N}(u_1,\ldots,u_k,\beta_{k+1}^{(N)},\ldots,\beta_N^{(N)})}{N}=Q_1(u_1),\qquad\forall\, k\geqslant 1,$$

$$\lim_{N\to\infty}\partial_1\partial_2\log\mathcal{S}_{\rho_N,B^N}(u_1,\ldots,u_k,\beta_{k+1}^{(N)},\ldots,\beta_N^{(N)})=Q_2(u_1,u_2),\quad\forall\, k\geqslant 1,$$

$$\lim_{N\to\infty}\sup_{1\leqslant i\leqslant N}|\beta_i^{(N)}-1|N=0$$

where $Q_1(z)$, $Q_2(z,w)$ are holomorphic functions, and the convergence is uniform in a complex neighborhood of unity. Then $\{\rho_N\}$ is a (CLT-)appropriate sequence of measures with

$$A_\rho(z)=Q_1(z),\qquad B_\rho(z,w)=Q_2(z,w)$$

Let t be a positive integer, and $a_t\in(0,1]$ be a real number. Let ρ_N be a probability measure on $\mathbb{GT}_N$. Define the random variables

$$p_{k,t}^{(N)}:=\sum_{i=1}^{[a_tN]}(\lambda_i^{(t)}+[a_tN]-i)^k,$$

where $k=1,2,\ldots$, and $\lambda=(\lambda_1,\ldots,\lambda_N)$ is ρ_N-distributed.

Theorem 7.7. — *Let $\rho=\{\rho_N\}_{N\geqslant 1}$ be an appropriate sequence of measures on signatures with limiting functions $A_\rho(z)$ and $C_\rho(z,w)$, as defined in Definition 7.5. Then the collection of random variables*

$$\{N^{-k}(p_{k,t}^{(N)}-\mathbb{E}p_{k,t}^{(N)})\}_{k\in\mathbb{N};t=1,\ldots,m}$$

converges, as $N\to\infty$, in the sense of moments, to the Gaussian vector with zero mean and covariance

$$\begin{aligned}\lim_{N\to\infty}&\frac{\mathrm{cov}(p_{k_1,t_1}^{(N)},p_{k_2,t_2}^{(N)})}{N^{k_1+k_2}}\\&=\frac{a_{t_1}^{k_1}a_{t_2}^{k_2}}{(2\pi\mathbf{i})^2}\oint_{|z|=\epsilon}\oint_{|w|=2\epsilon}\left(\frac{1}{z}+1+(1+z)A_{t_1}(1+z)\right)^{k_1}\\&\qquad\times\left(\frac{1}{w}+1+(1+w)A_{t_2}(1+w)\right)^{k_2}C_{t_2}(z,w)\mathrm{d}z\mathrm{d}w,\end{aligned}$$

where

- *for $i=1,2$, $A_{t_i}(z)$ and $C_{t_i}(z)$ are analytic functions in a complex neighborhood of 1 obtained by Definition 7.5 and Lemma 7.6 from the induced measure on signatures of the $2[a_tN]$-th row of the contracting square-hexagon lattice.*

- *the z- and w-contours of integration are counter-clockwise and $\epsilon \ll 1$.*
- $1 \leqslant t_1 \leqslant t_2 \leqslant m$.

Definition 7.8. — *Let $B = (\beta_1, \beta_2, \dots, \beta_n, \dots)$. Let m be a positive integer and $\epsilon > 0$. Let Λ_ϵ^m be the space of analytic symmetric functions in the region*

$$\{(z_1, \dots, z_m) \in \mathbb{C}^m \mid z_i \in \cup_{i=1}^{\infty} O_\epsilon(\beta_i)\},$$

where

$$O_\epsilon(\beta_i) = \{z \mid |z - \beta_i| < \epsilon\}.$$

The space Λ_ϵ^m can be considered as a topological space with topology of uniform convergence in the region above. Let

$$\Lambda^m := \cup_{\epsilon>0} \Lambda_\epsilon^m$$

the topological space endowed with the topology of direct limit.

Let $\mathbf{p}_{m,n,B} : \Lambda^m \to \Lambda^n$ be a map with the properties below

(1) *$\mathbf{p}_{m,n,B}$ is a continuous linear map.*

(2) *For every $\lambda \in \mathbb{GT}_m$,*

$$\mathbf{p}_{m,n,B}\left(\frac{s_\lambda(u_1, \dots, u_m)}{s_\lambda(\beta_1, \dots, \beta_m)}\right) = \sum_{\mu \in \mathbb{GT}_n} c_{\lambda,\mu}^{\mathbf{p}_{m,n,B}} \frac{s_\mu(u_1, \dots, u_n)}{s_\mu(\beta_1, \dots, \beta_n)}, \quad c_{\lambda,\mu}^{\mathbf{p}_{m,n,B}} \geqslant 0.$$

(3) *For any $f \in \Lambda^m$ we have*

$$f(\beta_1, \dots, \beta_m) = \mathbf{p}_{m,n,B}(f)(\beta_1, \dots, \beta_n).$$

It follows from (2) and (3) that

$$\sum_{\mu \in \mathbb{GT}_n} c_{\lambda,\mu}^{\mathbf{p}_{m,n,B}} = 1.$$

Recall that we have a probability measure on dimer configurations of the square-hexagon lattice defined by (2.10), where the square-hexagon lattice has $(2N+1)$ rows and on the bottom row, the configuration has signature ω. Define a mapping $\phi : \{1, \dots, 2N+1\} \to \{\mu^{(i)}, \nu^{(j)} : i, j \in \{1, 2, \dots, N\}\}$ as follows

$$\phi(n) = \begin{cases} \mu^{\left(\frac{n-1}{2}\right)} & \text{if } n \text{ is odd,} \\ \nu^{\left(\frac{n}{2}\right)} & \text{if } n \text{ is even.} \end{cases}$$

Let $1 \leqslant n_1 \leqslant n_2 \leqslant \cdots \leqslant n_s = 2N+1$ be positive row numbers of the square-hexagon lattice, counting from the top. For $1 \leqslant i \leqslant s$, let ρ_{n_i} be the induced probability measure on dimer configurations of the n_ith row. Then the induced probability measure on the state space

$$\mathbb{GT}_{\lfloor \frac{n_1}{2} \rfloor} \times \mathbb{GT}_{\lfloor \frac{n_2}{2} \rfloor} \times \cdots \times \mathbb{GT}_{\lfloor \frac{n_s}{2} \rfloor}$$

by the measure (2.10) can be expressed as follows:

$$\text{Prob}\left(\phi(n_s),\dots,\phi(n_1)\right)=\rho_{n_s}\left((\phi(n_s)\right)\prod_{i=2}^{k}c^{\mathbf{p}_{n_i,n_{i-1},B}}_{\phi(n_i),\phi(n_{i-1})}, \tag{7.5}$$

where $c^{\mathbf{p}_{n_i,n_{i-1},B}}_{\phi(n_i),\phi(n_{i-1})}$ is the probability of $\phi(n_{i-1})$ conditional on $\phi(n_i)$. In particular, we have

$$c^{\mathbf{p}_{2t,2t-1,B}}_{\nu^{(t)},\mu^{(t-1)}}=\mathrm{pr}_B(\nu^{(t)}\to\mu^{(t-1)}),\qquad c^{\mathbf{p}_{2t+1,2t,B}}_{\mu^{(t)},\nu^{(t)}}=\mathrm{st}_B(\nu^{(t)}\to\mu^{(t-1)}),$$

where B depends on the edge weights. Moreover, we have

$$\mathbf{p}_{2t,2t-1,B}\left(\frac{s_{\nu^{(t)}}(u_1,\dots,u_{t-1},\beta_t)}{s_{\nu^{(t)}}(\beta_1,\dots,\beta_t)}\right)=\frac{s_{\nu^{(t)}}(u_1,\dots,u_{t-1},\beta_t)}{s_{\nu^{(t)}}(\beta_1,\dots,\beta_t)};$$

$$\mathbf{p}_{2t+1,2t,B}\left(\frac{s_{\mu^{(t)}}(u_1,\dots,u_{t-1},u_t)}{s_{\mu^{(t)}}(\beta_1,\dots,\beta_t)}\right)=\left(\prod_{i=1}^{t}\frac{1+u_i}{1+\beta_i}\right)\cdot\left(\frac{s_{\mu^{(t)}}(u_1,\dots,u_{t-1},u_t)}{s_{\mu^{(t)}}(\beta_1,\dots,\beta_t)}\right);$$

and for $2\leqslant i\leqslant s$

$$\mathbf{p}_{n_i,n_{i-1},B}=\mathbf{p}_{n_i,n_i-1,B}\circ\mathbf{p}_{n_i-1,n_i-2,B}\circ\cdots\mathbf{p}_{n_{i-1}+1,n_{i-1},B}.$$

For simplicity, we will denote the Schur generating function $\mathcal{S}_{\rho_N,B}$ from Definition 3.2 simply by:

$$S_{N,B^{(N)}}:=\mathcal{S}_{\rho_N,B^{(N)}}.$$

For $1\leqslant i\leqslant s$, let $S_{N,B^{(N)},n_i}$ be the Schur generating function for the induced measure ρ_{n_i} on the n_ith row of the $\mathcal{R}(N,\Omega,m)$.

We use also the following notation $\mathbf{u}_a=(u_1,\dots,u_a)$ for a positive integer a.

Lemma 7.9. — *Let $m_1,\dots,m_k$ be positive integers. Let $n_1,\dots,n_k$, $\mathbf{p}_{n_2,n_1,B},\dots,\mathbf{p}_{n_k,n_{k-1},B}$ be defined as in Definition 7.8. Assume that $(\phi(n_s),\dots,\phi(n_1))$ has distribution (7.5). Let $\mathcal{D}_k^{(m)}$ be the k-th order differential operator defined by (3.6) on functions in Λ^m. Then*

$$\begin{aligned}\mathcal{D}^{(n_1)}_{m_1}\mathbf{p}_{n_2,n_1,B}\mathcal{D}^{(n_2)}_{m_2}\mathbf{p}_{n_3,n_2,B}\cdots\mathbf{p}_{n_k,n_{k-1},B}&\\ \times\mathcal{D}^{(n_k)}_{m_k}S_{N,B^{(N)}}(u_1,\dots,u_{n_k})|_{(u_1,\dots,u_{n_k})=(\beta_1,\dots,\beta_{n_k})}&\\ =\mathbb{E}\left(\prod_{j=1}^{k}\left(\sum_{i_j=1}^{\lfloor\frac{n_j}{2}\rfloor}(\phi(n_j)_{i_j}+\lfloor\frac{n_j}{2}\rfloor-i_j)^{m_j}\right)\right)&.\end{aligned}$$

Proof. — The theorem follows from the fact that Schur functions are eigenfunctions for the operators $\mathcal{D}^{(n)}_m$, and explicit computations as in the proof of Proposition 4.3 of [7]. □

Let f be a function with r complex variables. Define its symmetrization:

$$\mathrm{Sym}_{u_1,\dots,u_r} f(u_1,\dots,u_r) := \frac{1}{r!}\sum_{\sigma\in S_r} f(u_{\sigma(1)},u_{\sigma(2)},\dots,u_{\sigma(r)}),$$

where S_r is the symmetric group of r symbols.

Let m be a positive integer and let $0 < a_1 \leqslant a_2 \leqslant \dots \leqslant a_m = 1$ be real numbers. For positive integers l and q satisfying $1 \leqslant q \leqslant m$, we introduce the notation

$$\begin{aligned}
F_{l,a_q^1}(\mathbf{u}_{[a_qN]}) :=&\ \frac{1}{S_{N,B^{(N)},2[a_qN]-1}(\mathbf{u}_{[a_qN]})V_{[a_qN]}(\mathbf{u}_{[a_qN]})}\\
&\times\sum_{i=1}^{[a_qN]}(u_i\partial_i)^l\left[S_{N,B^{(N)},2[a_qN]-1}(\mathbf{u}_{[a_qN]})V_{[a_qN]}(\mathbf{u}_{[a_qN]})\right]\\
=&\ \frac{1}{S_{N,B^N,2[a_qN]-1}(\mathbf{x})}\mathcal{D}_l S_{N,B^{(N)},2[a_qN]-1}(\mathbf{x}),\\
F_{l,a_q^2}(\mathbf{u}_{[a_qN]}) :=&\ \frac{1}{S_{N,B^{(N)},2[a_qN]}(\mathbf{u}_{[a_qN]})V_{[a_qN]}(\mathbf{u}_{[a_qN]})}\\
&\times\sum_{i=1}^{[a_qN]}(u_i\partial_i)^l\left[S_{N,B^{(N)},2[a_qN]}(\mathbf{u}_{[a_qN]})V_{[a_qN]}(\mathbf{u}_{[a_qN]})\right]\\
=&\ \frac{1}{S_{N,B^N,2[a_qN]}(\mathbf{x})}\mathcal{D}_l S_{N,B^{(N)},2[a_qN]}(\mathbf{x})
\end{aligned}$$

where $\mathcal{D}_l$ is defined by (3.6), and V_N is the Vandermonde determinant on N variables $x_1,\dots,x_N$.

For a positive integer s, let $[s] = \{1,2,\dots,s\}$.

Lemma 7.10. — *Assume that for each $r = 1,2,\dots$ $\xi_r(\mathbf{u})$ is an analytic function of $\mathbf{u}$ in an open neighborhood of 1^r. Then for any indices $b_1,\dots,b_{q+1}$ the function*

$$\mathrm{Sym}_{b_1,b_2,\dots,b_{q+1}}\left(\frac{\xi_r(\mathbf{u})}{(u_{b_1}-u_{b_2})\dots(u_{b_1}-u_{b_{q+1}})}\right) \tag{7.6}$$

is analytic in a (possibly smaller) open neighborhood of 1^r. If the degree of r in $\xi_r(\mathbf{u})$ is at most D (less than D), then the sequence (7.6) *has r-degree at most D (less than D).*

Lemma 7.11. — *Let $\{\rho_N\}_{N\geqslant 1}$ be an appropriate sequence of measures on $\mathbb{GT}_N$ with the corresponding Schur generating function $\mathcal{S}_{N,B^{(N)}}$. Let $\tau\in\{1,2\}$, then*

$$\partial_i F_{l,a_q^\tau}(\mathbf{u}) = \partial_i\left[\sum_{r=0}^{l}\binom{l}{r}(r+1)! \times \sum_{\{b_1,\dots,b_{r+1}\}\subset[N]} \mathrm{Sym}_{b_1,\dots,b_{r+1}}\left(\frac{u_{b_1}^l(\partial_{b_1}[\log S_{N,B^{(N)},2[a_qN]+\tau-2}])^{l-r}}{(u_{b_1}-u_{b_2})\dots(u_{b_1}-u_{b_{r+1}})}\right)\right]+\widehat{T}_l(\mathbf{u})$$

where the degree of N in $\partial_i F_{l,a_q^\tau}(\mathbf{u})$ is at most l, and the degree of N in $\widehat{T}(\mathbf{u})$ is less than l.

Proof. — This lemma follows from the same arguments as in the proof of Lemma 5.5 in [7]. $\square$

For positive integers l_1, l_2, q, s satisfying $1\leqslant q\leqslant s\leqslant m$, and $\tau_1,\tau_2\in\{1,2\}$, we define

$$\begin{aligned}(7.7)\quad & G_{l_1,l_2;a_q^{\tau_1},a_s^{\tau_2}}(\mathbf{u}) \\ &= l_1\sum_{r=0}^{l_1-1}\binom{l_1-1}{r}\sum_{\{b_1,\dots,b_{r+1}\}\subset[N]}(r+1)! \\ &\quad\times \mathrm{Sym}_{b_1,\dots,b_{r+1}}\frac{x_{b_1}^{l_1}\partial_{b_1}[F_{l_2,a_s^{\tau_2}}](\partial_{b_1}[\log S_{N,B^{(N)},2[a_qN]+\tau_1-2}])^{l_1-1-r}}{(u_{b_1}-u_{b_2})\dots(u_{b_1}-u_{b_{r+1}})}.\end{aligned}$$

For a subset $\{j_1,\dots,j_p\}\subseteq[s]$, let $\mathcal{P}^s_{j_1,\dots,j_p}$ be the set of all pairings of the set $[s]\setminus\{j_1,\dots,j_p\}$. Note that $\mathcal{P}^s_{j_1,\dots,j_p}=\emptyset$ if $s-p$ is odd. For a pairing P, let $\prod_{(a,b)\in P}$ be the product over all pairs (a,b) from this pairing. Finally, for each positive integer l let

$$E_{l,B^{(N)},a_q^\tau} = F_{l,a_q^\tau}\left(\beta_1^{(N)},\dots,\beta_N^{(N)}\right).$$

Proposition 7.12. — *Let ρ_N be an appropriate sequence of measures on $\mathbb{GT}_N$, $N=1,2,\dots$ Then for any positive integer m, any positive integers*

$l_1,\dots,l_m$, and $\tau_i\in\{1,2\}$, for $1\leqslant i\leqslant m$, we have

$$\lim_{N\to\infty}\frac{1}{N^{l_1+\dots+l_m}}\frac{1}{V_{[a_mN]}}\left(\sum_{i_1=1}^{[a_1N]}(u_{i_1}\partial_{i_1})^{l_1}-E_{l_1,B^{(N)},a_1^{\tau_1}}\right)\frac{S_{N,B^{(N)},2[a_1N]+\tau_1-2}}{S_{N,B^{(N)},2[a_2N]+\tau_2-2}}$$
$$\times\left(\sum_{i_1=1}^{[a_2N]}(u_{i_2}\partial_{i_2})^{l_2}-E_{l_2,B^{(N)},a_2^{\tau_2}}\right)\frac{S_{N,B^{(N)},2[a_2N]+\tau_2-2}}{S_{N,B^{(N)},2[a_3N]+\tau_3-2}}$$
$$\times\dots\times\left(\sum_{i_m=1}^{[a_mN]}(u_{i_m}\partial_{i_m})^{l_m}-E_{l_m,B^{(N)},a_m^{\tau_m}}\right)V_{[a_mN]}S_{N,B^{(N)},2[a_mN]+\tau_m-2}\Bigg|_{\mathbf{u}=B_N^{(N)}}$$
$$=\lim_{N\to\infty}\frac{1}{N^{l_1+\dots+l_m}}\sum_{P\in\mathcal{P}_\emptyset^s}\prod_{(s,t)\in P}G_{l_s,l_t;a_s^{\tau_s},a_t^{\tau_t}}(\mathbf{u})\Bigg|_{\mathbf{u}=B_N^{(N)}}.$$

Proof. — The proposition follows from the same technique as the proof of Proposition 5.12 in [7], although our definition of Schur generating functions $S_{N,B^{(N)}}$ is slightly different. □

Proof of Theorem 7.7. — The case when $\beta_i^{(N)}=1$ for all N and $1\leqslant i\leqslant N$ was proved in [7, Theorem 2.8]. We will prove it under the assumption that

$$\lim_{N\to\infty}\sup_i|\beta_i^{(N)}-1|N=0.$$

By Lemma 7.9 and Proposition 7.12, it suffices to show that

$$\lim_{N\to\infty}\frac{G_{l_1,l_2;a_1^{\tau_1},a_2^{\tau_2}}(\mathbf{u})}{N^{l_1+l_2}}\Bigg|_{\mathbf{x}=B^{(N)}}=\lim_{N\to\infty}\frac{G_{l_1,l_2;a_1^{\tau_1},a_2^{\tau_2}}(\mathbf{u})}{N^{l_1+l_2}}\Bigg|_{\mathbf{u}=1^N},\tag{7.8}$$

where by the result in [7], the right hand side of (7.8) is known to be

$$\lim_{N\to\infty}\frac{G_{l_1,l_2,a_1^{\tau_1},a_2^{\tau_2}}(\mathbf{u})}{N^{l_1+l_2}}\Bigg|_{\mathbf{x}=1^N}$$
$$=\frac{a_1^{l_1}a_2^{l_2}}{(2\pi\mathbf{i})^2}\oint_{|z|=\epsilon}\oint_{|w|=2\epsilon}\left(\frac{1}{z}+1+(1+z)A_1(1+z)\right)^{k_1}$$
$$\times\left(\frac{1}{w}+1+(1+w)A_2(1+w)\right)^{k_2}C_2(z,w)\mathrm{d}z\mathrm{d}w$$

By definition of an appropriate sequence, see Definitions 7.4, 7.5, we have

$$\lim_{N\to\infty} \frac{\partial_i[\log S_{N,B^{(N)},a_t^{\tau_t}}]}{N}\bigg|_{\mathbf{x}=B^{(N)}} = \lim_{N\to\infty} \frac{\partial_i[\log S_{N,1^N},a_t^{\tau_t}]}{N}\bigg|_{\mathbf{x}=1^N}$$

$$\lim_{N\to\infty} \partial_i\partial_j[\log S_{N,B^{(N)},a_t^{\tau_t}}]\Big|_{\mathbf{x}=B^{(N)}} = \lim_{N\to\infty} \partial_i\partial_j[\log S_{N,1^N,a_t^{\tau_t}}]\Big|_{\mathbf{x}=1^N}$$

From the expression of G_{l_1,l_2} in (7.7) and the expression of $\partial_i F_l$ in Lemma 7.11, as well as Lemma 7.10, (7.8) follows. □

Lemma 7.13. — *Assume that $\lambda(N) \in \mathbb{GT}_N$, $N = 1, 2, \ldots$ is a regular sequence of signatures such that*

$$\lim_{N\to\infty}[m(\lambda(N))] = \mathbf{m}.$$

Then

$$\lim_{N\to\infty} \partial_1\partial_2 \log\left(\frac{s_{\lambda(N)}(u_1, u_2, \ldots, u_k, 1^{N-k})}{s_{\lambda(N)}(1^N)}\right) = \partial_1\partial_2\left(1 - (u_1-1)(u_2-1)\frac{u_1 H'_{\mathbf{m}}(u_1) - u_2 H'_{\mathbf{m}}(u_2)}{u_1 - u_2}\right),$$

where the convergence is uniform over an open complex neighborhood of $(u_1, \ldots, u_k) = (1^k)$.

Proof. — See Theorem 6.7 of [8]. □

Proposition 7.14. — *Assume that $\lambda(N) \in \mathbb{GT}_N$, $N = 1, 2, \ldots$ is a regular sequence of signatures such that*

$$\lim_{N\to\infty}[m(\lambda(N))] = \mathbf{m}.$$

Let $B^{(N)} = (\beta_1^{(N)}, \ldots, \beta_N^{(N)})$ such that

$$\lim_{N\to\infty} \sup_{1\leqslant i\leqslant N} |\beta_i^{(N)} - 1|N = 0. \tag{7.9}$$

Then

$$\lim_{N\to\infty} \partial_1\partial_2 \log\left(\frac{s_{\lambda(N)}(u_1, u_2, \ldots, u_k, \beta_{k+1}^{(N)}, \ldots, \beta_N^{(N)})}{s_{\lambda(N)}\left(B^{(N)}\right)}\right) = \partial_1\partial_2\left(1 - (u_1-1)(u_2-1)\frac{u_1 H'_{\mathbf{m}}(u_1) - u_2 H'_{\mathbf{m}}(u_2)}{u_1 - u_2}\right),$$

where the convergence is uniform over an open complex neighborhood of $(u_1, \ldots, u_k) = (1^k)$.

Proof. — Note that

$$\partial_1\partial_2\log\left(\frac{s_{\lambda(N)}(u_1,\dots,u_k,\beta_{k+1}^{(N)},\dots,\beta_N^{(N)})}{s_{\lambda(N)}\left(B^{(N)}\right)}\right)=\partial_1\partial_2\log\left(\frac{s_{\lambda(N)}(u_1,\dots,u_k,\beta_{k+1}^{(N)},\dots,\beta_N^{(N)})}{s_{\lambda(N)}\left(1^N\right)}\right),$$

by Lemma 7.13, it suffices to show that

$$\lim_{N\to\infty}\partial_1\partial_2\log\left(\frac{s_{\lambda(N)}(u_1,u_2,\dots,u_k,\beta_{k+1}^{(N)},\dots,\beta_N^{(N)})}{s_{\lambda(N)}\left(1^N\right)}\right)=\lim_{N\to\infty}\partial_1\partial_2\log\left(\frac{s_{\lambda(N)}(u_1,u_2,\dots,u_k,1^{N-k})}{s_{\lambda(N)}(1^N)}\right). \tag{7.10}$$

Indeed, it suffices to show that the left hand side of (7.10) is independent of $B^{(N)}$. Recall that by Lemma 2.24, we have:

$$\frac{s_{\lambda(N)}\left(u_1,\dots,u_k,\beta_{\overline{k+1}}^{(N)},\dots,\beta_{\overline{N}}^{(N)}\right)}{s_{\lambda(N)}(1,\dots,1)} \tag{7.11}$$
$$=\left[\prod_{1\leqslant i<j\leqslant k}\frac{\log(u_i)-\log(u_j)}{u_i-u_j}\prod_{1\leqslant i\leqslant k,k+1\leqslant j\leqslant N}\frac{\log(u_i)-\log\left(\beta_{\overline{j}}^{(N)}\right)}{u_i-\beta_{\overline{j}}^{(N)}}\right]$$
$$\times\left[\prod_{k+1\leqslant i<j\leqslant N}\frac{\log\left(\beta_{\overline{i}}^{(N)}\right)-\log\left(\beta_{\overline{j}}^{(N)}\right)}{\beta_{\overline{i}}^{(N)}-\beta_{\overline{j}}^{(N)}}\right]\int_{U(N)}e^{z_N N\,\mathrm{tr}(U^*P_{N,B,k}UQ_N)}\mathrm{d}U.$$

We can compute

$$\partial_1\partial_2\log\left(\frac{s_{\lambda(N)}(u_1,u_2,\dots,u_k,\beta_{k+1}^{(N)},\dots,\beta_N^{(N)})}{s_{\lambda(N)}\left(1^N\right)}\right)=\partial_1\partial_2\left[\frac{\log(u_1)-\log(u_2)}{u_1-u_2}\right]+\partial_1\partial_2\log I_{N,B,k}(z_N),$$

where

$$I_{N,B,k}(z):=\int_{U(N)}e^{zN\,\mathrm{tr}(U^*P_{N,B,k}UQ_N)}\mathrm{d}U=\int_{U(N)}e^{zN\sum_{i,j=1}^N P_{N,B,k}(i,i)Q_N(j,j)|U(i,j)|^2}\mathrm{d}U, \tag{7.12}$$

Hence it suffices to show that

$$\lim_{N\to\infty}\partial_1\partial_2\log I_{N,B,k}(z_N)=\lim_{N\to\infty}\partial_1\partial_2\log I_{N,k}(z_N),$$

where $I_{N,k}(z)$ is given by

$$\begin{aligned}I_{N,k}(z) &:= \int_{U(N)} e^{zN\,\mathrm{tr}(U^* P_{N,k} U Q_N)} \mathrm{d}U \\ &= \int_{U(N)} e^{zN\sum_{i,j=1}^{N} P_{N,k}(i,i) Q_N(j,j) |U(i,j)|^2} \mathrm{d}U.\end{aligned} \tag{7.13}$$

Using the same technique as in the proof of Theorem 2.1 in [17], we have

$$\begin{aligned}\log I_{N,B,k}(z_N) = \sum_{g=0}^{\infty} \frac{1}{N^{2g-2}} \sum_{d=0}^{\infty} \frac{1}{d!} \sum_{|\alpha|=|\psi|=d} (-1)^{l(\alpha)+l(\psi)} \\ \times \prod_{i=1}^{l(\alpha)} \left(\frac{\sum_{t=1}^{k} (\log u_t)^{\alpha_i} + \sum_{t=k+1}^{N} (\log \beta_t)^{\alpha_i}}{N} \right) \\ \times \prod_{j=1}^{l(\psi)} \left(\frac{1}{N} \sum_{t=1}^{N} \left(\frac{\lambda_t + N - t}{N} \right)^{\psi_j} \right) H_g(\alpha, \psi)\end{aligned} \tag{7.14}$$

$$\begin{aligned}\log I_{N,k}(z_N) = \sum_{g=0}^{\infty} \frac{1}{N^{2g-2}} \sum_{d=0}^{\infty} \frac{1}{d!} \sum_{|\alpha|=|\psi|=d} (-1)^{l(\alpha)+l(\psi)} \\ \times \prod_{i=1}^{l(\alpha)} \left(\frac{\sum_{t=1}^{k} (\log u_t)^{\alpha_i}}{N} \right) \\ \times \prod_{j=1}^{l(\psi)} \left(\frac{1}{N} \sum_{t=1}^{N} \left(\frac{\lambda_t + N - t}{N} \right)^{\psi_j} \right) H_g(\alpha, \psi)\end{aligned} \tag{7.15}$$

where the $H_g(\alpha, \beta)$ are the double Hurwitz number, see [37]. Under the assumption that $\lambda(N)$ is a regular sequence of signatures and (7.9), we have:

$$\begin{aligned}\Bigg| \prod_{i=1}^{l(\alpha)} \left(\frac{\sum_{t=1}^{k} (\log u_t)^{\alpha_i} + \sum_{t=k+1}^{N} (\log \beta_t)^{\alpha_i}}{N} \right) \\ \times \prod_{j=1}^{l(\psi)} \left(\frac{1}{N} \sum_{t=1}^{N} \left(\frac{\lambda_t + N - t}{N} \right)^{\psi_j} \right) \Bigg| \\ \leqslant C_4^d \left[\max_{1 \leqslant t \leqslant k} |\log u_t| \right]^d \left(\frac{k}{N} \right)^{l(\alpha)}.\end{aligned}$$

Let

$$\Phi_{d,g,B,N}(u_1,\dots,u_k) = \frac{1}{d!}\sum_{|\alpha|=|\psi|=d}(-1)^{l(\alpha)+l(\psi)}\prod_{i=1}^{l(\alpha)}\left(\frac{\sum_{t=1}^{k}(\log u_t)^{\alpha_i}+\sum_{t=k+1}^{N}(\log\beta_t)^{\alpha_i}}{N}\right) \times\prod_{j=1}^{l(\psi)}\left(\frac{1}{N}\sum_{t=1}^{N}\left(\frac{\lambda_t+N-t}{N}\right)^{\psi_j}\right)H_g(\alpha,\psi),$$

then $\Phi_{d,g,B,N}(u_1,\dots,u_k)$ is an analytic function in an open complex neighborhood of 1^k. Let

$$\Phi_{d,g,N}(u_1,\dots,u_k)=\Phi_{d,g,1^N,N}(u_1,\dots,u_k).$$

We will compute $\partial_1\partial_2\Phi_{d,g,N}(u_1,\dots,u_k)$. Note that only those signatures α satisfying $l(\alpha)\geqslant 2$ contributes to the derivative $\partial_1\partial_2\Phi_{d,g,N}(u_1,\dots,u_k)$. In a sufficiently small complex neighborhood of 1^k, and when N is sufficiently large, by Lemma 7.15 below, we have

$$|\partial_1\partial_2\Phi_{d,g,B,N}(u_1,\dots,u_k)|\leqslant\frac{1}{N^2}\left(\frac{1}{2}\right)^d$$

Therefore for any $\epsilon>0$ there exists an integer $E\geqslant 1$, such that for any $(u_1,\dots,u_k)$ in an small open complex neighborhood of 1^k, and for any $B^{(N)}$ satisfying (7.9), for any $g\geqslant 0$, we have

$$\sum_{d\geqslant E+1}N^2|\partial_1\partial_2\Phi_{d,g,B,N}(u_1,\dots,u_k)|<\frac{\epsilon}{8}. \tag{7.16}$$

By the uniform convergence of the derivative we have

$$\partial_1\partial_2\log I_{N,B,k}(z_N)=\sum_{g=0}^{\infty}\frac{1}{N^{2g-2}}\sum_{d=0}^{\infty}\partial_1\partial_2\Phi_{d,g,B,N}(u_1,\dots,u_k).$$

As a result,

$$\begin{aligned}&|\partial_1\partial_2\log I_{N,B,k}(z_N)-\partial_1\partial_2\log I_{N,k}(z_N)| \\ &\leqslant\sum_{g=0}^{\infty}\frac{1}{N^{2g-2}}\sum_{d=1}^{E}\left|\partial_1\partial_2\left[\Phi_{d,g,B,N}(u_1,u_2,\dots,u_k)-\Phi_{d,g,N}(u_1,u_2,\dots,u_k)\right]\right| \\ &\quad+\sum_{g=0}^{\infty}\frac{1}{N^{2g-2}}\sum_{d\geqslant E+1}\left|\partial_1\partial_2\left[\Phi_{d,g,B,N}(u_1,u_2,\dots,u_k)-\Phi_{d,g,N}(u_1,u_2,\dots,u_k)\right]\right|\end{aligned} \tag{7.17}$$

By (7.16), we have

$$\sum_{g=0}^{\infty}\frac{1}{N^{2g-2}}\sum_{d\geqslant E+1}\left|\partial_1\partial_2\left[\Phi_{d,g,B,N}(u_1,u_2,\dots,u_k)-\Phi_{d,g,N}(u_1,u_2,\dots,u_k)\right]\right| \leqslant \frac{\epsilon N^2}{4(N^2-1)}<\frac{\epsilon}{2}, \tag{7.18}$$

when N is large. Moreover,

$$\begin{aligned}
&\left|\partial_1\partial_2\left[\Phi_{d,g,B,N}(u_1,u_2,\dots,u_k)-\Phi_{d,g,N}(u_1,u_2,\dots,u_k)\right]\right| \\
&= \Bigg| \frac{1}{d!}\sum_{|\alpha|=|\psi|=d}(-1)^{l(\alpha)+l(\psi)}\sum_{i=1}^{l(\alpha)}\sum_{1\leqslant j\leqslant l(\alpha),j\neq i}\frac{[\log u_1]^{\alpha_i-1}[\log u_2]^{\alpha_j-1}}{N^2u_1u_2} \\
&\quad\times\Bigg(\prod_{1\leqslant r\leqslant l(\alpha),r\neq i,r\neq j}\left(\frac{\sum_{t=1}^{k}(\log u_t)^{\alpha_r}+\sum_{t=k+1}^{N}(\log \beta_t)^{\alpha_r}}{N}\right) \\
&\qquad -\prod_{1\leqslant r\leqslant l(\alpha),r\neq i,r\neq j}\left(\frac{\sum_{t=1}^{k}(\log u_t)^{\alpha_r}}{N}\right)\Bigg) \\
&\quad\times\prod_{j=1}^{l(\psi)}\left(\frac{1}{N}\sum_{t=1}^{N}\left(\frac{\lambda_t+N-t}{N}\right)^{\psi_j}\right)H_g(\alpha,\psi)\Bigg|
\end{aligned}$$

$$= o\left(\frac{1}{N^3}\right), \tag{7.19}$$

given that $B^{(N)}$ satisfies (7.9). By (7.17), (7.18), (7.19), we have

$$\lim_{N\to\infty}\left|\partial_1\partial_2\log I_{N,B,k}(z_N)-\partial_1\partial_2\log I_{N,k}(z_N)\right|=0,$$

and the proof is complete. □

We now state the technical lemma about the generating function of the double Hurwitz numbers:

Lemma 7.15. — *Let $H_g(\alpha,\beta)$ be the double Hurwitz number as given in* (7.14) *and* (7.15)*. For each $g\geqslant 0$, let*

$$\mathbf{H}_g(z)=\sum_{d=1}^{\infty}\frac{z^d}{d!}\sum_{|\alpha|=|\beta|=d}H_g(\alpha,\beta).$$

Then the series $\mathbf{H}_g(z)$ has a radius of convergence of at least $\frac{1}{54}$ and at most $\frac{2}{27}$.

Proof. — See Theorem 3.4 of [17]. □

Let $\mathcal{R}(\Omega, \check{a})$ be a contracting square-hexagon lattice. Let $[2\kappa N]$ be the row number of a row of $\mathcal{R}(\Omega, \check{a})$ counting from the bottom. Then the induced probability measure of dimer configurations on that row is also a measure on Young diagrams given by $[(1-\kappa)N]$-tuples. We define the moment function by

$$p_j^{\kappa} := \sum_{i=1}^{(1-\kappa)N} \left(\lambda_i^{([(1-\kappa)N])} + [(1-\kappa)N] - i\right)^j . \tag{7.20}$$

Theorem 7.16. — *The collection of random variables*

$$\{N^{-j}(p_j^{\kappa} - \mathbb{E}p_j^{\kappa})\}_{0<\kappa\leqslant 1; j\in N},$$

defined by (7.20)*, is asymptotically Gaussian with limit covariance*

$$\begin{aligned}\lim_{N\to\infty} & N^{-(j_1+j_2)} \operatorname{cov}(p_{j_1}^{\kappa_1}, p_{j_2}^{\kappa_2}) \\ &= \frac{(1-\kappa_1)^{j_1}(1-\kappa_2)^{j_2}}{(2\pi\mathbf{i})^2} \oint_{|z|=\epsilon} \oint_{|w|=2\epsilon} \left(\frac{1}{z}+1+(1+z)A_{\kappa_1}(1+z)\right)^{j_1} \\ &\qquad \times \left(\frac{1}{w}+1+(1+w)A_{\kappa_2}(1+w)\right)^{j_2} Q(z,w)\mathrm{d}z\mathrm{d}w,\end{aligned}$$

where $\epsilon \ll 1$ *and* $1 \geqslant \kappa_1 \geqslant \kappa_2 > 0$,

$$A_{\kappa}(1+z) = \frac{1}{1-\kappa} H'_{\mathbf{m}_\omega}(1+z) + \frac{\kappa}{(1-\kappa)n} \sum_{l=1}^{n} \frac{1}{z+c_l+1}.$$

Here $c_l = \frac{1}{x_{2l-1}}$ *is the reciprocal of edge weights. Moreover,*

$$Q(z,w) = \partial_z \partial_w \left(\log\left(1 - \frac{zw[(1+z)H'_{\mathbf{m}_\omega}(1+z) - (1+w)H'_{\mathbf{m}_\omega}(1+w)]}{z-w}\right)\right) + \frac{1}{(z-w)^2}$$

Proof. — The theorem follows directly from Proposition 7.14 and Theorem 7.7. □

Proof of Theorem 7.3. — Theorem 7.3 follows from Theorem 7.16 in a similar way as the proof of Theorem 6.3 in [8]. □

8. Examples

In this section, we study a few examples of contracting square-hexagon lattices. By applying the theory developed in previous work and this paper, we explicitly find the limit shapes and the frozen boundaries of perfect matching on these lattices. The examples with parameters $x_i = 1$ are studied in Section 8.1; while the examples with general periodic parameters $x_i \neq 1$ are studied in Section 8.2.

8.1. Examples with $x_i = 1$

8.1.1. An Aztec rectangle

Figure 8.1 shows a domino tiling of a large Aztec rectangle, sampled from the Boltzmann measure with 1×4 periodic weights given by $x_1 = x_2 = 1$, $y_1 = 4$, $y_2 = \frac{1}{4}$. Here the number of distinct y_i's in a period is equal to $m = 2$ and the number of distinct parts for the boundary partition ω is equal to $s = 4$.

This typical tiling, as explained in Section 5, exhibits a spatial phase separation between frozen regions close to the boundary, and a liquid region in the middle. The *frozen boundary* separating the phases converges in probability to an algebraic curve which, as one can see from Figure 8.1, has $(m+1)s-1 = 11$ tangent points with the bottom boundary. In each interval $[a_j, b_j]$ for $1 \leqslant j \leqslant s = 4$ of the bottom boundary, the frozen boundary has 2 tangent points, while in each interval (b_j, a_{j+1}) for $1 \leqslant j \leqslant s-1 = 3$, the frozen boundary has 1 tangent point.

8.1.2. A simple square-hexagon lattice

Consider a square-hexagon lattice $\mathrm{SH}(\check{a})$ in which

$$a_i = \begin{cases} 0 & \text{if } i \text{ is odd} \\ 1 & \text{if } i \text{ is even.} \end{cases} \tag{8.1}$$

See the left graph of Figure 8.2 for a subgraph of the original square-hexagon lattice, and the right graph Figure 8.2 for a subgraph by translating every row to start from the same vertical line.

Assume that the contracting square-hexagon lattice $\mathcal{R}(\Omega, \check{a})$ with $\check{a}$ satisfying (8.1) has boundary condition specified by

$$\Omega = (A_1, A_1+1, \dots, B_1-1, B_1, A_2, A_2+1, \dots, B_2-1, B_2,\\ A_3, A_3+1, \dots, B_3-1, B_3).$$

i.e. the boundary partition ω has three distinct parts of macroscopic size, repeated a macroscopic number of times.

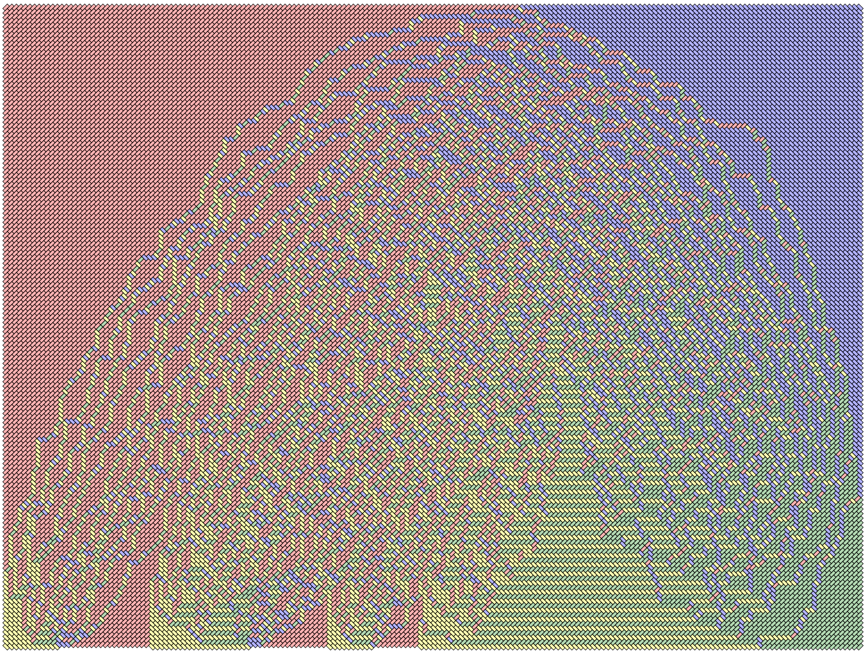


Figure 8.1. A domino tiling of a large Aztec rectangle with 1×4 periodic weights, and a boundary partition with parts taking 4 distinct values. It exhibits a frozen boundary with 11 tangent points on the boundary

Figure 8.2. A subgraph of a square hexagon lattice: the left graph represents a subgraph of the original lattice; the right graph represents the lattice obtained by letting all the rows of the left graph start from the same vertical line.

We give two realizations of the dimer model on a large graph $\mathcal{R}(\Omega, \check{a})$ on Figures 8.3 and 8.4. Instead of drawing dimers on the graph, we draw only the particles from the bijective correspondence with finite Maya diagrams of Section 2.3.

Figure 8.3 shows the uniform dimer model (i.e. all the edge weights are 1) on a contracting square-hexagon lattice $\mathcal{R}(\Omega, \check{a})$, whereas Figure 8.4 shows a random dimer configuration of the same graph but with periodic weights with a fundamental domain consisting of 8 rows, and edge weights $x_1 = x_2 = x_3 = x_4 = 1$, $y_1 = 3, y_3 = 0.5$. We can see that in the periodic case, the frozen boundary has more tangent points with the horizontal line $\kappa = 0$ compared to the uniform case.

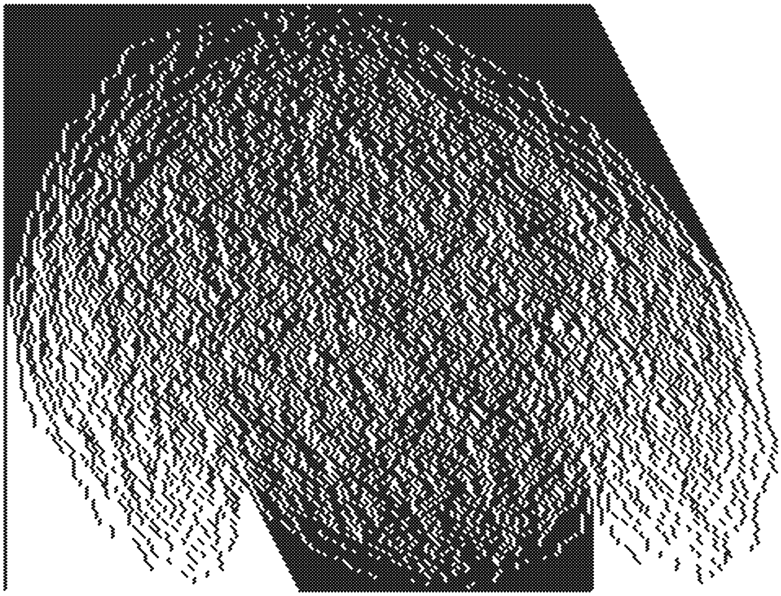


Figure 8.3. Limit shape of perfect matchings on the square-hexagon lattice with uniform weights $x_1 = x_2 = x_3 = x_4 = 1 = y_1 = y_3$.

We may also consider contracting square hexagon lattices with $\check{a}$ satisfying (8.1) and other boundary conditions. For example, we may consider the counting measures corresponding to the signatures on the bottom boundary converge to a uniform measure $\mathbf{m}_\omega$ on $[0, 2]$ as $N \to \infty$. Assume that each fundamental domain consists of four rows, $x_1 = x_2 = 1$, and $c_1 = \frac{1}{y_1} = 1$. In this case we have

$$\mathrm{St}_{\mathbf{m}_\omega}(t) = -\frac{1}{2} \log\left(1 - \frac{2}{t}\right).$$

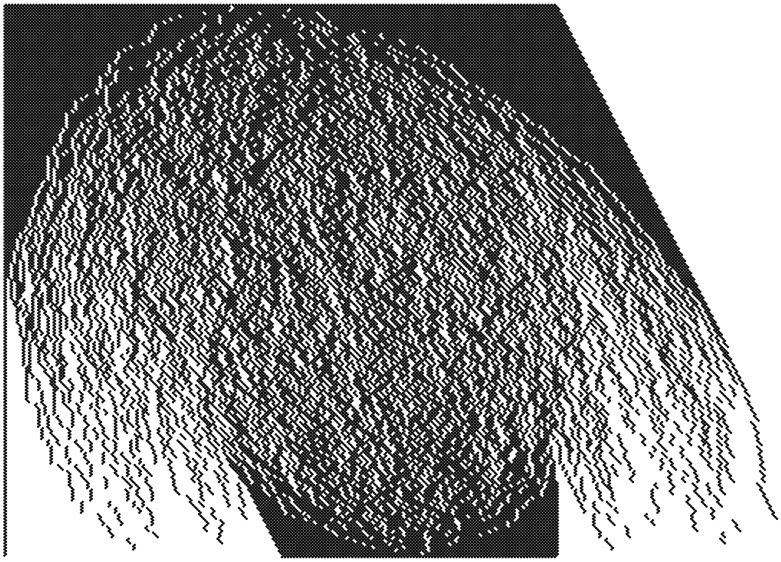


Figure 8.4. Limit shape of perfect matchings on the square-hexagon lattice with periodic weights $x_1 = x_2 = x_3 = x_4 = 1, y_1 = 3, y_3 = 0.5$.

Then we solve the equation $\mathrm{St}_{\mathbf{m}_\omega}(t) = \log z$ for t and substitute it into (4.3), we have

$$\frac{z[(4-\kappa)z - 3\kappa]}{2(z+1)(z-1)} = \chi. \tag{8.2}$$

Hence the frozen boundary is given by the condition that the discriminant of (8.2) is 0, which gives

$$16\chi^2 + 9\kappa^2 + 8\chi\kappa - 32\chi = 0;\ \kappa \geqslant 0.$$

See Figure 8.5 for the frozen boundary in that case, which is not a full closed real algebraic curve inscribed in the domain, contrary to the "stepped case" above.

8.2. An example with $\lim_{N\to\infty} x_i^{(N)} \neq 1$

We show on a particular example how computations above can be adapted to take into account situations when the weights $x_i^{(N)}$ do not go to 1. The boundary condition considered is given by a staircase partition where the steps have constant height, not depending on N. More precisely, consider a contracting square-hexagon lattice $\mathcal{R}(\Omega, \check{a})$ with edge weights

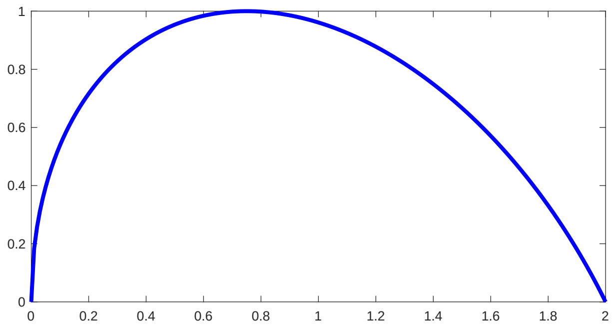


Figure 8.5. Frozen boundary of contracting square-hexagon lattices with $x_1 = x_2 = y_1 = 1$, a_{2i-1}=0, $a_{2i} = 1$, and uniform counting measure on $[0, 2]$ for the signatures on the lower boundary

assigned as in Proposition 2.18. Assume the configuration on the boundary row is given by the following very specific partition:

$$\lambda(N) = ((M-1)(N-1), (M-1)(N-2), \dots, (M-1), 0), \tag{8.3}$$

where $M \geqslant 1$ is a positive integer. In other words, there are N vertices remaining in the boundary row in total; the leftmost vertex and the rightmost vertex are remaining vertices in the boundary row; between each pair of nearest remaining vertices on the boundary row, there are $(M-1)$ removed vertices. This distribution in the limit as N goes to infinity converges to a uniform measure on the whole interval $[0, M]$. By [33, Example 1.3.7], we have

$$s_{\lambda(N)}(x_1, \dots, x_N) = \prod_{1 \leqslant i < j \leqslant N} \frac{x_i^M - x_j^M}{x_i - x_j}$$

when all the x_i's are distinct, and extended by continuity when some x_i's are equal.

We further assume that the edge weights are assigned periodically with a finite period n. That is, for each $i \in \mathbb{N}$, and $\bar{i} = i \bmod n$

$$x_i = x_{\bar{i}}; \tag{8.4}$$

and if $i \in I_2$,

$$y_i = y_{\bar{i}}. \tag{8.5}$$

By Proposition 2.18, the partition function of dimer configurations on the graph $\mathcal{R}(\Omega, \check{a})$ can be computed by the following formula

$$Z_N = \left[\prod_{i\in I_2}\prod_{t=i+1}^{N}(1+y_i x_t)\right] s_{\lambda(N)}(x_1,\ldots,x_N).$$

When the edge weights satisfy (8.4) and (8.5), we may compute the free energy as follows, distinguishing the cases where $x_i = x_j$ or not in the expression above:

$$\begin{aligned}\mathcal{F} &:= \lim_{N\to\infty}\frac{1}{N^2}\log Z_N \\ &= \frac{1}{n^2}\left[\sum_{1\leqslant i<j\leqslant n}\log\left(\frac{x_i^M - x_j^M}{x_i - x_j}\right) + \frac{1}{2}\sum_{1\leqslant i\leqslant n}\log(M x_i^{M-1})\right. \\ &\qquad\qquad\left. + \frac{1}{2}\sum_{i\in I_2\cap\{1,2,\ldots,n\}}\sum_{t=1}^{n}\log(1+y_i x_t)\right].\end{aligned}$$

Here we assumed that all the weights x_i's in a fundamental domain are distinct. The case when some weights coincide is obtained by continuity.

We can also compute the Schur generating function for the random partition corresponding to the random dimer configuration on each row of the graph, and obtain in this particular case an explicit analogue of Proposition 3.7 giving the moments of the limiting distribution of particles $\mathbf{m}^{\kappa}$, at macroscopic height $\kappa \in [0,1]$.

Let ρ^k be the distribution of the random partition corresponding to the random dimer configuration on the kth row. By Lemma 3.6, we have

$$\begin{aligned}&\mathcal{S}_{\rho^k, X^{(N-t)}}(u_1,\ldots,u_{N-t}) \\ &= \frac{s_{\lambda(N)}(u_1,\ldots,u_{N-t},x_1,\ldots,x_t)}{s_{\lambda(N)}(x_1,\ldots,x_N)}\prod_{i\in\{1,2,\ldots,t/t+1\}\cap I_2}\prod_{j=1}^{N-t}\left(\frac{1+y_i u_j}{1+y_i x_{t+j}}\right).\end{aligned}$$

for $k = 2t+1$ or $k = 2t+2$. When the edge weights are assigned periodically as in (8.4) and (8.5), and letting $N\to\infty$, $\frac{t}{N}\to\kappa\in[0,1)$, we have

$$\begin{aligned}\lim_{(1-\kappa)N\to\infty}\frac{1}{(1-\kappa)N}\log\mathcal{S}_{\rho^k, X^{(N-t)}}(u_1,\ldots,u_l,x_{t+1+l},\ldots,x_{N-t}) \\ = \sum_{1\leqslant i\leqslant l}\left[Q_\kappa(u_i) - Q_\kappa(x_{t+i})\right],\end{aligned}$$

where

$$Q_\kappa(u) = \frac{1}{1-\kappa}\left[\frac{1}{n}\sum_{1\leqslant j\leqslant n}\log\left(\frac{u^M - x_j^M}{u-x_j}\right) + \frac{\kappa}{n}\sum_{i\in\{1,2,\dots,n\}\cap I_2}\log(1+y_i u)\right].$$

Let $p \geqslant 1$ be a positive integer. Let $\rho_{\lfloor(1-\kappa)N\rfloor}$ be the probability measure on the row of the square-hexagon lattice with $\lfloor(1-\kappa)N\rfloor$ present V-edges, and let $\mathbf{m}_{\rho_{\lfloor(1-\kappa)N\rfloor}}$ be the corresponding random counting measure. Let $\mathcal{N} = \lfloor(1-\kappa)N\rfloor$. Let $U = (u_1,\dots,u_N)$ and $X = (x_1,\dots,x_N)$ satisfying $x_{i \bmod n} = x_i$.

Following similar computations as the proof of [6, Theorem 5.1], we have that the leading term for

$$\mathcal{N}^{p+1}\int_{\mathbb{R}} x^p \mathbf{m}_{\rho_{\lfloor(1-\kappa)N\rfloor}}$$

is given by

$$\mathcal{M}_{p,\mathcal{N}} := \lim_{U\to X}\sum_{i=1}^{\mathcal{N}}\sum_{l=0}^{p}\mathcal{N}^{p-l}\binom{p}{l}u_i^p[Q'_\kappa(u_i)]^{p-l}\left(\sum_{j\in\{1,2,\dots\mathcal{N}\}\setminus\{i\}}\frac{1}{u_i-u_j}\right)^l.$$

First we assume that the edge weights $x_1,\dots,x_n$ are pairwise distinct. Let

$$S_{\mathcal{N}}(i) = \{j\in\{1,2,\dots,\mathcal{N}\} : j \mod n = i\} = \{an+i; 0\leqslant a\leqslant\lfloor\mathcal{N}/n\rfloor\}.$$

Then

$$\begin{aligned}\mathcal{M}_{p,\mathcal{N}} = \lim_{U\to X}&\sum_{i=1}^{\mathcal{N}}\sum_{l=0}^{p}\mathcal{N}^{p-l}\binom{p}{l}u_i^p[Q'_\kappa(u_i)]^{p-l}\\ &\times\left[\sum_{k=0}^{l}\binom{l}{k}\left(\sum_{j\in\{1,2,\dots,\mathcal{N}\}\setminus S_{\mathcal{N}}(i)}\frac{1}{u_i-u_j}\right)^{l-k}\left(\sum_{j\in S_{\mathcal{N}}(i)\setminus\{i\}}\frac{1}{u_i-u_j}\right)^k\right],\end{aligned}$$

and

$$\begin{aligned}\lim_{N\to\infty}\frac{\mathcal{M}_{p,\mathcal{N}}}{\mathcal{N}^{p+1}} = \lim_{U\to X}&\frac{1}{n}\sum_{i=1}^{n}\sum_{k=0}^{p}\frac{p!}{k!(p-k)!}\frac{1}{n^k(k+1)!}\\ &\times\frac{\partial^k}{\partial u^k}\left[u^p\left(Q'_\kappa(u)+\frac{1}{n}\sum_{1\leqslant j\leqslant n, j\neq i}\frac{1}{u-x_j}\right)^{p-k}\right]\Bigg|_{u=x_i}.\end{aligned}$$

Note that

$$Q_\kappa'(u) = \frac{1}{n(1-\kappa)} \sum_{1\leqslant j\leqslant n} \left(\frac{Mu^{M-1}}{u^M - x_j^M} - \frac{1}{u-x_j} \right) + \frac{\kappa}{n(1-\kappa)} \sum_{i\in\{1,2,\dots,n\}\cap I_n} \frac{y_i}{1+y_i u}.$$

Using residue and following similar computations, we have

$$\int_{\mathbb{R}} x^p \mathbf{m}^\kappa(\mathrm{d}x) = \frac{1}{2(p+1)\pi\mathbf{i}} \oint_{C_{x_1,\dots,x_n}} \frac{\mathrm{d}z}{z} \left(zQ_\kappa'(z) + \sum_{j=1}^n \frac{z}{n(z-x_j)} \right)^{p+1},$$

where $C_{x_1,\dots,x_n}$ is a simple, closed, positively oriented, contour containing only the poles $x_1,\dots,x_n$ of the integrand, and no other singularities.

In the case that some of the edge weights $x_1,\dots,x_n$ may be equal, one has to be separate terms in $\mathcal{M}_{p,\mathcal{N}}$ and introduce instead of $S_{\mathcal{N}}(i)$ the set $T_{\mathcal{N}}(i) = \{j \in \{1,2,\dots,\mathcal{N}\} : x_j = x_i\}$ but at the end, we arrive to the same expression for the moment of $\mathbf{m}^\kappa$.

In [32], it is proved that when the boundary partition differs from (8.3) by at most one component at the beginning, the limit shape is the same as when the boundary partition is given by (8.3).

Define $F_{\kappa,M}(z) = zQ_\kappa'(z) + \sum_{j=1}^n \frac{z}{n(z-x_j)}$. Adapting again the computations in [6], we can compute the Stieltjes transform of the measure $\mathbf{m}^\kappa$ when x is in a neighborhood of infinity by

$$\begin{aligned}\mathrm{St}_{\mathbf{m}^\kappa}(x) &= \sum_{j=0}^\infty x^{-(j+1)} \int_{\mathbb{R}} y^j \mathbf{m}^\kappa(\mathrm{d}y) \\ &= \sum_{j=0}^\infty \frac{1}{2(j+1)\pi\mathbf{i}} \oint_{C_{x_1,\dots,x_n}} \left(\frac{F_{\kappa,M}(z)}{x}\right)^{j+1} \frac{\mathrm{d}z}{z} \\ &= -\frac{1}{2\pi\mathbf{i}} \oint_{C_{x_1,\dots,x_n}} \log\left(1 - \frac{F_{\kappa,M}(z)}{x}\right) \frac{\mathrm{d}z}{z}.\end{aligned}$$

Integration by parts gives

$$\mathrm{St}_{\mathbf{m}^\kappa}(x) = \frac{1}{2\pi\mathrm{i}} \left[\oint_{C_{x_1,\dots,x_n}} \log z \frac{\frac{\mathrm{d}}{\mathrm{d}z}\left(1 - \frac{F_{\kappa,M}(z)}{x}\right)}{1 - \frac{F_{\kappa,M}(z)}{x}} \mathrm{d}z - \oint_{C_{x_1,\dots,x_n}} \mathrm{d}\left(\log z \log\left(1 - \frac{F_{\kappa,M}(z)}{x}\right)\right)\right].$$

Because $F_{\kappa,M}(z)$ has a Laurent series expansion in a neighborhood of x_i given by

$$F_{\kappa,M}(z) = \frac{x_j}{n(z-x_j)} + \sum_{k=0}^{\infty} \alpha_k (z-x_j)^k,$$

$F_{\kappa,M}(z) = x$ has exactly one root in a neighborhood of x_i for $1 \leqslant i \leqslant n$, and thus, we can find a unique composite inverse Laurent series given by

$$G_{\kappa,M,j}(w) = x_j + \sum_{i=1}^{\infty} \frac{\beta_i^{(j)}}{w^i},$$

such that $F_{\kappa,M}(G_{\kappa,M,j}(w)) = w$ when w is in a neighborhood of infinity. Then

$$z_i(x) = G_{\kappa,M,j}(x) \tag{8.6}$$

is the unique root of $F_{\kappa,M}(z) = x$ in a neighborhood of x_i.

Since $1 - \frac{F_{\kappa,M}}{x}$ has exactly one zero $z_i(x)$ and one pole x_i in a neighborhood of x_i, we have

$$\oint_{x_i} \mathrm{d}\left(\log z \log\left(1 - \frac{F_{\kappa,M}(z)}{x}\right)\right) = 0;$$

and therefore

$$\mathrm{St}_{\mathbf{m}_\kappa}(x) = \sum_{j=1}^{n} \log(z_j(x)) - \log x_j$$

when x is in a neighborhood of infinity. By the complex analyticity of both sides of (8.2), we infer that (8.2) holds whenever x is outside the support of $\mathbf{m}_\kappa$.

Lemma 8.1. — *Assume the liquid region is nonempty, and assume that for any $x \in \mathbb{R}$, $F_{\kappa,M}(z)$ has at most one pair of complex conjugate roots. Then for any point (χ,κ) lying on the frozen boundary, the equation $F_{\kappa,M} = \frac{\chi}{1-\kappa}$ has double roots.*

Proof. — By Lemma 4.10, the continuous density $f_{\mathbf{m}_\kappa}(x)$ of the measure $\mathbf{m}_\kappa(x)$ with respect to the Lebesgue measure is given by

$$f_{\mathbf{m}_\kappa}(x) = -\lim_{\epsilon\to 0+} \frac{1}{\pi} \Im[\mathrm{St}_{\mathbf{m}_\kappa}(x + \mathbf{i}\epsilon)]$$

By (8.2), we have

$$f_{\mathbf{m}_\kappa}(x) = -\lim_{\epsilon\to 0+} \frac{1}{\pi} \operatorname{Arg}(\prod_{i=1}^{n} z_i(x + \mathbf{i}\epsilon)).$$

If the liquid region is nonempty, and for any $x \in \mathbb{R}$, $F_{\kappa,M} = x$ has at most one pair of complex conjugate roots, then for each point (χ,κ) in the liquid region, there is exactly one of roots $z_j\left(\frac{\chi}{1-\kappa}+\mathbf{i}\epsilon\right)$ from (8.6) converging to a non-real root of $F_{\kappa,M}(z)=\frac{\chi}{1-\kappa}$; and all the others converge to real roots of $F_{\kappa,M}(z)=\frac{\chi}{1-\kappa}$. Then, the lemma follows from the complex analyticity of the density of the limit measure with respect to (χ,κ). □

Let us now be more specific with the particular cases when M is equal to 1 or 2.

8.2.1. $M=1$

When $M=1$, all the vertices on the bottom row are V-vertices. Let

$$\begin{aligned}F_\kappa(z):=F_{\kappa,M=1}(z)&=\frac{\kappa z}{n(1-\kappa)}\sum_{i\in\{1,2,\dots,n\}\cap I_2}\frac{y_i}{1+y_iz}+\sum_{j=1}^{n}\frac{z}{n(z-x_j)}\\&=K-\frac{\kappa}{n(1-\kappa)}\sum_{i=1}^{m}\frac{n_i\gamma_i}{z+\gamma_i}+\frac{1}{n}\sum_{j=1}^{n}\frac{x_j}{z-x_j}.\end{aligned}$$

where $\gamma_1,\dots,\gamma_m$ are the distinct values of $\frac{1}{y_i}, i\in I_2\cap\{1,\dots,n\}$, with respective multiplicities $n_1,\dots,n_m$, and K the constant $\frac{1}{1-\kappa}-(n-r)\frac{\kappa}{1-\kappa}$.

Let us call m' the number of distinct values of x_j's for $j\in\{1,\dots,n\}$. The equation

$$F_\kappa(z)=\frac{\chi}{1-\kappa}\tag{8.7}$$

has at least a real solution for z between two consecutive $-\gamma_i$, and between two consecutive (distinct) x_j. This gives at least $m+m'-2$ real roots. As it can be written as a polynomial equation in z of degree $m+m'$, we obtain then that Equation (8.7) as at most a pair of complex conjugate roots.

The frozen boundary C is given by the condition that the two complex conjugate roots of (8.7) merge to a double root. More precisely, let

$$U(z)=\frac{z}{n}\sum_{i\in\{1,2,\dots,n\}\cap I_2}\frac{y_i}{1+y_iz},\qquad V(z)=\sum_{j=1}^{n}\frac{z}{n(z-x_j)}.$$

Then the frozen boundary has a parametric equation (with parameter z) as follows

$$\begin{cases}\chi=\kappa U(z)+(1-\kappa)V(z)\\0=\kappa U'(z)+(1-\kappa)V'(z).\end{cases}$$

Hence we have

$$(\chi,\kappa)=\left(\frac{U(z)V'(z)-U'(z)V(z)}{V'(z)-U'(z)},\frac{V'(z)}{V'(z)-U'(z)}\right);\tag{8.8}$$

for (χ, κ) on the frozen boundary. The dual curve of the frozen boundary has parametric equation

$$(x, y) = \left(-\frac{1}{V(z)}, \frac{U(z) - V(z)}{V(z)}\right). \tag{8.9}$$

The values of z for which the ratio x/y is 0 correspond to horizontal tangency points of the frozen boundary.

For $1 \leqslant j \leqslant m$, $z = -\gamma_j$ corresponds to the point $(V(-\gamma_j), 0)$ of the frozen boundary, and the slope x/y for the corresponding point on the dual curve is 0, because $U(-\gamma_j) = \infty$. This gives thus m points on the line $\kappa = 0$ with horizontal tangent for the frozen boundary.

When z is one of the x_i's, the corresponding point on the frozen boundary is $(U(x_i), 1)$ and the corresponding tangent line is also horizontal. If m' is the number of distinct x_i's in a fundamental domain, then we get m' horizontal tangency points on the line $\kappa = 1$ for the frozen boundary.

By slightly adapting the proof of Proposition 5.4, and checking the definition of cloud curve for this case, one obtains the following

Proposition 8.2. — *The frozen boundary given by the parametric equation* (8.8) *is a cloud curve of rank $m + m'$.*

8.2.2. $M = 2$

When $M = 2$, the counting measure on the bottom row converges to the uniform measure on $[0, 2]$ when $N \to \infty$. Let

$$F_\kappa(z) := F_{\kappa, M=2}(z) = \frac{\kappa z}{n(1-\kappa)} \sum_{i \in \{1,2,\ldots,n\} \cap I_2} \frac{y_i}{1 + y_i z} + \sum_{j=1}^{n} \frac{z}{n(z - x_j)} + \frac{z}{n(1-\kappa)} \sum_{1 \leqslant j \leqslant n} \frac{1}{z + x_j}.$$

As in the case $M = 1$ discussed above, Equation (8.7) for $M - 2$ has at most one pair of complex conjugate roots, and the parameters z for which the two complex conjugate roots merge to a double root correspond to the frozen boundary. More precisely, let

$$W(z) = \frac{z}{n} \sum_{1 \leqslant j \leqslant n} \frac{1}{z + x_j},$$

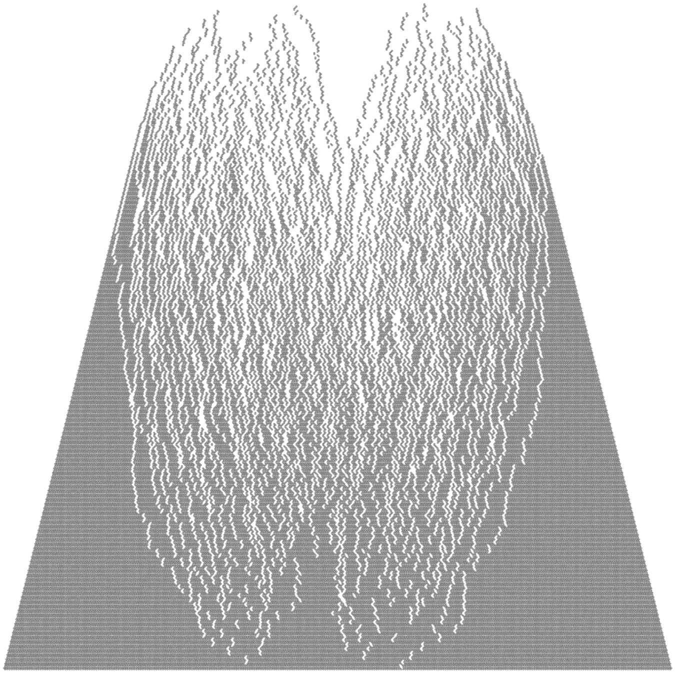


Figure 8.6. Limit shape of perfect matchings on the square-hexagon lattice with weights $y_1 = 3, x_1 = 2ma, x_2 = 0.8, y_3 = 0.5, x_3 = 1.4, x_4 = 1.8$ and $M = 1$

then

$$(\chi,\kappa) = \left(\frac{W'(z)U(z)+V'(z)U(z)-U'(z)V(z)-W'(z)V(z)}{V'(z)-U'(z)} + W(z), \frac{V'(z)+W'(z)}{V'(z)-U'(z)}\right),$$

for (χ,κ) on the frozen boundary. The dual curve of the frozen boundary has parametric equation

$$(x,y) = \left(-\frac{1}{V(z)+W(z)}, \frac{U(z)-V(z)}{V(z)+W(z)}\right).$$

If we have m' distinct values of x_i's in the fundamental domain, then for $z = x_i$, we get that the points $(U(x_j)+W(x_j),1)$ are m' tangent points of the frozen boundary to the line $\kappa = 1$.

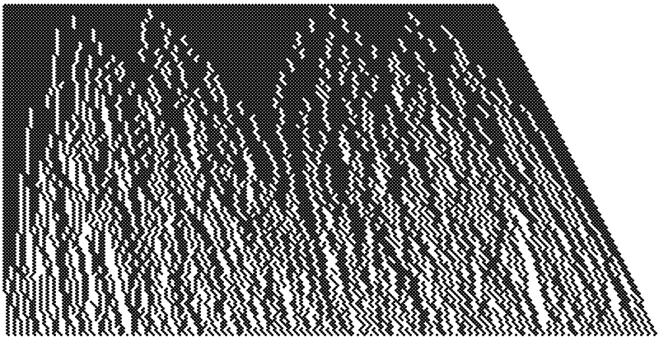


Figure 8.7. Limit shape of perfect matchings on the square-hexagon lattice with weights $y_1 = 3, x_1 = 10, x_2 = 0.1, y_3 = 0.5, x_3 = 3.0, x_4 = 0.3$ and $M = 2$

8.3. General Case

In general, it is possible that the equation $f_{\kappa,M}(z) = \frac{\chi}{1-\kappa}$ has more than one pair of complex conjugate roots. For example, let $M = 3$, $I_2 = \emptyset$, $x_1 = 1$ and $x_2 = 2$. In this case, we have

$$F_{\kappa,3}(z) = \frac{z}{2}\left(\frac{1}{z-1} + \frac{1}{z-2}\right) + \frac{z}{2(1-\kappa)}\left(\frac{2z+1}{z^2+z+1} + \frac{2z+2}{z^2+2z+4}\right).$$

For $\chi = 1$ and $\kappa = 0.5$, it is not hard to check that the equation $F_{\kappa,3} = \frac{\chi}{1-\kappa}$ has 2 real roots and 4 non-real roots.

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