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## EQUIVARIANT CALLIAS INDEX THEORY VIA COARSE GEOMETRY

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*Dedicated to the memory of John Roe (1959–2018)*

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ABSTRACT. — The equivariant coarse index is well-understood and widely used for actions by discrete groups. We first extend the definition of this index to general locally compact groups. We use a suitable notion of admissible modules over  $C^*$ -algebras of continuous functions to obtain a meaningful index. Inspired by a work of Roe, we then develop a localised variant, with values in the  $K$ -theory of a group  $C^*$ -algebra. This generalises the Baum–Connes assembly map to non-cocompact actions. We show that an equivariant index for Callias-type operators is a special case of this localised index, obtain results on existence and non-existence of Riemannian metrics of positive scalar curvature invariant under proper group actions, and show that a localised version of the Baum–Connes conjecture is weaker than the original conjecture, while still giving a conceptual description of the  $K$ -theory of a group  $C^*$ -algebra.

RÉSUMÉ. — L'indice grossier équivariant est bien compris et utilisé pour les actions par les groupes discrets. On commence par étendre la définition de cet indice aux groupes localement compacts généraux. On utilise une notion de modules admissibles sur des  $C^*$ -algèbres de fonctions continues, pour obtenir un indice utile. Inspirés par le travail de Roe, nous développons une variante localisée, à valeurs dans la  $K$ -théorie de la  $C^*$ -algèbre d'un groupe, généralisant l'assembly map de Baum–Connes aux actions non-cocompactes. On montre qu'un indice pour des opérateurs de type Callias est un cas spécial de cet indice localisé; on obtient des résultats sur l'existence et la non-existence de métriques Riemanniennes à courbure scalaire positive, invariants par des actions propres; et on montre qu'une version localisée de la conjecture de Baum–Connes est plus faible que la conjecture originale, et on donne une description conceptuelle de la  $K$ -théorie des  $C^*$ -algèbres de groupes.

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*Keywords:* Roe algebra, equivariant index, proper group action, locally compact group, Callias-type operator.

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## 1. Introduction

### Background

*Coarse geometry* is the study of large-scale structures of metric spaces. Important invariants in this area are various versions of the *Roe algebra* and their  $K$ -theory groups. The *coarse index*, with values in such  $K$ -theory groups, is a powerful tool that has been studied and applied by many authors. A standard introduction is [34]. A central problem is the coarse Baum–Connes conjecture [33], which has a range of important consequences. Important areas of applications of coarse index theory are obstructions to Riemannian metrics of positive scalar curvature (see [38] for a survey) and the Novikov conjecture (see for example [42, 43]).

Equivariant versions of the Roe algebra and the coarse index have been developed for proper actions by discrete groups. Another refinement is a localised variant of the coarse index developed by Roe [37], generalising an index defined by Gromov and Lawson [16]. This localised coarse index applies, in a certain precise sense, to operators that are invertible outside a given subset of the space. For actions by discrete groups, an equivariant version of this localised index theory in terms of coarse  $K$ -homology was recently developed by Bunke and Engel [10].

Equivariant coarse index theory for general locally compact groups would be useful in the study of such groups and their actions. In particular, a localised approach to this index, which takes values in  $K$ -theory of the group  $C^*$ -algebra, offers greater flexibility compared to the standard equivariant index for actions with compact quotients [4]. However, the topology of a non-discrete group poses some technical challenges in the development of equivariant coarse index theory for such groups.

### Results

Our main goal in this paper is to develop equivariant coarse index theory for proper actions by general locally compact groups  $G$ , and in particular a localised version with values in  $K_*(C_{\text{red}}^*(G))$ . Here  $C_{\text{red}}^*(G)$  is the reduced group  $C^*$ -algebra of  $G$ . Our secondary goal is to demonstrate the usefulness of this theory by showing how it simultaneously generalises other versions of index theory, and by obtaining applications to Riemannian metrics of positive scalar curvature invariant under proper actions. Our main results are:

- (1) constructions of equivariant Roe algebras and an equivariant coarse index, and in particular localised versions of these objects (Definitions 2.10, 2.17, 3.4 and 3.6);
- (2) a proof that the analytic assembly map [4] and the equivariant index of Callias-type operators in [17] are special cases of the localised equivariant coarse index (Theorem 4.2 and Corollary 4.3);
- (3) obstructions to and existence of Riemannian metrics with positive scalar curvature, invariant under proper group actions (Proposition 4.4 and Theorem 4.6);
- (4) the formulation of a localised version of the surjectivity part of the Baum–Connes conjecture [4], and relations between the localised and original conjectures (Conjecture 4.8 and Proposition 4.9).

In forthcoming work, we will give further applications of the index theory we develop in this paper. In [18], we show that it refines the indices in [6, 24], and obtain an application to the quantisation commutes with reduction problem. And in [26, 27], this index theory is used to obtain an equivariant Atiyah–Patodi–Singer index theorem for proper actions.

### The localised equivariant coarse index

Let  $M$  be a complete Riemannian manifold, on which a locally compact group  $G$  acts properly and isometrically. Let  $D$  be an elliptic differential operator on  $M$ . For the definition of the localised index, we assume that there is a  $G$ -invariant subset  $Z \subset M$  such that  $D^2$  has a uniform positive lower bound outside  $Z$ . Then we obtain the localised equivariant coarse index

$$\text{index}_G^Z(D) \in K_*(C^*(Z)^G).$$

(See Definition 3.6.) Here  $C^*(Z)^G$  is the equivariant Roe algebra of  $Z$ . We are particularly interested in the case where  $Z/G$  is compact, so that  $C^*(Z)^G$  is stably isomorphic to  $C_{\text{red}}^*(G)$ . While there is no technical reason a priori to restrict to the case where  $Z/G$  is compact, this special case is interesting for several reasons, namely, in this case:

- (1) the localised equivariant coarse index and the  $K$ -theory group it lies in are independent of  $Z$ ;
- (2) the receptacle of the index, namely  $K_*(C_{\text{red}}^*(G))$ , is a rich and relevant object (in particular, nonzero); there exist many tools to extract information from it, such as traces and higher cyclic cohomology classes on (smooth subalgebras of)  $C_{\text{red}}^*(G)$ ;

- (3) various existing indices, including the analytic assembly map, are special cases, as we discuss below.

Operators  $D$  to which the localised equivariant index applies include the following three important special cases:

- (1) Callias-type operators of the form  $D = \tilde{D} + \Phi$ , where  $\tilde{D}$  is a Dirac operator and  $\Phi$  is a vector bundle endomorphism making  $D^2$  uniformly positive outside  $Z$ . The study of these operators, their indices and their applications was initiated by Callias [11], and extended in various directions by many authors, see e.g. [2, 3, 5, 7, 8, 9, 12, 13, 28, 29, 31, 41]. The equivariant case for proper actions was treated in [17];
- (2) Dirac operators  $D$  whose curvature term  $R$  in the Weitzenböck formula  $D^2 = \Delta + R$  is uniformly positive outside  $Z$  [16, 37];
- (3) Dirac operators  $D$  on manifolds with boundary that are invertible on the boundary, extended to cylinders attached to these boundaries.

For Callias-type operators, the coarse-geometric approach in this paper may already be useful in the case of trivial groups. In addition, we can now consider the lift of a (non-equivariant) Callias-type operator on a manifold  $M$  to the universal cover of  $M$ , and obtain an equivariant index in the  $K$ -theory of the fundamental group of  $M$ . This is a more refined invariant than the Fredholm index of the initial operator.

In the case where  $Z/G$  is compact, the localised equivariant index generalises the Baum–Connes assembly map from the case of actions with compact quotients to the cases above. This allows us to formulate a localised version of the surjectivity direction of the Baum–Connes conjecture. We will show that this localised surjectivity is implied by standard Baum–Connes surjectivity.

One of the technical challenges in constructing a meaningful index in this context is to develop the appropriate notion of an *admissible module*. For actions by discrete groups, this was done in Definition 2.2 in [44]. The Roe algebra of a metric space  $X$  acted on by a locally compact group  $G$  is defined in terms of operators on a Hilbert space  $H_X$  with compatible actions by  $C_0(X)$  and  $G$ . The resulting algebra should ideally be independent of the choice of  $H_X$ , and its  $K$ -theory should contain relevant information about  $G$ , and possibly also  $X$ . (A natural initial choice would be  $H_X = L^2(X)$  for a Borel measure on  $X$ , but this does not contain enough information if, for example,  $X$  is a point or if  $G$  is compact and acts trivially on  $X$ .) We achieve these two things by taking  $H_X$  to be an admissible module, in the

sense that we define in Sections 2 and 5. We indicate how Roe algebras and the associated  $K$ -theory groups and indices defined in terms of admissible modules on the one hand, and more geometric, but non-admissible modules on the other, are related in Subsection 3.4.

### Outline of this paper

We introduce admissible modules and the associated Roe algebras in Section 2. We use these notions in Section 3 to define the equivariant coarse index and its localised version. In Section 4 we apply these notions to show that the equivariant Callias-type index in [17] is a special case, and we establish results on positive scalar curvature, as well as state a localised Baum–Connes conjecture. Proofs of the properties of admissible modules and Roe algebras from Section 2 are given in Section 5. Proofs of the results in Section 4 are given in Sections 6 and 7.

## 2. Equivariant Roe algebras and admissible modules

A key idea in this paper is to use coarse geometry and Roe algebras to construct a localised equivariant index for proper actions with values in the  $K$ -theory of a group  $C^*$ -algebra. We start by discussing the necessary background in coarse geometry. Much of the material in this section is well-known in the case of discrete groups, but we will see that the generalisation to general locally compact groups requires some work.

Throughout this section,  $(X, d)$  will denote a proper metric space, i.e. a metric space in which closed balls are compact, and  $G$  a locally compact group acting properly and isometrically on  $X$ . We assume  $G$  to be unimodular and fix a Haar measure  $dg$  on  $G$ . Throughout this paper, we will use left and right invariance of  $dg$  without mentioning this explicitly, in arguments involving substitutions in integrals over  $G$ . (The contents of this paper can likely be generalised to non-unimodular group if the modular function is inserted where appropriate.) We will sometimes assume  $X/G$  to be compact, but not always. We always view  $L^2(G)$  as a unitary representation of  $G$  via the left-regular representation.

The two properties of the modules and algebras we define here that are most important to the construction of the equivariant localised coarse index in Section 3 are Theorems 2.7 and 2.11. These are proved in Subsections 5.3 and 5.4, respectively.

## 2.1. Admissible modules

We will construct the reduced and maximal equivariant Roe algebras of  $X$  in terms of *admissible  $C_0(X)$ -modules*, and a particularly useful type of such modules we call *geometric admissible modules*. Using admissible modules ensures that the algebras constructed are independent of the choice of module. Their purpose is also to ensure that the Roe algebras we use contain sufficient information about the group  $G$ , as illustrated in Examples 2.16 and 3.8. Admissible modules were first defined in the case of discrete groups by Guoliang Yu in Definition 2.2 in [44]. For non-discrete  $G$ , the definition needs to take into account the topology of  $G$ .

Admissible modules are special cases of ample, equivariant  $C_0(X)$ -modules.

**DEFINITION 2.1.** — *An equivariant  $C_0(X)$ -module is a Hilbert space  $H_X$  with a unitary representation of  $G$ , together with a  $*$ -homomorphism*

$$\pi: C_0(X) \rightarrow \mathcal{B}(H_X)$$

such that for all  $g \in G$  and  $f \in C_0(X)$ ,

$$\pi(g \cdot f) = g \circ \pi(f) \circ g^{-1}.$$

Here  $g \cdot f$  is the function mapping  $x \in X$  to  $f(g^{-1}x)$ .

An equivariant  $C_0(X)$ -module is nondegenerate if  $\pi(C_0(X))H_X$  is dense in  $H_X$ . It is standard if  $\pi(f)$  is a compact operator only if  $f = 0$ . The module is ample if it is nondegenerate and standard.

We will usually omit the homomorphism  $\pi$  from the notation, and write  $f \cdot \xi := \pi(f)\xi$  for  $f \in C_0(X)$  and  $\xi \in H_X$ .

**Example 2.2.** — The space  $H_G = L^2(G) \otimes H$ , for a separable infinite-dimensional Hilbert space  $H$  equipped with the trivial  $G$ -representation, is an ample equivariant  $C_0(G)$ -module with respect to the multiplicative action of  $C_0(G)$  on  $L^2(G)$ .

The action by  $C_0(X)$  on any ample, equivariant  $C_0(X)$ -module  $H_X$  has a unique extension to an action by the algebra  $L^\infty(X)$  of bounded Borel functions, characterised by the property that for a uniformly bounded sequence in  $L^\infty(X)$  converging pointwise, the corresponding operators on  $H_X$  converge strongly. All functions we will apply this extension to are bounded, continuous functions on closed sets in  $X$ , such as the indicator function  $\mathbb{1}_Y$  of a closed subset  $Y \subset X$ .

DEFINITION 2.3. — Let  $H_X$  be an ample  $C_0(X)$ -module. An operator  $T \in \mathcal{B}(H_X)$  has finite propagation if there is an  $r > 0$  such that for all  $f_1, f_2 \in C_0(X)$  whose supports are further than  $r$  apart, we have  $f_1 T f_2 = 0$ .

An operator  $T \in \mathcal{B}(H_X)$  is locally compact if for all  $f \in C_0(X)$ , the operators  $fT$  and  $Tf$  are compact.

We now formulate the general notion of admissible module over a space.

DEFINITION 2.4. — Let  $G$  is a locally compact group. Let  $H_X$  be an ample, equivariant  $C_0(X)$ -module. If  $X/G$  is compact,  $H_X$  is said to be admissible if there is a  $G$ -equivariant unitary isomorphism

$$\Psi : L^2(G) \otimes H \cong H_X,$$

for a separable infinite-dimensional Hilbert space  $H$  equipped with the trivial  $G$ -representation, such that for any bounded,  $G$ -equivariant operator  $T$  on  $H_X$ ,

- (1)  $T$  has finite propagation with respect to the action by  $C_0(X)$  if and only if  $\Psi^{-1} \circ T \circ \Psi$  has finite propagation with respect to the action by  $C_0(G)$ ;
- (2)  $T$  is locally compact with respect to the action by  $C_0(X)$  if and only if  $\Psi^{-1} \circ T \circ \Psi$  is locally compact (with respect to the action by  $C_0(G)$ ).

If  $X/G$  is not necessarily compact, then an ample, equivariant  $C_0(X)$ -module is admissible if for every closed,  $G$ -invariant subset  $Y \subset X$  such that  $Y/G$  is compact,  $\mathbf{1}_Y H_X$  is an admissible module over  $C_0(Y)$ .

Remark 2.5. — If  $G = \Gamma$  is a discrete group, Definition 2.4 is an alternative to the definition of admissible modules given in the conditions in Definition 2.4 are implied by the definition of admissible modules given in [44]. In particular, if  $H_X$  is an admissible module in the sense of Definition 2.2 in [44], then it is also admissible in our sense.

Remark 2.6. — The idea behind admissible modules is to provide a class of general spaces on which to carry out analysis of operators on  $X$ , but which also allow us to define an equivariant index of these operators. For example, we will use Conditions (1) and (2) of Definition 2.4 in an essential way<sup>(1)</sup> to define the equivariant coarse index and the localised equivariant coarse index in  $K_*(C^*(G))$  (see Subsections 6.1 and 6.2).

In this paper, we will mostly work with a particular geometric type of admissible module.

<sup>(1)</sup>See the proof of Proposition 5.12.

Consider a covering of  $X$  by sets of the form  $G \times_{K_j} Y_j$ , for compact subgroups  $K_j < G$  and compact,  $K_j$ -invariant slices  $Y_j \subset X$ . Suppose that the intersections between these sets have measure zero. For each  $j$ , fix a  $K$ -invariant measure  $dy_j$  on  $Y_j$ . Together with the Haar measure  $dg$ , they induce a  $G$ -invariant measure  $dx$  on  $X$ . We call such a measure *induced from slices*. Such measures are natural choices; see for example Lemma 4.1 in [25].

**THEOREM 2.7.** — *Suppose  $X/G$  is compact. Suppose that at least one of the sets  $G/K$  and  $X/G$  is infinite. Let  $H_X = L^2(E) \otimes L^2(G)$ , for a Hermitian  $G$ -vector bundle  $E \rightarrow X$ , defined with respect to the a measure on  $X$  induced from slices. Then  $H_X$ , equipped with the diagonal representation of  $G$  and the  $C_0(X)$ -action on the factor  $L^2(E)$  by pointwise multiplication, is an admissible  $C_0(X)$  module.*

**DEFINITION 2.8.** — *If  $X/G$  is compact, then  $L^2(E) \otimes L^2(G)$ , for a Hermitian  $G$ -vector bundle  $E \rightarrow X$ , defined with respect to the a measure on  $X$  induced from slices is a geometric admissible  $C_0(X)$ -module.*

Note that the difference between a general admissible module in Definition 2.4 and a geometric module is that on a geometric admissible module, the group  $G$  acts diagonally, whereas on a general admissible module in the form  $L^2(G) \otimes H$ ,  $G$  acts only on the first factor. The advantage of working with a geometric module is that the action by  $C_0(X)$  is explicit. We will in fact usually work with geometric admissible modules.

The notion of a geometric admissible module, and hence Theorem 2.7, is a key component of our construction of (localised) coarse indices of elliptic operators in Subsections 3.2 and 3.3.

*Remark 2.9.* — The condition in Definition 2.4 that  $H$  is infinite-dimensional, and the corresponding condition in Theorem 2.7 that  $G/K$  or  $X/G$  is infinite, are assumed to ensure that the equivariant coarse index theory of Section 3 is rich enough to capture information about the group  $G$  (see Examples 2.16 and 3.8). More specifically, infinite-dimensionality of  $H$  guarantees that the algebras  $D^*(X)^G$ , which are used to define the coarse index (see Subsection 3.1) exist and have the properties needed to define a useful index. It also implies that the localised Roe algebra is independent of the choice of admissible module, as in (2.6). (Although for finite-dimensional  $H$ , the factor  $\mathcal{K}$  in (2.6) would become a finite-dimensional matrix algebra, which makes no difference at the level of  $K$ -theory.)

In Theorem 2.7, if both  $G/K$  and  $X/G$  are finite, then one can still form the admissible module  $L^2(E) \otimes L^2(G) \otimes l^2(\mathbb{N})$ .

### 2.2. Equivariant Roe algebras

Fix an equivariant  $C_0(X)$ -module  $H_X$ . We denote the algebra of  $G$ -equivariant bounded operators  $H_X$  by  $\mathcal{B}(H_X)^G$ .

DEFINITION 2.10. — *The algebraic equivariant Roe algebra for  $H_X$  of  $X$  is the algebra  $C_{\text{alg}}^*(X; H_X)^G$  consisting of the locally compact operators in  $\mathcal{B}(H_X)^G$  with finite propagation. The equivariant Roe algebra for  $H_X$  of  $X$  is the closure  $C^*(X; H_X)^G$  of  $C_{\text{alg}}^*(X; H_X)^G$  in  $\mathcal{B}(H_X)$ .*

*If  $H_X$  is an admissible module and  $X/G$  is compact, then  $C_{\text{alg}}^*(X)^G := C_{\text{alg}}^*(X; H_X)^G$  is the algebraic equivariant Roe algebra of  $X$ , and  $C^*(X)^G := C^*(X; H_X)^G$  is the equivariant Roe algebra of  $X$ .*

THEOREM 2.11. — *If  $X/G$  is compact and  $H_X$  is admissible, then  $C_{\text{alg}}^*(X)^G$  is  $*$ -isomorphic to a dense subalgebra of  $C_{\text{red}}^*(G) \otimes \mathcal{K}$ , where  $\mathcal{K}$  is the algebra of compact operators on a separable Hilbert space.*

The proof of this theorem, given in Subsection 5.4, uses a subalgebra of the Roe algebra consisting of continuous  $G$ -equivariant Schwartz kernels with finite propagation, denoted by  $C_{\text{ker}}^*(X)^G$  (see Definition 5.11). We will see that there is a  $*$ -isomorphism

$$(2.1) \quad C_{\text{ker}}^*(X)^G \cong C_c(G) \otimes \mathcal{K}(H).$$

One consequence of this is that we can define a maximal version of the equivariant Roe algebra, in the following sense. For any  $*$ -algebra  $A$ , and any  $a \in A$ , write

$$\|a\|_{\text{max}} := \sup_{\pi} \|\pi(a)\|_{\mathcal{B}(H_{\pi})},$$

where the supremum runs over all irreducible  $*$ -representations  $\pi$  of  $A$  in Hilbert spaces  $H_{\pi}$ . While this supremum may in general be infinite, it is always finite for the group convolution algebra  $A = C_c(G)$ . As a result, we have:

PROPOSITION 2.12. — *If  $X/G$  is compact, then  $\|a\|_{\text{max}} < \infty$  for all  $a \in C_{\text{ker}}^*(X)^G$ .*

DEFINITION 2.13. — *If  $X/G$  is compact, then the maximal equivariant Roe algebra of  $X$ , denoted by  $C_{\text{max}}^*(X)^G$ , is the completion of  $C_{\text{ker}}^*(X)^G$  in the maximal norm  $\|\cdot\|_{\text{max}}$ .*

Remark 2.14. — Gong, Wang and Yu [15] proved that if  $G = \Gamma$  is discrete and acts freely on  $X$  with  $X/\Gamma$  compact, then the maximal equivariant Roe algebra is always well-defined. This is extended to cases where  $X/\Gamma$

is noncompact in [20], under additional hypotheses. We expect this to be true also when  $G$  is locally compact. (In [15], the Roe algebras are defined in terms of kernels; see Subsection 5.4.)

Summarising, if  $X/G$  is compact then we have  $*$ -isomorphisms

$$(2.2) \quad C^*(X)^G \cong C_{\text{red}}^*(G) \otimes \mathcal{K};$$

$$(2.3) \quad C_{\text{max}}^*(X)^G \cong C_{\text{max}}^*(G) \otimes \mathcal{K}.$$

*Remark 2.15.* — The relations (2.2) and (2.3) in particular imply that the reduced and maximal algebras are independent of the choice of the admissible  $C_0(X)$ -module  $H_X$ . We expect this to be true even if  $X/G$  is not compact. In the case of  $G = \Gamma$  discrete and  $X/\Gamma$  compact, a proof can be found in the forthcoming book [40].

*Example 2.16.* — Suppose that  $G = K$  is compact, and  $X = \text{pt}$  is a point. Then  $l^2(\mathbb{N})$  is an ample module over  $C_0(\text{pt}) = \mathbb{C}$ . It is equivariant if we equip it with the trivial action by  $K$ . The algebraic, reduced and maximal equivariant Roe algebras defined with respect to this module all equal  $\mathcal{K}(l^2(\mathbb{N}))$ , which contains no group-theoretic information about  $K$ . This is because the module  $l^2(\mathbb{N})$  is not admissible.

### 2.3. Localised Roe algebras

The localised index that we will define in Definition 3.3 involves a localised version of Roe algebras.

Let  $H_X$  be an equivariant  $C_0(X)$ -module. Let  $Z \subset X$  be a  $G$ -invariant, closed subset.

**DEFINITION 2.17.** — *An operator  $T \in \mathcal{B}(H_X)$  is supported near  $Z$  if there is an  $r > 0$  such that for all  $f \in C_0(X)$  whose support is at least a distance  $r$  away from  $Z$ , we have  $fT = Tf = 0$ .*

The algebraic equivariant Roe algebra for  $H_X$  of  $X$ , localised at  $Z$ , denoted by  $C_{\text{alg}}^*(X; Z, H_X)^G$ , consists of the operators in  $C_{\text{alg}}^*(X, H_X)^G$  supported near  $Z$ .

The equivariant Roe algebra for  $H_X$  of  $X$ , localised at  $Z$ , denoted by  $C^*(X; Z, H_X)^G$ , is the closure of  $C_{\text{alg}}^*(X; Z, H_X)^G$  in  $\mathcal{B}(H_X)$ .

If  $Z/G$  is compact, then we call  $C_{\text{alg}}^*(X; H_X)_{\text{loc}}^G := C_{\text{alg}}^*(X; Z, H_X)^G$  the localised algebraic equivariant Roe algebra for  $H_X$  of  $X$ , and  $C^*(X; H_X)_{\text{loc}}^G := C^*(X; Z, H_X)^G$  the localised equivariant Roe algebra for  $H_X$  of  $X$ . If  $H_X$  is an admissible module, then we omit it from the notation

and terminology, and obtain the localised algebraic equivariant Roe algebra  $C_{\text{alg}}^*(X)_{\text{loc}}^G$  and the localised equivariant Roe algebra  $C^*(X)_{\text{loc}}^G$  of  $X$ .

Of the terms in Definition 2.17, the localised equivariant Roe algebra  $C^*(X)_{\text{loc}}^G$  is the one we are most interested in. Note that if  $Z/G$  is compact, then the algebras  $C_{\text{alg}}^*(X; Z, H_X)^G$  and  $C^*(X; Z, H_X)^G$  are independent of  $Z$ , as long as  $Z/G$  is compact.

For  $r > 0$ , we write

$$(2.4) \quad \text{Pen}(Z, r) := \{x \in X; d(x, Z) \leq r\}.$$

In terms of these sets, we have

$$(2.5) \quad \begin{aligned} C_{\text{alg}}^*(X; Z, H_X)^G &= \varinjlim_r C_{\text{alg}}^*(\text{Pen}(Z, r); H_X)^G; \\ C^*(X; Z, H_X)^G &= \varinjlim_r C^*(\text{Pen}(Z, r); H_X)^G. \end{aligned}$$

Theorem 2.11 and (2.5) imply that

$$(2.6) \quad C^*(X)_{\text{loc}}^G \cong C_{\text{red}}^*(G) \otimes \mathcal{K}.$$

If  $H_X$  is admissible, then the algebra  $C_{\text{ker}}^*(\text{Pen}(Z, r))^G$  has a well-defined maximal norm by (2.1), for all  $r$ . Hence so does its injective limit as  $r \rightarrow \infty$ . The completion of this injective limit in the maximal norm will be called the *maximal localised equivariant Roe algebra of  $X$* , and denoted by  $C_{\text{max}}^*(X)_{\text{loc}}^G$ . By (2.3), it equals

$$C_{\text{max}}^*(X)_{\text{loc}}^G \cong C_{\text{max}}^*(G) \otimes \mathcal{K}.$$

### 3. The localised equivariant index

#### 3.1. Indices of abstract operators

The (non-localised) equivariant coarse index is defined completely analogously to the case for discrete groups, but with Roe algebras defined in terms of the admissible modules from Subsection 2.1.

Let  $H_X$  be an equivariant  $C_0(X)$ -module. Let  $C^*(X; H_X)^G$  denote the reduced or maximal version (if it exists<sup>(2)</sup>) of the equivariant Roe algebra for  $H_X$ . Let  $D^*(X)^G$  be any  $C^*$ -algebra containing  $C^*(X; H_X)^G$  as a two-sided ideal. For example, we can take  $D^*(X)^G$  to be the multiplier algebra of  $C^*(X; H_X)^G$ , or the  $C^*$ -algebra generated by  $C^*(X; H_X)^G$  and a single operator in  $\mathcal{B}(H_X)^G$ .

<sup>(2)</sup>For a general space, one first needs to show finiteness of the maximal norm to know that  $C_{\text{max}}^*(X; H_X)^G$  is well-defined.

*Remark 3.1.* — In the reduced case, a natural choice for  $D^*(X)^G$  is the algebra  $D_{\text{red}}^*(X)^G$ , an equivariant version of the algebra  $D^*(X)$  used in [37]. This is defined as the closure in  $\mathcal{B}(H_X)$  of the algebra of operators  $T \in \mathcal{B}(H_X)^G$  with finite propagation such that  $[T, f]$  is compact for all  $f \in C_0(X)$ .

Let

$$(3.1) \quad \partial: K_{*+1}(D^*(X)^G/C^*(X)^G) \rightarrow K_*(C^*(X)^G)$$

be the boundary map associated to the short exact sequence

$$0 \rightarrow C^*(X)^G \rightarrow D^*(X)^G \rightarrow D^*(X)^G/C^*(X)^G \rightarrow 0.$$

**DEFINITION 3.2.** — Let  $F \in D^*(X)^G$ , and suppose that  $F - F^*$  and  $F^2 - 1$  lie in  $C^*(X; H_X)^G$ .

- If no grading on  $H_X$  given, consider the projection  $P = \frac{1}{2}(F + 1)$  in  $D^*(X)^G/C^*(X; H_X)^G$  and the class  $[P] \in K_0(D^*(X)^G/C^*(X; H_X)^G)$ . Then the equivariant coarse index of  $F$  is

$$\text{index}_G(F) = \partial[P] \in K_1(C^*(X; H_X)^G).$$

- If a grading on  $H_X$  is given that is preserved by  $C_0(X)$  and  $G$ , and interchanged by  $F$ , let  $F_+$  be the restriction of  $F$  to the even-degree part of  $H_X$ . Then  $F_+$  is invertible modulo  $C^*(X)^G$ , and we have  $[F_+] \in K_1(D^*(X)^G/C^*(X; H_X)^G)$ . The equivariant coarse index of  $F$  is

$$\text{index}_G(F) = \partial[F_+] \in K_0(C^*(X; H_X)^G).$$

More generally, we will also write  $\text{index}_G$  for the boundary map (3.1).

Let  $Z \subset X$  be a closed  $G$ -invariant subset. Let  $D^*(X)^G \subset \mathcal{B}(H_X)^G$  be a  $C^*$ -algebra containing  $C^*(X; Z, H_X)^G$  as a two-sided ideal. The algebra  $D_{\text{red}}^*(X)^G$  defined above has this property.

**DEFINITION 3.3.** — Let  $F \in D^*(X)^G$ , and suppose that  $F - F^*$  and  $F^2 - 1$  lie in  $C^*(X; Z, H_X)^G$ . The equivariant coarse index of  $F$ , localised at  $Z$ ,

$$\text{index}_G^Z(F) \in K_*(C^*(X; Z, H_X)^G)$$

is defined analogously to  $\text{index}_G(F)$  in Definition 3.2, with  $C^*(X, H_X)^G$  replaced by  $C^*(X; Z, H_X)^G$  everywhere.

If  $Z/G$  is compact and  $H_X$  is an admissible module, we write

$$\text{index}_G^{\text{loc}}(F) := \text{index}_G^Z(F) \in K_*(C^*(X)_{\text{loc}}^G) = K_*(C^*(G)),$$

and call this the localised equivariant coarse index of  $F$ . Here  $C^*(G)$  denotes either the reduced or maximal group  $C^*$ -algebra.

We will also denote the boundary map (3.1), with  $C^*(X, H_X)^G$  replaced by  $C^*(X; Z, H_X)^G$ , by  $\text{index}_G^Z$ , or by  $\text{index}_G^{\text{loc}}$  if  $Z/G$  is compact and  $H_X$  is an admissible module.

### 3.2. The equivariant coarse index of elliptic operators

In the rest of this section, we work with the reduced version of the equivariant Roe algebra and specialise to the geometric setting that we are most interested in. Let  $X = M$  be a Riemannian manifold, and  $d$  the Riemannian distance. Suppose, as before, that  $G$  acts properly and isometrically on  $M$ . Let  $E \rightarrow M$  be a  $G$ -equivariant Hermitian vector bundle. Let  $D$  be a  $G$ -equivariant, first order elliptic differential operator on  $E$  that is essentially self-adjoint on  $L^2(E)$ .

To apply Definitions 3.2 and 3.3 to  $D$ , we embed  $L^2(E)$  into the (geometric) admissible module  $H_M := L^2(E) \otimes L^2(G)$  of Theorem 2.7. We will illustrate why using admissible modules is necessary in Example 3.8 (see also Example 2.16).

Let  $\chi \in C^\infty(M)$  be a cutoff function, in the sense that its support has compact intersections with all  $G$ -orbits, and that for all  $m \in M$ ,

$$(3.2) \quad \int_G \chi(gm)^2 \, dg = 1.$$

The map

$$(3.3) \quad j: L^2(E) \rightarrow H_M,$$

given by

$$(j(s))(m, g) = \chi(g^{-1}m)s(m),$$

for  $s \in L^2(E)$ ,  $m \in M$  and  $g \in G$ , is a  $G$ -equivariant, isometric embedding. It intertwines the actions by  $C_0(M)$  on  $L^2(E)$  and  $H_M$ . Define the maps

$$(3.4) \quad \oplus 0, \oplus 1: \mathcal{B}(L^2(E)) \rightarrow \mathcal{B}(H_M)$$

by identifying operators on  $L^2(E)$  with operators on  $j(L^2(E))$  via conjugation by  $j$ , and extending them by zero or the identity operator, respectively, on the orthogonal complement of  $j(L^2(E))$  in  $H_M$ .

Let  $b \in C_b(\mathbb{R})$  be a normalising function, i.e. an odd function with values in  $[-1, 1]$  such that  $\lim_{x \rightarrow \infty} b(x) = 1$ . Let  $D^*(M; L^2(E))^G$  be a unital  $*$ -subalgebra of  $\mathcal{B}(L^2(E))$  containing  $b(D)$ , and  $C^*(M; L^2(E))^G$  as a two-sided ideal. And, as at the start of Subsection 3.1, let  $D^*(M)^G$  be any unital  $*$ -subalgebra of  $\mathcal{B}(H_M)$  containing  $C^*(M; H_M)^G$  as a two-sided ideal. We now assume in addition that the image of  $D^*(M; L^2(E))^G$  under the

map  $\oplus 1$  lies inside  $D^*(M)^G$ . For example, we may take  $D^*(M)^G$  to be the multiplier algebra of  $C^*(M; H_M)^G$ , or the algebra  $D_{\text{red}}^*(M)^G$  as in Remark 3.1.

If  $F \in D^*(M; L^2(E))^G$  and  $F^* - F$  and  $F^2 - 1$  lie in  $C^*(M; L^2(E))^G$ , then  $F \oplus 1 \in D^*(M)^G$  by assumption, and  $(F \oplus 1)^* - F \oplus 1$  and  $(F \oplus 1)^2 - 1$  lie in  $C^*(M; H_M)^G$ . Hence  $F \oplus 1$  has an index in  $K_*(C^*(M; H_M)^G)$  as in Definition 3.2.

DEFINITION 3.4. — *The equivariant coarse index of  $D$  is*

$$\text{index}_G(D) := \text{index}_G(b(D) \oplus 1) \in K_*(C^*(M; H_M)^G),$$

where the index on the right hand side is as in Definition 3.2.

As in Definition 3.2,  $\text{index}_G(D)$  lies in even or odd  $K$ -theory depending on the presence of a grading on  $E$  with respect to which  $D$  is odd.

### 3.3. The localised index of elliptic operators

Again, let  $Z \subset M$  be a closed,  $G$ -invariant subset. Suppose that there is a constant  $c > 0$  such that for all  $s \in \Gamma_c^\infty(E)$  supported outside  $Z$ ,

$$\|Ds\|_{L^2} \geq c\|s\|_{L^2}.$$

Let  $b \in C^\infty(\mathbb{R})$  be an odd, increasing function taking values in  $\{\pm 1\}$  on  $\mathbb{R} \setminus [-c, c]$ . Form the operator  $b(D)$  by functional calculus. The following result by Roe is the basis of the index theory we develop in this paper.

PROPOSITION 3.5. — *The operator  $b(D) \in \mathcal{B}(L^2(E))$  satisfies*

$$b(D)^2 - 1 \in C^*(M; Z, L^2(E))^G.$$

See Lemma 2.3 in [37]. As above Definition 3.4 This proposition implies that  $b(D) \oplus 1$  satisfies the conditions of Definition 3.3.

DEFINITION 3.6. — *The equivariant coarse index of  $D$ , localised at  $Z$  is*

$$(3.5) \quad \text{index}_G^Z(D) := \text{index}_G^Z(b(D) \oplus 1) \in K_*(C^*(X; Z, H_M)^G),$$

where the index on the right hand side is as in Definition 3.3.

If  $Z/G$  is compact, then the index (3.5) is by definition the localised equivariant coarse index of  $D$ , and denoted by

$$\text{index}_G^{\text{loc}}(D) := \text{index}_G^{\text{loc}}(b(D) \oplus 1) \in K_*(C_{\text{red}}^*(G)).$$

As before,  $\text{index}_G^Z(D)$  and  $\text{index}_G^{\text{loc}}(D)$  lie in even or odd  $K$ -theory depending on the presence of a grading.

The localised equivariant coarse index of an elliptic operator is the object we are most interested in here. It is a natural generalisation of the Baum–Connes analytic assembly map [4] from cocompact to non-cocompact actions; see Corollary 4.3.

### 3.4. Admissible and non-admissible modules

Let us clarify the relevance of using admissible modules in the definition of the equivariant coarse index. This will lead to an equivalent definition of the localised equivariant coarse index in the graded case, (3.8) below.

The map  $\oplus 0$  preserves finite propagation, local compactness and having support near  $Z$ , and hence restricts to an injective  $*$ -homomorphism

$$(3.6) \quad \oplus 0: C^*(M; Z, L^2(E))^G \rightarrow C^*(M; Z, H_M)^G.$$

We denote the map induced on  $K$ -theory by  $\oplus 0$  as well.

Viewing  $C^*(M; Z, L^2(E))^G$  as a subalgebra of  $C^*(M; Z, H_M)^G$  via the map (3.6), we find that the map  $\oplus 1$  descends to a multiplicative but non-linear map

$$(3.7) \quad \oplus 1: D^*(M; L^2(E))^G / C^*(M; L^2(E))^G \rightarrow D^*(M)^G / C^*(M; H_M)^G.$$

This induces a homomorphism on odd  $K$ -theory, which we still denote by  $\oplus 1$ .

LEMMA 3.7. — *The following diagram commutes:*

$$\begin{CD} K_1(D^*(M; L^2(E))^G / C^*(M; Z, L^2(E))^G) @>\partial>> K_0(C^*(M; Z, L^2(E))^G) \\ @V\oplus 1VV @VV\oplus 0V \\ K_1(D^*(M)^G / C^*(M; Z, H_M)^G) @>\partial>> K_0(C^*(M; Z, H_M)^G), \end{CD}$$

where the maps  $\partial$  are boundary maps in the respective six-term exact sequences.

*Proof.* — The boundary map can be described explicitly as follows. Suppose that  $u$  is an invertible element in  $D^*(M; L^2(E))^G / C^*(X; Z, L^2(E))^G$ , and let  $v$  be its inverse. Let  $U$  and  $V$  respectively be representatives of  $u$  and  $v$  in  $D^*(M; L^2(E))^G$ . Define

$$W = \begin{pmatrix} 1 & U \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -V & 1 \end{pmatrix} \begin{pmatrix} 1 & U \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$\partial[u] = \left[ W \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W^{-1} \right] - \left[ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] \in K_0(C^*(M; Z, L^2(E))^G).$$

Using the analogous description of  $\text{index}_G([u] \oplus 1)$ , one can explicitly prove the claim.  $\square$

Let  $D$  be as in Subsection 3.3, and suppose that  $E$  has a  $\mathbb{Z}_2$  grading with respect to which  $D$  is odd. Let

$$\text{index}_G^{Z, L^2(E)}(D) := \partial[b(D)] \in K_0(C^*(M; Z, L^2(E))^G)$$

be the image of  $[b(D)] \in K_1(D^*(M; L^2(E))^G / C^*(M; Z, L^2(E))^G)$  under the boundary map. Lemma 3.7 implies that the localised equivariant index of  $D$  equals

$$(3.8) \quad \text{index}_G^Z(D) = \text{index}_G^{Z, L^2(E)}(D) \oplus 0.$$

*Example 3.8.* — The importance of using admissible modules in the definition of the (localised) equivariant coarse index is clear in the simplest case, where  $G$  is compact,  $M$  is a point,  $E = V$ , an irreducible representation of  $G$  (with the trivial grading), and  $D = 0_V$  is the zero operator on  $V$ . We take  $Z = M$ , so the localised index equals the non-localised index. We have  $L^2(E) = V$ , and<sup>(3)</sup>

$$(3.9) \quad [b(D)] = [0_V] \in K_1(D^*(M; V)^G / C^*(M; V)^G).$$

In that case, Schur’s lemma implies that

$$C^*(M; V)^G = \text{End}(V)^G = \mathbb{C}I_V,$$

where  $I_V$  is the identity operator on  $V$ . The map  $j$  in (3.3) is now given by

$$j(v) = 1 \otimes v,$$

for  $v \in V$ , where 1 is the constant function 1 on  $G$ . The map

$$\oplus 0: C^*(M; V)^G = \mathbb{C}I_V \rightarrow \bigoplus_{W \in \hat{G}} \text{End}(W) = C^*(G)$$

is given by the inclusion map  $\mathbb{C}I_V \hookrightarrow \text{End}(V)$ . At the level of  $K$ -theory, the map  $\oplus 0$  is the map

$$K_0(C^*(M; V)^G) = \mathbb{Z} \rightarrow R(G) = K_0(C^*(G))$$

---

<sup>(3)</sup> There is a technical subtlety in the case where  $M$  is a finite set: finite-dimensionality of  $V$  implies that it is not a standard module (see Definition 2.1). The conditions in Theorem 2.7 are not satisfied, since  $G/K$  and  $M/G$  are both finite sets. Furthermore, we now have  $D^*(M; V)^G = C^*(M; V)^G$  if  $D^*(M; V)^G$  is a subalgebra of  $\mathcal{B}(V)^G$ . Then  $D^*(M; V)^G / C^*(M; V)^G$  is the zero algebra. All of these issues can be solved by tensoring  $V$  by  $l^2(\mathbb{N})$ , see also Remark 2.9.

mapping  $k \in \mathbb{Z}$  to  $k[V] \in R(G)$ , the representation ring of  $G$ . The image under  $\text{index}_G^V$  of the class (3.9) is the Fredholm index of  $0_V$ , which is  $[V] \in K_0(\mathbb{C}I_V)$ . Under the identification of this  $K$ -theory group with  $\mathbb{Z}$ , that class is mapped to 1. Hence

$$\text{index}_G^{\text{pt},V}(0_V) \oplus 0 = [V] \in R(G).$$

On the other hand,  $\text{index}_G(0_V \oplus 1)$  is the equivariant Fredholm index of the operator  $0_V \oplus 1$ , which also equals  $[V] \in R(G)$ .

This example shows that:

- (1) we need to use an admissible module to obtain a single  $K$ -theory group  $K_0(C^*(X)^G) = R(G)$  containing all localised,  $G$ -equivariant indices on  $X$ ;
- (2) commutativity of the diagram in Lemma 3.7 means that  $\text{index}_G^{L^2(E)}$ , defined via a natural  $C_0(X)$ -module, but landing in a non-canonical  $K$ -theory group, determines the localised equivariant index with values in  $K_0(C^*(G))$ . In this example,  $\text{index}_G^{\text{pt},V}(0_V)$  is just the integer 1 when viewed as an element of  $\mathbb{Z} = K_0(\mathbb{C}I_V)$ , but the representation theoretic information about this index is encoded in the map  $\oplus 0$ .

### 3.5. Special cases

Operators  $D$  as in Subsection 3.3 occur naturally in at least three settings.

#### Callias-type operators

First of all, let  $\tilde{D}$  be a Dirac-type operator on  $E$ . Let  $\Phi$  be a  $G$ -equivariant vector bundle endomorphism of  $E$  such that  $\tilde{D}\Phi + \Phi\tilde{D}$  is a vector bundle endomorphism, and

$$(3.10) \quad \tilde{D}\Phi + \Phi\tilde{D} + \Phi^2 \geq c^2$$

outside a cocompact subset  $Z \subset M$ , for a constant  $c > 0$ . This can, for example, be guaranteed by constructing  $\Phi$  from projections in the Higson corona algebra as in [17]. (In that case, the pointwise norm of  $\tilde{D}\Phi + \Phi\tilde{D}$  goes to zero at infinity, while the norm of  $\Phi^2$  goes to one.) Then the *Callias-type operator*

$$(3.11) \quad D = D_\Phi := \tilde{D} + \Phi$$

has the properties in Subsection 3.3.

If  $G$  is the trivial group, i.e. in the non-equivariant case, index theory of these operators was studied and applied in various places [9, 11, 31]. The coarse geometric viewpoint we develop in this paper could already be useful in the case of trivial groups. Possibly more useful in the non-equivariant setting is to consider the lift of a Callias-type operator  $D_\Phi$  on a manifold  $M$ , with fundamental group  $\Gamma$ , to a  $\Gamma$ -equivariant Callias-type operator on the universal cover of  $M$ . The localised equivariant coarse index in  $K_*(C_{\text{red}}^*(\Gamma))$  of this lift is a more refined invariant than the Fredholm index of  $D_\Phi$  itself. This could for example yield more refined obstructions to Riemannian metrics of positive scalar curvature. In future work, we intend to show that the Fredholm index of  $D_\Phi$  can be recovered from the localised equivariant coarse index of its lift via an application of the von Neumann trace, or the summation trace in the context of maximal group  $C^*$ -algebras.

In [13], Callias-type index theory is extended to operators on bundles of Hilbert modules over  $C^*$ -algebras  $A$ , with indices in  $K_*(A)$ . If  $A = C_{\text{red}}^*(G)$  this seems related to the index we study here; in fact we suspect that the two coincide if  $G$  is the fundamental group of  $M/G$ , and  $M$  is its universal cover, and the Hilbert module bundle in question is constructed from the natural bundle  $M \times_G C_{\text{red}}^*(G) \rightarrow M/G$ . In the general equivariant case, index theory of Callias-type operators was developed in [17]; we will see in Theorem 4.2 that Definition 3.3 generalises the index of [17].

### Positive curvature at infinity

Secondly, suppose that  $D$  is a Dirac-type operator satisfying a Weitzenböck-type formula

$$D^2 = \nabla^* \nabla + R,$$

for a vector bundle endomorphism  $R$  satisfying  $R \geq c^2$  outside  $Z$ . Then  $D$  satisfies the conditions in Subsection 3.3, and therefore has a well-defined index in  $K_0(C_{\text{red}}^*(G))$ . The case where  $G$  is trivial (so that  $Z$  is compact) was studied by Gromov and Lawson [16] and applied to questions about Riemannian metrics of positive scalar curvature. The case where  $G$  is trivial and  $Z$  may be noncompact was treated by Roe [37] using coarse geometry.

### Manifolds with boundary

Finally, let  $\widetilde{M}$  be a Riemannian manifold with boundary, on which  $G$  acts properly, isometrically and cocompactly. Suppose that a neighbourhood

$U$  of  $\widetilde{\partial M}$  is  $G$ -equivariantly and isometrically diffeomorphic to a collar  $\widetilde{\partial M} \times [0, \varepsilon)$ . Let  $\widetilde{E} \rightarrow \widetilde{M}$  be a  $G$ -equivariant,  $\mathbb{Z}_2$ -graded, Hermitian vector bundle, and a module over the Clifford bundle of  $\widetilde{M}$ . Suppose that  $\widetilde{E}|_U \cong \widetilde{E}|_{\widetilde{\partial M}} \times [0, \varepsilon)$  as equivariant, Hermitian vector bundles. Suppose that  $\widetilde{D}$  is a Dirac-type operator on  $M$ , and that on  $U$  it is of the form

$$(3.12) \quad D|_U = \sigma \circ \left( \frac{\partial}{\partial t} + D_{\partial \widetilde{M}} \right),$$

where  $\sigma: \widetilde{E}_+|_{\partial \widetilde{M}} \rightarrow \widetilde{E}_-|_{\partial \widetilde{M}}$  is an equivariant vector bundle isomorphism,  $t$  is the coordinate in  $[0, \varepsilon)$ , and  $D_{\partial \widetilde{M}}$  is a Dirac operator on  $\widetilde{E}_+|_{\partial \widetilde{M}}$ .

Form  $M$  by attaching a cylinder  $\partial \widetilde{M} \times [0, \infty)$  to  $\widetilde{M}$ . Extend the Riemannian metric and the action by  $G$  to  $M$  in the natural way. Let  $E \rightarrow M$  be the natural extension of  $\widetilde{E}$ , and let  $D$  be the extension of  $\widetilde{D}$  to  $E$  equal to (3.12) on  $\partial \widetilde{M} \times [0, \infty)$ .

Suppose that  $D_{\partial \widetilde{M}}$  is invertible. Then  $D_{\partial \widetilde{M}}^2 \geq c^2$  for some  $c > 0$ , and  $D^2 \geq c^2$  outside  $Z = \widetilde{M}$ . Hence  $D$  satisfies the conditions in Subsection 3.3. The index of Definition 3.6 is now an equivariant Atiyah–Patodi–Singer type index for proper actions, and reduces to the original APS index if  $G$  is trivial. An index theorem for this index is proved in [26, 27]. As in the case of Callias operators, a special case is the lift of an operator on a compact manifold with boundary to the universal cover, in which case one obtains a refinement of the Atiyah–Patodi–Singer index in the  $K_*(C_{\text{red}}^*(\pi))$ , where  $\pi$  is the fundamental group of the compact manifold.

## 4. Results

We will show that the index of Definition 3.6 generalises the equivariant index of Callias-type operators introduced in [17] (Theorem 4.2). As applications, we obtain results on existence and non-existence of Riemannian metrics of positive scalar curvature in Subsection 4.2, and discuss a localised version of the Baum–Connes conjecture (Conjecture 4.8).

### 4.1. The equivariant Callias index

Suppose that  $D = D_{\Phi}$  is a Callias-type operator as in (3.11). Let  $\mathcal{E}$  denote the Hilbert  $C_{\text{red}}^*(G)$ -module defined by completing the space  $\Gamma_c^\infty(E)$  with respect to the  $C_c(G)$ -valued inner product

$$\langle s, t \rangle(g) := \langle s, gt \rangle_{L^2(E)}$$

and the right action of  $C_c(G)$  defined by

$$s \cdot b := \int_G g^{-1}(b(g)s) \, dg,$$

for  $s_1, s_2 \in \Gamma_c^\infty(E)$  and  $g \in G$ . One can find a continuous,  $G$ -invariant, cocompactly supported function  $f$  on  $M$  such that  $D_\Phi^2 + f$  is invertible in the sense of the Hilbert  $C_{\text{red}}^*(G)$ -modules  $\mathcal{E}^j$  as in Definition 1 in [17]. We can then form the normalised  $G$ -Callias-type operator

$$(4.1) \quad F := D_\Phi(D_\Phi^2 + f)^{-1/2}.$$

Then  $F$  lies in the  $C^*$ -algebra  $\mathcal{L}(\mathcal{E})$  of bounded adjointable operators on  $\mathcal{E}$ . It was shown in Theorem 25 in [17] that  $F$  is invertible modulo the algebra  $\mathcal{K}(\mathcal{E})$  of compact operators on  $\mathcal{E}$ , and thus defines a class

$$[F] \in K_1(\mathcal{L}(\mathcal{E})/\mathcal{K}(\mathcal{E})).$$

Here, as in [17], we assume that  $E$  is  $\mathbb{Z}_2$ -graded and  $D_\Phi$  is odd with respect to the grading, and the above  $K$ -theory class is defined in terms of the even part of  $F$ , as in the second point in Definition 3.2.

Let

$$(4.2) \quad \partial: K_1(\mathcal{L}(\mathcal{E})/\mathcal{K}(\mathcal{E})) \rightarrow K_0(\mathcal{K}(\mathcal{E})) = K_0(C_{\text{red}}^*(G))$$

be the boundary map associated to the short exact sequence

$$0 \rightarrow \mathcal{K}(\mathcal{E}) \rightarrow \mathcal{L}(\mathcal{E}) \rightarrow \mathcal{L}(\mathcal{E})/\mathcal{K}(\mathcal{E}) \rightarrow 0.$$

In (4.2), we have used the Morita equivalence  $\mathcal{K}(\mathcal{E}) \sim C_{\text{red}}^*(G)$ .

In [17], the following index was constructed and applied.

DEFINITION 4.1. — *The equivariant Callias-index of  $D_\Phi$  is*

$$\text{index}_G^C(D_\Phi) := \partial[F] \in K_0(C_{\text{red}}^*(G)).$$

One of our main results in this paper is that this index is a special case of the localised equivariant index. This gives a new approach to Callias index theory.

THEOREM 4.2. — *We have*

$$(4.3) \quad \text{index}_G^{\text{loc}}(D_\Phi) = \text{index}_G^C(D_\Phi) \in K_0(C_{\text{red}}^*(G)).$$

This result is proved in Section 6.

If  $M/G$  is compact, then we may take  $\Phi = 0$ , and  $\text{index}_G^C$  equals the analytic assembly map [4]. Therefore, Theorem 4.2 has the following immediate consequence.

COROLLARY 4.3. — *If  $M/G$  is compact, then the localised equivariant index of an elliptic operator is its image under the analytic assembly map.*

Note that if  $M/G$  is compact, then the localised equivariant coarse index equals the usual equivariant coarse index. In the case of discrete groups, Corollary 4.3 is the well-known fact that the equivariant coarse index for a proper action equals the analytic assembly map for such groups [35].

## 4.2. Positive scalar curvature

In the second special case in Subsection 3.5, if  $R$  is uniformly positive, i.e.  $Z = \emptyset$ , then its localised coarse index vanishes by standard arguments. Thus  $\text{index}_G^{\text{loc}}(D) \in K_*(C_{\text{red}}^*(G))$  is an obstruction to  $G$ -invariant Riemannian metrics of positive scalar curvature. There are many techniques for extracting more concrete, numerical obstructions from this  $K$ -theory class, such as pairing with traces and higher cyclic cocycles on (smooth subalgebras of)  $C_{\text{red}}^*(G)$ .

In the case where  $Z/G$  is non-compact, the localised equivariant coarse index allows us to use the following method to find obstructions to  $G$ -invariant Riemannian metrics of positive scalar curvature. This generalises the comments at the start of Section 3 in [37].

PROPOSITION 4.4. — *Let  $M$  be a complete Riemannian manifold, with a proper, isometric action by a locally compact group  $G$ . Let  $D$  be a Dirac-type operator whose curvature term  $R$  in the Weitzenböck-type formula  $D^2 = \Delta + R$  is uniformly positive outside a  $G$ -invariant subset  $Z \subset M$ , for which the inclusion map  $C^*(M; Z)^G \rightarrow C^*(M)^G$  induces the zero map on  $K$ -theory, with respect to an admissible  $C_0(M)$ -module  $H_M$  and its restriction  $H_Z := \mathbb{1}_Z H_M$ . Then  $\text{index}_G(D) = 0$ .*

When  $Z$  is cocompact, it is clear that the inclusion

$$K_*(C^*(M; Z, H_M)^G) = K_*(C^*(Z, \mathbb{1}_Z H_M)^G).$$

induces the identity map on  $K$  theory. More generally, we expect that, as in the discrete group case,<sup>(4)</sup> the equivariant Roe algebras of coarsely equivalent spaces have canonically isomorphic  $K$ -theory, and hence that this identity holds for general  $Z$  (see e.g. Lemma 1 in Section 5 of [23], or Proposition 6.4.7 in [22] for the non-equivariant case.) Then the condition

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<sup>(4)</sup> See the forthcoming book [40].

on the set  $Z$  in the above proposition is satisfied for example if  $Z$  is contained in a subset  $Y \subset M$  such that  $K_*(C^*(Y)^G) = 0$ . In future work, we aim to prove index theorems that allow us to deduce concrete topological obstructions to positive scalar curvature from Proposition 4.4.

We now turn to an existence result. Recall the following theorem from [1], which we need only for Lie groups:

**THEOREM 4.5 (Abels).** — *If  $M$  is a proper  $G$ -manifold, where  $G$  is an almost connected Lie group, then there exists a global slice  $N$  which is a  $K$ -manifold, in the  $G$ -manifold  $M$ , where  $K$  is a maximal compact subgroup of  $G$ .*

By this theorem,  $M$  is  $G$ -equivariantly diffeomorphic to  $G \times_K N$ .

**THEOREM 4.6.** — *Let  $G$  be an almost connected Lie group, and let  $K$  be a maximal compact subgroup of  $G$ . If  $N$  is a bounded geometry manifold with a  $K$ -invariant Riemannian metric of uniform positive scalar curvature, then  $M = G \times_K N$  is a bounded geometry manifold with a  $G$ -invariant Riemannian metric of uniform positive scalar curvature.*

### 4.3. A localised Baum–Connes conjecture

The Baum–Connes conjecture [4] describes  $K_*(C_{\text{red}}^*(G))$  in terms of equivariant indices of elliptic operators for cocompact actions by  $G$ . The surjectivity part of this conjecture is a particularly hard problem. Using the localised equivariant index of Definition 3.3, we will formulate a localised version of Baum–Connes surjectivity. We show that this is implied by Baum–Connes surjectivity in the usual sense (Proposition 4.9 below). It is therefore a weaker statement (and potentially easier to prove because one is allowed to use equivariant indices for *non-cocompact* actions) but which nevertheless describes the group  $K_*(C_{\text{red}}^*(G))$ .

Let  $D_{\text{red}}^*(X)^G$  be the algebra defined in Subsection 3.1.

**DEFINITION 4.7.** — *The localised equivariant  $K$ -homology of  $X$  is*

$$K_*^G(X)_{\text{loc}} := K_{*+1}(D_{\text{red}}^*(X)^G / C^*(X)_{\text{loc}}^G).$$

This terminology is motivated by Paschke duality (see e.g. page 85 of [36] and Theorem 8.4.3 in [22]), which implies that  $K_*^G(X)_{\text{loc}}$  equals the usual equivariant  $K$ -homology of  $X$  in the opposite degree, if  $X/G$  is compact. The index of Definition 3.3 defines a localised equivariant index

$$\text{index}_G^{\text{loc}} : K_*^G(X)_{\text{loc}} \rightarrow K_*(C_{\text{red}}^*(G)).$$

Now let  $\underline{E}G$  be a universal example for proper  $G$ -actions [4].

CONJECTURE 4.8 (Localised Baum–Connes surjectivity). — *The map*

$$\text{index}_G^{\text{loc}} : K_*^G(\underline{EG})_{\text{loc}} \rightarrow K_*(C_{\text{red}}^*(G))$$

*is surjective.*

Recall that the representable equivariant  $K$ -homology of  $X$  is

$$RK_*^G(X) := \varinjlim_Z K_G^*(Z),$$

where  $Z$  runs over the  $G$ -invariant closed subsets of  $X$  such that  $Z/G$  is compact. The Baum–Connes conjecture is the statement that the analytic assembly map

$$\mu_G : RK_*^G(\underline{EG}) \rightarrow K_*(C_{\text{red}}^*(G))$$

is bijective.

PROPOSITION 4.9. — *Surjectivity of  $\mu_G$  implies Conjecture 4.8.*

The converse of Proposition 4.9 is directly related to the question of whether the equivariant localised index of Definitions 3.3 and 3.6 lands in the image of the Baum–Connes assembly map. This question was posed for the equivariant Callias index in [17], and is open in general. See also Remark 7.3 below.

## 5. Proofs of properties of equivariant Roe algebras

In Subsections 5.1–5.3, we prove Theorem 2.7, which guarantees the existence of geometric admissible modules. We then use this in Subsection 5.4 to prove Theorem 2.11.

### 5.1. An isomorphism of $G$ -representations

We start by constructing the isomorphism  $\Psi$  as in Definition 2.4. We will use the following fact, whose proof is straightforward.

LEMMA 5.1 (Fell absorption). — *If  $\pi : G \rightarrow \text{U}(\mathcal{H})$  is a unitary representation, and  $\lambda : G \rightarrow \text{U}(L^2(G))$  is the left-regular representation, then the map*

$$\Phi : L^2(G) \otimes \mathcal{H} \rightarrow L^2(G) \otimes \mathcal{H},$$

*defined by*

$$\Phi(f)(g) = \pi(g)f(g),$$

*for  $f \in L^2(G, \mathcal{H})$  and  $g \in G$ , is a unitary isomorphism intertwining the representations  $\lambda \otimes \pi$  and  $\lambda \otimes 1$ .*

Suppose first that  $X = G \times_K Y$ , for a  $K$ -space  $Y$ . Let  $dx$  be a measure on  $X$  induced from the measure  $dg$  on  $G$  and a  $K$ -invariant measure  $dy$  on  $Y$ . Consider the measure  $d(Kg)$  on  $K \backslash G$ . Choose a measurable section  $\phi: K \backslash G \rightarrow G$ .

LEMMA 5.2. — *The map*

$$\psi: X \times G = (G \times_K Y) \times G \cong G \times K \backslash G \times Y,$$

given by

$$\psi([g, y], h) = (h(\phi(Kg^{-1}h)^{-1}), Kg^{-1}h, (\phi(Kg^{-1}h)h^{-1}g)y)$$

is a  $G$ -equivariant, measurable bijection. It relates the measures  $dx \times dg$  and  $dg \times d(Kg) \times dy$  to each other.

*Proof.* — There is a  $K$ -equivariant isomorphism of measure spaces (by which we mean a measurable bijection relating the given measures on the two spaces)

$$G \rightarrow K \times (K \backslash G), \quad g \mapsto (g(\phi(Kg)^{-1}), Kg).$$

Thus we have a  $G$ -equivariant isomorphism of measure spaces

$$\begin{aligned} G \times_K (G \times Y) &\cong G \times_K (K \times K \backslash G \times Y), \\ [(g, (h, y))] &\mapsto [(g, h(\phi(Kh)^{-1}), Kh, y)]. \end{aligned}$$

Combining this with the  $K$ -equivariant isomorphism

$$K \times Y \rightarrow K \times Y, \quad (k, y) \mapsto (k, k^{-1}y)$$

(where  $K$  acts diagonally on the left and only on the first factor on the right), this gives a  $G$ -equivariant isomorphism of measure spaces

$$\begin{aligned} G \times_K (G \times Y) &\cong G \times_K (K \times K \backslash G \times Y), \\ [(g, (h, y))] &\mapsto [(g, h(\phi(Kh)^{-1}), Kh, (\phi(Kh)h^{-1}y))], \end{aligned}$$

where  $K$  now acts trivially on  $Y$  on the right. Using the identification  $G \times_K K \cong G$ ,  $[(g, k)] \mapsto gk$ , we get the  $G$ -equivariant isomorphism

$$\begin{aligned} G \times_K (G \times Y) &\cong G \times K \backslash G \times Y, \\ [(g, (h, y))] &\mapsto (gh(\phi(Kh)^{-1}), Kh, (\phi(Kh)h^{-1}y)). \end{aligned}$$

Here  $G$  acts only on the first factor of both sides. Note that

$$\begin{aligned} (G \times_K Y) \times G &\cong G \times_K (G \times Y), \\ [(g, y), h] &\mapsto [(g, (g^{-1}h, y))] \end{aligned}$$

(where  $G$  acts diagonally on the left and only on the first factor on the right). The first claim then follows.  $\square$

Let  $E \rightarrow X$  be a  $G$ -equivariant, Hermitian vector bundle. Set  $H := L^2(K \backslash G) \otimes L^2(E|_Y)$ . We have

$$\psi^*(G \times K \backslash G \times E|_Y) \cong G \times E \rightarrow G \times X,$$

as  $G$ -equivariant vector bundles. So pulling back sections along the map  $\psi$  in Lemma 5.2 induces a unitary,  $G$ -equivariant isomorphism

$$\psi^*: L^2(G) \otimes H \rightarrow L^2(E) \otimes L^2(G).$$

Suppose that  $X/G$  is compact. Then, in general,  $X$  is a finite union of sets of the form  $U_j = G \times_{K_j} Y_j$  for compact subgroups  $K_j < G$  and compact subsets  $Y_j$ . These can be chosen so that the overlaps between these sets have measure zero. If the measure  $dx$  is induced from slices, then we can choose the slices  $Y_j$ , and measures  $dy_j$  on them, such that on each set  $U_j$ , the measure  $dx$  is induced by  $dg$  and  $dy_j$  as in the setting of Lemma 5.2. Then that lemma yields isomorphisms

$$\psi_j^*: L^2(G) \otimes H_j \rightarrow L^2(E|_{U_j}) \otimes L^2(G),$$

where  $H_j = L^2(K_j \backslash G) \otimes L^2(Y_j)$ . They combine into a global,  $G$ -equivariant, unitary isomorphism

$$(5.1) \quad \Psi: L^2(G) \otimes H \xrightarrow{\cong} \bigoplus_j L^2(E|_{U_j}) \otimes L^2(G) \cong L^2(E) \otimes L^2(G),$$

where  $H = \bigoplus_j H_j$ .

We have proved the following.

**PROPOSITION 5.3.** — *If the measure  $dx$  on  $X$  is induced from slices, then there is a  $G$ -equivariant, unitary isomorphism  $\Psi : L^2(G) \otimes H \rightarrow L^2(E) \otimes L^2(G)$  of the form (5.1), for a separable Hilbert space  $H$ . The space  $H$  is infinite-dimensional if  $X/G$  is infinite or if  $G/K_j$  is infinite for any  $j$ .*

*Remark 5.4.* — The conditions in Proposition 5.3 that  $X/G$  or  $G/K_j$  are infinite are mild: the set  $G/K_j$  is infinite for any noncompact group  $G$ , for example. Apart from this, one can tensor  $H$  by an infinite-dimensional separable Hilbert space to obtain an infinite-dimensional Hilbert space if desired.

### 5.2. Propagation in $X$ and in $G$

Next, we show that  $\Psi$  coarsely relates propagation on  $L^2(G) \otimes H$  with respect to  $C_0(G)$  to propagation on  $L^2(E) \otimes L^2(G)$  with respect to  $C_0(X)$ .

From now on, we fix a left-invariant distance function  $d_G$  on  $G$  that generates the topology on  $G$ , such that every closed ball is compact. This always exists if  $G$  is second-countable [21].

PROPOSITION 5.5. — *Let  $\Psi$  be the isomorphism in Proposition 5.3. An operator  $T$  on  $L^2(E) \otimes L^2(G)$  has finite propagation in  $X$  if and only if  $\Psi^{-1} \circ T \circ \Psi$  has finite propagation in  $G$ .*

To show this, we suppose first that  $X$  consists of just one slice. That is,  $X = G \times_K Y$ , for a compact  $K$ -space  $Y$ . We will reduce the general case to this case. Let  $\text{diam}(K)$  be the diameter of  $K$ .

LEMMA 5.6. — *Let*

$$\psi: G \times_K Y \times G \rightarrow G \times K \backslash G \times Y$$

be the bijective map from Lemma 5.2. Let  $\psi_1$  be its first component, mapping into  $G$ . Then for all  $g, g', h, h' \in G$  and  $y, y' \in Y$ ,

$$\begin{aligned} d_G(g, g') - 2 \text{diam}(K) &\leq d_G(\psi_1([g, y], h), \psi_1([g', y'], h')) \\ &\leq d_G(g, g') + 2 \text{diam}(K). \end{aligned}$$

*Proof.* — If  $g, h \in G$  and  $y \in Y$ , then

$$\psi_1([g, y], h) = h\phi(Kg^{-1}h)^{-1},$$

where  $\phi: K \backslash G \rightarrow G$  is a section. This means that there is a  $k \in K$  such that  $\phi(Kg^{-1}h) = kg^{-1}h$ . Hence

$$\psi_1([g, y], h) = gk^{-1}.$$

Let  $g, g', h, h' \in G$  and  $y, y' \in Y$ . Then we have just seen that there are  $k, k' \in K$  such that

$$d_G(\psi_1([g, y], h), \psi_1([g', y'], h')) = d_G(gk^{-1}, g'k'^{-1}).$$

This lies in the range specified by the triangle inequality and left invariance of  $d$ .  $\square$

LEMMA 5.7. — *For all  $s > 0$  there are  $r, r' > 0$  such that for all  $g, g' \in G$  and  $y, y' \in Y$ ,*

$$\begin{aligned} d_G(g, g') \leq s &\Rightarrow d(gy, g'y') \leq r. \\ d(gy, g'y') \leq s &\Rightarrow d_G(g, g') \leq r'. \end{aligned}$$

*Proof.* — Let  $s > 0$  be given. Define

$$r := \max\{d(gy, y'); y, y' \in Y, g \in G, d_G(g, e) \leq s\}.$$

Here we use compactness of  $Y$ . Then for all  $g, g' \in G$  with  $d_G(g, g') \leq s$ , we have  $d_G(g'^{-1}g, e) \leq s$ , so for all  $y, y' \in Y$ ,

$$d(gy, g'y') = d(g'^{-1}gy, y') \leq r.$$

To prove the second claim, note that properness of the action by  $G$  on  $X$  and compactness of  $Y$  imply that the set

$$A_s := \{g \in G; gY \cap \text{Pen}(Y, s) \neq \emptyset\}$$

is compact (with notation as in (2.4)). Set

$$r' := \max\{d_G(g, e); g \in A_s\}.$$

Then for all  $g, g' \in G$  and  $y, y' \in Y$  with  $d(gy, gy') \leq s$ , we have  $g'^{-1}g \in A_s$ , so  $d_G(g, g') \leq r'$ . □

LEMMA 5.8. — *If an operator  $T$  on  $L^2(X) \otimes L^2(G)$  has finite propagation in  $X$ , then  $\Psi^{-1} \circ T \circ \Psi$  has finite propagation in  $G$ .*

*Proof.* — Suppose that  $T$  is an operator on  $L^2(X) \otimes L^2(G)$  with finite propagation  $s$  in  $X$ . By the second part of Lemma 5.7, there is an  $r > 0$  such that for all  $g, g' \in G$  and  $y, y' \in Y$ ,

$$d_G(g, g') \geq r \Rightarrow d(gy, g'y') \geq s + 1.$$

Let  $\chi_1, \chi_2 \in C_c(G)$  be given, with  $d_G(\text{supp}(\chi_1), \text{supp}(\chi_2)) \geq r + 2 \text{diam}(K)$ . For  $j = 1, 2$ , let  $g_j \in \text{supp}(\chi_j)$ ,  $h_j \in G$  and  $y_j \in Y$  be given. Write

$$(\tilde{g}_j \tilde{y}_j, \tilde{h}_j) = \psi^{-1}(g_j, Kh_j, y_j),$$

for  $\tilde{g}_j, \tilde{h}_j \in G$  and  $\tilde{y}_j \in Y$ . Then by Lemma 5.6,

$$d_G(\tilde{g}_1, \tilde{g}_2) \geq d_G(g_1, g_2) - 2 \text{diam}(K) \geq r.$$

So  $d(\tilde{g}_1 \tilde{y}_1, \tilde{g}_2 \tilde{y}_2) \geq s + 1$ . Let  $\pi_X: X \times G \rightarrow X$  be the projection onto the first factor. We have just seen that

$$d(\pi_X(\psi^{-1}(\text{supp} \chi_1 \times K \setminus G \times G)), \pi_X(\psi^{-1}(\text{supp} \chi_2 \times K \setminus G \times G))) \geq s + 1.$$

Hence we can choose  $\varphi_j \in C_c(X)$ , for  $j = 1, 2$ , such that  $\varphi_j \equiv 1$  on  $\pi_X(\psi^{-1}(\text{supp} \chi_j \times K \setminus G \times G))$ , and

$$d(\text{supp}(\varphi_1), \text{supp}(\varphi_2)) \geq s.$$

We conclude that

$$\begin{aligned} &\chi_1(\Psi^{-1} \circ T \circ \Psi)\chi_2 \\ &= \Psi^{-1} \circ (\psi^*(\chi_1 \otimes 1_{K \setminus G \times Y})\varphi_1 T \varphi_2 \psi^*(\chi_2 \otimes 1_{K \setminus G \times Y})) \circ \Psi = 0, \end{aligned}$$

since  $\varphi_1 T \varphi_2 = 0$ . □

LEMMA 5.9. — *If an operator  $\tilde{T}$  on  $L^2(G) \otimes H$  has finite propagation in  $G$ , then  $\Psi \circ \tilde{T} \circ \Psi^{-1}$  has finite propagation in  $X$ .*

*Proof.* — Let  $\tilde{T}$  be an operator on  $L^2(G) \otimes H$  with finite propagation  $s$  in  $G$ . The first part of Lemma 5.7 implies that there is an  $r > 0$  such that for all  $g, g' \in G$  and  $y, y' \in Y$ ,

$$(5.2) \quad d(gy, g'y') \geq r \quad \Rightarrow \quad d_G(g, g') \geq s + 1 + 2 \operatorname{diam}(K).$$

For  $j = 1, 2$ , let  $\varphi_j \in C_c(X)$  be such that  $d(\operatorname{supp}(\varphi_1), \varphi_2) \geq r$ . Let  $\pi_G: G \times K \setminus G \times Y \rightarrow G$  be the projection onto the first factor. By Lemma 5.6, we have, for all  $g_j, h_j \in G$  and  $y_j \in Y$  such that  $g_j y_j \in \operatorname{supp}(\varphi_j)$ ,

$$d_G(\pi_G(\psi(g_1 y_1, h_1)), \pi_G(\psi(g_2 y_2, h_2))) \geq d_G(g_1, g_2) - 2 \operatorname{diam}(K) \geq s + 1,$$

where we have used (5.2). So we can choose  $\chi_j \in C_c(G)$  such that  $\chi_j \equiv 1$  on  $\pi_G(\psi(\operatorname{supp}(\varphi_j) \times G))$  and  $d_G(\operatorname{supp}(\chi_1), \operatorname{supp}(\chi_2)) \geq s$ . Then

$$\varphi_1(\Psi \circ \tilde{T} \circ \Psi^{-1})\varphi_2 = \Psi \circ ((\psi^{-1})^*(\varphi_1 \otimes 1_G)\chi_1 \tilde{T} \chi_2 (\psi^{-1})^*(\varphi_2 \otimes 1_G)) \circ \Psi^{-1} = 0,$$

since  $\chi_1 \tilde{T} \chi_2 = 0$ . □

*Proof of Proposition 5.5.* — If  $X = G \times_K Y$  for a single, compact slice  $Y \subset X$ , then the claim is precisely Lemmas 5.8 and 5.9.

In the general case when the cocompact space  $X$  consists of finitely many slices,  $L^2(E) \otimes L^2(G)$  is a finite direct sum  $\bigoplus_i L^2(G) \otimes H_i$  as in (5.1). An operator  $T$  on this space can be written as a finite matrix  $(T_{i,j})$ , where each entry  $T_{i,j}$  is an operator

$$T_{i,j} : L^2(G) \otimes H_i \rightarrow L^2(G) \otimes H_j.$$

The result then follows from the case of a single slice. □

### 5.3. Proof of Theorem 2.7

The remaining step in the proof of Theorem 2.7 is to show that  $\Psi$  relates local compactness of operators on  $H_X$  with respect to  $C_0(X)$  to local compactness of operators on  $L^2(G) \otimes H$  with respect to  $C_0(G)$ .

PROPOSITION 5.10. — *A bounded operator  $T$  on  $H_X$  is locally compact with respect to the action by  $C_0(X)$  if and only if the bounded operator  $\Psi^{-1} \circ T \circ \Psi$  on  $L^2(G) \otimes H$  is locally compact with respect to the action by  $C_0(G)$ .*

*Proof.* — First, suppose that  $X = G \times_K Y$  for a single slice  $Y$ .

Suppose  $T$  is a bounded operator on  $L^2(E) \otimes L^2(G)$  that is locally compact with respect to multiplication by  $C_0(X)$ . Let  $\chi \in C_c(G)$  be given. As in the proof of Lemma 5.8, Lemmas 5.6 and 5.7 imply that the subset  $\pi_X(\psi^{-1}(\text{supp } \chi \times K \backslash G \times G))$  of  $X$  is bounded. Hence we can choose  $\phi \in C_c(X)$  such that  $\phi \equiv 1$  on  $\pi_X(\psi^{-1}(\text{supp } \chi \times K \backslash G \times G))$ . Thus

$$(\Psi^{-1} \circ T \circ \Psi)\chi = \Psi^{-1} \circ (T\phi\psi^*(\chi \otimes 1_{K \backslash G \times Y})) \circ \Psi \in \mathcal{K}(L^2(G) \otimes H),$$

since  $T\phi \in \mathcal{K}(L^2(E) \otimes L^2(G))$ .

Now suppose  $\tilde{T} = \Psi^{-1} \circ T \circ \Psi$  is a bounded operator on  $L^2(G) \otimes H$  that is locally compact with respect to the multiplicative action of  $C_0(G)$  on the first factor. Let  $\phi \in C_c(X)$  be given. As in the proof of Lemma 5.9, Lemmas 5.6 and 5.7 imply that the subset  $\pi_G(\psi(\text{supp}(\phi) \times G))$  of  $G$  is bounded. Hence we can choose  $\chi \in C_c(G)$  such that  $\chi \equiv 1$  on  $\pi_G(\psi(\text{supp}(\phi) \times G))$ . Then

$$T\phi = (\Psi \circ \tilde{T} \circ \Psi^{-1})\phi = \Psi \circ (\tilde{T}\chi(\psi^{-1})^*(\phi \otimes 1_G)) \circ \Psi^{-1} \in \mathcal{K}(L^2(E) \otimes L^2(G)),$$

since  $\tilde{T}\chi \in \mathcal{K}(L^2(G) \otimes H)$ .

The general case for cocompact  $X$  and more than one slice follows from the single slice case as in the proof of Proposition 5.5. □

Theorem 2.7 is the combination of Propositions 5.5 and 5.10. Under the assumption in Theorem 2.7 that  $G/K$  or  $X/G$  is infinite, the space  $H$  in Proposition 5.3 is infinite-dimensional.

### 5.4. Kernels and group $C^*$ -algebras

The reduced and maximal equivariant Roe algebras can alternatively be described in terms of continuous Schwartz kernels of operators. We work this out in detail this subsection and use it to prove Theorem 2.11.

Let  $X$  and  $G$  be as before. Suppose that  $X/G$  is compact. Let  $H_X$  be any admissible equivariant  $C_0(X)$ -module over  $X$ . In this subsection, we will always identify  $H_X$  with  $L^2(G) \otimes H$  for an infinite-dimensional separable Hilbert space  $H$ , using the isomorphism  $\Psi$  in Definition 2.4.

**DEFINITION 5.11.** — *Let  $C_{\text{ker}}^*(X)^G$  denote the algebra of bounded operators on  $H_X$  defined by continuous  $G$ -invariant Schwartz kernels*

$$\kappa: G \times G \rightarrow \mathcal{K}(H)$$

*that have finite propagation in  $G$ .*

By the ‘if’ part of Proposition 5.5, we have  $C_{\ker}^*(X)^G \subset C_{\text{alg}}^*(X)^G$ . Our goal is to prove the following proposition.

PROPOSITION 5.12. —  $C_{\ker}^*(X)^G$  is dense in  $C_{\text{alg}}^*(X)^G$  with respect to the operator norm on  $H_X$ .

Because of this proposition, the Roe algebra  $C^*(X)^G$  can alternatively be defined as the closure of  $C_{\ker}^*(X)^G$  in  $\mathcal{B}(H_X)$ . This immediately implies Theorem 2.11, since

$$C_{\ker}^*(X)^G \cong C_c(G) \otimes \mathcal{K}(H)$$

via the isomorphism sending  $\kappa \in C_{\ker}^*(X)^G$  to the map  $g \mapsto \kappa(g^{-1}, e)$ .

Let  $T \in C_{\text{alg}}^*(X)^G$ . We will prove Proposition 5.12 by showing that  $T$  can be approximated by elements of  $C_{\ker}^*(X)^G$  in the operator norm.

Fix  $\chi \in C_c^\infty(G)$  such that for all  $h \in G$ ,

$$\int_G \chi(g^{-1}h) \, dg = 1.$$

Let  $H$  be as in Definition 2.4. By the ‘only if’ part of Proposition 5.5,  $T$  has finite propagation in  $G$ . So there exist functions  $\chi_1, \chi_2 \in C_c(G)$  such that

$$\begin{aligned} \chi_1 T \chi &= T \chi, \\ \chi \chi_2 &= \chi \end{aligned}$$

when  $T$  is viewed as an operator on  $L^2(G) \otimes H$ .

Let  $\{e_j\}_{j=1}^\infty$  be a Hilbert basis for  $H_X \cong L^2(G, H)$  such that  $e_j \in C_c(G, H)$  for every  $j$ . Let  $\{e^k\}_{k=1}^\infty$  be the dual basis. We view  $e^k$  as the element of  $C_c(G, H^*)$  such that for all  $g \in G$  and  $v \in H$ ,

$$e^k(g)(v) = (e_k(g), v)_H.$$

By the definition of  $C_{\text{alg}}^*(X)^G$  and Proposition 5.10, we have  $T \chi \in \mathcal{K}(H_X)$ . Thus we can write

$$(5.3) \quad T \chi = \sum_{j,k} a_k^j e_j \otimes e^k$$

for some constants  $a_k^j$ , with the sum converging in operator norm in  $\mathcal{B}(L^2(G) \otimes H)$ . Define  $T_k^j \in \mathcal{B}(L^2(G) \otimes H)$  to be the operator given by the Schwartz kernel

$$(5.4) \quad \begin{aligned} &\kappa_k^j : G \times G \rightarrow \mathcal{K}(H), \\ (h, h') &\mapsto a_k^j \int_G \chi_1(g^{-1}h) \chi_2(g^{-1}h') e_j(g^{-1}h) \otimes e^k(g^{-1}h') \, dg, \end{aligned}$$

where  $h, h' \in G$ . Since  $e_j(g) \otimes e^k(g')$  (for  $g, g' \in G$ ) is a finite-rank operator on  $H$  and the integrand in (5.4) is compactly supported, we find that indeed  $\kappa_k^j(h, h') \in \mathcal{K}(H)$  for all  $h, h' \in G$ . Furthermore,  $\kappa_k^j$  is continuous,  $G$ -invariant, and has finite propagation in  $G$ .

LEMMA 5.13. — For every  $f \in L^2(G, H)$  and  $h \in G$ ,

$$(Tf)(h) = \sum_{j,k=1}^{\infty} T_k^j f(h).$$

Proof. — Let  $f \in L^2(G, H)$ . Then for every  $g \in G$  we have

$$\begin{aligned} T \circ (g \cdot \chi) &= g(T\chi)g^{-1} \\ &= g\chi_1 T\chi\chi_2 g^{-1}. \end{aligned}$$

Thus for all  $h \in G$ ,

$$\begin{aligned} (Tf)(h) &= \int_G (T(g \cdot \chi)f)(h) \, dg \\ &= \int_G ((g\chi_1 T\chi\chi_2 g^{-1})f)(h) \, dg \\ &= \sum_{j,k} a_k^j \int_G \chi_1(g^{-1}h)e_j(g^{-1}h) \left( \int_G (e_k(l), \chi_2(l)f(gl))_H \, dl \right) \, dg \\ &= \sum_{j,k} \int_G \kappa_k^j(h, m)f(m) \, dm, \end{aligned}$$

where we substitute  $m = gl$ . Note that all integrands are continuous and compactly supported, so we may indeed interchange integrals and sums.  $\square$

LEMMA 5.14. — The sum

$$\sum_{j,k=1}^{\infty} T_k^j$$

converges in  $\mathcal{B}(L^2(G) \otimes H)$  with respect to the operator norm.

Proof. — We have for all  $j, k \in \mathbb{N}$  and  $f \in C_c(G, H)$ ,

$$\begin{aligned} (5.5) \quad T_k^j f &= a_k^j \int_G (g \cdot (\chi_1 e_j))(g \cdot (\chi_2 e_k), f)_{L^2(G, H)} \, dg \\ &= a_k^j \int_G (g \cdot \chi_1)(g \circ (e_j \otimes e^k) \circ g^{-1})(g \cdot \chi_2) f \, dg. \end{aligned}$$

Hence for all  $M, N, M', N' \in \mathbb{N}$  with  $M \leq M'$  and  $N \leq N'$ ,

$$\begin{aligned} & \left\| \sum_{j=M}^{M'} \sum_{k=N}^{N'} T_k^j f \right\|_{L^2(G, H)} \\ &= \left\| \sum_{j=M}^{M'} \sum_{k=N}^{N'} a_k^j \int_G (g \cdot \chi_1)(g \circ (e_j \otimes e^k) \circ g^{-1})(g \cdot \chi_2) f \, dg \right\|_{L^2(G, H)} \\ &= \left\| \int_G (g \cdot \chi_1) \left( g \circ \left( \sum_{j=M}^{M'} \sum_{k=N}^{N'} a_k^j e_j \otimes e^k \right) \circ g^{-1} \right) (g \cdot \chi_2) f \, dg \right\|_{L^2(G, H)}. \end{aligned}$$

Write

$$T_{M, N}^{M', N'} := \sum_{j=M}^{M'} \sum_{k=N}^{N'} a_k^j e_j \otimes e^k.$$

Define  $F: G \rightarrow L^2(G, H)$  by

$$F(g) = (g \cdot \chi_1)(g \circ T_{M, N}^{M', N'} \circ g^{-1})(g \cdot \chi_2) f,$$

for  $g \in G$ . If  $g, g' \in G$ , and  $(F(g), F(g'))_{L^2(G, H)} \neq 0$ , then

$$g \operatorname{supp}(\chi_1) \cap g' \operatorname{supp}(\chi_1) \neq \emptyset.$$

By properness of the action, this means that  $g^{-1}g'$  lies in a compact set  $S \subset G$ , only depending on  $\chi_1$ .

By Lemma 1.5 in [14], this implies that

$$\left\| \int_G F(g) \, dg \right\|_{L^2(G, H)}^2 \leq \operatorname{vol}(S) \int_G \|F(g)\|_{L^2(G, H)}^2 \, dg.$$

Hence, since  $G$  acts unitarily on  $L^2(G, H)$ ,

$$\begin{aligned} & \left\| \sum_{j=M}^{M'} \sum_{k=N}^{N'} T_k^j f \right\|_{L^2(G, H)}^2 \\ & \leq \operatorname{vol}(S) \|\chi_1\|_\infty^2 \|T_{M, N}^{M', N'}\|_{\mathcal{B}(L^2(G, H))}^2 \int_G \|(g \cdot \chi_2) f\|_{L^2(G, H)}^2 \, dg \\ & \leq \operatorname{vol}(S) \|\chi_1\|_\infty^2 \|T_{M, N}^{M', N'}\|_{\mathcal{B}(L^2(G, H))}^2 \|\chi_2\|_G^2 \|f\|_{L^2(G, H)}^2, \end{aligned}$$

where

$$\begin{aligned} \|\chi_1\|_\infty &:= \max_{g \in G} \|\chi_1(g)\|_{\mathcal{B}(H)}, \\ \|\chi_2\|_G &:= \sqrt{\max_{h \in G} \int_G \|\chi_2(g^{-1}h)\|_{\mathcal{B}(H)}^2 \, dg}. \end{aligned}$$

We conclude that the operator

$$\sum_{j=M}^{M'} \sum_{k=N}^{N'} T_k^j$$

on  $L^2(G, H)$  is bounded, with norm at most

$$\text{vol}(S)^{1/2} \|\chi_1\|_\infty \|\chi_2\|_G \|T_{M,N}^{M',N'}\|_{\mathcal{B}(L^2(G,H))}.$$

Since the sum (5.3) converges in the operator norm and  $\mathcal{B}(L^2(G, H))$  is complete, the claim follows.  $\square$

*Proof of Proposition 5.12.* — By Lemmas 5.13 and 5.14, we have

$$T = \sum_{j,k=1}^{\infty} T_k^j,$$

where the sum converges in the operator norm. Hence  $C_{\text{ker}}^*(X)^G$  is dense in  $C_{\text{alg}}^*(X)^G$ .  $\square$

## 6. The equivariant Callias index

In this section, we prove Theorem 4.2, showing that the equivariant index of a  $G$ -Callias-type operator, as defined in [17], identifies naturally with its localised equivariant index given by Definition 3.6. We begin in Subsection 6.1 by relating the equivariant coarse index defined in Subsection 3.1 for cocompact actions to the usual  $G$ -equivariant index obtained through the assembly map, before relating the localised equivariant index to the non-cocompact  $G$ -equivariant index in Subsection 6.2. With respect to the notation in Subsection 3.1, we are working with  $D^*(X)^G = \mathcal{M}(C^*(X)^G)$  or  $\mathcal{M}(C^*(X)_{\text{loc}}^G)$ , depending on context.

The results in the first two subsections of this section are of a general nature and apply to both the maximal and reduced versions of the index, and we will use  $C^*(G)$  will denote either  $C_{\text{red}}^*(G)$  or  $C_{\text{max}}^*(G)$ , and  $C^*(X)^G$  (resp.  $C^*(X)_{\text{loc}}^G$ ) for either the reduced or maximal version of the Roe algebra (resp. localised Roe algebra).

### 6.1. Index maps in the cocompact case

Suppose that  $X$  is  $G$ -cocompact.

Equip the dense subspace  $C_c(G, L^2(E))$  of  $L^2(E) \otimes L^2(G)$  with the  $C_c(G)$ -valued inner product

$$\langle s, t \rangle(g) := \langle s, gt \rangle_{L^2(E) \otimes L^2(G)}$$

and the right action of  $C_c(G)$  defined by

$$s \cdot b := \int_G g^{-1}(b(g)s) dg.$$

Taking the completion gives rise to a Hilbert  $C^*(G)$ -module  $\mathcal{E}_{C^*(G)}$ .

LEMMA 6.1. —  $\mathcal{E}_{C^*(G)}$  is isomorphic to the standard Hilbert  $C^*(G)$ -module  $C^*(G) \otimes H$ , for a separable Hilbert space  $H$ .

*Proof.* — Let  $H$  be the Hilbert space in the isomorphism

$$(6.1) \quad L^2(E) \otimes L^2(G) \cong L^2(G) \otimes H$$

from Proposition 5.3. Let  $\mathcal{E}'_{C^*(G)}$  denote the Hilbert  $C^*(G)$ -module completion of  $C_c(G) \otimes H$  with respect to the  $C_c(G)$ -valued inner product and right  $C_c(G)$ -action

$$\langle s, t \rangle(g) := \langle s, gt \rangle_{L^2(G) \otimes H}, \quad s \cdot b := \int_G g^{-1}(b(g)s) dg,$$

where  $s, t \in L^2(G, H)$ . Then the isomorphism (6.1), restricted to the dense subspace  $C_c(G, L^2(E)) \subseteq L^2(E) \otimes L^2(G)$ , extends to an isomorphism  $\mathcal{E}_{C^*(G)} \cong \mathcal{E}'_{C^*(G)}$ . Further, one can check that the map

$$\mathcal{E}'_{C^*(G)} \rightarrow C^*(G) \otimes H, \quad s \mapsto \tilde{s},$$

where  $\tilde{s}$  takes  $g \mapsto s(g^{-1})$ , is an isometric isomorphism of  $\mathcal{E}'_{C^*(G)}$  onto the standard Hilbert  $C^*(G)$ -module equipped with its usual inner product and right  $C^*(G)$ -action. □

Using Lemma 6.1, we can write down an identification

$$U : \mathcal{K}(\mathcal{E}_{C^*(G)}) \cong \mathcal{K}(C^*(G) \otimes H).$$

Now let  $C^*(X)^G$  denote the  $G$ -equivariant Roe algebra on  $L^2(E) \otimes L^2(G) \cong L^2(G) \otimes H$ , and let  $C^*_{\ker}(X)^G$  be its dense subalgebra of  $G$ -invariant kernels from Definition 5.11. Let  $W$  denote the identification

$$W : C^*_{\ker}(X)^G \cong C_c(G) \otimes \mathcal{K}(H)$$

below Proposition 5.12. This map identifies  $C^*_{\ker}(X)^G$  with a subalgebra of the compact operators on the standard Hilbert  $C^*(G)$ -module  $C^*(G) \otimes H$ :

$$W : C^*_{\ker}(X)^G \xrightarrow{\cong} \underbrace{C_c(G) \otimes \mathcal{K}(H)}_{\subseteq \mathcal{K}(\mathcal{E}'_{C^*(G)})} \xrightarrow{\cong} \underbrace{C_c(G) \otimes \mathcal{K}(H)}_{\subseteq \mathcal{K}(C^*(G) \otimes H)}.$$

This extends to an identification  $W : C^*(X)^G \xrightarrow{\cong} \mathcal{K}(C^*(G) \otimes H)$ .

Let  $\mathcal{M}$  be the multiplier algebra of  $C^*(X)^G$ , and let  $\mathcal{L} := \mathcal{L}(\mathcal{E}_{C^*(G)})$  be the algebra of adjointable operators on  $\mathcal{E}_{C^*(G)}$ .

COROLLARY 6.2. — *We have an isomorphism*

$$U^{-1} \circ W : C^*(X)^G \xrightarrow{\cong} \mathcal{K}(\mathcal{E}_{C^*(G)}).$$

*This induces an isomorphism on the multiplier algebras and an isomorphism on  $K$ -theory of the quotient algebras:*

$$(U^{-1} \circ W)_* : K_1(\mathcal{M}/C^*(X)^G) \xrightarrow{\cong} K_1(\mathcal{L}/\mathcal{K}(\mathcal{E}_{C^*(G)})).$$

Now let

$$\eta : K_0(\mathcal{K}(C^*(G) \otimes H)) \rightarrow K_0(C^*(G))$$

be the stabilisation isomorphism on  $K$ -theory, and write

$$\phi := \eta \circ W_*,$$

where  $W_*$  is the map on  $K$ -theory induced by  $W$ . After making these identifications, the following proposition follows directly from naturality of boundary maps with respect to  $*$ -homomorphisms.

PROPOSITION 6.3. — *The following diagram commutes:*

$$(6.2) \quad \begin{array}{ccc} K_1(\mathcal{M}/C^*(X)^G) & \xrightarrow{\text{index}} & K_0(C^*(X)^G) \\ \downarrow (U^{-1} \circ W)_* & & \searrow \phi \\ K_1(\mathcal{L}/\mathcal{K}(\mathcal{E}_{C^*(G)})) & \xrightarrow{\text{index}} & K_0(\mathcal{K}(\mathcal{E}_{C^*(G)})) \end{array} \quad \begin{array}{c} \\ \\ \nearrow \eta \circ U_* \end{array} \rightarrow K_0(C^*(G)),$$

where  $U_*$  is the map induced by  $U$  on  $K$ -theory.

The map  $(\eta \circ U_* \circ \text{index})$  is the usual  $G$ -index map for Fredholm operators in the sense of Hilbert  $C^*(G)$ -modules on the module  $\mathcal{E}_{C^*(G)}$ . Thus Proposition 6.3 provides an identification of the index map in the Roe algebra picture with the usual notion of  $G$ -index for operators on a  $G$ -cocompact space.

**6.2. The localised equivariant index**

Now suppose that  $X/G$  is possibly noncompact. As before, let  $E$  be a  $G$ -vector bundle over  $X$ . Similar to the previous subsection, equip the dense subspace  $C_c(G, L^2(E))$  of  $L^2(E) \otimes L^2(G)$  with the  $C_c(G)$ -valued inner product and right  $C_c(G)$ -action given by

$$\langle s, t \rangle(g) := \langle s, gt \rangle_{L^2(E) \otimes L^2(G)}, \quad s \cdot b := \int_G g^{-1}(b(g)s) \, dg.$$

Taking the completion gives rise to a Hilbert  $C^*(G)$ -module  $\mathcal{E}_{C^*(G)}$ .

Let  $Z \subset X$  be closed and  $G$ -invariant. Let  $H_X$  be an admissible equivariant  $C_0(X)$ -module. The restriction map  $C_0(X) \rightarrow C_0(Z)$  allows us to view  $H_X$  as a  $C_0(Z)$ -module, which will be degenerate (see Definition 2.1) in general. Let  $H_Z = \mathbf{1}_Z H_X$ , an admissible  $C_0(Z)$ -module. Write  $H_{X \setminus Z}$  for the orthogonal complement of  $H_Z$  in  $H_X$ . The map

$$(6.3) \quad \varphi_Z^X : \mathcal{B}(H_Z) \rightarrow \mathcal{B}(H_X),$$

defined by extending operators by zero on  $H_{X \setminus Z}$ , restricts to a  $*$ -homomorphism

$$(6.4) \quad \varphi_Z^X : C^*(Z)^G \rightarrow C^*(X; H_X)^G,$$

whose image lies in  $C^*(X; Z, H_X)^G$ .

Let  $C^*(X)_{\text{loc}}^G$  be the localised equivariant Roe algebra of Definition 2.17.

PROPOSITION 6.4. — We have

$$C^*(X)_{\text{loc}}^G \cong \mathcal{K}(\mathcal{E}_{C^*(G)}).$$

*Proof.* — Fix  $Z \subseteq X$  a closed,  $G$ -invariant cocompact subset. For  $i > 0$ , let  $\text{Pen}(Z, i)$  be as in (2.4). Then  $\text{Pen}(Z, i)$  is cocompact and  $G$ -stable, so by (2.2), Let  $\varphi_i$  be the map

$$\varphi_i := \varphi_{\text{Pen}(Z, i)}^{\text{Pen}(Z, i+1)} : C^*(\text{Pen}(Z, i))^G \rightarrow C^*(\text{Pen}(Z, i+1))^G$$

as in (6.4). Then

$$\{C^*(\text{Pen}(Z, i))^G, \varphi_i\}_{i \in \mathbb{N}}$$

is a directed system of  $C^*$ -algebras whose direct limit is  $C^*(X)_{\text{loc}}^G$ . Hence

$$C^*(X)_{\text{loc}}^G \cong C^*(G) \otimes \mathcal{K}(H),$$

where  $H$  is the Hilbert space from the isomorphism  $L^2(E) \otimes L^2(G) \cong L^2(G) \otimes H$  in Definition 2.4.

Now let  $\mathcal{E}|_{\text{Pen}(Z,i)}$  be the restriction of the Hilbert module  $\mathcal{E}_{C^*(G)}$  to  $\text{Pen}(Z,i)$ . By Corollary 6.2, for each  $i$ , we have an isomorphism  $\mathcal{K}(\mathcal{E}|_{\text{Pen}(Z,i)}) \cong C^*(\text{Pen}(Z,i))^G$ . These maps fit into a commutative diagram

$$\begin{CD} C^*(\text{Pen}(Z,i))^G @>\cong>> \mathcal{K}(\mathcal{E}|_{\text{Pen}(Z,i)}) \\ @V\varphi_iVV @VVV \\ C^*(\text{Pen}(Z,i+1))^G @>\cong>> \mathcal{K}(\mathcal{E}|_{\text{Pen}(Z,i+1)}). \end{CD}$$

Finally, note that each element of  $\mathcal{K}(\mathcal{E}_{C^*(G)})$  is a limit of finite-rank operators, hence  $\mathcal{K}(\mathcal{E}_{C^*(G)}) = \lim_i \mathcal{K}(\mathcal{E}|_{\text{Pen}(Z,i)})$ . □

It follows that we have an isomorphism

$$\mathcal{L}(\mathcal{E}_{C^*(G)})/\mathcal{K}(\mathcal{E}_{C^*(G)}) \cong \mathcal{M}(C^*(X)_{\text{loc}}^G)/C^*(X)_{\text{loc}}^G.$$

Applying Proposition 6.3 to each of the  $G$ -cocompact spaces  $\text{Pen}(Z,i)$  and taking the direct limit, we have shown:

PROPOSITION 6.5. — *The following two index maps are equal:*

$$\text{index}_G : K_1(\mathcal{L}/\mathcal{K}(\mathcal{E}_{C^*(G)})) \xrightarrow{\lim(\eta \circ U_* \circ \text{index})} K_0(C^*(G))$$

and

$$\text{index}_G : K_1(\mathcal{M}/(C^*(X)_{\text{loc}}^G)) \xrightarrow{\lim(\phi \circ \text{index})} K_0(C^*(G)).$$

Here  $C^*(G)$  and  $C^*(X)_{\text{loc}}^G$  can be taken to be either the reduced or maximal version of the group  $C^*$  and Roe algebras.

### 6.3. $G$ -Callias-type operators and Roe’s localised index

We now relate the reduced version of the equivariant Callias-type index defined in [17] to (the reduced version of) the localised coarse index.

Recall the setting of Subsection 4.1, where  $D = D_\Phi$  is a Callias-type operator. The operator  $F$  in (4.1) defines a class  $[F] \in K_1(\mathcal{L}(\mathcal{E})/\mathcal{K}(\mathcal{E}))$ , whose image under the boundary map for the six-term exact sequence corresponding to the ideal  $\mathcal{L}(\mathcal{E}) \subset \mathcal{K}(\mathcal{E})$  is by definition  $\text{index}_G^C(D_\Phi) \in K_0(C_{\text{red}}^*(G))$ . Consider the embedding

$$\mathcal{E} \hookrightarrow \mathcal{E}_{C_{\text{red}}^*(G)},$$

defined on the dense subspace  $C_c(E)$  by the map  $j$  in (3.3). The image of  $\mathcal{E}$  is a complemented submodule of  $\mathcal{E}_{C_{\text{red}}^*(G)}$ . Extend the operator  $F$  to all of  $\mathcal{E}_{C_{\text{red}}^*(G)}$  by defining the extension to be the identity on the orthogonal complement. We denote this extended operator by  $F_1$ .

The assumption (3.10) on  $\Phi$  and  $\tilde{D}$  implies that there is a  $G$ -cocompact subset  $Z \subset X$  such that  $D_\Phi^2 \geq c^2$  outside  $Z$ , for some  $c > 0$ . By replacing  $D_\Phi$  by the operator  $\frac{1}{c}D_\Phi$ , which has the same index as  $D_\Phi$ , we may alternatively make the slightly more convenient assumption that  $D_\Phi^2 \geq 1$  outside  $Z$ . As in Definition 3.6, the localised equivariant coarse index of  $D_\Phi$  is

$$(6.5) \quad \text{index}_G(D_\Phi) = \text{index}_G([b(D_\Phi)] \oplus 1) \in K_0(C_{\text{red}}^*(G)),$$

for an odd, continuous function  $b$  on  $\mathbb{R}$  with

$$\text{supp}(b^2 - 1) \subseteq [-1, 1],$$

We now make a specific choice for the function  $b$ :

$$b(x) = \begin{cases} -1 & \text{if } x \in (-\infty, -1]; \\ x & \text{if } x \in (-1, 1); \\ 1 & \text{if } x \in [1, \infty). \end{cases}$$

We will write  $F_0 := b(D_\Phi) \oplus 1$ , for this function  $b$ . Then we have a class

$$[F_0] \in K_1((\mathcal{M}/C^*(X))_{\text{loc}}^G).$$

The index (6.5) equals the image of  $[F_0]$  under the relevant boundary map,

$$\partial[F_0] \in K_0(C^*(X)_{\text{loc}}^G).$$

Here the localised equivariant Roe algebra  $C^*(X)_{\text{loc}}^G$  is realised on the admissible module  $L^2(E) \otimes L^2(G)$ .

The operators  $F_0$  and  $F_1$  define elements of  $K_1(\mathcal{M}/C^*(X)_{\text{loc}}^G)$  and  $K_1(\mathcal{L}/\mathcal{K}(\mathcal{E}_{C_{\text{red}}^*(G)}))$  respectively. By Proposition 6.5, their indices in  $K_0(C_{\text{red}}^*(G))$  can be viewed equivalently through either of these pictures, and they equal the two sides of (4.3). To prove Theorem 4.2, it therefore suffices to prove the following equality.

PROPOSITION 6.6. — *We have*

$$\text{index}_G[F_0] = \text{index}_G[F_1] \in K_0(C_{\text{red}}^*(G))$$

### 6.4. Proof of Theorem 4.2

We now prove Proposition 6.6, and hence Theorem 4.2.

For  $s > 0$ , define the functions  $b_s \in C_b(\mathbb{R})$  and  $\psi_s \in C_0(\mathbb{R})$  by

$$b_s(x) = \frac{x}{(|x|^{1/s} + 1)^s};$$

$$\psi_s(x) = \frac{1}{(|x|^{1/s} + 1)^s},$$

for  $x \in \mathbb{R}$ . Then

$$(6.6) \quad \lim_{s \downarrow 0} \|b_s - b\|_\infty = 0.$$

Let  $\zeta: (0, 1] \rightarrow (0, 1]$  be a continuous function such that  $\zeta(1) = 1$  and

$$(6.7) \quad \lim_{s \downarrow 0} \zeta(s) \|\psi_{s/2}(D_\Phi)\| = 0.$$

For  $s \in (0, 1]$ , consider the operator

$$\tilde{F}_s := b_{s/2}(D_\Phi) + \zeta(s)\psi_{s/2}(D_\Phi)\Phi$$

on  $\mathcal{E}$ . For  $s \in (0, 1]$ , the operator  $D_\Phi + \zeta(s)\Phi$  is of  $G$ -Callias type. Hence there is a continuous,  $G$ -invariant, cocompactly supported function  $f_s$  on  $M$  such that

$$\frac{D_\Phi + \zeta(s)\Phi}{\sqrt{(D_\Phi + \zeta(s)\Phi)^2 + f_s}}$$

is invertible modulo  $\mathcal{K}(\mathcal{E})$ . Since the operator  $\sqrt{(D_\Phi + \zeta(s)\Phi)^2 + f_s}\psi_{s/2}(D_\Phi)$  is invertible, the operator

$$\tilde{F}_s = \left( \frac{D_\Phi + \zeta(s)\Phi}{\sqrt{(D_\Phi + \zeta(s)\Phi)^2 + f_s}} \right) \sqrt{(D_\Phi + \zeta(s)\Phi)^2 + f_s}\psi_{s/2}(D_\Phi)$$

is invertible modulo  $\mathcal{K}(\mathcal{E})$  as well.

We have

$$\tilde{F}_1 = \frac{D_\Phi + \Phi}{\sqrt{(D_\Phi + \Phi)^2 + f_1}} \frac{\sqrt{(D_\Phi + \Phi)^2 + f_1}}{\sqrt{(D_\Phi + \Phi)^2 + 1}}.$$

Hence  $\tilde{F}_1$  has the same index as

$$\frac{D_\Phi + \Phi}{\sqrt{(D_\Phi + \Phi)^2 + f_1}},$$

which equals the index of  $D_\Phi$ .

Finally, (6.6) and (6.7) imply that

$$\lim_{s \downarrow 0} \|\tilde{F}_s \oplus 1 - F_0\| = 0.$$

So  $s \mapsto \tilde{F}_s$  is a continuous path of operators that are invertible modulo  $\mathcal{K}(\mathcal{E})$  connecting  $F_0$  to the operator  $\tilde{F}_1$ , which has the same index in  $K_0(C_{\text{red}}^*(G))$  as  $F_1$ . This implies Proposition 6.6.

## 7. Positive scalar curvature and the localised Baum–Connes conjecture

### 7.1. Positive scalar curvature

*Proof of Proposition 4.4.* — In the setting of the proposition, the operator  $D$  has a well-defined localised index

$$\text{index}_G^Z(D) \in K_*(C^*(M; Z)^G).$$

Then  $\text{index}_G(D) \in K_*(C^*(M)^G)$  is the image of  $\text{index}_G^Z(D)$  under the map

$$K_*(C^*(Z)^G) = K_*(C^*(M; Z)^G) \rightarrow K_*(C^*(M)^G),$$

and hence equal to zero. □

To prove Theorem 4.6, we use the following equivariant version of a theorem of Vilms [39] that was proved in [19].

**THEOREM 7.1.** — *Let  $\pi: M \rightarrow B$  be a fibre bundle with fibre  $N$  and structure group  $K$ . Suppose that  $M$  and  $B$  both have bounded geometry and proper, isometric  $G$ -actions making  $\pi$   $G$ -equivariant. Let  $g_N$  be a  $K$ -invariant Riemannian metric on  $N$ . Then there is a  $G$ -invariant Riemannian metric  $g_M$  on  $M$  such that  $\pi$  is a  $G$ -equivariant Riemannian submersion with totally geodesic fibres.*

*Proof of Theorem 4.6.* — Let  $\kappa_{G/K}$  denote the scalar curvature of the  $G$ -invariant Riemannian metric  $g_{G/K}$  on the base of the fibre bundle  $M \rightarrow G/K$ . Note that since  $G/K$  is a homogeneous space,  $\kappa_{G/K}$  is a finite constant. Let  $H \subseteq TM$  be an Ehresmann connection. Then as in the proof of Theorem 7.1 above, we may lift  $g_{G/K}$  to a  $G$ -invariant metric  $g_H$  on  $H$ , as well as lift the  $K$ -invariant Riemannian metric  $g_N$  on  $N$  to a metric on the vertical subbundle  $V \subseteq TM$ . Define a  $G$ -invariant metric on  $M$  by  $g_M := g_H \oplus g_V$ .

Since  $N$  has uniformly positive scalar curvature  $\kappa_N$ , it satisfies  $\inf\{\kappa_N\} =: \kappa_0 > 0$ . Now let  $T$  and  $A$  denote the O’Neill tensors of the submersion  $\pi$  (their definitions can be found in [32]). By Theorem 7.1 above, the fibres of  $M$  are totally geodesic, so  $T = 0$ . Pick an orthonormal basis of horizontal vector fields  $\{X_i\}$ . A result of Kramer ([30], p. 596), relates the scalar curvatures by

$$\kappa_M(p) = \kappa_{G/K} + \kappa_N(p) - \sum_{i,j} \|A_{X_i}(X_j)\|_p.$$

Since both  $M$  and  $N$  have bounded geometry, it follows that their scalar curvatures  $\kappa_M$  and  $\kappa_N$  are uniformly bounded. Therefore

$$\sup_{p \in M} \sum_{i,j} \|A_{X_i}(X_j)\|_p \leq A_0 < \infty$$

for some positive constant  $A_0$ . Upon scaling the fibre metric on  $N$  by a positive factor  $t$ , we obtain

$$\kappa_M(p) \geq \kappa_{G/K} + t^{-2}\kappa_0 - A_0 > 0 \quad \text{whenever} \quad 0 < t < \sqrt{\frac{\kappa_0}{-\kappa_{G/K} + A_0}},$$

where we choose  $A_0 > 0$  large enough such that  $-\kappa_{G/K} + A_0 > 0$ . Thus  $g_M$  is a  $G$ -invariant metric of uniform positive scalar curvature on  $M$ .  $\square$

### 7.2. The localised Baum–Connes conjecture

Let  $Z \subset X$  be a closed,  $G$ -invariant, cocompact subset. Let  $H_X$  be an admissible equivariant  $C_0(X)$ -module. The map  $\varphi_Z^X$  in (6.3) restricts to a map

$$\varphi_Z^X : D_{\text{red}}^*(Z)^G \rightarrow D_{\text{red}}^*(X)^G$$

that maps  $C^*(Z)^G$  into  $C^*(X; Z)^G$ . Hence we obtain

$$(7.1) \quad \varphi_Z^X : D_{\text{red}}^*(Z)^G / C^*(Z)^G \rightarrow D_{\text{red}}^*(X)^G / C^*(X; Z)^G.$$

By Paschke duality, the analytic  $K$ -homology of  $Z$  equals

$$K_*^G(Z) = K_{*+1}(D_{\text{red}}^*(Z)^G / C^*(Z)^G).$$

So (7.1) induces

$$(7.2) \quad (\varphi_Z^X)_* : K_*^G(Z) \rightarrow K_*^G(X)_{\text{loc}}.$$

Using the maps (7.2), we obtain

$$(7.3) \quad \varphi_*^X : RK_*^G(X) \rightarrow K_*^G(X)_{\text{loc}}.$$

LEMMA 7.2. — *The following diagram commutes, where  $\mu_G^X$  denotes the analytic assembly map for  $X$ :*

$$\begin{array}{ccc} RK_*^G(X) & \xrightarrow{\mu_G^X} & K_*(C_{\text{red}}^*G). \\ \varphi_*^X \downarrow & \nearrow \text{index}_G^{\text{loc}} & \\ K_*^G(X)_{\text{loc}} & & \end{array}$$

*Proof.* — If  $Z \subset X$  is  $G$ -cocompact, then naturality of boundary maps with respect to  $*$ -homomorphisms implies that the diagram

$$\begin{array}{ccc} K_*^G(Z) & \xrightarrow{\text{index}_G} & K_*(C_{\text{red}}^*G). \\ (\varphi_Z^X)_* \downarrow & \nearrow \text{index}_G^{\text{loc}} & \\ K_*^G(X)_{\text{loc}} & & \end{array}$$

commutes. By Corollary 4.3, the top horizontal arrow equals  $\mu_G^Z$ , so the claim follows after we take direct limits.  $\square$

This lemma directly implies Proposition 4.9.

*Remark 7.3.* — The arguments in this subsection have two more consequences.

- (1) If the map (7.3) is injective, then injectivity of the Baum–Connes assembly map implies injectivity of the map in Conjecture 4.8.
- (2) If the map (7.3) is surjective, then Conjecture 4.8 implies surjectivity of the Baum–Connes assembly map.

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