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ON A MEASURABLE ANALOGUE OF SMALL TOPOLOGICAL FULL GROUPS II

by François LE MAÎTRE (*)

ABSTRACT. — We pursue the study of L^1 full groups of graphings and of the closures of their derived groups, which we call derived L^1 full groups. Our main result shows that aperiodic probability measure-preserving actions of finitely generated groups have finite Rokhlin entropy if and only if their derived L^1 full group has finite topological rank. We further show that a graphing is amenable if and only if its L^1 full group is, and explain why various examples of (derived) L^1 full groups fit very well into Rosendal's geometric framework for Polish groups. As an application, we obtain that every abstract group isomorphism between L^1 full groups of amenable ergodic graphings must be a quasi-isometry for their respective L^1 metrics. We finally show that L^1 full groups of rank one transformations have topological rank 2.

RÉSUMÉ. — Nous poursuivons l'étude des groupes pleins L^1 de graphages ainsi que des adhérences de leurs groupes dérivés, que nous appelons groupes pleins L^1 dérivés. Notre résultat principal montre que toute action préservant la mesure de probabilité d'un groupe de type fini est d'entropie de Rokhlin finie si et seulement si son groupe plein L^1 dérivé est de rang topologique fini. Nous montrons également que tout graphage est moyennable si et seulement si son groupe plein L^1 l'est, et présentons des exemples variés de groupes pleins L^1 (parfois dérivés) qui rentrent dans le cadre de la géométrie des groupes polonais construits par Rosendal. On en déduit que tout isomorphisme abstrait entre des groupes pleins L^1 de graphages ergodiques moyennables doit être une quasi-isométrie pour leurs distances L^1 respectives. Enfin, on montre que les groupes pleins L^1 des transformations de rang 1 sont de rang topologique 2.

1. Introduction

In his pioneering work [9], Dye defined *full* subgroups of the group $\text{Aut}(X, \mu)$ of all measure-preserving transformations of a standard probability space (X, μ) as those subgroups \mathbb{G} which are stable under cutting

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and pasting their elements along a countable measurable partition: for every measurable partition $(A_n)_{n \in \mathbb{N}}$ of X and every sequence $(T_n)_{n \in \mathbb{N}}$ of elements of \mathbb{G} such that $(T_n(A_n))_{n \in \mathbb{N}}$ is also a partition of X , the measure preserving transformation T defined by $T(x) = T_n(x)$ for all $x \in A_n$ also belongs to \mathbb{G} . Then, given a measure-preserving action of a countable group Γ on a standard probability space (X, μ) (i.e. a homomorphism $\alpha : \Gamma \rightarrow \text{Aut}(X, \mu)$), the smallest full subgroup of $\text{Aut}(X, \mu)$ containing $\alpha(\Gamma)$ is called the *full group* $[\alpha(\Gamma)]$ of the action α .

The full group of the action α can be concretely understood as the group of all measure-preserving transformations T such that $T(x)$ belongs to the Γ -orbit of x for all $x \in X$. Moreover, if two actions are conjugate, then their full groups are isomorphic. It was then realized that the full group was exactly capturing the partition of the space into orbits induced by the action, leading to the notion of *orbit equivalence* which is by now a fairly large subject on its own (see for instance the survey [10]).

The present paper is about a family of smaller subgroups which one can associate to any measure-preserving action of any *finitely generated* group. These were defined in [22] and are called L^p full groups, although they are not full as subgroups of $\text{Aut}(X, \mu)$, but only *finitely full*, which means that they are stable under cutting and pasting their elements along a *finite* measurable partition.

Their definition uses the extra geometric structure provided by the Schreier graph of the action. To be more precise, given a measure preserving action $\alpha : \Gamma \rightarrow \text{Aut}(X, \mu)$ of a finitely generated group Γ and a finite symmetric generating set S for Γ , let us first denote by d_S the path metric associated to the Schreier graph of the action with respect to the set S . For $p \in [1, +\infty[$, we then define the L^p full group of (α, S) be the group of all $T \in [\alpha(\Gamma)]$ such that the map

$$x \mapsto d_S(x, T(x))$$

is in $L^p(X, \mu)$. Changing the generating set S does not change the L^p full group, so the L^p full group of the action α itself is well-defined, and we denote it by $[\alpha(\Gamma)]_p$. If two actions are conjugate, then their L^p full groups are isomorphic. Moreover, the natural L^p metric on L^p full groups is complete, separable, right-invariant and endows the L^p full groups with a group topology which does not depend on the choice of the finite generating set. It is thus natural to ask the following.

QUESTION. — *How are properties of the action reflected in the L^p full group as a topological group?*

One of the natural invariants of topological groups is the *topological rank*, namely the smallest number of elements needed to generate a dense subgroup, so it is natural to ask how the action influences the topological rank of the associated L^p full group.

In [22] we showed that the topological rank of the L^1 full group of an ergodic \mathbb{Z} -action was finite if and only if the action had finite entropy. The main goal of the present paper is to extend this result to general measure-preserving actions of finitely generated groups. In order to do so, the right notion of entropy is *Rokhlin entropy*, which was recently introduced by Seward [32]. It is the infimum of the entropies of generating partitions and, for \mathbb{Z} -actions, it coincides with the usual definition of entropy of a measure-preserving transformation by a result of Rokhlin [29]. A crucial result of Seward is that the finiteness of the Rokhlin entropy characterizes measure-preserving subshifts, generalizing Krieger's theorem to the widest imaginable setup (see Section 2.4 for more on this).

Our proof that ergodic \mathbb{Z} -actions of finite entropy have a topologically finitely generated L^1 full group used crucially the *derived L^1 full group*, defined as the closure of the subgroup generated by commutators in the L^1 full group of the action⁽¹⁾. Indeed, the latter is far easier to understand, in particular it is topologically generated by naturally defined involutions arising from the generating set of Γ (see Theorem 2.14 below). The main result from [22] is then a direct consequence of the following two statements:

- (1) The derived L^1 full group is topologically finitely generated as soon as the action has finite entropy.
- (2) The topological abelianization of the L^1 full group of an ergodic \mathbb{Z} -action is isomorphic to \mathbb{Z} , in particular it always has finite topological rank.

In the present paper, we generalize the first item from the above list as follows (see Section 4 for the proof).

THEOREM A. — *Let Γ be a finitely generated group, let $\alpha : \Gamma \rightarrow \text{Aut}(X, \mu)$ be a measure-preserving aperiodic Γ -action on (X, μ) . The following are equivalent.*

- (i) *The action has finite Rokhlin entropy.*
- (ii) *The derived L^1 full group of the action has finite topological rank.*

⁽¹⁾The quotient of the L^1 full group by the derived L^1 full group is the largest abelian continuous quotient of the L^1 full group, and is thus called the *topological abelianization* of the L^1 full group.

Apart from Seward's characterization of subshifts, another key ingredient in the proof is Nekrashevych's work on alternating topological full groups [26], which provides dense finitely generated subgroups in the derived L^1 full groups. The reader is referred to Section 2.9 for the definition of alternating topological full group and Section 3 for the proof of their density in L^1 full groups.

In the above result, it is natural to ask whether one can remove the "derived" adjective in item (ii), as in the \mathbb{Z} -case. However, we have no idea what the topological abelianization of the L^1 full group of a general measure-preserving ergodic action of a finitely generated group looks like, and in particular we do not know when it happens to be topologically finitely generated. We don't have any example or non example, even for \mathbb{Z}^2 -actions.

Let us now consider another invariant of topological groups: amenability. In [22] we showed that if the L^1 full group of a free action of a finitely generated group Γ was amenable, then Γ itself was amenable, and left the question of the converse open. Here we show that the converse holds so as to obtain the following result (see Section 5 for definitions and a more general statement in terms of *graphings*).

THEOREM B. — *Let Γ be a finitely generated group acting freely on a standard probability space (X, μ) . Then the following are equivalent:*

- (i) Γ is amenable;
- (ii) The L^1 full group of the action is amenable;
- (iii) The derived L^1 full group of the action is amenable;
- (iv) The derived L^1 full group of the action is extremely amenable;
- (v) The derived L^1 full group of the action is whirly amenable.

Note that the whole L^1 full group of any free ergodic \mathbb{Z} -action factors onto \mathbb{Z} [22, Corollary 4.20] and hence cannot be extremely amenable. So we cannot have a stronger notion of amenability for the whole L^1 full group in the above equivalences.

We also note that L^1 full groups provide a rich playground for Rosendal's extension of geometric group theory to Polish groups: the L^1 metric can often canonically be obtained (up to quasi-isometry) purely from the topological group structure. We refer the reader to Section 6 for precise statements, and single out the following application, proved in Section 6.4.

THEOREM C. — *Let Γ and Λ be two finitely generated amenable groups acting in a measure-preserving and ergodic manner on (X, μ) . Then every*

abstract group isomorphism between their L^1 full groups must be a quasi-isometry for their respective L^1 metrics.

The conclusion of the theorem is somehow optimal: if we replace quasi-isometry by bilipschitz equivalence, the theorem fails (see Example 6.16). We don't know how to generalize Theorem C to the nonamenable setup, but we conjecture that the following stronger statement holds.

CONJECTURE. — *Let Φ be a graphing. Then the L^1 metric on the L^1 full group of Φ is maximal.*

Moreover, the following natural problem arises.

PROBLEM. — *Classify (derived) L^1 full groups up to quasi-isometry.*

A simpler instance of the above problem is to decide whether the derived L^1 full groups of the 2-odometer and of the 3-odometer are quasi-isometric, which has a natural reformulation in terms of quasi-isometry of the dyadic and triadic symmetric groups for an L^1 -like metric (see the paragraph following Question 6.9 for details).

Finally, we make progress on the computation of the exact topological rank of L^1 full groups of measure-preserving ergodic transformations. As reminded before, we had shown in [22] that the topological rank of L^1 full groups of a measure-preserving ergodic transformations is finite if and only if the transformation has finite entropy. Using Marks' work [23], we moreover had obtained that the topological rank of the L^1 full group of irrational rotations is equal to two.

Here, we extend this result to the class of *rank one* transformations, which we will properly define in Section 7. Let us for now mention that every compact transformation is rank one (in particular every irrational rotation is rank one), and that every rank one transformation has entropy zero.

THEOREM D. — *Let T be a rank-one measure-preserving transformation. Then the topological rank of its L^1 full group is equal to 2.*

It would be very interesting to have an example of a measure-preserving ergodic transformation such that the topological rank of its L^1 full group is finite, but greater than 2.

The paper ends with further remarks and questions related to these results. We introduce the notion of L^p -full orbit equivalence and ask whether it is equivalent to L^p orbit equivalence, we mention a natural extension of Theorem A to *symmetric* L^p full groups. We finally make a few more remarks on the topological abelianization of L^1 full groups.

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2. Preliminaries

In what follows Γ and Λ will always denote countable *infinite* groups.

2.1. Standard probability spaces

Recall that a measurable space is *standard* if its σ -algebra comes from the Borel σ -algebra induced by the topology of some Polish space. Measurable maps between standard Borel spaces are simply called Borel maps. One of the highlights of standard Borel spaces is the fact that the range of any injective Borel map is Borel so that Borel bijections automatically have a Borel inverse [17, Theorem 15.1]. Another important fact is that in any standard Borel space we have a countable family of Borel subsets (C_n) which *separates points*: for every distinct $x, y \in X$ there is $n \in \mathbb{N}$ such that $x \in C_n \not\leftrightarrow y \in C_n$.

Given a countable group Γ , a Γ -action on X is called Borel if the bijections induced by the elements of Γ are Borel maps. Two Borel actions $\Gamma \curvearrowright X$ and $\Gamma \curvearrowright Y$ are *conjugate* if there is a Borel Γ -equivariant bijection between X and Y .

The *support* of a Borel bijection $T : X \rightarrow X$ is the Borel set $\text{supp}T := \{x \in X : T(x) \neq x\}$. Here are two useful lemmas on Borel bijections of standard Borel spaces which only rely on the existence of a countable family of Borel subsets which separates points.

LEMMA 2.1 (see e.g. [22, Proposition 2.7]). — *Let T be a Borel bijection of a standard Borel space X . Then there is a partition of $\text{supp}T$ in three Borel subsets A_1, A_2, A_3 such that for all $i \in \{1, 2, 3\}$ the set $T(A_i)$ disjoint from A_i .*

By applying the above lemma n times we have.

LEMMA 2.2. — *Let X be a standard Borel space, let $(T_i)_{i=1}^n$ be a finite family of Borel bijections of X . Then there is a finite Borel partition $(A_j)_{j=1}^m$ of $\bigcap_{i=1}^n \text{supp } T_i$ such that for each $i \in \{1, \dots, n\}$ and each $j \in \{1, \dots, m\}$ we have that $T_i(A_j)$ is disjoint from A_j .*

Recall that given a measurable map $f : X \rightarrow Y$ between measurable spaces, if μ is a probability measure on X then one can define the *pushforward measure* $f_*\mu$ on Y by $f_*\mu(A) = \mu(f^{-1}(A))$ for all every $A \subseteq Y$.

We will work most of the time in a *standard probability space* (X, μ) , which means that X is still a standard Borel space but now equipped with a nonatomic Borel probability measure μ . Given two standard probability spaces (X, μ) and (Y, ν) , there is a Borel bijection $S : X \rightarrow Y$ such that $S_*\mu = \nu$ [17, Theorem 17.41], so we may as well work in a fixed standard probability space (X, μ) .

2.2. The measure algebra and its automorphisms

Let (X, μ) be a standard probability space, we denote by $\text{MAlg}(X, \mu)$ the quotient of the algebra $\mathcal{B}(X)$ of Borel subsets of X by the ideal of measure zero sets; in other words in the measure algebra we identify two Borel sets A and B as soon as $\mu(A \triangle B) = 0$. Unless specified otherwise, we will always be working in the measure algebra when considering Borel sets, so for instance when we write $A \subseteq B$ we mean $\mu(A \setminus B) = 0$.

A *measure-preserving transformation* of (X, μ) is a Borel bijection $T : X \rightarrow X$ such that $T_*\mu = \mu$. We denote by $\text{Aut}(X, \mu)$ be the group of all measure-preserving transformations, two such transformations being identified if they coincide on a full measure subset of X . We say that a measure-preserving transformation is *periodic* if all its orbits are finite.

Every element of $\text{Aut}(X, \mu)$ defines a measure-preserving automorphism of the measure algebra, and conversely every automorphism of the measure algebra comes from a measure-preserving transformation which is unique up to measure zero [17, Theorem 15.11]. In particular, since every measure-preserving Borel bijection between full measure Borel subsets of (X, μ) defines an automorphism of its measure algebra, it also defines a measure-preserving transformation. This will be used implicitly in many proofs.

Given $T \in \text{Aut}(X, \mu)$ and $A \subseteq X$ of positive measure, we define the *T-return time* to A as the function $n_{T,A} : X \rightarrow \mathbb{N} \setminus \{0\}$ given by: for all $x \in A$, $n_{A,T}(x)$ is the smallest $n \in \mathbb{N}$ such that $T^n(x) \in A$. The Poincaré recurrence theorem ensures us that the return time is well defined up to

measure zero, also called the *return time* to A . We can then define the measure-preserving transformation T_A induced by T on A by:

$$T_A(x) = \begin{cases} T^{n_A, \tau}(x) & \text{if } x \in A \\ x & \text{otherwise.} \end{cases}$$

The measure algebra $\text{MAlg}(X, \mu)$ can be equipped with a complete metric d_μ defined by $d_\mu(A, B) = \mu(A \triangle B)$. The following lemma which was used implicitly in [22].

LEMMA 2.3. — *Let (X, μ) be a standard probability space, let \mathcal{D} be a set of Borel subsets of X such that the image of \mathcal{D} in $\text{MAlg}(X, \mu)$ is dense. Then there is a full measure Borel subset $X' \subseteq X$ such that \mathcal{D} separates the points of X' .*

Proof. — Let $(C_n)_{n \in \mathbb{N}}$ separate the points of X , by changing our enumeration we may as well assume each C_n appears infinitely often. Now for each $n \in \mathbb{N}$ we pick $D_n \in \mathcal{D}$ such that $\mu(C_n \triangle D_n) < 2^{-n}$. By the Borel–Cantelli lemma, there is a full measure Borel subset $X' \subseteq X$ such that for all $x \in X'$, there is $N \in \mathbb{N}$ such that for all $n \geq N$, $x \notin C_n \triangle D_n$. By construction the set X' is as wanted. \square

2.3. Ergodic theory

An action of a countable group Γ on a standard probability space (X, μ) is *measure-preserving* if it is a Borel action and for every $\gamma \in \Gamma$ we have $\gamma_*\mu = \mu$. Every such action yields a homomorphism $\Gamma \rightarrow \text{Aut}(X, \mu)$ which only depends on the action up to measure zero and conversely, every such homomorphism comes from a measure-preserving action which is unique up to measure zero (see [13, Theorem 2.15]).

Two measure-preserving actions of Γ are *conjugate* if they are conjugate as Borel actions up to restricting to full measure Γ -invariant Borel subsets. Viewing actions as homomorphisms $\alpha, \beta : \Gamma \rightarrow \text{Aut}(X, \mu)$, this is equivalent to the existence of $T \in \text{Aut}(X, \mu)$ such that for all $\gamma \in \Gamma$ we have $\beta(\gamma) = T\alpha(\gamma)T^{-1}$. The L^1 full groups that we are interested in only depend on the action up to measure zero, so we can and will forget about what happens on measure zero sets most of the time. By doing so we will implicitly use the following : if we have a Borel set A of full measure, then it contains the full measure Γ -invariant set $\bigcap_{\gamma \in \Gamma} \gamma A$.

Two basic invariants of conjugacy of measure preserving actions are *aperiodicity* (the Γ -orbit of almost every $x \in X$ is infinite) and the stronger

notion of *ergodicity* (every Γ -invariant Borel subset has measure 0 or 1). We will only be interested in aperiodic actions, and sometimes we will restrict to ergodic ones. Another important invariant is *freeness*, which means that for almost all $x \in X$ and all $\gamma \in \Gamma$ we have $\gamma x \neq x$. Note that aperiodicity follows from freeness since all the groups we consider are infinite.

Example 2.4. — Let I be a countable set and let ν be a probability measure on I which is not a Dirac measure, then the measure preserving action $\Gamma \curvearrowright (I^\Gamma, \nu^{\otimes \Gamma})$ given by $(\gamma \cdot x)_g = x_{\gamma^{-1}g}$ is free and ergodic. Such an action is called a *Bernoulli shift*.

2.4. Subshifts and Rokhlin entropy

Let X be a standard Borel space, let $\Gamma \curvearrowright X$ be a Borel action. An *observable* on X is a Borel map $\varphi : X \rightarrow I$ where I is a countable set. To every such observable we can associate a Borel partition of X indexed by I defined as $(\varphi^{-1}\{i\})_{i \in I}$, and conversely if \mathcal{P} is a countable partition of X then the map $\varphi : X \rightarrow \mathcal{P}$ which sends every $x \in X$ to the unique $P \in \mathcal{P}$ such that $x \in P$ is an observable. An observable φ is called (dynamically) *generating* if for all $x, y \in X$ with $x \neq y$ there is $\gamma \in \Gamma$ such that $\varphi(\gamma x) \neq \varphi(\gamma y)$.

A Borel action $\Gamma \curvearrowright X$ on a standard Borel space is called a *subshift* if it admits a generating observable with *finite range*. A measure-preserving Γ -action on a standard probability space is called a *measure-preserving subshift* if there is a full measure Borel Γ -invariant set Y such that the restriction of the action to Y is a subshift. Note that Bernoulli shifts over finite spaces are examples of measure-preserving subshifts.

Finally, an action by homeomorphisms on a compact Hausdorff space is called a *topological subshift* if it has a *continuous* generating observable (such an observable must then take values in a finite set by compactness and continuity).

We have the following well-known characterization of subshifts up to conjugacy (in the respective Borel and topological categories).

PROPOSITION 2.5. — *The following hold.*

- (1) *A Borel action $\Gamma \curvearrowright X$ is a subshift if and only if it is conjugate to a restriction of $\Gamma \curvearrowright I^\Gamma$ to an invariant Borel subset where I is a finite set.*
- (2) *A Γ -action by homeomorphisms on a compact zero-dimensional topological space K is a topological subshift if and only if it is*

conjugate (via a homeomorphism) to the restriction of $\Gamma \curvearrowright I^\Gamma$ to an invariant closed subset, where I is a finite set.

Proof. — For (1), let $\varphi : X \rightarrow I$ be a generating observable where I is finite, and consider the Γ -equivariant map $\Phi_I : X \rightarrow I^\Gamma$ given by $x \mapsto (\varphi_I(\gamma^{-1}x))_{\gamma \in \Gamma}$. Since φ is generating and Borel, Φ_I is an injective Borel map. Its range is thus a Borel Γ -invariant subset of I^Γ .

For (2), consider the same map and observe that it now has to be a homeomorphism onto its image. \square

If we are now given a measure-preserving Γ -action on (X, μ) , we say that an observable φ is *generating* if we can find a Borel Γ -invariant subset $X' \subseteq X$ of full measure such that the restriction of φ to X' is generating for $\Gamma \curvearrowright X'$. The following analogue of Proposition 2.5 is immediate.

PROPOSITION 2.6. — *A measure-preserving Γ -action on (X, μ) is a measure-preserving subshift if and only if it admits a generating observable whose range is finite.*

Let (X, μ) be standard probability space. The *entropy* of an observable $\varphi : X \rightarrow I$ is the non-negative real number $H(\varphi)$ defined by the equation

$$H(\varphi) = - \sum_{i \in I} \mu(\varphi^{-1}(\{i\})) \ln \mu(\varphi^{-1}(\{i\})),$$

where we make the convention that $0 \ln(0) = 0$. Here are two fundamental properties of entropy:

- Subadditivity: for any two observables φ and ψ we have

$$H(\varphi \times \psi) \leq H(\varphi) + H(\psi)$$

where (φ, ψ) is the observable $x \mapsto (\varphi(x), \psi(x))$.

- Concavity: if we fix $k \in \mathbb{N}$, the observables taking value in $\{1, \dots, k\}$ which maximize entropy are exactly those which give the same probability to every $i \in \{1, \dots, k\}$ (and their entropy is thus $\ln k$)

DEFINITION 2.7 (Seward). — *The Rokhlin entropy of a measure-preserving aperiodic Γ -action is the infimum of the entropies of its generating observables.*

Every aperiodic Borel action admits a generating observable by [15, Theorem 5.4], so the above quantity is an element of $[0, +\infty]$. The following remarkable result of Seward is a far-reaching generalization of Krieger's generator theorem which will allow us to work with arbitrary countable groups rather than \mathbb{Z} .

THEOREM 2.8 (Seward [31]). — *Let Γ be a countable group, let $\Gamma \curvearrowright (X, \mu)$ be a measure-preserving action. The action is a measure-preserving subshift if and only if it has finite Rokhlin entropy.*

Seward also has a quantitative version of the above result, yielding a generating observable whose range is as small as possible (see [32], where he also defines Rokhlin entropy).

2.5. Full groups

If (X, μ) is a standard probability space, and A, B are Borel subsets of X , a *partial isomorphism* of (X, μ) of domain A and range B is a Borel bijection $f : A \rightarrow B$ which is measure-preserving for the measures induced by μ on A and B respectively. We denote by $\text{dom } f = A$ its domain, and by $\text{rng } f = B$ its range. Given two partial isomorphisms $\varphi_1 : A \rightarrow B$ and $\varphi_2 : C \rightarrow D$, we define their composition $\varphi_2 \circ \varphi_1$ as the map $\varphi_1^{-1}(B \cap C) \rightarrow \varphi_2(B \cap C)$ given by $\varphi_2 \circ \varphi_1(x) = \varphi_2(\varphi_1(x))$. We also define the *inverse* φ^{-1} of a partial isomorphism $\varphi : A \rightarrow B$ as the unique map $\varphi^{-1} : B \rightarrow A$ such that $\varphi^{-1} \circ \varphi = \text{id}_A$.

By definition a *graphing* is a countable set of partial isomorphisms. Every graphing Φ generates a *measure-preserving equivalence relation* \mathcal{R}_Φ , defined to be the smallest equivalence relation containing $(x, \varphi(x))$ for every $\varphi \in \Phi$ and $x \in \text{dom } \varphi$. Given a graphing Φ , a Φ -word is a composition of finitely many elements of Φ or their inverses. Observe that $(x, y) \in \mathcal{R}_\Phi$ iff and only if there exists a Φ -word w such that $y = w(x)$. Moreover we have a natural path metric d_Φ on the \mathcal{R}_Φ -classes: for $(x, y) \in \mathcal{R}_\Phi$ we let $d_\Phi(x, y)$ be the minimum length of a Φ -word w such that $y = w(x)$.

A graphing is called *aperiodic* when the \mathcal{R}_Φ -class of μ -almost every $x \in X$ is infinite. Given a Borel set A , we say that A is \mathcal{R}_Φ -invariant if whenever $x \in A$ and $(x, y) \in \mathcal{R}_\Phi$, we have $y \in A$. Equivalently, this means that for all $x \in A$ and all $\varphi \in \Phi$, we have $\varphi(x) \in A$ and $\varphi^{-1}(x) \in A$. A graphing Φ is *ergodic* if every \mathcal{R}_Φ -invariant Borel set has measure 0 or 1. Every ergodic graphing is automatically aperiodic.

The *full group* of a measure-preserving equivalence relation \mathcal{R} is the group $[\mathcal{R}]$ of automorphisms of (X, μ) which induce a permutation on every \mathcal{R} -class, that is

$$[\mathcal{R}] = \{\varphi \in \text{Aut}(X, \mu) : \forall x \in X, \varphi(x) \mathcal{R} x\}.$$

It is a separable group when equipped with the complete metric d_u defined by

$$d_u(T, U) = \mu(\{x \in X : T(x) \neq U(x)\}).$$

The metric d_u is called the *uniform metric* and the topology it induces is called the uniform topology. One also defines the *pseudo full group* of \mathcal{R} , denoted by $[[\mathcal{R}]]$, which consists of all partial isomorphisms φ such that $\varphi(x) \mathcal{R} x$ for all $x \in \text{dom } \varphi$.

2.6. L^1 full groups

DEFINITION 2.9 ([22]). — *Let Φ be a graphing on a standard probability space (X, μ) . Its L^1 full group $[\Phi]_1$ is the group of all $T \in [\mathcal{R}_\Phi]$ such that*

$$\int_X d_\Phi(x, T(x)) \, d\mu(x) < +\infty.$$

The L^1 full group of Φ is a Polish group for the topology induced by the right-invariant complete metric d_Φ^1 defined by

$$d_\Phi^1(T, U) = \int_X d_\Phi(T(x), U(x)) \, d\mu(x).$$

When Γ is a finitely generated group acting on a standard probability space (X, μ) by measure-preserving transformations, we can also define the associated L^1 full group as follows: let S be a finite generating set of the group Γ , then S is a graphing whose associated equivalence relation is the equivalence relation \mathcal{R}_Γ whose equivalence classes are the Γ -orbits. Observe that up to bilipschitz equivalence, the associated metric d_S on the Γ -orbits does not depend on S , so that two finite generating sets of Γ yield the same L^1 full group with the same topology. This is by definition the L^1 full group of the Γ -action, which we will often simply denote by $[\Gamma]_1$ without referring explicitly to the action.

Here is another way to think of the L^1 full group of a Γ -action, similar to what was done in [5, Section 3.2]. Consider the Polish space $L^1(X, \mu, \Gamma)$ of integrable maps $X \rightarrow \Gamma$, i.e. maps $f : X \rightarrow \Gamma$ such that

$$\int_X d_S(1, f(x)) \, d\mu(x) < +\infty.$$

To each element f of $L^1(X, \mu, \Gamma)$ we associate a map $T_f : X \rightarrow X$ defined by $T_f(x) = f(x) \cdot x$. The subspace

$$\widetilde{[\Gamma]_1} = \{f \in L^1(X, \mu, \Gamma) : T_f \in [\mathcal{R}_\Gamma]\}$$

is closed, and we define on it a law $*$ which is compatible with the group law on $[\mathcal{R}_\Gamma]$ by letting $f * g(x) = f(T_g(x))g(x)$. Then $(\widetilde{[\Gamma]_1}, *)$ is a Polish group, and $[\Gamma]_1$ is topologically isomorphic to its quotient by the closed normal subgroup $\{f \in \widetilde{[\Gamma]_1} : T_f = \text{id}_X\}$ via the map $f \mapsto T_f$.

If we are given $T \in [\Gamma]_1$, a lift of T to $\widetilde{[\Gamma]}_1$ is called a *cocycle* associated to T . When the Γ action is free, $f \mapsto T_f$ is injective and so each $T \in [\Gamma]_1$ has a unique associated cocycle $c_T : X \rightarrow \Gamma$ given by the equation

$$T(x) = c_T(x) \cdot x.$$

An important feature of cocycles associated to elements of L^1 full groups is that they have finite entropy.

PROPOSITION 2.10 (Austin, see [1, Lemma 2.1]). — *Let Γ be a finitely generated group. Then every element of $L^1(X, \mu, \Gamma)$ has finite entropy.*

We finally note the following lemma about cocycles which are measurable with respect to an invariant sub- σ -algebra.

LEMMA 2.11. — *Let $\Gamma \curvearrowright (X, \mu)$ be a measure-preserving action of a finitely generated group Γ , suppose \mathcal{A} is a Γ -invariant sub- σ -algebra of the Borel subsets of X . Let $[\Gamma]_1^{\mathcal{A}}$ denote the set consisting of all elements of $[\Gamma]_1$ which admit an \mathcal{A} -measurable cocycle. Then $[\Gamma]_1^{\mathcal{A}}$ is a group.*

Proof. — Let $T, U \in [\Gamma]_1^{\mathcal{A}}$, and let c_T and c_U be cocycles for T and U which are \mathcal{A} -measurable. Then the map $c : X \rightarrow \Gamma$ defined by

$$c(x) = c_T(U(x))c_U(x)$$

is a cocycle for TU , and $c(x) = \gamma$ if and only if there are $\gamma_1, \gamma_2 \in \Gamma$ such that $\gamma_1\gamma_2 = \gamma$, $c_U(x) = \gamma_2$, $c_T(\gamma_2x) = \gamma_1$. So c is clearly \mathcal{A} -measurable as wanted. □

Remark 2.12. — The same proof yields a similar statement for the whole full group of the action.

2.7. Derived L^1 full groups

Contrarily to many Polish groups that have been studied, L^1 full groups are not topologically simple in general, and can actually happen to have a non trivial topological abelianization. The closure of their derived group (i.e. the group generated by commutators) is thus distinct from them a priori. Let us give it a shorter name.

DEFINITION 2.13. — *The derived L^1 full group of a graphing Φ is the closure of the derived group of the L^1 full group of Φ . It is denoted by $[\Phi]_1'$.*

Given $\varphi \in \Phi$ and a Borel set $A \subseteq X$, let us define an involution $I_{\varphi,A}$ by:

$$I_{\varphi,A}(x) = \begin{cases} \varphi(x) & \text{if } x \in A \setminus \varphi(A) \\ \varphi^{-1}(x) & \text{if } x \in \varphi(A) \setminus A \\ x & \text{otherwise.} \end{cases}$$

We can now give two fundamental properties of the derived L^1 full group. The second one suggests to rather call it the *symmetric L^1 full group* (see Section 8.2).

THEOREM 2.14 ([22, Lemma 3.11 and Theorem 3.15]). — *Let Φ be an aperiodic graphing. Then the following hold:*

- (a) *every periodic element of the L^1 full group of Φ actually belongs to the derived L^1 full group of Φ ;*
- (b) *the derived L^1 full group of Φ is topologically generated by the set of involutions $\{I_{\varphi,A} : \varphi \in \Phi, A \subseteq X\}$.*

Observe that since $I_{\varphi,A}$ depends continuously on $A \in \text{MAlg}(X, \mu)$ and $\text{MAlg}(X, \mu)$ is connected, we can deduce that $[\Phi]'_1$ is connected (cf. [22, Corollary 3.14]). We still don't know whether it is simply connected. Another important question is to understand what is the topological abelianization of L^1 full groups, i.e. what is $[\Phi]_1/[\Phi]'_1$. The only known case is that of ergodic \mathbb{Z} -actions, for which we then always have $[\mathbb{Z}]_1/[\mathbb{Z}]'_1 = \mathbb{Z}$ [22]. See the last section of the paper for more on this.

Using (b) from the above statement along with its full group version [19, Section 4], we also have the following.

COROLLARY 2.15. — *Let Φ be an aperiodic graphing. Then the derived L^1 full group of Φ is dense in the full group $[\mathcal{R}_\Phi]$ for the uniform topology.*

Given an aperiodic graphing Φ and an \mathcal{R}_Φ -invariant Borel set A , consider the subgroup $[\Phi_A]'_1$ consisting of all the elements of $[\Phi]'_1$ supported in A . It is a closed normal subgroup of $[\Phi]'_1$, and the following result says that these are the only ones.

THEOREM 2.16 ([22, Theorem 3.24]). — *Let Φ be an aperiodic graphing. Then every closed normal subgroup of $[\Phi]'_1$ is of the form $[\Phi_A]'_1$, where A is \mathcal{R}_Φ -invariant.*

As a corollary, one can show that a graphing is ergodic if and only if its derived L^1 full group is topologically simple [22, Corollary 3.25].

2.8. Conditional measures

Given a measure preserving action $\Gamma \curvearrowright (X, \mu)$ of a finitely generated group Γ , it will be important for us to know for which Borel sets A and B we can find an element of T of the L^1 full group of the action such that $T(A)$ is close to B . Because the L^1 full group is dense in the full group $[\mathcal{R}_\Gamma]$, it suffices to understand the $[\mathcal{R}_\Gamma]$ -orbits in the measure algebra. It is a well-known fact that when the action is ergodic, there is $T \in [\mathcal{R}_\Gamma]$ such that $T(A) = B$ if and only if $\mu(A) = \mu(B)$. In the non ergodic case, one needs to refine the measure μ to a *conditional measure* with respect to the σ -algebra of Γ -invariant sets.

DEFINITION 2.17. — *Let \mathcal{R} be a measure-preserving equivalence relation. Denote by $M_{\mathcal{R}}$ the σ -algebra of \mathcal{R} -invariant Borel subsets. The conditional measure of $A \in \text{MAlg}(X, \mu)$ is the $M_{\mathcal{R}}$ -measurable function $\mu_{\mathcal{R}}$ defined by*

$$\mu_{\mathcal{R}}(A) = \pi_{\mathcal{R}}(\chi_A),$$

where $\pi_{\mathcal{R}}$ is orthogonal projection onto the closed subspace of $L^2(X, \mu)$ consisting of the $M_{\mathcal{R}}$ -measurable functions.

Note that $\mu_{\mathcal{R}}$ is well-defined up to measure zero. We now have the following.

PROPOSITION 2.18 (Dye, see [22, Lemma 2.11]). — *Let \mathcal{R} be a measure-preserving equivalence relation and $A, B \in \text{MAlg}(X, \mu)$. Then $\mu_{\mathcal{R}}(A) = \mu_{\mathcal{R}}(B)$ if and only if there is an involution $T \in [\mathcal{R}]$ such that $T(A) = B$.*

Aperiodicity has an important reformulation in terms of conditional measures, which is also due to Dye who calls it Maharam’s lemma. For a proof in our setup, see [21, Proposition 2.2].

PROPOSITION 2.19 (Maharam’s lemma). — *A measure-preserving equivalence relation \mathcal{R} is aperiodic if and only if for any $A \in \text{MAlg}(X, \mu)$, and for any $M_{\mathcal{R}}$ -measurable function f such that $0 \leq f \leq \mu_{\mathcal{R}}(A)$, there exists $B \subseteq A$ such that the \mathcal{R} -conditional measure of B equals f .*

2.9. Alternating topological full groups

Alternating topological full groups were defined by Nekrashevych and have been the source of outstanding new examples of finitely generated groups [27]. They provide the right setup for generalizing Matui’s results

on finite generatedness of derived groups of topological full groups of homeomorphisms of the Cantor space [24]. Topological full groups were first studied and defined by Giordano, Putnam and Skau [12]. Here we give a restricted definition of (alternating) topological full groups which will be sufficient for our purposes.

Let $2^{\mathbb{N}}$ be the Cantor space and let ν be a Borel probability measure on it. Recall from [22] that a *continuous graphing* on the measured Cantor space $(2^{\mathbb{N}}, \nu)$ is a graphing Φ such that for all $\varphi \in \Phi$, the sets $\text{dom } \varphi$, $\text{rng } \varphi$ are clopen and $\varphi : \text{dom } \varphi \rightarrow \text{rng } \varphi$ is a homeomorphism.

DEFINITION 2.20. — *Let Φ be a continuous graphing on $(2^{\mathbb{N}}, \nu)$. The topological full group of Φ , denoted by $[\Phi]_c$, is the group of all homeomorphisms T of $2^{\mathbb{N}}$ such that for all $x_0 \in 2^{\mathbb{N}}$ there is a neighborhood U of x_0 and a Φ -word w such that for all $x \in U$, $T(x) = w(x)$.*

Say that an element T of the topological full group of Φ is a *basic 3-cycle* if there are a three clopen subsets U_1, U_2, U_3 which partition the support of T and two Φ -words w_1, w_2 such that $w_1(U_1) = U_2$ and $w_2(U_2) = U_3$ and for all $x \in X$ we have

$$T(x) = \begin{cases} w_1(x) & \text{if } x \in U_1 \\ w_2(x) & \text{if } x \in U_2 \\ w_1^{-1}w_2^{-1}(x) & \text{if } x \in U_3 \\ x & \text{otherwise.} \end{cases}$$

The *alternating topological full group* of a continuous graphing Φ is the group $\mathfrak{A}[\Phi]$ generated by basic 3-cycles. One can check that the alternating topological full group is a normal subgroup contained in the derived group of the topological full group.

Let us observe that when Γ is a countable group acting by homeomorphisms on the Cantor space, then if S is any generating set for Γ , the elements of S define a continuous graphing whose topological full group is the topological full group of the groupoid of germs of the Γ -action in the sense of [26, Example 2.2] (see also [26, Definition 2.6] and the two paragraphs which follow it). Moreover, the alternating topological full group of the continuous graphing S is equal to the alternating topological full group of the groupoid of germs of the Γ -action.

Remark 2.21. — Since we do not require the action to be free or essentially free in the topological sense, the étale groupoid of germs of the action is not Hausdorff in general, in particular its unit space may not be closed, although it is compact. As a consequence, Lemma 2.2 from [25] does not

apply and elements of the topological full group may very well have a non clopen support⁽²⁾, as opposed to elements of the alternating topological full group.

We have the following straightforward consequence of [26, Theorem 1.2] and [26, Proposition 5.7] which will be crucial in our proof of Theorem A.

THEOREM 2.22 (Nekrashevych). — *Let Γ be a finitely generated group, suppose Γ acts on the Cantor space by homeomorphism and that every Γ -orbit has cardinality at least 5. Then the action is a topological subshift if and only if its alternating topological full group is finitely generated.*

3. Density of the alternating topological full group

In this section, we show that the alternating topological full group of a continuous graphing is always dense in its derived L^1 full group. We first need to work a bit on the general setup, so (X, μ) is still a standard probability space.

A measure-preserving transformation is called a 3-cycle when it has order 3, or equivalently when all its nontrivial orbits have cardinality 3. Given a 3-cycle T , there is $A \subseteq X$ such that $\text{supp } T = A \sqcup T(A) \sqcup T^2(A)$, and the identity $(1 \ 2 \ 3) = (1 \ 2)(1 \ 3)(1 \ 2)(1 \ 3)$ yields that T is a commutator, so the 3-cycles of the L^1 full group actually belong to the derived L^1 full group.

We need to ensure that there are many 3-cycles. This is a consequence of the following two lemmas, which are straightforward generalizations of [22, Lemma 3.6] and [22, Lemma 2.12] respectively.

Recall the following notation: given a measure-preserving transformation T and $A \subseteq X$, we denote by T_A the transformation induced by T on A (see Section 2.2). In what follows, we only use this construction when A is T -invariant, and then T_A is the map which acts as T on A and trivially outside of A .

LEMMA 3.1. — *Let Φ be a graphing. Then for every periodic $U \in [\mathcal{R}_\Phi]$, there exists an increasing sequence of U -invariant sets $A_n \subseteq \text{supp } U$ such that $\text{supp } U = \bigcup_{n \in \mathbb{N}} A_n$ and for all $n \in \mathbb{N}$, $U_{A_n} \in [\Phi]_1$.*

⁽²⁾In the topological setup, the support of a homeomorphism $T : X \rightarrow X$ is rather defined as the closure as the set of $x \in X$ such that $T(x) \neq x$.

Proof. — Let $U \in [\mathcal{R}_\Phi]$ be periodic and for all $n \in \mathbb{N}$, let $A_n = \{x \in \text{supp } U : d_\Phi(x, U^k(x)) < n \text{ for all } k > 0\}$. By definition each A_n is U -invariant, and since U is periodic, $\bigcup_{n \in \mathbb{N}} A_n = \text{supp } U$, so $\mu(\text{supp } U \setminus A_n) \rightarrow 0$. By the definition of A_n , each U_{A_n} belongs to $[\Phi]_1$. \square

LEMMA 3.2. — *Let Φ be an aperiodic graphing. Then for every Borel $A \subseteq X$, there is a 3-cycle $U \in [\mathcal{R}_\Phi]$ whose support is equal to A .*

Proof. — By Maharam’s Lemma (Proposition 2.19), one can write $A = A_1 \sqcup A_2 \sqcup A_3$ where $\mu_{\mathcal{R}_\Phi}(A_1) = \mu_{\mathcal{R}_\Phi}(A_2) = \mu_{\mathcal{R}_\Phi}(A_3) = \mu_{\mathcal{R}_\Phi}(A)/3$. By a result of Dye (see Proposition 2.18), there are $T_1, T_2 \in [\mathcal{R}_\Phi]$ such that $T_1(A_1) = A_2$ and $T_2(A_2) = A_3$. We then define $T \in [\mathcal{R}_\Phi]$ by

$$T(x) = \begin{cases} T_1(x) & \text{if } x \in A_1 \\ T_2(x) & \text{if } x \in A_2 \\ T_1^{-1}T_2^{-1}(x) & \text{if } x \in A_3 \\ x & \text{otherwise.} \end{cases}$$

The transformation T is the desired 3-cycle. \square

As an immediate consequence of the two previous lemmas, we have:

PROPOSITION 3.3. — *Let Φ be an aperiodic graphing. Then for every Borel $A \subseteq X$, there is a sequence of 3-cycles $U_n \in [\Phi]_1$ such that $(\text{supp } U_n)$ is an increasing sequence of subsets of A and $\mu(A \setminus \text{supp } U_n) \rightarrow 0$.*

PROPOSITION 3.4. — *Let Φ be a graphing. Then $[\Phi]'_1$ is topologically generated by 3-cycles.*

Proof. — Let X_p be the periodic part of the graphing, i.e. the set of x whose \mathcal{R}_Φ -class is finite, and let $X_a = X \setminus X_p$. Then if one lets Φ_p (resp. Φ_a) be the restriction of Φ to X_p (resp. X_a), the derived L^1 full group of Φ splits naturally as a direct product $[\Phi_p]'_1 \times [\Phi_a]'_1$. So it suffices to establish the theorem in the aperiodic and in the periodic case.

Let us start by supposing Φ is aperiodic. The group generated by 3-cycles is normal in $[\Phi]'_1$ so its closure N is a closed normal subgroup of $[\Phi]'_1$. By [22] we conclude that N is equal to the group of $T \in [\Phi]'_1$ whose support is contained in the reunion of the supports of the 3-cycles. Using the previous proposition, we see that the reunion of such supports is equal to X , so $N = [\Phi]'_1$, which end the proof of the theorem in the aperiodic case.

Now suppose Φ is periodic. For every $n \in \mathbb{N}$ let X_n be the set of $x \in X$ whose \mathcal{R}_Φ -class has d_Φ -diameter less than n . Then $X = \bigcup_{n \in \mathbb{N}} X_n$, the X_n 's are Φ -invariant and non-decreasing so if we let $G_n = \{T \in [\Phi]_1 : \text{supp } T \subseteq X_n\}$ then the reunion of the G_n 's is dense in $[\Phi]_1$. It thus suffices to show that commutator of elements of G_n is a product of 3-cycles belonging to G_n , which comes from the fact that by boundedness of the orbits, each G_n is a full group and every periodic element of a full group which is a commutator is a product of 3-cycles. \square

PROPOSITION 3.5. — *Let Φ be a continuous graphing on $(2^{\mathbb{N}}, \nu)$. Then its alternating topological full group is dense in its derived L^1 full group.*

Proof. — By the previous proposition, it suffices to show that every 3-cycle in $[\Phi]_1$ can be approximated by an element of the alternating topological full group. So let T be a 3-cycle in the L^1 full group of Φ . Let $A \subseteq X$ such that $\text{supp } T = A \sqcup T(A) \sqcup T^2(A)$.

Since the set of Φ -words is countable, we have a partition (A_n) of A and Φ -words $w_{1,n}, w_{2,n}$ such that for all $x \in X$ we have

$$T(x) = \begin{cases} w_{1,n}(x) & \text{if } x \in A_n \text{ for some } n \in \mathbb{N} \\ w_{2,n}(x) & \text{if } x \in T(A_n) \text{ for some } n \in \mathbb{N} \\ w_{1,n}^{-1}w_{2,n}^{-1}(x) & \text{if } x \in T^2(A_n) \text{ for some } n \in \mathbb{N} \\ x & \text{otherwise.} \end{cases}$$

In particular, for every $n \in \mathbb{N}$ the set $A_n \sqcup T(A_n) \sqcup T^2(A_n)$ is T -invariant, and if we let $T_n := T_{A_n \sqcup T(A_n) \sqcup T^2(A_n)}$, then by the dominated convergence theorem we conclude $T = \lim_{m \rightarrow +\infty} \prod_{i=0}^m T_m$. So it suffices to prove that each T_n is a limit of elements of the alternating topological full group.

Let $n \in \mathbb{N}$, then there is a sequence of clopen sets $(U_i)_{i \in \mathbb{N}}$ such that $\mu(A_n \triangle U_i) \rightarrow 0$ [$i \rightarrow +\infty$]. If we let $V_i = U_i \setminus (T(U_i) \cup T^2(U_i))$, then by continuity we also have $\mu(A_n \triangle V_i) \rightarrow 0$. We now define $T_{n,i}$ by

$$T_{n,i}(x) = \begin{cases} w_{1,n}(x) & \text{if } x \in V_i \\ w_{2,n}(x) & \text{if } x \in T(V_i) \\ w_{1,n}^{-1}w_{2,n}^{-1}(x) & \text{if } x \in T^2(V_i) \\ x & \text{otherwise.} \end{cases}$$

Then $T_{n,i}$ is clearly a basic 3-cycle, and hence belongs to the alternating topological full group of Φ . Moreover by the dominated convergence theorem we have $T_{n,i} \rightarrow T_n$ [$n \rightarrow +\infty$], which finishes the proof. \square

4. Topological rank of derived L^1 full groups

In this section, we prove that the derived L^1 full group of a measure-preserving Γ -action has finite topological rank if and only if it is a subshift (Theorem A). We want to use Nekrashevych's result on finite generatedness of alternating topological full groups, and the next section prepares the ground for this.

4.1. From measure-preserving subshifts to topological subshifts

In this preparatory section we will show that every aperiodic measure-preserving subshift is conjugate to a topological subshift on the Cantor space where each orbit has cardinality at least 5. For this, we will freely use the characterization of subshifts provided by Proposition 2.5.

Given a group Γ and a generating set S for Γ , we equip Γ with the graph metric induced by its Cayley graph with respect to S . The closed ball of radius n around the identity for this metric is denoted by $B_S(e, n)$.

LEMMA 4.1. — *Let Γ be a finitely generated group, suppose $S \subseteq \Gamma$ is a finite generating set and let $\Gamma \curvearrowright X$ be a subshift with every orbit of cardinality at least 5. Then we can find a finite generating Borel partition \mathcal{P} such that for every $P \in \mathcal{P}$ there are $\gamma_1, \dots, \gamma_4 \in B_S(e, 4)$ such that $P, \gamma_1 P, \dots, \gamma_4 P$ are disjoint.*

Proof. — Let us first show that for every $x \in X$ there are $\gamma_1, \dots, \gamma_4 \in B_S(e, 4)$ such that $x, \gamma_1 x, \dots, \gamma_4 x$ are all distinct. Observe that if for some $n \in \mathbb{N}$ we have $|B_S(e, n)x| = |B_S(e, n+1)x|$ then $B_S(e, n)x$ is a finite set which is invariant under each generator of Γ , hence Γ -invariant. In particular we must have $|B_S(e, 4)x| \geq 5$ so for every $x \in X$ there are $\gamma_1, \dots, \gamma_4 \in B_S(e, 4)$ such that $x, \gamma_1 x, \dots, \gamma_4 x$ are all distinct.

Now apply Lemma 2.2 to the family $(\gamma\lambda^{-1})_{\gamma, \lambda \in B_S(e, 4)}$ to get a partition \mathcal{P} such that for every $P \in \mathcal{P}$ there are $\gamma_1, \dots, \gamma_4 \in B_S(e, 4)$ such that $P, \gamma_1 P, \dots, \gamma_4 P$ are disjoint. The desired partition is obtained by taking the join of \mathcal{P} with a finite generating partition. \square

PROPOSITION 4.2. — *Let Γ be a finitely generated group acting on (X, μ) by an aperiodic measure-preserving subshift. Then the action is conjugate to a topological subshift on the Cantor space where every orbit has cardinality at least 5.*

Proof. — Let \mathcal{P} be as in the previous lemma and consider $\Omega = \bigvee_{\gamma \in B_S(e,4)} \gamma \mathcal{P}$. Observe that for μ -almost every $x \in X$, there are $\gamma_1, \dots, \gamma_4 \in B_S(e,4)$ such that $x, \gamma_1 x, \dots, \gamma_4 x$ all belong to distinct elements of Ω . Let $\varphi_\Omega : X \rightarrow \Omega$ be the associated observable.

Now consider the Γ -equivariant map $\Phi_\Omega : X \rightarrow \Omega^\Gamma$ given by $x \mapsto (\varphi_\Omega(\gamma^{-1}x))_{\gamma \in \Gamma}$. Observe that the set of sequences $(\omega_\gamma)_{\gamma \in \Gamma}$ such that for all $\gamma \in \Gamma$ there are $\gamma_1, \dots, \gamma_4 \in B_S(e,4)$ such that $\omega_\gamma, \omega_{\gamma\gamma_1^{-1}}, \dots, \omega_{\gamma\gamma_4^{-1}}$ are all distinct is a closed Γ -invariant set containing the essential range of Φ_Ω , so the Γ -action on the essential range of Φ_Ω is a topological subshift all whose orbits have cardinality at least 5.

Moreover the measure μ has no atoms because the Γ -orbits are almost all infinite. So the essential range of Φ_Ω contains no isolated points and thus must be homeomorphic to the Cantor space by a theorem of Brouwer (see [17, Theorem 7.4]). We conclude that the Γ -action on the essential range of Φ_Ω is a topological subshift on the Cantor space which has the desired properties. □

Remark 4.3. — I do not know whether one can improve the above result so as to get an aperiodic topological subshift.

4.2. Proof of Theorem A

Let us recall the statement that we are aiming at.

THEOREM 4.4. — *Let Γ be a finitely generated group and consider a measure-preserving aperiodic Γ -action on a standard probability space (X, μ) . Then the following are equivalent:*

- (i) *the Γ -action has finite Rokhlin entropy and*
- (ii) *the derived L^1 full group of the Γ -action has finite topological rank.*

Proof. — Let us start by the direct implication (i) \Rightarrow (ii). Suppose the measure-preserving Γ -action has finite Rokhlin entropy. By Seward’s theorem (Theorem 2.8), we may assume that the action is a measurable subshift, and by Proposition 4.2 we can further assume that it is a topological subshift all whose orbits have cardinality at least 5. Nekrashevych’s result (Theorem 2.22) yields that the associated alternating topological full group is finitely generated. So the alternating topological full group is the desired dense finitely generated subgroup of the derived L^1 full group of the action by Proposition 3.5.

We now prove the reverse implication (ii) \Rightarrow (i). Suppose that for some $n \in \mathbb{N}$ we have $T_1, \dots, T_n \in [\Gamma]_1'$ which generate a dense subgroup of $[\Gamma]_1'$. By

definition for each $i \in \{1, \dots, n\}$ the transformation T_i admits an associated cocycle $c_i \in L^1(X, \mu, \Gamma)$. By Lemma 2.10 each cocycle c_i has finite entropy. Finally, let c be the observable associated to a partition (A_1, A_2) of X into two disjoint sets such that for all $x \in X$, $\mu_{\mathcal{R}_\Gamma}(A_1)(x) = \mu_{\mathcal{R}_\Gamma}(A_2)(x) = \frac{1}{2}$, as provided by Maharam’s lemma (cf. Proposition 2.19).

The observable $c_1 \times \dots \times c_n \times c$ has finite entropy by subadditivity, let us show that it is generating so that (i) holds. Let $\tilde{\mathcal{A}}$ be the σ -algebra generated by the Γ -translates of the partition associated to the observable $c_1 \times \dots \times c_n \times c$, we will show that it separates the points of a full measure subset of X , thus finishing the proof that the action has finite Rokhlin entropy.

First, by Lemma 2.11, every element of the group generated by T_1, \dots, T_n admits a cocycle which is $\tilde{\mathcal{A}}$ -measurable. Furthermore, the following holds.

CLAIM. — *The σ -algebra $\tilde{\mathcal{A}}$ is invariant under every element of the group generated by T_1, \dots, T_n .*

Proof of claim. — We just saw that every element of the group generated by T_1, \dots, T_n admits a cocycle which is $\tilde{\mathcal{A}}$ -measurable. But such elements must preserve $\tilde{\mathcal{A}}$ because if $A \in \tilde{\mathcal{A}}$ and U is an element of the full group of \mathcal{R}_Γ admitting a cocycle which is $\tilde{\mathcal{A}}$ -measurable, we find a partition $(A_\gamma)_{\gamma \in \Gamma}$ of A such that each A_γ belongs to $\tilde{\mathcal{A}}$ and for each $x \in A_\gamma$, $U(x) = \gamma x$. So $U(A) = \bigsqcup_{\gamma \in \Gamma} \gamma(A_\gamma)$ and since $\tilde{\mathcal{A}}$ is a Γ -invariant σ -algebra which contains each A_γ we conclude that $U(A) \in \tilde{\mathcal{A}}$. □

Let \mathcal{A} be the closure of the image of $\tilde{\mathcal{A}}$ in $\text{MAlg}(X, \mu)^{(3)}$, which is then also invariant under every element of the group generated by T_1, \dots, T_n . By Lemma 2.3, in order to show that $\tilde{\mathcal{A}}$ separates the points of a full measure subset of X , we only need to show that $\mathcal{A} = \text{MAlg}(X, \mu)$. We first prove that \mathcal{A} satisfies the following property with respect to the \mathcal{R}_Γ -conditional measure $\mu_{\mathcal{R}_\Gamma}$:

$$(4.1) \quad \forall A \in \mathcal{A} \quad \forall B \in \text{MAlg}(X, \mu), \mu_{\mathcal{R}_\Gamma}(A) = \mu_{\mathcal{R}_\Gamma}(B) \Rightarrow B \in \mathcal{A}.$$

Indeed let $A \in \mathcal{A}$, and suppose $B \in \text{MAlg}(X, \mu)$ satisfies $\mu_{\mathcal{R}_\Gamma}(A) = \mu_{\mathcal{R}_\Gamma}(B)$. By Proposition 2.18 we may find an involution $U \in [\mathcal{R}_\Gamma]$ such that $U(A) = B$. Using Lemma 3.1, we then find a sequence $(U_k)_{k \in \mathbb{N}}$ of elements of the group generated by T_1, \dots, T_n such that $\mu(U_k(A) \Delta B) \rightarrow 0$. Since \mathcal{A} is closed and invariant under elements of the group generated by T_1, \dots, T_n , we conclude that $B \in \mathcal{A}$.

(3) Actually $\tilde{\mathcal{A}}$ is automatically closed because $\tilde{\mathcal{A}}$ is a σ -algebra, but we won’t need that.

Now observe recall that by the definition of our observable c , the algebra \mathcal{A} contains a set B whose \mathcal{R}_Γ -conditional measure is constant equal to $1/2$. By Maharam’s lemma, whenever $A \in \text{MAlg}(X, \mu)$ satisfies $\mu_{\mathcal{R}_\Gamma}(A)(x) \leq 1/2$ for all $x \in X$, we can write A as the intersection of two elements of $\text{MAlg}(X, \mu)$ whose \mathcal{R}_Γ -conditional measure is constant equal to $1/2$. So by (4.1), \mathcal{A} contains all sets whose \mathcal{R}_Γ -conditional measure is everywhere at most $1/2$. Using Maharam’s lemma once more, we see that every element of $\text{MAlg}(X, \mu)$ can be written as the union of two elements of $\text{MAlg}(X, \mu)$ whose \mathcal{R}_Γ -conditional measure is everywhere at most $1/2$, so we conclude $\mathcal{A} = \text{MAlg}(X, \mu)$ as wanted. \square

5. Amenability and L^1 full groups

Recall that a Polish group G is *extremely amenable* when every continuous G -action on a compact Polish space has a fixed point, while G is *amenable* if every continuous G -action on a compact Polish space admits an invariant Borel probability measure. Following [28] we say that a Polish group G is *whirly amenable* if it is amenable and, given any continuous G -action on a compact Polish space K , the support of every invariant Borel probability measure is contained in the set of fixed points.

One has the following implications: whirly amenability implies extreme amenability which implies amenability. The first examples of whirly amenable groups were found by Y. Glasner, B. Tsirelson and B. Weiss, who showed that *Lévy groups* are whirly amenable [14, Theorem 1.1]. Note that if G is a Polish group which contains a dense increasing union of whirly amenable groups, then G is whirly amenable.

Let Φ be an amenable graphing, our aim is to show that its derived L^1 full group is whirly amenable. Recall that a measure-preserving equivalence relation is called *finite* if all its classes are finite. The two following lemmas are essentially contained in [8], but we include a full proof for the reader’s convenience.

LEMMA 5.1. — *Let Φ be an aperiodic graphing, suppose \mathcal{R}_Φ is written as an increasing union of equivalence relations \mathcal{R}_n . Then $\bigcup_{n \in \mathbb{N}} [\mathcal{R}_n] \cap [\Phi]'_1$ is dense in $[\Phi]'_1$.*

Proof. — By Theorem 2.14 we only need to be able to approximate involutions from $[\Phi]'_1$ by elements from $\bigcup_{n \in \mathbb{N}} [\mathcal{R}_n] \cap [\Phi]'_1$. Let $U \in [\Phi]'_1$ be an involution, then the U -invariant sets $A_n = \{x \in X : x\mathcal{R}_n U(x)\}$ satisfy

$\bigcup_{n \in \mathbb{N}} A_n = X$ since $\bigcup_n \mathcal{R}_n = \mathcal{R}_\Phi$. By the dominated convergence theorem, we then have $U_{A_n} \rightarrow U$. Moreover U_{A_n} is an involution so $U_{A_n} \in [\Phi]'_1$ and by construction $U_{A_n} \in [\mathcal{R}_n]$ so $U_{A_n} \in [\mathcal{R}_n] \cap [\Phi]'_1$ and the lemma is proved. \square

LEMMA 5.2. — *Let Φ be an aperiodic graphing, let \mathcal{R} be a finite subequivalence relation of \mathcal{R}_Φ whose equivalence classes have a uniformly bounded Φ -diameter. Then $[\mathcal{R}]$ is a closed subgroup of $[\Phi]'_1$.*

Proof. — The fact that there is a uniform bound M on the diameter of the \mathcal{R} -classes ensures us that $[\mathcal{R}]$ is a subgroup of $[\Phi]_1$. Moreover, the inclusion map is M -lipschitz (for the uniform distance on $[\mathcal{R}]$ and d_Φ^1 on $[\Phi]_1$) and its inverse is clearly 1-lipschitz, so $[\mathcal{R}]$ is closed in $[\Phi]_1$. Finally, $[\mathcal{R}]$ only consists of periodic elements, and since these belong to $[\Phi]'_1$ (see [22, Lemma 3.10]), we conclude that $[\mathcal{R}]$ is a closed subgroup of $[\Phi]'_1$. \square

THEOREM 5.3. — *Let Φ be an amenable aperiodic graphing. Then $[\Phi]'_1$ is whirly amenable and $[\Phi]_1$ is amenable.*

Proof. — Since $[\Phi]_1$ is an extension of $[\Phi]'_1$ by an abelian (hence amenable) group, the second statement follows from the first, so we only need to prove that $[\Phi]'_1$ is whirly amenable.

By the Connes–Feldmann–Weiss theorem [6], we may write \mathcal{R}_Φ as an increasing union of finite equivalence relations \mathcal{R}_n . Let us now change these so that their classes have uniformly bounded diameter.

We first find an increasing sequence of integers $(\varphi(n))_{n \in \mathbb{N}}$ such that for all $n \in \mathbb{N}$,

$$\mu(\{x \in X : \text{diam}_\Phi([x]_{\mathcal{R}_n}) > \varphi(n)\}) < \frac{1}{2^n}.$$

Then by the Borel–Cantelli lemma, for almost all $x \in X$ there are only finitely many $n \in \mathbb{N}$ such that $\text{diam}_\Phi([x]_{\mathcal{R}_n}) > \varphi(n)$. So if we define new equivalence relations \mathcal{S}_n by $(x, y) \in \mathcal{S}_n$ if $(x, y) \in \mathcal{R}_n$ and $\forall m \geq n$, $\text{diam}_\Phi([x]_{\mathcal{R}_m}) \leq \varphi(n)$, we still have $\bigcup_{n \in \mathbb{N}} \mathcal{S}_n = \mathcal{R}$ and the \mathcal{S}_n -classes have a uniformly bounded diameter as wanted.

By Lemma 5.1, $\bigcup_{n \in \mathbb{N}} [\mathcal{S}_n] \cap [\Phi]'_1$ is dense in $[\Phi]'_1$. Then Lemma 5.2 yields that $[\mathcal{S}_n] \cap [\Phi]'_1 = [\mathcal{S}_n]$ and that the induced topology on $[\mathcal{S}_n]$ is the uniform topology. Now recall that full groups of finite equivalence relations are whirly amenable for the uniform topology (they are products of groups of the form $L^0(Y, \nu, \mathfrak{S}_n)$ where (Y, ν) is non-atomic, and those are Levy groups by work of Y. Glasner, generalised by T. Giordano and V. Pestov [11, Corollary 2.10]). We conclude that each $[\mathcal{S}_n]$ is whirly amenable so that $[\Phi]'_1$ also is. \square

In [22, Theorem 5.1] it was shown that non amenable graphings have non amenable L^1 full groups and non amenable derived L^1 full groups. So Theorem B is established, and more generally the following holds.

THEOREM 5.4. — *Let Φ be an aperiodic graphing on a standard probability space (X, μ) . The following are equivalent:*

- (1) *the graphing Φ is amenable;*
- (2) *the L^1 full group $[\Phi]_1$ is amenable;*
- (3) *the derived L^1 full group $[\Phi]_1'$ is amenable.*
- (4) *the derived L^1 full group $[\Phi]_1'$ is extremely amenable.*
- (5) *the derived L^1 full group $[\Phi]_1'$ is whirly amenable.*

The above argument can be refined so as to show that derived L^1 full groups of amenable graphings are actually Lévy groups, and that one can find Levy families made of finite subgroups. The details are however tedious and the increasing chain of finite subgroups is not explicit so we won't provide a proof here. It was asked in a first version of this paper whether the following metrics (which are induced by the L^1 metric on the L^1 full group of a 2-odometer on the subgroups \mathfrak{S}_{2^n}) could provide a concrete Lévy family for the derived L^1 full group of the odometer.

DEFINITION 5.5. — *Let \mathfrak{S}_n be the symmetric group over $\{1, \dots, n\}$. Define Spearman's footrule metric on it by*

$$d_{L^1}^n(\sigma, \tau) = \frac{1}{n} \sum_{i=1}^n |\sigma(i) - \tau(i)|.$$

Let us however point out that, as suggested by the referee, the family $(\mathfrak{S}_n, d_{L^1}^n)$ is not a Lévy family. Indeed, by a result of Diaconis and Graham, the random variable $d_{L^1}^n(\text{id}, \sigma)$ where σ is a uniform element of \mathfrak{S}_n has asymptotically mean $n/3$ and variance $2n/45$ [7, Theorem 1] (moreover, suitably normalized, its law converges to the normal distribution). In particular, for n large enough, the ϵ -neighborhood of the ball of radius $n/3$ will be very close in measure to that same ball, thus violating concentration of measure.

6. Large scale geometry and L^1 full groups

6.1. Large scale geometry of Polish groups

In this preliminary section, we recall how Rosendal's framework allow one to study Polish groups as geometric objects. First, a map $f : X \rightarrow Y$

between two metric spaces (X, d_X) and (Y, d_Y) is called a *quasi-isometric embedding* if there is a constant $C > 0$ such that for all $x_1, x_2 \in X$,

$$-C + \frac{1}{C}d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq C + Cd_Y(f(x_1), f(x_2)).$$

It is *coarsely surjective* if there is $C > 0$ such that every element of Y is at distance at most C from $f(X)$. Finally f is a *quasi-isometry* if it is a coarsely surjective quasi-isometric embedding. We say that two metric spaces (X, d_X) and (Y, d_Y) are quasi-isometric if there is a quasi-isometry between them. It is not hard to check that quasi-isometry is an equivalence relation between metric spaces. The equivalence class of a metric space is called its *quasi-isometry type*.

Two metrics d and d' on the same set X are called *quasi-isometric* if the identity map is a quasi-isometry between (X, d) and (X, d') . Note that the coarse surjectivity condition is then automatic so this just means that there is $C > 0$ such that for all $x_1, x_2 \in X$,

$$-C + \frac{1}{C}d(x_1, x_2) \leq d'(x_1, x_2) \leq C + Cd(x_1, x_2).$$

A continuous right-invariant metric d on a Polish group G is called *maximal* if for every continuous right-invariant metric d' on G , there is $C > 0$ such that for all $g_1, g_2 \in G$, $d'(g_1, g_2) \leq C + Cd(g_1, g_2)$. It follows from the definition that any two maximal right-invariant metrics on a Polish group are quasi-isometric, so if a Polish group admits such a metric, then it has a well-defined quasi-isometry type.

We finally give two conditions on a right-invariant metric which together are equivalent to the maximality of the metric by [30, Proposition 2.52].

DEFINITION 6.1. — *A metric d on a set X is called large-scale geodesic if there is $K > 0$ such that for any $x, y \in X$, there are $x_0, \dots, x_n \in X$ with $x_0 = x$ and $x_n = y$ such that $d(x_i, x_{i+1}) \leq K$ and*

$$\sum_{i=0}^{n-1} d(x_i, x_{i+1}) \leq Kd(x, y).$$

DEFINITION 6.2. — *A right-invariant metric d on a topological group G is called coarsely proper if for every neighborhood of the identity V and every $N \in \mathbb{N}$, there are a finite subset $F \subseteq G$ and $n \in \mathbb{N}$ such that*

$$B_d(e, N) \subseteq (FV)^n.$$

6.2. Derived L^1 full groups of amenable graphings

In this section, we show that derived L^1 full groups of amenable graphings fit rather well into C. Rosendal’s framework, enabling their study as geometric objects.

To be more precise, we show that the L^1 metric on derived L^1 full groups of amenable graphings is maximal among left-invariant continuous metrics up to quasi-isometry, which implies that derived L^1 full groups of amenable graphings have a well defined quasi-isometry type which is moreover induced by the L^1 metric.

PROPOSITION 6.3. — *Let Φ be an amenable graphing. Then every element of $[\Phi]’_1$ is a limit of periodic elements.*

Proof. — Using the Connes-Feldmann-Weiss theorem [6], we write \mathcal{R}_Φ as an increasing union of finite equivalence relations \mathcal{R}_n . Then every element of $[\mathcal{R}_n]$ is periodic and by Lemma 5.1 the union $\bigcup_{n \in \mathbb{N}} [\mathcal{R}_n] \cap [\Phi]’_1$ is dense in $[\Phi]’_1$ so we are done. □

The next lemma is key to our results on the geometry of L^1 full groups.

LEMMA 6.4. — *Let Φ be a graphing. Let $T \in [\Phi]_1$ be periodic. Then for every $N \in \mathbb{N}$, there are periodic elements $V_1, \dots, V_N \in [\Phi]_1$ such that $T = V_1 \cdots V_N$ and for every $i \in \{1, \dots, N\}$,*

$$d^1_\Phi(V_i, \text{id}_X) = \frac{1}{N} d^1_\Phi(T, \text{id}_X).$$

Proof. — Let $M = d^1_\Phi(T, \text{id}_X)$. Let $(A_t)_{t \in [0, M]}$ be a continuous increasing path from \emptyset to $\text{supp } T$, and consider $B_t = \bigcup_{n \in \mathbb{N}} (TU^{-1})^n(A_t)$. Since TU^{-1} is periodic, the map $t \mapsto B_t$ is still a continuous increasing path from \emptyset to $\text{supp } T$ now consisting of T -invariant sets.

Define a continuous increasing map $\psi : [0, M] \rightarrow [0, M]$ by

$$\psi(t) = \int_{B_t} d_\Phi(x, T(x)).$$

We have $\psi(0) = 0$ and $\psi(M) = M$, so by the intermediate value theorem we find $t_0 = 0 < t_1 < \dots < t_{N-1} < t_N = M$ such that for all $i \in \{0, \dots, N\}$, $\psi(t_i) = \frac{iM}{N}$. We then let $A_i = B_{t_i} \setminus B_{t_{i-1}}$ for each $i \in \{1, \dots, N\}$. By construction each A_i is T -invariant and $\text{supp } T = \bigsqcup_{i=1}^n A_i$. So if we define $V_i = T_{A_i}$, we get $T = \prod_{i=1}^n V_i$. Finally for each $i \in \{1, \dots, n\}$ the equality

$A_i = B_{t_i} \setminus B_{t_{i-1}}$ yields

$$\begin{aligned} d_{\Phi}^1(\text{id}_X, V_i) &= \int_{B_{t_i}} d_{\Phi}(x, T(x)) - \int_{B_{t_{i-1}}} d_{\Phi}(x, T(x)) \\ &= \psi(t_i) - \psi(t_{i-1}) = \frac{M}{N} \end{aligned}$$

as wanted. □

Remark 6.5. — Observe that the above proof also yields the existence of a geodesic segment between any periodic element and the identity map. Indeed, if ψ^{-1} is a right inverse of ψ defined above, we let $W_t = T_{B_{\psi^{-1}(t)}}$, and then $(W_t)_{t \in [0, M]}$ is a geodesic segment from id_X to T .

PROPOSITION 6.6. — *Let Φ be an amenable graphing, let $K > 0$ and let $N \in \mathbb{N}$. Then we have the following inclusions in the derived L^1 full group of Φ :*

$$B_{d_{\Phi}^1}(1, K/N)^N \subseteq B_{d_{\Phi}^1}(1, K) \subseteq B_{d_{\Phi}^1}(1, K/N)^{N+1}.$$

Proof. — The first inclusion is a straightforward consequence of the triangle inequality and right invariance. For the second one, let $T \in B_{d_{\Phi}^1}(1, K)$, then since periodic elements are dense we can find $U \in B_{d_{\Phi}^1}(1, K/N)$ such that TU^{-1} is periodic and $TU^{-1} \in B_{d_{\Phi}^1}(\text{id}_X, K)$.

By the previous proposition, there are $V_1, \dots, V_N \in B_{d_{\Phi}^1}(\text{id}_X, K/N)$ such that $TU^{-1} = V_1 \cdots V_N$. We conclude that $T = TU^{-1}U = (\prod_{k=1}^N V_k)U$ so $T \in B_{d_{\Phi}^1}(1, K/N)^{N+1}$ as wanted. □

COROLLARY 6.7. — *Let Φ be an amenable graphing, then the L^1 metric on the derived L^1 full group of Φ is maximal.*

Proof. — By the previous proposition, the L^1 metric is both large-scale geodesic and coarsely proper. So [30, Proposition 2.52] yields that the L^1 metric is maximal. □

We finally show that although derived L^1 full groups of amenable graphings do have a geometry, the latter is fundamentally infinite-dimensional.

PROPOSITION 6.8. — *Suppose Γ is an amenable group containing an element of infinite order, and let $\Gamma \curvearrowright (X, \mu)$ be a free measure-preserving action. Then for every $n \in \mathbb{N}$, the derived L^1 full group of the Γ action contains a quasi-isometric copy of \mathbb{R}^n .*

Proof. — It suffices to produce a quasi-isometric copy of \mathbb{Z}^n since the latter is quasi-isometric to \mathbb{R}^n . Let T be the aperiodic measure-preserving transformation which corresponds to the element of Γ of infinite order.

Using Rokhlin’s lemma for T , we find $A \subseteq X$ which intersect almost every T -orbit and such that all sets $A, T(A), \dots, T^{2n-1}(A)$ are disjoint. For each $i \in \{0, \dots, n - 1\}$ let $T_i = T_{T^{2i}(A)} T_{T^{2i+1}(A)}^{-1}$. Observe that since $T T_{T^{2i}(A)} T^{-1} = T_{T^{2i+1}(A)}$, each T_i is a commutator and thus belongs to the derived L^1 full group. It is now straightforward to check that the marked group generated by T_1, \dots, T_n is quasi-isometric to \mathbb{Z}^n . \square

As pointed out by the referee, it would be interesting to know whether some infinite dimensional metric spaces such as ℓ^1 can be coarsely embedded into derived L^1 full groups. We end this section with a basic open question on the geometry of derived L^1 full groups.

QUESTION 6.9. — *Are the derived L^1 full groups of the 2-odometer and of the 3-odometer quasi-isometric?*

Note that by [8], the derived L^1 full groups of the 2-odometer and of the 3-odometer are not isomorphic as abstract groups. Also, the countable group of dyadic permutations \mathfrak{S}_{2^∞} (resp. \mathfrak{S}_{3^∞}) is dense in the derived L^1 full group of the 2-odometer (resp. 3-odometer), and the induced metric is the inductive limit of the metrics from Definition 5.5, so this is really a question about quasi-isometry for the non finitely generated countable groups \mathfrak{S}_{2^∞} and \mathfrak{S}_{3^∞} equipped with these natural metrics.

6.3. L^1 full groups of ergodic \mathbb{Z} -actions

For measure-preserving \mathbb{Z} -actions, we have a much better understanding of elements of the L^1 full group. This will allow us to show that the L^1 metric on the whole L^1 full group is maximal, and hence defines its quasi isometry type. Since a measure-preserving \mathbb{Z} -action is nothing but a single measure-preserving transformation, we rather speak of L^1 full groups of measure-preserving transformations. Let us start by reminding Belinskaya’s decomposition results for elements of the L^1 full group [4].

Given a measure-preserving transformation T , we denote by $[T]_1$ its L^1 full group. If T is aperiodic then every T -orbit is endowed with a linear ordering \leq_T defined by $x \leq_T y$ if and only if there is $n \geq 0$ such that $y = T^n(x)$. An element U of the full group of T is *almost positive* if for all $x \in X$, there is $N \in \mathbb{N}$ such that for all $n \geq N$ we have $U^n(x) \geq_T x$. It is *almost negative* if for all $x \in X$, there is $N \in \mathbb{N}$ such that for all $n \geq N$ we have $U^n(x) \leq_T x$. Every element of the L^1 full group has a natural decomposition with respect to such elements.

LEMMA 6.10. — *Let T be a measure-preserving aperiodic transformation, let $U \in [T]_1$. Then there is a partition $X = X_- \sqcup X_{per} \sqcup X_+$ of X into U -invariant sets such that U_{X_-} is almost negative, U_{X_+} is almost positive and $U_{X_{per}}$ is periodic.*

Proof. — This is a direct consequence of [4, Theorem 3.2]. □

Observe that since the sets from the above decomposition are U -invariant and partition X , we then have $U = U_{X_-} U_{X_{per}} U_{X_+}$ and

$$d_T^1(\text{id}_X, U) = d_T^1(\text{id}_X, U_{X_-}) + d_T^1(\text{id}_X, U_{X_{per}}) + d_T^1(\text{id}_X, U_{X_+}).$$

Recall that if T is an aperiodic measure-preserving transformation, every $U \in [T]_1$ has an associated cocycle $c_U : X \rightarrow \mathbb{Z}$ uniquely defined by the equation $U(x) = T^{c_U(x)}(x)$. We can then defined the *index map* $I : [T]_1 \rightarrow \mathbb{R}$ by $I(U) = \int_X c_U(x) d\mu(x)$. In [22, Corollary 4.20], we prove that if T is ergodic, the kernel of the index map is the derived L^1 full group of T and the index map takes values into \mathbb{Z} . The conjunction of Proposition 4.15 and Proposition 4.17 from [22] implies the following result.

LEMMA 6.11. — *Let $U \in [T]_1$ be almost positive, let $n = I(U)$. Then there is a family $(A_i)_{i=1}^n$ of subsets of X such that $U = T_{A_1} \cdots T_{A_n} V$, where V is periodic.*

THEOREM 6.12. — *Let T be an ergodic measure-preserving transformation. Then the L^1 metric on the L^1 full group of T is maximal.*

Proof. — We first show that for every $N \in \mathbb{N}$, every element of $[T]_1$ of L^1 norm⁽⁴⁾ less than N can be written as a product of at most $11N$ elements, all of which are either T , T^{-1} , or periodic of L^1 norm less than 1.

Let $X = X_{per} \sqcup X_+ \sqcup X_-$ be the partition of X into U -invariant sets provided by Lemma 6.10. We start by noting that $d_T^1(U_{X_+}, \text{id}_x) \leq d_T^1(U, \text{id}_x)$ and thus $I(U_{X_+}) < N$. We may thus write

$$U_{X_+} = T_{A_1} \cdots T_{A_n} V,$$

where $n = I(U_{X_+}) < N$ and V is periodic. By Kac's return time theorem [16, Theorem 2[]], each T_{A_i} is at distance 1 from id_X , so by the triangle inequality, we have $d_T^1(V, \text{id}_X) < 2N$.

Now for each $i \in \{1, \dots, n\}$ we have $T_{A_i} = (T_{A_i} T^{-1}) T$. Moreover $T_{A_i} T^{-1}$ is periodic and $d_T^1(T_{A_i} T^{-1}, \text{id}_X) < 2$. Since U_{X_+} is equal to the product $(T_{A_1} T^{-1}) T \cdots (T_{A_n} T^{-1}) T V$ and $d_T^1(V, \text{id}_X) < 2N$, Lemma 6.4 applied to

⁽⁴⁾By L^1 norm, we mean the L^1 distance to id_X .

both V and the $T_{A_i}T^{-1}$ yields that U_{X_+} is a product of at most $5N$ elements, all of which are either periodic and in the open unit ball or equal to T .

The same reasoning for U_{X_-} yields that U_{X_-} can be written as a product of at most $5N$ elements, all of which are either periodic of norm < 1 or equal to T^{-1} .

Finally, Lemma 6.4 yields that $U_{X_{per}}$ can be written as the product of at most N periodic elements of norm < 1 . We can thus conclude that $U = U_{X_+}U_{X_{per}}U_{X_-}$ can be written as the product of at most $11N$ elements, all of which are either periodic of norm < 1 or belong to $\{T, T^{-1}\}$.

This fact already yields that the L^1 metric is large-scale geodesic, and we now need to show that d_T^1 is coarsely proper. So let $N \in \mathbb{N}$, let V be a neighborhood of the identity. We find $k \in \mathbb{N}$ such that $B_{d_T^1}(\text{id}_X, 1/k) \subseteq V$ and we deduce from the above fact along with Lemma 6.4 that

$$B_{d_T^1}(\text{id}_X, N) \subseteq (\{T, T^{-1}\}V)^{11kN}.$$

The metric d_T^1 is thus both large-scale geodesic and coarsely proper, so using [30, Proposition 2.52] we conclude that d_T^1 is maximal as wanted. \square

6.4. Automatic quasi-isometry: proof of Theorem C

In this section we prove the following result, which directly yields Theorem C.

THEOREM 6.13. — *Let Φ and Ψ be two amenable ergodic graphings on a standard probability space (X, μ) . Then every abstract group isomorphism between their L^1 full groups must be a quasi-isometry for their respective L^1 metrics.*

Proof. — Let $\rho : [\Phi]_1 \rightarrow [\Psi]_1$ be an abstract group isomorphism. First observe that by [22, Corollary 3.19], there is a measure-preserving transformation S such that for all $U \in [\Phi]_1$, $\rho(U) = SUS^{-1}$. In other words, we can assume that ρ is the conjugacy by some $S \in \text{Aut}(X, \mu)$, and it follows that ρ is d_u -isometric, in particular it is continuous for the uniform topology. On $[\Phi]_1$ the uniform topology is refined by the L^1 topology, and since it separates points we get that it generates the Borel σ -algebra of $[\Phi]_1$ induced by the L^1 topology. We conclude that ρ is a Borel group isomorphism, and it follows from Pettis' lemma (see e.g. [3, Theorem 1.2.5]) that ρ is a topological group isomorphism.

In particular, ρ induces a topological group isomorphism between $[\Phi]'_1$ and $[\Psi]'_1$, and since their respective L^1 metrics are maximal (Corollary 6.7), we conclude that ρ induces a quasi-isometry for their respective L^1 metrics: there is $C > 0$ such that for all $T_1, T_2 \in [\Phi]'_1$,

$$-C + \frac{1}{C}d_{\Phi}^1(T_1, T_2) \leq d_{\Psi}^1(\rho(T_1), \rho(T_2)) \leq Cd_{\Phi}^1(T_1, T_2) + C$$

In order to establish the same inequality for elements of the whose L^1 full group, first note that by right-invariance, one may as well assume $T_1 = \text{id}_X$, and then note that by symmetry, it suffices to find $C' > 0$ such that for all $T \in [\Phi]_1$,

$$d_{\Phi}^1(\text{id}_X, \rho(T)) \leq C'd_{\Psi}^1(\text{id}_X, T) + C'.$$

We will show that $C' = 6C$ works.

Let $T \in [\Phi]_1$, then by a well-known lemma (see e.g. [22, Proposition 2.7]), there is a partition $(A_1, A_2, B_1, B_2, B_3)$ of $\text{supp } T$ such that

$$T(A_1) = A_2, T(B_1) = B_2 \text{ and } T(B_2) = B_3.$$

We use this decomposition to define three involutions U_1, U_2, U_3 as follows:

$$U_1(x) = \begin{cases} T(x) & \text{if } x \in A_1 \sqcup B_1 \\ T^{-1}(x) & \text{if } x \in A_2 \sqcup B_2 \\ x & \text{otherwise,} \end{cases}$$

$$U_2(x) = \begin{cases} T(x) & \text{if } x \in B_2 \\ T^{-1}(x) & \text{if } x \in B_3 \\ x & \text{otherwise and} \end{cases}$$

$$U_3(x) = \begin{cases} T(x) & \text{if } x \in A_2 \sqcup B_3 \\ T^{-1}(x) & \text{if } x \in T(A_2) \sqcup T(B_3) \\ x & \text{otherwise.} \end{cases}$$

By construction for each $i = 1, 2, 3$ we have $d_{\Phi}^1(\text{id}_X, U_i) \leq 2d_{\Phi}^1(\text{id}_X, T)$, moreover since U_1, U_2 and U_3 are involutions they actually belong to the derived L^1 full group of Φ . We can then decompose T as

$$T = U_1 \upharpoonright_{A_1 \sqcup B_1} \sqcup U_2 \upharpoonright_{B_2} \sqcup U_3 \upharpoonright_{A_2 \sqcup B_3} \sqcup \text{id}_{X \setminus \text{supp } T}.$$

Because ρ is the conjugacy by the transformation S , we then have

$$\rho(T) = \rho(U_1) \upharpoonright_{S(A_1 \sqcup B_1)} \sqcup \rho(U_2) \upharpoonright_{S(B_2)} \sqcup \rho(U_3) \upharpoonright_{S(A_2 \sqcup B_3)} \sqcup \text{id}_{X \setminus S(\text{supp } T)}.$$

We thus have $d_{\Psi}^1(\text{id}_X, \rho(T)) \leq \sum_{i=1}^3 d_{\Psi}^1(\text{id}_X, U_i)$. Since for $i = 1, 2, 3$,

$$d_{\Psi}^1(\text{id}_X, \rho(U_i)) \leq Cd_{\Phi}^1(\text{id}_X, U_i) + C \leq 2Cd_{\Phi}^1(\text{id}_X, T) + C,$$

we conclude that

$$d_{\Psi}^1(\text{id}_X, \rho(T)) \leq \sum_{i=1}^3 d_{\Psi}^1(\text{id}_X, U_i) \leq 6Cd^1(\Phi(\text{id}_X, T)) + 3C$$

as wanted. □

Note that if we knew that the L^1 metric on L^1 full groups of amenable graphings is maximal, then the first two paragraphs of the above proof would suffice to get the conclusion of Theorem 6.13. As explained in the introduction, we actually conjecture that the following is true, which would then imply that one can remove both the amenability and the ergodicity hypothesis in Theorem 6.13.

CONJECTURE 6.14. — *Let Φ be a graphing. Then the L^1 metric on the L^1 full group of Φ is maximal.*

We end this section by explaining why in Theorem 6.13 the conclusion cannot be strengthened to bilipschitzness. Recall that in a topological space, a subset is G_{δ} if it can be written as a countable intersection of open sets, and that by the Baire category theorem, in a complete metric space every countable intersection of dense open sets has to be dense.

LEMMA 6.15. — *Let T be an aperiodic transformation. Then the set of all $A \in \text{MAlg}(X, \mu)$ such that the return time to A is unbounded is dense G_{δ} in $\text{MAlg}(X, \mu)$.*

Proof. — The set \mathcal{B} of all $A \in \text{MAlg}(X, \mu)$ such that the return time to A is unbounded is the intersection over $n \in \mathbb{N}$ of the sets

$$\mathcal{B}_n := \left\{ A \in \text{MAlg}(X, \mu) : \mu(T^{-n}(A) \setminus \bigcup_{i=0}^{n-1} T^{-i}(A)) > 0 \right\},$$

so it is G_{δ} . Let us now fix $n \in \mathbb{N}$ and show that \mathcal{B}_n is dense so as to finish the proof. Let $\epsilon > 0$, let $N \in \mathbb{N}$ such that $\frac{n}{N} < \epsilon$. By aperiodicity we find $B \subseteq X$ with positive measure such that $B, T(B), \dots, T^{Nn}(B)$ are disjoint, and so in particular $\mu(B) < \frac{1}{Nn}$. Now observe that for any $A \in \text{MAlg}(X, \mu)$, the set

$$(A \cup B \cup T^n(B)) \setminus \bigcup_{i=1}^{n-1} T^i(B)$$

is in \mathcal{B}_n and is $n\mu(B) < \epsilon$ -close to A , which shows that \mathcal{B}_n is dense. □

Example 6.16. — Let T be an aperiodic measure-preserving transformation, and consider $A \subseteq X$ with unbounded return time, as provided by the previous lemma. Then the conjugacy by the induced transformation

T_A is not bilipschitz: for each n we may find $B \subseteq A$ such that $T(B), \dots, T^n(B)$ are all disjoint from A , and then if U is the involution defined by $U = T|_B \sqcup T|_{T(B)}^{-1} \sqcup \text{id}_{X \setminus (B \sqcup T(B))}$, we have $d_T^1(\text{id}_X, T_A U T_A^{-1}) \geq n d_T^1(\text{id}_X, U)$.

7. Topological rank two for rank one transformations

In this section we prove that every rank one measure-preserving transformation has an L^1 full group whose topological rank is equal to two. Let us recall the definition of such transformations and first introduce some useful terminology.

DEFINITION 7.1. — *Let T be a measure-preserving transformation and $A \subseteq X$ Borel. Let $N_T(A) \in \mathbb{N}$ be the greatest integer such that $A, T(A), \dots, T^{N_T(A)-1}(A)$ are all disjoint. We then have a Rokhlin partition $\mathcal{P}_{A,T} = (A, T(A), \dots, T^{N_T(A)-1}(A))$ of $Y_{A,T} := \bigcup_{i=0}^{N_T(A)-1} T^i(A)$ and we let $M_{A,T}$ be the sets of finite unions of elements of $\mathcal{P}_{A,T}$.*

Note that Rokhlin’s lemma guarantees the existence of Rokhlin partitions $\mathcal{P}_{A,T}$ with arbitrarily small atoms and $\mu(Y_{A,T})$ arbitrarily close to 1.

DEFINITION 7.2. — *A measure-preserving transformation T is rank one if for every $\epsilon > 0$ and every finite family $(B_i)_{i=1}^k$ of Borel sets there is a Borel set A and Borel sets $A_1, \dots, A_k \in M_{A,T}$ such that for all $i \in \{1, \dots, k\}$,*

$$\mu(A_i \triangle B_i) < \epsilon.$$

It is not hard to check from the definition that rank one transformations are ergodic and form a G_δ set in $\text{Aut}(X, \mu)$ for the weak topology (see [20, Lemme 5.26] for a detailed proof). An easy example of a rank one transformation is provided by the odometer, and so by Halmos’ conjugacy lemma (see [18, Theorem 2.4]), rank one transformations form a dense G_δ set in $\text{Aut}(X, \mu)$.

The following theorem of Baxter says that rank one transformations are actually quite close to odometers.

THEOREM 7.3 ([2]). — *Let T be a rank one transformation. Then there is a decreasing sequence (A_n) of Borel subsets of X such that $M_{A_n,T}$ is increasing and its union is μ -dense in the Borel σ -algebra.*

A sequence (A_n) as in the previous theorem will be called a *rank one basis* of T , so in our terminology a measure-preserving is rank one if and only if it admits a rank one basis. Observe that if (A_n) is a rank one basis for T then the measure of A_n then tends to zero and $N_T(A_n) \rightarrow +\infty$.

Example 7.4. — Let T_0 be the dyadic odometer on $X = \{0, 1\}^{\mathbb{N}}$, and for every $n \in \mathbb{N}$ we let A_n denote the set of sequences $(x_i) \in \{0, 1\}^{\mathbb{N}}$ such that $x_0 = \dots = x_n = 0$. Then (A_n) is a rank one basis for T_0 with $N_{T_0}(A_n) = 2^n$ and the additional property that $Y_{A_n, T} = X$ for every $n \in \mathbb{N}$.

We will now generalize to rank one transformations the fact that dyadic transformations are dense in the derived L^1 full group of the odometer. Let us first see how Definition 7.1 provides us natural embeddings of the symmetric group in the derived L^1 full group.

Let T be a measure-preserving transformation and $A \subseteq X$ be Borel. We then define the embedding

$$\rho_A : \mathfrak{S}(\{0, \dots, N_T(A) - 1\}) \rightarrow [T]'_1$$

by: for all $\sigma \in \mathfrak{S}(\{0, \dots, N(A) - 1\})$ and all $x \in Y_{A, T}$, if $i \in \{0, \dots, N(A) - 1\}$ is such that $x \in T^i(A)$, then

$$\rho_A(\sigma)(x) = T^{\sigma(i)-i}(x),$$

and $\rho_A(\sigma)(x)$ for $x \notin Y_{A, T}$. Observe that by construction for all $i \in \{0, \dots, N(A) - 1\}$ we have $\rho_A(\sigma)(T^i(A)) = T^{\sigma(i)}(A)$. We then let

$$\mathfrak{S}_{A, T} := \rho_A(\mathfrak{S}(\{0, \dots, N(A) - 1\})) \text{ and } \mathfrak{A}_{A, T} := \rho_A(\mathfrak{A}(\{0, \dots, N(A) - 1\})).$$

Observe that if $A \subseteq B$ and $M_{A, T} \subseteq M_{B, T}$ then $\mathfrak{S}_{A, T} \leq \mathfrak{S}_{B, T}$ and $\mathfrak{A}_{A, T} \leq \mathfrak{A}_{B, T}$. The following generalizes to L^1 full groups a result from the thesis of the author [20, Lemme 5.31].

PROPOSITION 7.5. — *Let (A_n) be a rank one basis of a measure-preserving transformation T . Then*

$$\bigcup_{n \in \mathbb{N}} \mathfrak{A}_{A_n, T}$$

is dense in $[T]'_1$.

Proof. — Let U_n be the involution defined by $U_n = \varphi_{A_n}((0 \ 1))$, then $d_T^1(U_n, 1) \rightarrow 0$. Since $\mathfrak{S}_{A_n, T} = \mathfrak{A}_{A_n, T} \cup U_n \mathfrak{A}_{A_n, T}$, we conclude that the density of $\bigcup_{n \in \mathbb{N}} \mathfrak{A}_{A_n, T}$ will follow from that of $\bigcup_{n \in \mathbb{N}} \mathfrak{S}_{A_n, T}$ which will thus prove instead.

Since $[T]'_1$ is topologically generated by involutions of the form $I_{T, A}$ (cf. Theorem 2.14), we only need to approximate such involutions by elements of $\bigcup_{n \in \mathbb{N}} \mathfrak{S}_{A_n, T}$. So let $A \subseteq X$, we then have a sequence (B_n) of elements of $M_{A_n, T}$ with $B_n \rightarrow A$. Furthermore we may assume that B_n is disjoint from $A_n \sqcup T^{N(A_n)-1}$ since the measure of $A_n \sqcup T^{N(A_n)-1}$ tends to zero as n tends to infinity. This implies $I_{B_n, T} \in \mathfrak{S}_{A_n, T}$ and since $I_{B_n, T} \rightarrow I_{A, T}$ we get the desired conclusion. □

LEMMA 7.6. — *Let T be a measure preserving transformation, let $A \subseteq X$ and let $n \in \mathbb{N}$ such that $3 \leq n \leq N_T(A) - 2$. Consider the element $U = \rho_A((0 \ 1 \ \cdots \ n - 1))$. Then the group generated by T and U contains $\mathfrak{A}_{A,T}$.*

Proof. — By conjugating U by powers of T , we see that the group generated by T and U contains all the $\rho_A((i \ i+1 \ \cdots \ i+n-1))$ for $i+n-1 \leq N(A)$. The lemma then follows from the fact that $\mathfrak{A}(\{0, \dots, N(A) - 1\})$ is contained in the group generated by the $(i \ i+1 \ \cdots \ i+n-1)$ for $0 \leq i \leq N(A) - n$. We briefly recall the proof of the latter fact for the reader's convenience.

Denote by G the group generated by the $(i \ i+1 \ \cdots \ i+n-1)$ for $0 \leq i \leq N(A) - n$. First note that G contains the permutation

$$(n \ n-1 \ \cdots \ 1)(0 \ 1 \ \cdots \ n-1) = (0 \ n \ n-1)$$

so it also contains $(0 \ n \ n-1)^2 = (0 \ n-1 \ n)$. Conjugating by a power of $(1 \ 2 \ \cdots \ n)$, we obtain that G contains the element $(0 \ 1 \ 2)$. Conjugating again by appropriate cycles we get that G contains all the $(i \ i+1 \ i+2)$ for $0 \leq i \leq n-3$, and since the latter generate the group $\mathfrak{A}(\{0, \dots, N(A) - 1\})$ the result follows. □

The usual way of disjointifying a countable family of sets (B_n) is to let $B'_n := B_n \setminus \bigcup_{m < n} B_m$, but note that letting $B'_n = B_n \setminus \bigcup_{m > n} B_m$ also works. We will need to apply this idea to measure-preserving transformations so as to make their support disjoint.

For $p \in \mathbb{N}$, let us call a p -cycle a transformation all whose orbits have cardinality either 1 or p . Given a countable family $(U_n)_{n \in \mathbb{N}}$ where each U_n is a p_n -cycle, we explain how to change them so that they have disjoint support. For each n , we let V_n be the transformation induced by U_n on the biggest U_n -invariant set contained in $\text{supp } U_n \setminus \bigcup_{m > n} \text{supp } U_m$. Note that since U_n is a p_n -cycle, we can easily compute the support of V_n which is

$$\text{supp } V_n = \text{supp } U_n \setminus \left(\bigcup_{i=0}^{p_n-1} U_n^i \left(\bigcup_{m > n} \text{supp } U_m \right) \right).$$

Of course this support might be empty, but observe that if the supports of the remaining U_m 's are very small then V_n will actually be very close to U_n . The sequence (V_n) obtained this way is called the *disjointification* of the sequence (U_n) .

We are now ready to prove the main result of this section.

THEOREM 7.7. — *Let T be a rank one transformation. Then the topological rank of its L^1 full group $[T]_1$ is equal to 2.*

Proof. — Let $(p_n)_{n \in \mathbb{N}}$ be the increasing enumeration of prime numbers. Let ϵ_n be a sequence of positive real numbers decreasing to zero, e.g. $\epsilon_n = 2^{-n}$. Let (A_n) be a one-dimensional ranking of T .

Using Lemma 7.6, one can now build by induction an increasing sequence of integers (k_n) and positive real numbers (δ_n) such that for all n , if we let

$$U_n := \varphi_{A_{k_n}}((0 \ 1 \cdots p_n - 1))$$

then the following conditions are satisfied.

- (i) Any element V_n which is δ_n -close to U_n satisfies that the group generated by T and V_n contains every element of $\mathfrak{A}_{A_{k_n}, T}$ up to an ϵ_n error.
- (ii) For every $m < n$, we have $p_m \sum_{k=m+1}^n \mu(\text{supp } U_k) < \delta_m$.
- (iii) We have $k_n \geq 4^{n2^n + 2^n}$.

Consider the sequence (V_n) defined as the disjointification of the U_n 's. Thanks to condition (ii) we have that V_n is δ_n -close to U_n . By condition (i) the group $\langle T, V_n \rangle$ contains every element of $\mathfrak{A}_{A_{k_n}, T}$ up to an ϵ_n error.

Now let $V := \prod_{n \in \mathbb{N}} V_n$. We first establish a general inequality which will yield that $V \in [T]_1$ and moreover the closed group generated by V contains every V_n .

CLAIM. — For all $n \in \mathbb{N}$ we have

$$(7.1) \quad \left(\prod_{i < n} p_i \right) \left(\sum_{i \geq n} \frac{p_i}{k_i} \right) \leq 2^{-n2^n}.$$

Proof of claim. — We first need an explicit bound on p_n . A well-known way to get such a bound is as follows: Euclid's argument that there are infinitely many prime numbers actually shows that $p_{n+1} \leq p_0 \cdots p_n + 1$. From this, a straightforward induction yields the bound

$$p_n \leq 2^{2^n}.$$

Now by condition (iii) we have $k_n \geq 4^{n2^n + 2^n}$, so for every $n \geq 1$ we have

$$(7.2) \quad \sum_{i \geq n} \frac{p_i}{k_i} \leq 4^{-n2^n} \sum_{i \geq n} \frac{2^{2^i}}{4^{2^i}}$$

$$(7.3) \quad \leq 4^{-n2^n}$$

Now $\prod_{i < n} p_i \leq (p_n)^n \leq 2^{n2^n}$ so combining this with the previous inequality we see that (7.1) holds. □

Now observe that for every $n \in \mathbb{N}$ we have $d_T^1(V_n, \text{id}_X)_1 \leq d_T^1(U_n, \text{id}_X) \leq \frac{2p_n}{k_n}$. The inequality (7.1) for $n = 0$ yields $\sum_{i \geq 0} \frac{2p_i}{k_i} < +\infty$ so that $V \in [T]_1$ as announced.

Fix $n \in \mathbb{N}$. For every $m \geq n + 1$ let $P_m = \prod_{i < m, i \neq n} p_i$ and let $t_m \in \{1, \dots, p_n - 1\}$ such that $P_m t_m \equiv 1 \pmod{p_n}$. Now observe that

$$V^{P_m t_m} = V_n \prod_{i \geq m} V_i^{P_m t_m}.$$

Since $t_m \leq p_n$, we have $P_m t_m \leq \prod_{i < m} p_i$ so that

$$d_T^1 \left(\prod_{i > m} V_i^{P_m t_m}, \text{id}_X \right) \leq \left(\prod_{i < m} p_i \right) \sum_{i \geq k} d_T^1(V_i, \text{id}_X).$$

But for every i we have $\|V_i\|_1 \leq \frac{2p_i}{k_i}$ so we conclude by inequality (7.1) that we have $d_T^1(\prod_{i > m} V_i^{P_m t_m}, \text{id}_X) \rightarrow 0$ and hence

$$V^{P_m t_m} \rightarrow V_n \quad [m \rightarrow +\infty].$$

So for every $n \in \mathbb{N}$ the closed subgroup generated by V contains V_n .

Since the group $\langle T, V_n \rangle$ contains every element of $\mathfrak{A}_{A_{k_n, T}}$ up to an ϵ_n error and $\epsilon_n \rightarrow 0$, we conclude by Proposition 7.5 that the group generated by T and V is dense in $[T]_1$ which has thus topological rank at most 2. Since the L^1 full group $[T]_1$ is not abelian, we conclude that $[T]_1$ has topological rank 2 as wanted. □

8. Further remarks and questions

8.1. L^p orbit equivalence and L^p full orbit equivalence

It is an instructive exercise to show that given two measure preserving actions $\alpha : \Gamma \rightarrow \text{Aut}(X, \mu)$, $\beta : \Lambda \rightarrow \text{Aut}(X, \mu)$ and a measure-preserving transformation T , the following are equivalent:

- (i) T sends $\alpha(\Gamma)$ -orbits to $\beta(\Lambda)$ -orbits ;
- (ii) $T\alpha(\Gamma)T^{-1}$ is a subgroup of $[\beta(\Lambda)]$ and $T^{-1}\beta(\Lambda)T$ is a subgroup of $[\alpha(\Gamma)]$;
- (iii) $T[\alpha(\Gamma)]T^{-1} = [\beta(\Gamma)]$.

If T satisfies any of the above three conditions, one says that T is an *orbit equivalence* between the actions α and β , and that the latter are *orbit equivalent*.

We then have two natural L^p versions of orbit equivalence, which are direct translations of item (ii) and (iii) from the above equivalence. The first one was defined by Austin in [1] and the second appears to be new.

DEFINITION 8.1. — *Let $p \in [1, +\infty]$. Two measure-preserving actions of finitely generated groups $\alpha : \Gamma \rightarrow \text{Aut}(X, \mu)$ and $\beta : \Lambda \rightarrow \text{Aut}(X, \mu)$ are called L^p -orbit equivalent if there is $T \in \text{Aut}(X, \mu)$ such that*

$$T\alpha(\Gamma)T^{-1} \leq [\beta(\Lambda)]_p \quad \text{and} \quad T^{-1}\beta(\Lambda)T \leq [\alpha(\Gamma)]_p.$$

Remark 8.2. — For $p = +\infty$, the group $[\alpha(\Gamma)]_p$ is the group of all $T \in [\alpha(\Gamma)]$ such that the distance from $T(x)$ to x is essentially bounded. The L^∞ metric on this group is the discrete metric, so it is not a Polish group.

DEFINITION 8.3. — *Let $p \in [1, +\infty]$. Two measure-preserving actions of finitely generated groups $\alpha : \Gamma \rightarrow \text{Aut}(X, \mu)$ and $\beta : \Lambda \rightarrow \text{Aut}(X, \mu)$ are called L^p -fully orbit equivalent if there is $T \in \text{Aut}(X, \mu)$ such that*

$$T[\alpha(\Gamma)]_p T^{-1} = [\beta(\Lambda)]_p.$$

It is clear that L^p full orbit equivalence implies L^p orbit equivalence, and that L^∞ orbit equivalence is equivalent to L^∞ full orbit equivalence. Moreover, for all $p < q$ we have that L^q orbit equivalence implies L^p orbit equivalence, but it is unclear that L^q full orbit equivalence implies L^p full orbit equivalence. Also note that all these notions collapse to flip-conjugacy for ergodic \mathbb{Z} -actions by Belinskaya’s theorem.

A natural intermediate question is to ask what it means for two graphings to share the same L^1 full group, hoping that the identity map has to be induce an L^∞ orbit equivalence between them. However, this is not so, as the following example of Sébastien Gouëzel shows.

THEOREM 8.4 (Gouëzel). — *Given an aperiodic $T \in \text{Aut}(X, \mu)$, there is $T' \in \text{Aut}(X, \mu)$ such that $[T]_1 = [T']_1$ but T' does not belong to the L^∞ full group of T .*

Proof. — First note that for every $S \in \text{Aut}(X, \mu)$ we have $[STS^{-1}]_1 = S[T]_1 S^{-1}$, so for every $S \in [T]_1$ we have $[STS^{-1}]_1 = [T]_1$. So it suffices to find $S \in [T]_1$ such that $T' = STS^{-1}$ does not belong to the L^∞ full group of T .

Now observe that given any $S \in [T]_1$, the function $x \mapsto d_T(x, STS^{-1}(x))$ is bounded if and only if the function $x \mapsto d_T(S(x), ST(x))$ is bounded. It thus suffices to find $S \in [T]_1$ such that the function $x \mapsto d_T(S(x), ST(x))$ is unbounded. This can be achieved by taking $A \subseteq X$ with unbounded return time via Lemma 6.15, and considering the induced transformation $S = T_A$. Indeed if $x \in A$ has return time $n_{T,A}(x) \geq 2$ then $ST(x) = T(x)$ and so $d_T(S(x), ST(x)) = n_{T,A}(x) - 1$. □

Gouëzel’s example arises as a conjugate of T by some $S \in [T]_1$, but it is natural to ask more generally which $S \in \text{Aut}(X, \mu)$ satisfy $[STS^{-1}]_1 = [T]_1$.

Note that such an S has to be an automorphism of the equivalence relation \mathcal{R}_T . Two sources of such S are provided by the group of measure-preserving transformations which conjugate T to $T^{\pm 1}$, and elements of the L^1 full group of T . Still, we do not have a good understanding of the group of $S \in \text{Aut}(X, \mu)$ satisfying $[STS^{-1}]_1 = [T^{\pm 1}]_1$, and the following question is open.

QUESTION 8.5. — *Let T be an ergodic measure-preserving transformation. Is the group of $S \in \text{Aut}(X, \mu)$ satisfying $[T]_1 = [STS^{-1}]_1$ Borel? If so, is it Polishable?*

Note that by Fremlin's reconstruction theorem, every abstract automorphism of $[T]_1$ has to be the conjugacy by some $S \in \text{Aut}(X, \mu)$, so the above question is actually about the automorphism group of $[T]_1$. Also note that the automorphism group of the full group of any measure preserving equivalence relation is Polishable, see [18, Theorem 6.1].

8.2. Symmetric L^p full groups

The notions of symmetric (and alternating) topological full groups have natural analogues in our setup. Recall from [22, Section 5.3] that for any $p \in [1, +\infty[$ and any graphing Φ , the L^p full group of Φ is defined as the group of all elements T of the full group of \mathcal{R}_Φ such that

$$\int_X d_\Phi(x, T(x))^p d\mu(x) < +\infty.$$

It is a Polish group for the metric d_Φ^p defined by

$$d_\Phi^p(T_1, T_2) = \left[\int_X d_\Phi(T_1(x), T_2(x))^p d\mu(x) \right]^{1/p},$$

and we define the following closed normal subgroups.

DEFINITION 8.6. — *The symmetric L^p full group of a graphing Φ is the closed subgroup of $[\Phi]_p$ generated by all n -cycles for $n \geq 2$. The alternating L^p full group is defined as the closed subgroup generated by all 3-cycles.*

Note that if $p = 1$, both subgroups coincide with the derived L^1 full group of Φ , and I don't know whether the same holds for $p > 1$. Nevertheless, if Φ is aperiodic, the same proof as Proposition 3.4 shows that the alternating L^p full group of Φ coincides with its symmetric L^p full group. One can also adapt the arguments from [22] to show that in the aperiodic case, closed normal subgroups of the symmetric L^p full group correspond to invariant

sets; in particular the symmetric L^p full group of Φ is topologically simple iff Φ is ergodic. Moreover, the proof of Theorem A can easily be adapted to show the following.

THEOREM 8.7. — *The topological rank of the symmetric L^p full group of a measure-preserving aperiodic action of a finitely generated group is finite if and only if the action has finite Rokhlin entropy.*

In a work in progress with C. Conley and R. Tucker-Drob, we show that for ergodic \mathbb{Z} -actions, the symmetric L^p full group coincides with the derived L^p full group, and that the topological abelianization of the whole L^p full group is \mathbb{Z} , just as in the L^1 case. In particular, we can deduce that the whole L^p full group of an ergodic measure-preserving transformation has finite topological rank if and only if the transformation has finite entropy, and answer positively Question 5.5 from [22] in the case the graphing comes from a \mathbb{Z} -action.

8.3. Topological rank of (derived) L^1 full groups

Theorem A is not satisfactory in two ways: first, as in the \mathbb{Z} case, a more quantitative statement would be desirable. In particular, establishing a lower bound on the topological rank in terms of the Rokhlin entropy would be very nice. Second, it would be more natural to have a similar statement for the whole L^1 full group, but in order to do so one has to understand the topological abelianization of L^1 full groups. We would like to ask the following question, noting that a positive answer would imply that Theorem A also holds for the whole L^1 full group.

QUESTION 8.8. — *Let Γ be a finitely generated group. Is the topological abelianization of the L^1 full group of any ergodic measure-preserving Γ -action always topologically finitely generated?*

At the moment, the question is open even for $\Gamma = \mathbb{Z}^2$. It was answered in the affirmative for $\Gamma = \mathbb{Z}$ in [22], and with Conley and Tucker-Drob we have a proof that the answer is also yes when Γ is the infinite dihedral group (we actually show that in this case the topological abelianization is always trivial, thus obtaining examples of topologically simple L^1 full groups). Also note that the topological rank of the topological abelianization can be greater than the rank of the acting group: for a free ergodic \mathbb{Z}^2 action, if the two canonical generators have diffuse ergodic decomposition then \mathbb{R}^2 can be obtained as a quotient of the topological abelianization by integrating the cocycles, but it has topological rank $3 > 2$.

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