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A GEOMETRIC CRITERION FOR PRESCRIBING RESIDUES AND SOME APPLICATIONS

by Hanlong FANG

ABSTRACT. — Let X be a compact complex manifold and D a $\mathbb{C}$-linear finite formal sum of divisors of X. A theorem of Weil and Kodaira says that if X is Kähler, then there is a closed logarithmic 1-form with residue divisor D if and only if D is homologous to zero in $H_{2n-2}(X, \mathbb{C})$. We generalized their theorem to general compact complex manifolds. The necessary and sufficient condition is described by a new invariant called $\mathcal{Q}$-flat class. In the second part of the paper, we classify all the pluriharmonic functions on a compact algebraic manifold with mild singularities.

RÉSUMÉ. — Soit X une variété complexe compacte et D une somme formelle finie $\mathbb{C}$-linéaire des diviseurs de X. Un théorème de Weil et Kodaira dit que si X est kählerienne, alors il existe une $1-$forme logarithmique fermé avec un diviseur résiduel D si et seulement si D est homologue à zéro dans $H_{2n-2}(X, \mathbb{C})$. Nous généralisons leur théorème aux variétès complexes compactes générales. La condition nécessaire et suffisante est décrite par un nouvel invariant appelé $\mathcal{Q}$-flat class. Dans la deuxième partie de l'article, nous classons toutes les fonctions pluriharmoniques sur une variété algébrique compacte avec des singularités douces.

1. Introduction

In this paper, we study the following two questions for compact complex manifolds.

QUESTION 1.1 (Inverse residue problem). — *Find closed meromorphic* 1*-forms (called abelian differentials in dimension one) with given residues.*

QUESTION 1.2 (Existence of pluriharmonic functions). — *Construct and classify pluriharmonic functions locally taking the form of*

$$g_1(z) + g_2(\overline{z}) + \sum_{i=1}^{l} a_i \log |f_i|^2, \tag{1.1}$$

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where $a_1, \dots, a_l \in \mathbb{C}$ *and* $g_1, g_2, f_1, \dots, f_l$ *are meromorphic functions.*

Integrals of meromorphic 1-forms on Riemann surfaces played an important role in the development of the theory of complex analysis in one variable. There are two well-known holomorphic invariants associated with each meromorphic 1-form called residues and pole orders as follows. Suppose a meromorphic 1-form g has the following Laurent expansion near the point $\xi = 0$ in a local chart (U, ξ).

$$g = h(\xi)\mathrm{d}\xi = \left(\sum_{i=-l}^{\infty} c_i \xi^i\right)\mathrm{d}\xi, \quad c_{-l} \neq 0,\ l \geqslant 1. \tag{1.2}$$

Then l is the pole order and c_{-1} is the residue at $\xi = 0$. This leads to the following classification of meromorphic 1-forms by types (see [19] for details).

- A differential is called an *abelian differential of the first kind* if it is regular on the Riemann surface, that is, if it has no poles.
- A differential is called an *abelian differential of the second kind* if it has at least one pole and if, in addition, its residue at each pole is zero.
- A differential is called an *abelian differential of the third kind* if it has at least one nonzero residue.

Regarding Question 1.1, the following classical theorem give a thorough understanding of the compact Riemann surfaces case.

Theorem 1.3 ([19]). — *Let X be a compact Riemann surface. The following properties hold.*

- *The sum of the residues of an abelian differential over X is always zero.*
- *For each point p on X and $l = 2, 3, \dots$ there exist abelian differentials of the second kind $\mathrm{d}E_l(p)$ and $\mathrm{d}F_l(p)$ with a single pole at p with pole order l. Furthermore, all the periods of the integral $E_l(p)$ are pure imaginary and those of $F_l(p)$ are real.*
- *For any two distinct points p and q on X there exists an abelian differential $\mathrm{d}E(p,q)$ of the third kind which is regular apart from simple poles p and q with residues 1 and -1. Also, all the periods of the integral $E(p,q)$ are pure imaginary.*

Recall that the integral $E_l(p)$ (resp. $F_l(p)$) in Theorem 1.3 is a multi-valued definite integral on $X \backslash p$ defined by

$$E_l(p)(x) = \int_{x_0}^{x} \mathrm{d}E_l(p) \qquad \left(\text{resp. } F_l(p)(x) = \int_{x_0}^{x} \mathrm{d}F_l(p)\right), \tag{1.3}$$

where $x_0 \in X\backslash p$ is a fixed point and the integral is taken along some rectifiable curve C on $X\backslash p$. The multi-valuedness of the integral comes from the choice of the curve and is described by the periods; for any closed curve Γ on $X\backslash p$, the period $E_l(p)(\Gamma)$ (resp. $F_l(p)(\Gamma)$) of the integral $E_l(p)$ (resp. $F_l(p)$) is defined by

$$E_l(p)(\Gamma) = \int_\Gamma \mathrm{d}E_l(p) \qquad \left(\text{resp. } F_l(p)(\Gamma) = \int_\Gamma \mathrm{d}F_l(p)\right). \tag{1.4}$$

Picard and Lefschetz generalized the concept of abelian differentials on algebraic surfaces. Notice that abelian differentials on Riemann surfaces are closed, and hence the line integrals are homotopic invariant and the residues are well-defined; when the complex dimension of a complex manifold is larger than or equal to two, the closeness property does not hold automatically. Therefore, in order to attach residues to a differential, it is natural to make the closeness property as an additional assumption.

In higher dimension, Hodge and Atiyah ([9]) generalized the concept of abelian differentials and the associated periods by sheaf theory (see Section 2 for more details).

To be more precise, we assume that X is a complex manifold, W is a reduced divisor of X and q, k are nonnegative integers. Denote by $\Omega^q(kW)$ the sheaf of germs of meromorphic q-forms having, as their only singularities, poles of order at most k on the components of W. (We view meromorphic functions as 0-forms, when $q = 0$.) Denote by $\Omega^q(*W)$ the direct limit of the sheaves $\Omega^q(kW)$ as $k \to \infty$, which is just the sheaf of germs of meromorphic q-forms with poles of any order on W. Similarly we denote by $\Omega^q(*)$ the direct limit of the sheaves $\Omega^q(*W)$ as W runs through all reduced divisors of X. Define a presheaf by

$$\mathrm{d}\Omega^q(kW)(U) := \{\mathrm{d}f \mid f \in \Omega^q(kW)(U)\} \qquad \text{for each open subset } U \text{ of } X; \tag{1.5}$$

the $\mathbb{C}$-sheaf $\mathrm{d}\Omega^q(kW)$ is its sheafification. Denote by $\mathrm{d}\Omega^q(*W)$ the direct limit of the sheaves $\mathrm{d}\Omega^q(kW)$ as $k \to \infty$; denote by $\mathrm{d}\Omega^q(*)$ the direct limit of the sheaves $\mathrm{d}\Omega^q(*W)$ as W runs through all reduced divisors of X. Denote by $\Phi^q(kW)$, $\Phi^q(*W)$ and $\Phi^q(*)$ the subsheaves of $\Omega^q(kW)$, $\Omega^q(*W)$ and $\Omega^q(*)$, respectively, consisting of germs of closed forms. Moreover, define the sheaves $R^q(W)$ and $R^q(*)$ by the following exact sequences, respectively.

$$\begin{gathered} 0 \longrightarrow \mathrm{d}\Omega^{q-1}(*W) \longrightarrow \Phi^q(*W) \longrightarrow R^q(W) \longrightarrow 0, \\ 0 \longrightarrow \mathrm{d}\Omega^{q-1}(*) \longrightarrow \Phi^q(*) \longrightarrow R^q(*) \longrightarrow 0. \end{gathered} \tag{1.6}$$

Notice that when $q = 1$ one can show that the sheaf cohomology group $H^0(X, R^1(W))$ is the space of $\mathbb{C}$-linear formal sums of divisors of X supported on W (see Lemma 2.4); $H^0(X, \Phi^1(*W))$ is the space of the closed meromorphic 1-forms with poles on W; $H^0(X, \mathrm{d}\Omega^0(*W))$ is the subspace of $H^0(X, \Phi^1(*W))$ consisting of locally exact meromorphic 1-forms.

$H^0(X, \mathrm{d}\Omega^0)$ is the space of d-closed holomorphic forms because, for holomorphic forms, being d-closed is equivalent to being locally d-exact. Consider the following Hodge–Atiyah short exact sequences of $\mathbb{C}$-sheaves.

$$0 \longrightarrow \mathbb{C} \longrightarrow \Omega^0(*W) \longrightarrow \mathrm{d}\Omega^0(*W) \longrightarrow 0, \tag{1.7}$$

$$0 \longrightarrow \mathrm{d}\Omega^0(*W) \longrightarrow \Phi^1(*W) \longrightarrow R^1(W) \longrightarrow 0, \tag{1.8}$$

and the corresponding long exact sequences of the cohomology groups,

$$\begin{aligned} H^0(X, \Omega^0(*W)) \longrightarrow H^0(X, \mathrm{d}\Omega^0(*W)) \longrightarrow H^1(X, \mathbb{C}) \\ \longrightarrow H^1(X, \Omega^0(*W)) \longrightarrow H^1(X, \mathrm{d}\Omega^0(*W)) \\ \longrightarrow H^2(X, \mathbb{C}) \longrightarrow H^2(X, \Omega^0(*W)) \longrightarrow \dots; \end{aligned} \tag{1.9}$$

$$\begin{aligned} H^0(X, \mathrm{d}\Omega^0(*W)) \longrightarrow H^0(X, \Phi^1(*W)) \xrightarrow{\mathrm{Res}} H^0(X, R^1(W)) \\ \xrightarrow{\Delta^0} H^1(X, \mathrm{d}\Omega^0(*W)) \longrightarrow \dots. \end{aligned} \tag{1.10}$$

Based on (1.10), the following higher dimensional analogue of the periods of a closed meromorphic 1-forms are defined.

Definition 1.4 ([9]). — *Let $\Phi \in H^0(X, \Phi^1(*W))$ be a closed meromorphic* 1*-form. We call the image* $\mathrm{Res}(\Phi)$ *of* Φ *under the homomorphism* Res *the residue divisor of* Φ.

As a consequence of the long exact sequence (1.10), one derives immediately the following abstract criterion for Question 1.1.

Theorem 1.5 ([9]). — *Suppose X is a compact complex manifold and W is a reduced divisor of X. Let D be an element of $H^0(X, R^1(W))$. Then there is a closed meromorphic* 1*-form $\Phi \in H^0(X, \Phi^1(*W))$ with residue divisor D if and only if $\Delta^0(D) \in H^1(X, \mathrm{d}\Omega^0(*W))$ is trivial in the long exact sequence* (1.10).

Recall that for a complex manifold X of complex dimension n and a reduced divisor W of X, we can define the sheaf of germs of logarithmic 1-forms as follows (see [16]). For each $x \in X$ take irreducible germs of holomorphic functions $f_j \in \mathcal{O}_{X,x}$, $1 \leqslant j \leqslant k$, so that $\{f_1 = 0\}, \dots, \{f_k = 0\}$

define the local irreducible components of W at x. Then we define the sheaf $\Omega^1(\log W)$ of germs of logarithmic 1-forms along W by

$$\Omega^1_{X,x}(\log W) = \sum_{j=1}^{k} \mathcal{O}_{X,x}\frac{\mathrm{d}f_j}{f_j} + \Omega^1_{X,x}. \tag{1.11}$$

It is easy to verify that if W is smooth at x the germ $\Omega^1_{X,x}(\log W)$ is an $\mathcal{O}_{X,x}$-module generated by $\frac{\mathrm{d}x_1}{x_1}$ and $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$ where $(x_1, \ldots, x_n)$ are certain local coordinates of X near x such that W is defined locally by $x_1 = 0$. Hence, the sheaves $\Omega^1(\log W)$, $\Omega^1(W)$ are different when $n \geqslant 2$. Moreover, one can show that the sheaf $\Phi^1(W)$ is a subsheaf of $\Omega^1(\log W)$; $H^0(X, \Phi^1(W))$ is the vector space of closed logarithmic 1-forms with poles on W; $H^0(X, \Omega^1(\log W))$ is the vector space of logarithmic 1-forms on X with poles on W.

Weil and Kodaira then derived the following geometric criterion for the existence of a closed logarithmic 1-forms with a prescribed residue divisor by further assuming X is a compact Kähler manifold.

Theorem 1.6 ([10] and [22]). — *Let X be a compact Kähler manifold of complex dimension n and W a reduced divisor of X. Let D be an element of $H^0(X, R^1(W))$. Then there is a closed logarithmic 1-form with residue divisor D if and only if D is homologous to zero in $H_{2n-2}(X, \mathbb{C})$.*

In the first part of this paper, we investigate Question 1.1 and generalize the above geometric criterion for general compact complex manifolds. First recall the following definition of the $\mathcal{Q}$-flat class of a holomorphic line bundle via Čech cohomology. (See Section 3.3 for the detailed definition of Čech cohomology groups.)

Definition 1.7 ([6]). — *Let X be a compact complex manifold and E a holomorphic line bundle on X. Take an open cover $\mathcal{U} := \{U_i\}_{i=1}^{M}$ of X and let $g := \{g_{ij}\}$ be a system of transition functions associated with a certain family of local trivializations of E with respect to $\mathcal{U}$. For indices $1 \leqslant i_1 < i_2 \leqslant M$, define an element $t_{i_1 i_2} \in \Gamma(U_{i_1 i_2}, \mathrm{d}\Omega^0)$ by*

$$t_{i_1 i_2} := \frac{g_{i_1 i_2}^{-1} \cdot \mathrm{d}g_{i_1 i_2}}{2\pi\sqrt{-1}}. \tag{1.12}$$

We can define a Čech 1-cocycle $\widehat{f}_{\mathcal{U}}(E) \in \check{H}^1(\mathcal{U}, \mathrm{d}\Omega^0)$ by

$$\widehat{f}_{\mathcal{U}}(E) := \bigoplus_{i_1 < i_2} t_{i_1 i_2} \in \bigoplus_{i_1 < i_2} \Gamma(U_{i_1 i_2}, \mathrm{d}\Omega^0). \tag{1.13}$$

Recall the following canonical homomorphism (see Definition 3.6 for details).

$$P\colon \check{H}^1(\mathcal{U}, \mathrm{d}\Omega^0) \longrightarrow \check{H}^1(X, \mathrm{d}\Omega^0) = \varinjlim_{\mathcal{U}} \check{H}^1(\mathcal{U}, \mathrm{d}\Omega^0) \longrightarrow H^1(X, \mathrm{d}\Omega^0). \tag{1.14}$$

Denote by $F(E)$ the image of $\widehat{f}_{\mathcal{U}}(E)$ in $H^1(X, \mathrm{d}\Omega^0)$ under the homomorphism P; call $F(E)$ the $\mathcal{Q}$-flat class of E.This definition is independent of the choice of the open cover $\mathcal{U}$, the trivialization of E and the system of transition functions g.

Remark 1.8. — Similarly, we can define the $\mathcal{Q}$-flat classes of $\mathbb{C}$-linear finite formal sums of divisors. Suppose $D = \sum_{i=1}^{l} a_i \cdot W_i$ where W_i is a divisor of X and $a_i \in \mathbb{C}$ for $i = 1, \dots, l$. Define the $\mathcal{Q}$-flat class of D by $F(D) := \sum_{i=1}^{l} a_i \cdot F(W_i) \in H^1(X, \mathrm{d}\Omega^0)$.

We now state the geometric criterion for general compact complex manifolds in terms of the $\mathcal{Q}$-flat class.

Theorem 1.9. — *Let X be a compact complex manifold and W a reduced divisor of X. Let $D \in H^0(X, R^1(W))$. The following statements are equivalent.*

- *The $\mathcal{Q}$-flat class of D is trivial in $H^1(X, \mathrm{d}\Omega^0)$;*
- *there is a closed logarithmic 1-form with residue divisor D.*

We call a divisor D flat if under a certain trivialization the transition functions of $[D]$, the line bundle associated with D, can be taken as constant functions. One can prove that the $\mathcal{Q}$-flat class of a divisor D is trivial if only if D is flat up to some positive multiple (see [6, Thm. 1.11]). On the other hand, a well-known fact says that (see [17, Prop. 3.1]), for every flat, reduced divisor D on a complex manifold X, there exists a closed logarithmic 1-form with simple poles along D and holomorphic on $X \backslash D$. Therefore, Theorem 1.9 actually shows that the well-known fact about the flat, reduced divisors is general enough to cover all cases up to some multiple.

As an application of Theorem 1.9, we have the following corollary.

Corollary 1.10. — *Let X be a compact complex manifold. Assume that W (resp. W') is a reduced divisor of X and $D \in H^0(X, R^1(W))$ (resp. $D' \in H^0(X, R^1(W'))$). Moreover, suppose that D and D' are $\mathbb{C}$-linearly equivalent; that is, there exist a nonnegative integer m, complex numbers $a_!, \dots, a_m$ and meromorphic functions $f_1, \dots, f_m$ on X such that*

$$D - D' + \sum_{i=1}^{m} a_i \cdot (f_i) = 0 \quad \textit{as a formal sum of divisors,} \tag{1.15}$$

where (f_i) is the principle divisor associated with f_i, $1 \leqslant i \leqslant m$. The following statements are equivalent.

- *There is a closed logarithmic 1-form with residue divisor D and with poles on W;*
- *there is a closed logarithmic 1-form with residue divisor D' and with poles on W'.*

We also refine Hodge and Atiyah's criterion for closed meromorphic 1-forms with poles of arbitrary order (Theorem 5.1). In particular, we derive the following topological constraint on the residue divisors.

THEOREM 1.11. — *Let X be a compact complex manifold. Then the residue divisor of a closed meromorphic 1-form on X is homologous to zero in $H_{2n-2}(X, \mathbb{C})$.*

Next, we compare different criteria. We say that a compact complex manifold X has Property (H) (see Definition 3.11) if and only if

$$\dim H^1(X, \mathbb{C}) = \dim H^0(X, \mathrm{d}\Omega^0) + \dim H^1(X, \mathcal{O}_X). \tag{1.16}$$

A consequence of the complex manifold X having Property (H) is that the $\mathcal{Q}$-flat class of D, a $\mathbb{C}$-linear finite formal sums of divisors of X, is trivial if and only if D is homologous to zero (see Lemma 3.17). Therefore, we reduce the holomorphic criterion in Theorem 1.9 to the following topological criterion as in Theorem 1.6.

THEOREM 1.12. — *Let X be a compact complex manifold of dimension n. Assume X has Property (H). Let W be a reduced divisor of X and $D \in H^0(X, R^1(W))$. The following statements are equivalent.*

- *D is homologous to zero in $H_{2n-2}(X, \mathbb{C})$;*
- *there is a closed logarithmic 1-form with residue divisor D.*

Theorem 1.12 gives an alternate proof of Theorem 1.6, for compact Kähler manifolds having Property (H). Moreover, when X is of complex dimension two, we conclude the following corollary.

COROLLARY 1.13. — *Let X be a compact complex surface, W a reduced divisor of X and $D \in H^0(X, R^1(W))$. The following statements are equivalent.*

- *D is homologous to zero in $H_{2n-2}(X, \mathbb{C})$;*
- *there is a closed logarithmic 1-form with residue divisor D and with poles on W.*

A well-known result of Deligne [3] says that each logarithmic form on projective manifolds is closed. This result later was generalized to Kähler manifolds and complex manifolds of Fujiki class $\mathcal{C}$ by Noguchi [16] and Winkelmann [23], respectively. We further generalize it in the case of 1-forms as follows.

Theorem 1.14. — *Let X be a compact complex manifold with Property* (H)*. Then each logarithmic* 1*-form on X is a sum of a holomorphic* 1*-form and a closed logarithmic* 1*-form. To be more precise, for each reduced divisor W of X we have*

$$H^0(X, \Omega^1(\log W)) = H^0(X, \Omega^1) + H^0(X, \Phi^1(W)). \tag{1.17}$$

A consequence of it is that, whenever the Frölicher spectral sequence of X degenerates at E_1, every logarithmic 1-form on X is closed.

Similarly, we have the corresponding decomposition for closed meromorphic 1-forms with arbitrary pole order as follows.

Theorem 1.15. — *Let X be a compact complex manifold with Property* (H)*. Then every closed meromorphic* 1*-form is a sum of a closed logarithmic* 1*-form and a locally exact meromorphic* 1*-form.*

Notice that this result is optimal, for there are non-closed holomorphic differential 1-forms on an Iwasawa manifold which is not Kähler but has Property (H) (see Example 4.6).

In the second part of the paper, we turn to the study of pluriharmonic functions on projective manifolds (Question 1.2). Recall that a pluriharmonic function f (possibly singular) is a solution of the following overdetermined system of partial differential equations:

$$\partial\bar{\partial} f = 0. \tag{1.18}$$

The only regular solutions of equation (1.18) on a compact complex manifold are constant functions; meromorphic functions and anti-meromorphic functions are its singular solutions. We first derive the following theorem.

Theorem 1.16. — *Let X be a compact algebraic manifold. Assume that W is a reduced divisor of X and there is an effective, ample divisor of X whose support is contained in W. Then for every closed meromorphic* 1*-form with poles on W, there exists a closed anti-meromorphic* 1*-form on X with poles on W, so that the integral of the sum of these two differentials is a single-valued function on $X\backslash W$. In particular, the integral is a pluriharmonic function with singularities on W.*

Denote by $K(X)$ the vector space of meromorphic functions on X; denote by $\overline{K}(X)$ the vector space of anti-meromorphic functions on X; denote by $\mathrm{d}K(X)$ the vector space of the differentials of meromorphic functions on X. Denote by $Ph(X)$ the vector space of the pluriharmonic functions on X of local form (1.1); denote by $Ph_0(X)$ the vector space of the pluriharmonic functions on X of local form (1.1) without log terms. Recall that $H^0(X, \Phi^1(*))$ is the vector space of closed meromorphic 1-forms on X; $H^0(X, \mathrm{d}\Omega^0(*))$ is the vector space of all locally exact meromorphic 1-forms.

Then there are natural linear maps k, k_0 between vector spaces induced by differentiation as follows.

$$\begin{aligned} \kappa : Ph(X)/(K(X) + \overline{K}(X)) &\longrightarrow H^0(X, \Phi^1(*))/\mathrm{d}K(X) \\ h &\longmapsto \partial h, \end{aligned} \tag{1.19}$$

and

$$\begin{aligned} \kappa_0 : Ph_0(X)/(K(X) + \overline{K}(X)) &\longrightarrow H^0(X, \mathrm{d}\Omega^0(*))/\mathrm{d}K(X) \\ h &\longmapsto \partial h, \end{aligned} \tag{1.20}$$

where ∂h is the canonical projection of the exterior differentiation dh of h onto the holomorphic cotangent space at each point.

As an application of Theorem 1.16, we have the following theorem classifying all the singular solutions of equation (1.18) with local form (1.1).

Theorem 1.17. — *Let X be a compact algebraic manifold. The natural homomorphisms* (1.19) *and* (1.20) *induced by differentiation are isomorphisms.*

We now briefly describe the organization of the paper and the basic ideas for the proof of theorems. A natural approach to prove Theorems 1.9 and 1.12 is to interpret sheaf cohomology as Čech cohomology. In order to establish the isomorphism between these two cohomologies of X, we shall prove that certain cohomology groups are trivial. However, the Hodge–Atiyah exact sequences are not sequences of coherent $\mathcal{O}_X$-sheaves, and hence Cartan theorem B does not apply. This difficulty is settled by three lemmas: truncation lemma, good cover lemma and acyclic lemma. Next, we construct explicitly the map from the residue divisor group to the obstruction group by diagram chasing and prove Theorems 1.9 and 1.12. Then, we show that, for manifolds with Property (H), the criterion for the existence of logarithmic 1-forms with given residue divisors coincides with the criterion for the existence of closed meromorphic 1-forms with given residue divisors; therefore, each logarithmic 1-form can be decomposed into two parts as in Theorem 1.14. Similar to the logarithmic case, we derive the criteria for prescribing residues for closed meromorphic 1-forms with arbitrary pole orders and as a consequence of which we prove Theorem 1.11.

In the second part of the paper, we investigate Question 1.2. Firstly, we describe the two kinds of obstructions for getting a single-valued function by integrating closed meromorphic 1-forms, namely, the long period vectors corresponding to the integrals along loops in $H_1(X, \mathbb{C})$, and the short period vectors corresponding to the integrals along small loops around irreducible components of the residue divisor. Since such an integral is single-valued if and only if all periods vanish and it is impossible to carry out a cancellation of periods merely in the holomorphic category except the trivial case, we produce closed anti-meromorphic 1-forms with opposite periods, and then sum up the pairs to get a single-valued function. At last, we will prove Theorems 1.16 and 1.17 by a careful cancellation of the periods.

The organization of the paper is as follows. In Section 2, we introduce Hodge and Atiyah's sheaf theoretical method. In Section 3.1, we establish a special truncation of Hodge–Atiyah exact sequence. In Section 3.2, we prove the acyclic lemma. In Section 3.3, diagram chasing method is used to show the geometric meaning of the map from the residue divisor group to the obstruction group. In Section 3.4, we derive some basic properties of manifolds with the Property (H). In Section 4.1, we prove Theorems 1.9 and 1.12. In Section 4.2, we prove Theorem 1.14. In Section 5, we prove Theorem 1.11. In Section 6, we prove Theorems 1.16 and 1.17.

For reader's convenience, we include in Appendix A a detailed proof for the existence of a (very) good cover for a compact complex manifold which is crucial in the comparison of two cohomologies. Also, we include in Appendix B a proof for the existence of a smooth, transversal two-chain which is used for calculating periods.

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2. Preliminaries

Let X be a complex manifold and Ω^1_X (or Ω^1 for short) the cotangent bundle of X. For each meromorphic 1-form f on X, we define an ideal sheaf on X characterizing the singularities of f as follows.

Definition 2.1. — *Define a presheaf $\mathcal{P}^{pre}_f$ as follows. For each open set (in the Euclidean topology) $U \subset X$,*

$$\Gamma(\mathcal{P}^{pre}_f, U) := \left\{ h \in \mathcal{O}_X(U) \,\middle|\, hf|_U \in H^0(U, \Omega^1) \right\}. \tag{2.1}$$

Its sheafification is called the denominator ideal sheaf associated with f and denoted by $\mathcal{P}_f$.

Lemma 2.2. — *The denominator ideal sheaf associated with f is locally free and of rank 1; that is, $\mathcal{P}_f$ defines a divisor.*

Proof. — We choose a Stein open set U of X with complex coordinates $(z_1, \ldots, z_n)$ such that

$$f = \frac{f_1}{g_1}\mathrm{d}z_1 + \frac{f_2}{g_2}\mathrm{d}z_2 + \frac{f_3}{g_3}\mathrm{d}z_3 + \cdots + \frac{f_n}{g_n}\mathrm{d}z_n, \tag{2.2}$$

where f_i, g_i are holomorphic functions over U. Without loss of generality, we assume f_i and g_i are coprime for $i = 1, \ldots, n$. Let h be a least common multiple of $g_1, g_2, \ldots, g_n$. It is easy to verify that $\mathcal{P}_f(U) = (h)$. This completes the proof of the lemma. □

Following [9], we introduce some notations and recall some important results therein. In what follows, we assume that W is a reduced divisor on X; we also assume that all the open sets are in the Euclidean topology.

First recall that, similar to the case of linear spaces $H^0(X, \Phi^1(*W))$ and $H^0(X, \mathrm{d}\Omega^0(*W))$, $H^0(X, \Phi^1(*))$ is the space of the closed meromorphic 1-forms and $H^0(X, \mathrm{d}\Omega^0(*))$ is the subspace of $H^0(X, \Phi^1(*))$ consisting of locally exact meromorphic 1-forms.

Also, we say a number of meromorphic 1-forms are independent if no linear combination of them is equal to the differential of a meromorphic function on X.

Define $\mathbb{C}$-sheaves $\mathcal{D}(W)$ and $\mathcal{D}(*)$. Let U be an open set of X. Denote by $\{W^U_h\}$ the irreducible components of W in U and by $\mathbb{C}_{W^U_h}$ the constant sheaf on W^U_h. Since $\mathbb{C}_{W^U_h}$ can be viewed as a $\mathbb{C}$-sheaf on U, we define a presheaf $\mathcal{D}^{pre}(W)$ by

$$\Gamma(\mathcal{D}^{pre}(W), U) := \sum_h \mathbb{C}_{W^U_h} = \left\{ \sum f_h \,\middle|\, f_h \in \Gamma(U, \mathbb{C}_{W^U_h}) \right\}. \tag{2.3}$$

Let $\mathcal{D}(W)$ be the sheafification of $\mathcal{D}^{pre}(W)$. We denote by $\mathcal{D}(*)$ the direct limit of the sheaf $\mathcal{D}(W)$, as W runs through all reduced divisors of X.

Remark 2.3. — In the remainder of this paper, we will use Ω^0 and $\mathcal{O}_X$ interchangeably.

Recall the following lemma due to Hodge and Atiyah.

Lemma 2.4 ([9, Lem. 8]). — *Let the sheaves $R^1(W)$ and $R^1(*)$ be defined as in* (1.6). *The following properties hold.*

- $R^1(W) \cong \mathcal{D}(W)$;
- $R^1(*) \cong \mathcal{D}(*)$.

Remark 2.5 ([9]*).* — There is an explicit isomorphism between $R^1(W)$ and $\mathcal{D}(W)$ as follows. Let x be any point of X, and suppose that $W_1, \dots, W_l$ are the local irreducible components of W which pass through x, $f_h = 0$ being a local equation of W_h. Then $\frac{1}{2\pi\sqrt{-1}} \frac{\mathrm{d}f_h}{f_h}$ defines an element r_h of $R^1(W)_x$. The isomorphism is given by

$$\alpha : \mathcal{D}(W) \longrightarrow R^1(W), \quad [1_{W_h}]_x \longmapsto r_h.$$

Remark 2.6. — Let $W = \bigcup_{i=1}^{l} W_i$ be the irreducible decomposition of W. Then

$$H^0(X, \mathcal{D}(W)) \cong \bigoplus_{i=1}^{l} \mathbb{C} \cdot 1_{W_i},$$

where 1_{W_i} is the function taking value 1 on W_i and 0 on $X \backslash W_i$. We also identify the above direct sum with $\bigoplus_{i=1}^{l} \mathbb{C} W_i$, the vector space consisting of $\mathbb{C}$-linear formal sums of divisors $W_1, \dots, W_l$.

Recall the following well-known Serre vanishing theorem.

Theorem 2.7 (See [9, Lem. 5, 6 and 7]). — *If W is ample, and if k is a sufficiently large integer, then $H^p(V, \Omega^q(kW)) = 0$ for $p \geqslant 1$.*

It is also proved in [9] that

Theorem 2.8 ([9, Thm. 1 in §3]). — *The number of independent locally exact meromorphic* 1*-forms is equal to the first Betti number of X.*

In a parallel manner, the following result holds.

Proposition 2.9 ([9, Prop. 1 in §3]). — *If W is ample, then*

$$H^0(X, \mathrm{d}\Omega^0(*W)) / \operatorname{Im} H^0(X, \Omega^0(*W)) \cong H^1(X, \mathbb{C}). \tag{2.4}$$

Hence a basis for locally exact meromorphic 1*-forms (modulo differentials of meromorphic functions) can be chosen from forms with singularities on any ample divisor W.*

In fact, we have the following effective version.

Proposition 2.10. — *If $H^1(X, \mathcal{O}_X(kW)) = 0$ for a certain positive integer k, then*

$$H^0(X, \mathrm{d}\Omega^0(kW))/\operatorname{Im} H^0(X, \Omega^0(kW)) \cong H^1(X, \mathbb{C}). \tag{2.5}$$

Moreover, a basis for locally exact meromorphic 1-forms (modulo differentials of meromorphic functions) can be chosen from forms with singularities on W and pole order (at most) $k+1$ along W.

Proof. — By definition, we have that

$$0 \longrightarrow \mathbb{C} \longrightarrow \Omega^0(kW) \longrightarrow \mathrm{d}\Omega^0(kW) \longrightarrow 0. \tag{2.6}$$

The corresponding long exact sequence is:

$$\begin{aligned} H^0(X, \Omega^0(kW)) &\longrightarrow H^0(X, \mathrm{d}\Omega^0(kW)) \\ &\longrightarrow H^1(X, \mathbb{C}) \longrightarrow H^1(X, \Omega^0(kW)). \end{aligned} \tag{2.7}$$

Since each element of $H^0(X, \mathrm{d}\Omega^0(kW))$ has pole orders no more than $k+1$ along W, we conclude Proposition 2.10. $\square$

We end this section with the following definitions.

Remark 2.11. — Let $\Phi \in H^0(X, \Phi^1(*W))$ be a closed meromorphic 1-form and $W = \bigcup_{i=1}^{l} W_i$ the irreducible decomposition of W. Thanks to Lemma 2.4 and Remarks 2.5 and 2.6, $\operatorname{Res}(\Phi)$ is a $\mathbb{C}$-linear formal sum of $W_1, \dots, W_l$ as follows.

$$\operatorname{Res}(\Phi) = \sum_{i=1}^{l} a_i W_i, \ a_i \in \mathbb{C} \text{ for } i = 1, \dots, l.$$

Remark 2.12. — When W is a normal crossing divisor, we can calculate the residue divisor $\operatorname{Res}(\Phi)$ by taking terms with the form $\frac{\mathrm{d}z_i}{z_i}$ in the Laurent series expansion of Φ. In general, we can calculate $\operatorname{Res}(\Phi)$ by taking the contour integrals along small loops around the components of W (see Definition 6.5).

3. A geometric interpretation of δ^1 and $\delta^1 \circ \Delta^0$

Since the original Hodge–Atiyah sequences involve infinite-dimensional cohomology groups, it is not effective to control the pole orders and is also abstract for the purpose of a geometric understanding. In this section, we will introduce a special truncation of short exact sequences (1.7) and (1.8). Then by using Čech cohomology theory, we are able to derive a geometric interpretation of the homomorphisms δ^1 and $\delta^1 \circ \Delta^0$.

3.1. Truncation lemma

Lemma 3.1 (Truncation lemma). — *Let X be a smooth complex manifold of complex dimension m and W a reduced divisor of X. There exist short exact sequences of $\mathbb{C}$-sheaves on X as follows.*

$$0 \longrightarrow \mathbb{C} \longrightarrow \Omega^0 \longrightarrow \mathrm{d}\Omega^0 \longrightarrow 0\,; \tag{3.1}$$

$$0 \longrightarrow \mathrm{d}\Omega^0 \xrightarrow{\phi} \Phi^1(W) \xrightarrow{\psi} R^1(W) \longrightarrow 0. \tag{3.2}$$

Proof. — It is easy to see that the sequence (3.1) is exact. To prove sequence (3.2) is exact, it suffices to prove Lemma 3.1 locally at each point $x \in X$. Since $R^1(W)|_x = 0$ and ϕ is an isomorphism (by Poincaré lemma) when $x \in X \backslash W$, Lemma (3.2) holds trivially for $x \notin W$.

Now let x be a point of W. Take a neighborhood U_x of x in X such that W is defined by the equation $f_1 f_2 \dots f_l = 0$, $l \geqslant 1$, where $f_1, \dots, f_l \in \mathcal{O}(U_x)$ are irreducible holomorphic functions vanishing at x and coprime to each other. We denote by W_i the zero locus of f_i in U_x for $i = 1, \dots, l$.

Since $(\mathrm{d}\Omega^0)_x \subset (\Omega^1)_x$, homomorphism ϕ is the inclusion of the sheaf of closed holomorphic 1-forms into the sheaf of closed meromorphic 1-forms.

Thanks to the fact that there is a natural homomorphism from $\Phi^1(W)_x$ to $\Phi^1(*W)_x$ and a homomorphism from $\Phi^1(*W)_x$ to $R^1(W)_x$ (see (1.6)), homomorphism ψ is well-defined. Moreover, ψ is surjective, for $R^1(W)_x$ is generated by $\frac{\mathrm{d}f_1}{f_1}, \frac{\mathrm{d}f_2}{f_2}, \dots, \frac{\mathrm{d}f_l}{f_l} \in \Phi^1(W)_x$ by Lemma 2.4 and Remark 2.5.

In order to show (3.2) is exact at the place $\Phi^1(W)_x$, it suffices to prove that if $r \in \Phi^1(W)_x$ and $\psi(r) = 0$, then r is the germ of a holomorphic 1-form. Notice that if a meromorphic 1-form has its poles on a subvariety of codimension at least two, then the 1-form is actually holomorphic (see Lemma 2.2). Therefore, it suffices to prove the exactness at smooth points of W.

Let x be a smooth point of W. Take $r \in \Phi^1(W)_x$ with $\psi(r) = 0$. In the following, we denote by U_t for $t > 0$ the polydisc $\big\{(z_1, \dots, z_m) \,\big|\, |z_i| < t \text{ for } i = 1, \dots, m\big\}$. By a holomorphic change of coordinates, we can assume that U_x is biholomorphic to U_1; W is defined by $z_1 = 0$ in U_1; r takes the form in U_1 as

$$r = \sum_{p=1}^{m} r_p \mathrm{d}z_p = \sum_{p=1}^{m} \left(\sum_{i_1=-1}^{\infty} z_1^{i_1} \cdot g_{pi_1}(z_2, \dots, z_m) \right) \mathrm{d}z_p, \tag{3.3}$$

where $g_{pi_1}(z_2, \dots, z_m)$ is holomorphic in variables $z_2, \dots, z_m$ for $1 \leqslant p \leqslant m$;

$$r_p \cdot z_1 = \sum_{i_1=-1}^{\infty} z_1^{1+i_1} g_{pi_1}(z_2, \dots, z_m) \tag{3.4}$$

is an absolutely convergent series in U_t for $0 < t < 1$ and $1 \leqslant p \leqslant m$.

Claim I. — *$g_{1(-1)}(z_2, \ldots, z_m) \equiv 0$ in $\widehat{U}_1$ where $\widehat{U}_1 = \{(z_2, \ldots, z_m) \,|\, |z_i| < 1,\, i = 2, \ldots, m\}$.*

Proof of the Claim I. — Since $\psi(r) = 0$ in U_1, r locally is a differential of a meromorphic function. Then, each line integral of r along a closed loop in $U_1 \backslash \{z_1 = 0\}$ is zero. For fixed $(z_2, \ldots, z_m) \in \widehat{U}_1$, define a loop $\gamma_{z_2 \ldots z_m}$ by $\left(\frac{e^{2\pi\sqrt{-1}t}}{2}, z_2, \ldots, z_m\right)$ for $t \in [0,1]$. Computing the line integral $\int_{\gamma_{z_2 \ldots z_m}} r$, we have

$$\int_{\gamma_{z_2\ldots z_m}} r = \int_{\gamma_{z_2\ldots z_m}} \left(\sum_{p=1}^{m} \sum_{i_1=-1}^{\infty} z_1^{i_1} \cdot g_{pi_1}(z_2, \ldots, z_m) \mathrm{d}z_p \right) = 2\pi\sqrt{-1} \cdot g_{1(-1)}(z_2, \ldots, z_m). \tag{3.5}$$

Therefore $g_{1(-1)}(z_2, \ldots, z_m) \equiv 0$ in $\widehat{U}_1$. □

Claim II. — *$g_{p(-1)}(z_2, \ldots, z_m) \equiv 0$ in $\widehat{U}_1$ for $p = 2, \ldots, m$.*

Proof of the Claim II. — By Claim I, we can rewrite formula (3.3) as

$$r = \sum_{p=1}^{m} \left(\sum_{i_1=0}^{\infty} z_1^{i_1} g_{pi_1}(z_2, \ldots, z_m) \right) \mathrm{d}z_p + \sum_{p=2}^{m} z_1^{-1} g_{p(-1)}(z_2, \ldots, z_m)\, \mathrm{d}z_p. \tag{3.6}$$

Taking the differential of r, we have

$$\begin{aligned} 0 &= \sum_{p=1}^{m} \mathrm{d}\left(\sum_{i_1=0}^{\infty} z_1^{i_1} g_{pi_1}(z_2, \ldots, z_m) \right) \wedge \mathrm{d}z_p \\ &\quad + \sum_{p=2}^{m} \mathrm{d}\left(\frac{g_{p(-1)}(z_2, \ldots, z_m)}{z_1} \right) \wedge \mathrm{d}z_p \\ &= -\sum_{p=2}^{m} \frac{g_{p(-1)}(z_2, \ldots, z_m)}{z_1^2} \mathrm{d}z_1 \wedge \mathrm{d}z_p + \ldots. \end{aligned} \tag{3.7}$$

Notice that the coefficient of $\mathrm{d}z_1 \wedge \mathrm{d}z_p$ in $\mathrm{d}r$ is

$$-\frac{g_{p(-1)}(z_2, \ldots, z_m)}{z_1^2} + h_p(z_1, \ldots, z_m), \tag{3.8}$$

where h_p is a holomorphic function in U_1 and $2 \leqslant p \leqslant m$. Then $g_{p(-1)} \equiv 0$ in $\widehat{U}_1$ for $2 \leqslant p \leqslant m$. □

As a conclusion, we proved that r is a holomorphic 1-form in U_1, and hence complex (3.2) is exact at the place $\Phi^1(W)_x$. This completes the proof of Lemma 3.1. $\square$

By a similar argument, we can prove the following truncation with high order poles.

Lemma 3.2 (Truncation lemma). — *Let X be a smooth complex manifold of complex dimension m and W a reduced divisor of X. There exist short exact sequences of $\mathbb{C}$-sheaves on X for $k \geqslant 1$ as follows.*

$$0 \longrightarrow \mathbb{C} \longrightarrow \Omega^0(kW) \longrightarrow \mathrm{d}\Omega^0(kW) \longrightarrow 0\,; \tag{3.9}$$

$$0 \longrightarrow \mathrm{d}\Omega^0(kW) \xrightarrow{\phi} \Phi^1((k+1)W) \xrightarrow{\psi} R^1(W) \longrightarrow 0. \tag{3.10}$$

Proof of Lemma 3.2. — We first prove the following Claim.

Claim. — *Let U be a open set of X. Let $f \in H^0(U, \mathcal{O}_X((k+1)W))$ such that f has pole order at most k in U outside an analytic subvariety of codimension at least two. Then $f \in H^0(U, \mathcal{O}_X(kW))$.*

Proof of Claim. — Thanks to the fact that the problem is local, we can assume that U is a complex ball and W is defined by a reduced holomorphic function $h \in H^0(U, \mathcal{O}_X)$. Then $f \cdot h^k$ is holomorphic in U outside an analytic subvariety of codimension at least two; hence, $f \cdot h^k$ is holomorphic in U. Since the pole order of $\frac{1}{h}$ on W is 1, this completes the proof. $\square$

By the above Claim, we can reduce the problem to a smooth point of W. The remainder of the proof is similar to the proof of Lemma 3.2, and hence we omit it here. $\square$

3.2. Acyclic lemma

We will prove the following acyclic lemma.

Lemma 3.3 (Acyclic lemma). — *Suppose that X is a compact complex manifold. Let $\mathcal{U} := \{U_i\}_{i=1}^M$ be a finite, good cover of X ensured by Lemma A.1. Denote by $U_{i_1 \dots i_p}$ the intersection $\bigcap_{j=1}^p U_{i_j}$ for $p \geqslant 1$ and $1 \leqslant i_1 < \cdots < i_p \leqslant M$. The following vanishing results hold for all integers $p, q \geqslant 1$.*

$$H^q(U_{i_1 \dots i_p}, \mathbb{C}) = 0, \quad H^q(U_{i_1 \dots i_p}, \Omega^0) = 0, \quad H^q(U_{i_1 \dots i_p}, \mathrm{d}\Omega^0) = 0. \tag{3.11}$$

Proof. — Without loss of generality, we can assume $U_{i_1 \dots i_p}$ is nonempty; otherwise, the above formulas hold trivially.

A result of [18] shows that if a space X is locally contractible and paracompact, then singular cohomology $H^*_{sing}(X;A)$ is isomorphic to sheaf cohomology $H^*(X;\underline{A})$, where A is any abelian groups and $\underline{A}$ is the constant sheaf on X with value A. Recall that X is locally contractible if any open subset $U \subset X$ may be covered by contractible open sets; X is paracompact if it is Hausdorff and any cover U of X has a locally finite refinement. Therefore, since $U_{i_1\dots i_p}$ is contractible and paracompact, the sheaf cohomology group $H^q(U_{i_1\dots i_p},\mathbb{C})$ equals the singular cohomology group $H^q_{sing}(U_{i_1\dots i_p},\mathbb{C})$ for $q \geqslant 0$. Then for all $p,q \geqslant 1$, $H^q(U_{i_1\dots i_p},\mathbb{C}) = H^q_{sing}(U_{i_1\dots i_p},\mathbb{C}) = 0$.

Since $\mathcal{U}$ is a good cover, $U_{i_1},\dots,U_{i_p}$ and hence $U_{i_1\dots i_p}$ is Stein. Combined with the fact that Ω^0 is coherent, we conclude that $H^q(U_{i_1\dots i_p},\Omega^0) = 0$ by Cartan theorem B for $p,q \geqslant 1$.

Consider the following long exact sequence associated with short exact sequence (3.1).

$$\begin{aligned} H^0(U_{i_1\dots i_p},\Omega^0) \longrightarrow H^0(U_{i_1\dots i_p},\mathrm{d}\Omega^0) \longrightarrow H^1(U_{i_1\dots i_p},\mathbb{C}) \longrightarrow H^1(U_{i_1\dots i_p},\Omega^0) \\ \longrightarrow H^1(U_{i_1\dots i_p},\mathrm{d}\Omega^0) \longrightarrow H^2(U_{i_1\dots i_p},\mathbb{C}) \longrightarrow H^2(U_{i_1\dots i_p},\Omega^0) \longrightarrow \dots, \end{aligned}$$

Since $H^q(U_{i_1\dots i_p},\Omega^0) = H^{q+1}(U_{i_1\dots i_p},\mathbb{C}) = 0$ for $q \geqslant 1$ and $p \geqslant 1$, we conclude that for $q \geqslant 1$ and $p \geqslant 1$ $H^q(U_{i_1\dots i_p},\mathrm{d}\Omega^0) = 0$. The proof of Lemma 3.3 is complete. □

Similarly, by Lemma 3.2 we have the short exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow \Omega^0(kW) \longrightarrow \mathrm{d}\Omega^0(kW) \longrightarrow 0, \quad k \geqslant 1, \tag{3.12}$$

and the following lemma.

Lemma 3.4 (Acyclic lemma). — *Suppose X is a compact complex manifold. Let $\mathcal{U} := \{U_i\}_{i=1}^M$ be a good cover of X ensured by Lemma A.1. The following vanishing results hold for $k,p,q \geqslant 1$.*

$$H^q(U_{i_1\dots i_p},\Omega^0(kW)) = 0, \quad H^q(U_{i_1\dots i_p},\mathrm{d}\Omega^0(kW)) = 0. \tag{3.13}$$

Proof. — The proof is similar to the proof of Lemma 3.3 and we omit it here. □

3.3. The Čech cohomological interpretation of homomorphisms

We first recall some notions in Čech cohomology (see [21, §4] for details). Let X be a compact complex manifold and W a reduced divisor of X. Let $\mathcal{U} := \{U_i\}_{i=1}^M$ be a finite cover of X. Denote by $j_*^{i_1\dots i_p}, p \geqslant 1$, the inclusion $U_{i_1\dots i_p} \xrightarrow{j_*^{i_1\dots i_p}} X$. Let $\mathcal{F}$ be a sheaf on X. Define sheaf $j_*^{i_1\dots i_p}\mathcal{F}$, for any sheaf of abelian groups $\mathcal{F}$ on $U_{i_1\dots i_p}$, by formula $j_*^{i_1\dots i_p}\mathcal{F}(V) := \mathcal{F}(V\cap U_{i_1\dots i_p})$. Define the sheaf $\mathcal{C}^k(\mathcal{U},\mathcal{F})$, for integer $k\geqslant 0$, by

$$\mathcal{C}^k(\mathcal{U},\mathcal{F}) := \bigoplus_{1\leqslant i_1<\dots<i_{k+1}\leqslant M} j_*^{i_1\dots i_{k+1}}\mathcal{F}. \tag{3.14}$$

Define the coboundary operator $d:\mathcal{C}^k(\mathcal{U},\mathcal{F})\to\mathcal{C}^{k+1}(\mathcal{U},\mathcal{F})$, for integer $k\geqslant 0$, by formula

$$(\mathrm{d}\sigma)_{i_1\dots i_{k+2}} = \sum_s (-1)^{s-1}\sigma_{i_1\dots\hat{i}_s\dots i_{k+2}}|_{V\cap U_{i_1\dots i_{k+2}}}, \qquad 1\leqslant i_1<\dots<i_{k+2}\leqslant M, \tag{3.15}$$

where $\sigma=(\sigma_{j_1\dots j_{k+1}})$, $\sigma_{j_1\dots j_{k+1}}\in j_*^{j_1\dots j_{k+1}}\mathcal{F}(V)=\mathcal{F}(V\cap U_{j_1\dots j_{k+1}}), 1\leqslant j_1<\dots<j_{k+1}\leqslant M$ and V is any open set of X. One also defines a homomorphism $j:\mathcal{F}\to\mathcal{C}^0(\mathcal{U},\mathcal{F})$ by $j(\sigma)_i=\sigma|_{V\cap U_i}$ for $\sigma\in\mathcal{F}(V)$.

We have the following proposition.

Proposition 3.5 ([21, Prop. 4.17]). — *The (Čech) complex*

$$0\longrightarrow\mathcal{C}^0(\mathcal{U},\mathcal{F})\xrightarrow{d}\mathcal{C}^1(\mathcal{U},\mathcal{F})\xrightarrow{d}\cdots\xrightarrow{d}\mathcal{C}^n(\mathcal{U},\mathcal{F}) \xrightarrow{d}\mathcal{C}^{n+1}(\mathcal{U},\mathcal{F})\xrightarrow{d}\dots \tag{3.16}$$

is a resolution of $\mathcal{F}$.

Then the Čech cohomology is defined as follows.

Definition 3.6. — *Define $\check{H}^q(\mathcal{U},\mathcal{F})$ to be the q^{th} cohomology group of the complex of global sections*

$$\Gamma(X,\mathcal{C}^q(\mathcal{U},\mathcal{F})) := \bigoplus_{1\leqslant i_1<\dots<i_{q+1}\leqslant M}\mathcal{F}(U_{i_1\dots i_{q+1}})$$

of the Čech complex (3.16) *associated with the cover $\mathcal{U}$. Define the Čech cohomology $\check{H}^q(X,\mathcal{F})$ to be the direct limit of $\check{H}^q(\mathcal{U},\mathcal{F})$ as $\mathcal{U}$ runs through all open covers of X,*

$$\check{H}^1(X,\mathrm{d}\Omega^0) := \varinjlim_{\mathcal{U}}\check{H}^1(\mathcal{U},\mathrm{d}\Omega^0). \tag{3.17}$$

Moreover, we have the following canonical homomorphism.

$$P\colon \check{H}^1(\mathcal{U},\mathrm{d}\Omega^0)\longrightarrow\check{H}^1(X,\mathrm{d}\Omega^0)=\varinjlim_{\mathcal{U}}\check{H}^1(\mathcal{U},\mathrm{d}\Omega^0)\longrightarrow H^1(X,\mathrm{d}\Omega^0). \tag{3.18}$$

In what follows, we will take $\mathcal{U}$ to be a finite, good cover of X (by Lemma A.1) and consider the Čech complexes (3.16) for the sheaves $\mathbb{C}$, Ω^0, $\mathrm{d}\Omega^0$, $R^1(W)$ and $\Phi^1(W)$, respectively. Since the derived functor of the global section functor $\Gamma(X,\cdot\,)$ is left exact, we have the following isomorphisms.

$$\begin{gathered} H^0(X,\mathbb{C}) = \check{H}^0(\mathcal{U},\mathbb{C}), \qquad H^0(X,\Omega^0) = \check{H}^0(\mathcal{U},\Omega^0), \\ H^0(X,\mathrm{d}\Omega^0) = \check{H}^0(\mathcal{U},\mathrm{d}\Omega^0), \quad H^0(X,R^1(W)) = \check{H}^0(\mathcal{U},R^1(W)), \\ H^0(X,\Phi^1(W)) = \check{H}^0(\mathcal{U},\Phi^1(W)). \end{gathered} \tag{3.19}$$

By Lemma 3.3 we have

$$H^q(U_{i_1\ldots i_p},\mathbb{C}) = H^q(U_{i_1\ldots i_p},\Omega^0) = H^q(U_{i_1\ldots i_p},\mathrm{d}\Omega^0) = 0$$

for $p \geqslant 1$ and $q \geqslant 1$. Hence we have the following isomorphisms between the sheaf cohomology groups and the corresponding Čech cohomology groups, where $q \geqslant 1$ (see [21, Thm. 4.41] for instance).

$$\begin{gathered} H^q(X,\mathbb{C}) = \check{H}^q(\mathcal{U},\mathbb{C}), \qquad H^q(X,\Omega^0) = \check{H}^q(\mathcal{U},\Omega^0), \\ H^q(X,\mathrm{d}\Omega^0) = \check{H}^q(\mathcal{U},\mathrm{d}\Omega^0). \end{gathered} \tag{3.20}$$

Applying the global section functor $\Gamma(X,\cdot\,)$ and its derived functor to the short exact sequence (3.1) and the long exact sequence (3.16) with $\mathcal{F} = \mathbb{C}$, Ω^0 and $\mathrm{d}\Omega^0$, we have the following commutative diagram, thanks to $H^1(U_{i_1\ldots i_p},\mathbb{C}) = 0$ for $p \geqslant 1$.

(3.21)

$$\begin{tikzcd} 0 \arrow[r] & \bigoplus_{i_1} \mathbb{C}(U_{i_1}) \arrow[r] \arrow[d, "\mathrm{d}^0_{\mathbb{C}}"] & \bigoplus_{i_1} \Omega^0(U_{i_1}) \arrow[r] \arrow[d, "\mathrm{d}^0_{\Omega}"] & \bigoplus_{i_1} \mathrm{d}\Omega^0(U_{i_1}) \arrow[r] \arrow[d, "\mathrm{d}^0_{\mathrm{d}\Omega}"] & 0 \\ 0 \arrow[r] & \bigoplus_{i_1<i_2} \mathbb{C}(U_{i_1i_2}) \arrow[r] \arrow[d, "\mathrm{d}^1_{\mathbb{C}}"] & \bigoplus_{i_1<i_2} \Omega^0(U_{i_1i_2}) \arrow[r] \arrow[d, "\mathrm{d}^1_{\Omega}"] & \bigoplus_{i_1<i_2} \mathrm{d}\Omega^0(U_{i_1i_2}) \arrow[r] \arrow[d, "\mathrm{d}^1_{\mathrm{d}\Omega}"] & 0 \\ 0 \arrow[r] & \bigoplus_{i_1<i_2<i_3} \mathbb{C}(U_{i_1i_2i_3}) \arrow[r] \arrow[d, "\mathrm{d}^2_{\mathbb{C}}"] & \bigoplus_{i_1<i_2<i_3} \Omega^0(U_{i_1i_2i_3}) \arrow[r] \arrow[d, "\mathrm{d}^2_{\Omega}"] & \bigoplus_{i_1<i_2<i_3} \mathrm{d}\Omega^0(U_{i_1i_2i_3}) \arrow[r] \arrow[d, "\mathrm{d}^2_{\mathrm{d}\Omega}"] & 0 \\ & \cdots & \cdots & \cdots & \end{tikzcd}$$

Here the horizontal lines of the above commutative diagram are exact. By (3.19) and (3.20), we conclude that $H^0(X,\mathbb{C}) = \check{H}^0(\mathcal{U},\mathbb{C}) = \operatorname{Ker}\mathrm{d}^0_{\mathbb{C}}$, $H^1(X,\mathbb{C}) = \check{H}^1(\mathcal{U},\mathbb{C}) = \frac{\operatorname{Ker}\mathrm{d}^1_{\mathbb{C}}}{\operatorname{Im}\mathrm{d}^0_{\mathbb{C}}}$ and $H^2(X,\mathbb{C}) = \check{H}^2(\mathcal{U},\mathbb{C}) = \frac{\operatorname{Ker}\mathrm{d}^2_{\mathbb{C}}}{\operatorname{Im}\mathrm{d}^1_{\mathbb{C}}}$;

$H^0(X,\Omega^0)=\check{H}^0(\mathcal{U},\Omega^0)=\operatorname{Ker}\mathrm{d}^0_\Omega$, $H^1(X,\Omega^0)=\check{H}^1(\mathcal{U},\Omega^0)=\frac{\operatorname{Ker}\mathrm{d}^1_\Omega}{\operatorname{Im}\mathrm{d}^0_\Omega}$ and $H^2(X,\Omega^0)=\check{H}^2(\mathcal{U},\Omega^0)=\frac{\operatorname{Ker}\mathrm{d}^2_\Omega}{\operatorname{Im}\mathrm{d}^1_\Omega}$; $H^0(X,\mathrm{d}\Omega^0)=\check{H}^0(\mathcal{U},\mathrm{d}\Omega^0)=\operatorname{Ker}\mathrm{d}^0_{\mathrm{d}\Omega}$ and $H^1(X,\mathrm{d}\Omega^0)=\check{H}^1(\mathcal{U},\mathrm{d}\Omega^0)=\frac{\operatorname{Ker}\mathrm{d}^1_{\mathrm{d}\Omega}}{\operatorname{Im}\mathrm{d}^0_{\mathrm{d}\Omega}}$.

Moreover, we also have the following natural commutative diagram.

$$\begin{tikzcd}
\longrightarrow \check{H}^1(\mathcal{U},\Omega^0) \arrow[r] \arrow[d, "\cong"] & \check{H}^1(\mathcal{U},\mathrm{d}\Omega^0) \arrow[r, "\check{\delta}^1"] \arrow[d, "\cong"] & \check{H}^2(\mathcal{U},\mathbb{C}) \arrow[r] \arrow[d, "\cong"] & \check{H}^2(\mathcal{U},\Omega^0) \longrightarrow \arrow[d, "\cong"] \\
\longrightarrow H^1(X,\Omega^0) \arrow[r] & H^1(X,\mathrm{d}\Omega^0) \arrow[r, "\delta^1"] & H^2(X,\mathbb{C}) \arrow[r] & H^2(X,\Omega^0) \longrightarrow .
\end{tikzcd} \tag{3.22}$$

Since $H^1(U_{i_1\dots i_p},\mathrm{d}\Omega^0)=0$ for $p\geqslant 1$, we derive the following commutative diagram associated with short exact sequence (3.2) in a similar way.

(3.23)

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{i_1}\mathrm{d}\Omega^0(U_{i_1}) \arrow[r] \arrow[d, "\mathrm{d}^0_{\mathrm{d}\Omega}"] & \bigoplus_{i_1}\Phi^1(W)(U_{i_1}) \arrow[r, "H"] \arrow[d, "\mathrm{d}^0_\Phi"] & \bigoplus_{i_1}R^1(W)(U_{i_1}) \arrow[r] \arrow[d, "\mathrm{d}^0_R"] & 0 \\
0 \arrow[r] & \bigoplus_{i_1<i_2}\mathrm{d}\Omega^0(U_{i_1i_2}) \arrow[r, "G"] \arrow[d, "\mathrm{d}^1_{\mathrm{d}\Omega}"] & \bigoplus_{i_1<i_2}\Phi^1(W)(U_{i_1i_2}) \arrow[r] \arrow[d, "\mathrm{d}^1_\Phi"] & \bigoplus_{i_1<i_2}R^1(W)(U_{i_1i_2}) \arrow[r] \arrow[d, "\mathrm{d}^1_R"] & 0 \\
0 \arrow[r] & \bigoplus_{i_1<i_2<i_3}\mathrm{d}\Omega^0(U_{i_1i_2i_3}) & \dots & \dots &
\end{tikzcd}$$

where the horizontal lines are exact. By (3.19) and (3.20), $H^0(X,\mathrm{d}\Omega^0)=\check{H}^0(\mathcal{U},\mathrm{d}\Omega^0)$ and $H^1(X,\mathrm{d}\Omega^0)=\check{H}^1(\mathcal{U},\mathrm{d}\Omega^0)$; $H^0(X,\Phi^1(W))=\check{H}^0(\mathcal{U},\Phi^1(W))$; $H^0(X,R^1(W))=\check{H}^0(\mathcal{U},R^1(W))$.

Moreover, we have the following commutative diagram.

(3.24)

$$\begin{tikzcd}
\to \check{H}^0(\mathcal{U},\mathrm{d}\Omega^0) \arrow[r] \arrow[d, "\cong"] & \check{H}^0(\mathcal{U},\Phi^1(W)) \arrow[r] \arrow[d, "\cong"] & \check{H}^0(\mathcal{U},R^1(W)) \arrow[r, "\check{\Delta}^0"] \arrow[d, "\cong"] & \check{H}^1(\mathcal{U},\mathrm{d}\Omega^0) \to \dots \arrow[d, "\cong"] \\
\to H^0(X,\mathrm{d}\Omega^0) \arrow[r] & H^0(X,\Phi^1(W) \arrow[r] & H^0(X,R^1(W)) \arrow[r, "\Delta^0"] & H^1(X,\mathrm{d}\Omega^0) \to \dots
\end{tikzcd}$$

Combining diagrams (3.22) and (3.24), we have a homomorphism $\delta^1\circ\Delta^0$ from $H^0(X,R^1(W))$ to $H^2(X,\mathbb{C})$ such that the following diagram is commutative.

$$
\begin{tikzcd}
 & & & H^0(X,R^1(W)) \arrow[d, "\Delta^0"] \arrow[dr, "\delta^1\circ\Delta^0"] & \\
0 \arrow[r] & H^0(X,\mathrm{d}\Omega^0) \arrow[r] & H^1(X,\mathbb{C}) \arrow[r] & H^1(X,\Omega^0) \xrightarrow{j} H^1(X,\mathrm{d}\Omega^0) \arrow[r, "\delta^1"] & H^2(X,\mathbb{C}).
\end{tikzcd}
\tag{3.25}
$$

The main result of this section is the following theorem.

Theorem 3.7. — *Let W be a reduced divisor on a compact complex manifold X. Let $W = \bigcup_{i=1}^{l} W_i$ be the irreducible decomposition of W. Then the map $\delta^1 \circ \Delta^0$ is induced by the first Chern classes as follows.*

$$
\begin{aligned}
&\delta^1 \circ \Delta^0 : H^0(X,R^1(W)) \longrightarrow H^2(X,\mathbb{C}),\\
&\sum_{i=1}^{l} a_i \cdot 1_{W_i} \cong \sum_{i=1}^{l} a_i W_i \longmapsto \sum_{i=1}^{l} c_1(W_i) \otimes_{\mathbb{Z}} a_i,
\end{aligned}
\tag{3.26}
$$

where $c_1(W_i)$ is the first Chern class of W_i for $i = 1,\ldots,l$. By a slight abuse of notation, we call $(\delta^1 \circ \Delta^0)(D)$ the first Chern class of D in the De Rham cohomology for each $D \in H^0(X,R^1(W))$.

Remark 3.8. — Since $H^2(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^2(X,\mathbb{C})$, $c_1(W_i) \otimes_{\mathbb{Z}} 1$ can be naturally viewed as an element of $H^2(X,\mathbb{C})$. Also recall that, by Lemma 2.4,

$$
H^0(X,R^1(W)) = \bigoplus_{i=1}^{l} H^0(X,\mathbb{C}_{W_i}) \cong \bigoplus_{i=1}^{l} \mathbb{C}\cdot 1_{W_i} \cong \bigoplus_{i=1}^{l} \mathbb{C}W_i.
$$

Proof of Theorem 3.7. — Since W is a reduced divisor, Lemma 3.1 holds; then the short exact sequences (3.1) and (3.1) hold. Moreover, Lemmas A.1 and 3.3 hold for X is a compact complex manifold. Hence we have the commutative diagrams (3.21), (3.22), (3.23), (3.24) and (3.25) with respect to a good cover $\mathcal{U} := \{U_i\}_{i=1}^{M}$ of X.

By the linearity of $\delta^1 \circ \Delta^0$, in order to prove (3.26), it suffices to prove it for 1_{W_1}. Notice that, by (3.22) and (3.24), the homomorphism $\delta^1 \circ \Delta^0$ is isomorphic to the homomorphism $\check{\delta}^1 \circ \check{\Delta}^0$ between the corresponding Čech cohomology groups. In what follows, we will do diagram chasing in (3.21) and (3.23).

Recall that the Čech 0-cocycle of 1_{W_1} with respect to $\mathcal{U}$ is given by

$$
\bigoplus_{1\leqslant i_1 \leqslant M} 1_{W_1}|_{U_{i_1}} \in \bigoplus_{1\leqslant i_1 \leqslant M} R^1(W)(U_{i_1}).
$$

Since U_i is Stein for $i = 1,\ldots,M$, W_1 is defined by a holomorphic function $f_i = 0$ on U_i. Define $g_{i_1 i_2} := \frac{f_{i_1}}{f_{i_2}} \in \mathcal{O}^*(U_{i_1 i_2})$ for $i_1, i_2 = 1,\ldots,M$.

Then $\{g_{i_1i_2}\}$ is a system of transition functions of holomorphic line bundle $[W_1]$ with respect to $\mathcal{U}$.

Thanks to Remark 2.5 and the diagram (3.23), we define a preimage σ of $\bigoplus_{1\leqslant i_1\leqslant M} 1_{W_1}|_{U_{i_1}}$ under the homomorphism $H : \bigoplus_{1\leqslant i_1\leqslant M} \Phi^1(W)(U_{i_1}) \xrightarrow{H} \bigoplus_{1\leqslant i_1\leqslant M} R^1(W)(U_{i_1})$ by

$$\sigma := \bigoplus_{1\leqslant i_1\leqslant M} \frac{1}{2\pi\sqrt{-1}}\frac{\mathrm{d}f_{i_1}}{f_{i_1}}\bigg|_{U_{i_1}} \in \bigoplus_{1\leqslant i_1\leqslant M} \Phi^1(W)(U_{i_1}). \tag{3.27}$$

Then $\mathrm{d}^0_\Phi(\sigma)$ is a Čech 1-cocycle as follows.

$$\begin{aligned}\mathrm{d}^0_\Phi(\sigma) = \bigoplus_{1\leqslant i_1<i_2\leqslant M}\left(\frac{1}{2\pi\sqrt{-1}}\frac{\mathrm{d}f_{i_1}}{f_{i_1}} - \frac{1}{2\pi\sqrt{-1}}\frac{\mathrm{d}f_{i_2}}{f_{i_2}}\right)\bigg|_{U_{i_1i_2}} \\ \in \bigoplus_{1\leqslant i_1<i_2\leqslant M} \Phi^1(W)(U_{i_1i_2}).\end{aligned} \tag{3.28}$$

Since $f_{i_1} = g_{i_1i_2}\cdot f_{i_2}$ on $U_{i_1i_2}$ and $g_{i_1i_2}\in\mathcal{O}^*(U_{i_1i_2})$, we have that

$$\begin{aligned}\mathrm{d}^0_\Phi(\sigma)(U_{i_1i_2}) &= \frac{1}{2\pi\sqrt{-1}}\frac{\mathrm{d}f_{i_1}}{f_{i_1}} - \frac{1}{2\pi\sqrt{-1}}\frac{\mathrm{d}f_{i_2}}{f_{i_2}} \\ &= \frac{1}{2\pi\sqrt{-1}}\frac{\mathrm{d}(g_{i_1i_2}f_{i_2})}{(g_{i_1i_2}f_{i_2})} - \frac{1}{2\pi\sqrt{-1}}\frac{\mathrm{d}f_{i_2}}{f_{i_2}} \\ &= \frac{1}{2\pi\sqrt{-1}}\frac{\mathrm{d}g_{i_1i_2}}{g_{i_1i_2}} + \frac{1}{2\pi\sqrt{-1}}\frac{\mathrm{d}f_{i_2}}{f_{i_2}} - \frac{1}{2\pi\sqrt{-1}}\frac{\mathrm{d}f_{i_2}}{f_{i_2}} \\ &= \frac{1}{2\pi\sqrt{-1}}\frac{\mathrm{d}g_{i_1i_2}}{g_{i_1i_2}} = \frac{\mathrm{d}(\log g_{i_1i_2})}{2\pi\sqrt{-1}}.\end{aligned} \tag{3.29}$$

Define a Čech 1-cocycle ξ by

$$\xi := \bigoplus_{1\leqslant i_1<i_2\leqslant M}\left(\frac{\mathrm{d}(\log g_{i_1i_2})}{2\pi\sqrt{-1}}\right)\bigg|_{U_{i_1i_2}} \in \bigoplus_{1\leqslant i_1<i_2\leqslant M} \mathrm{d}\Omega^0(U_{i_1i_2}).$$

It is easy to see that ξ is a preimage of $\mathrm{d}^0_\Phi(\sigma)$ under the homomorphism $G : \bigoplus_{i_1<i_2}\mathrm{d}\Omega^0(U_{i_1i_2}) \xrightarrow{G} \bigoplus_{i_1<i_2}\Phi^1(W)(U_{i_1i_2})$.

Fix a base point $a_{i_1i_2}\in U_{i_1i_2}$ for $1\leqslant i_1<i_2\leqslant M$. Let τ be a Čech 1-cochain given by

$$\tau := \bigoplus_{1\leqslant i_1<i_2\leqslant M} \tau_{i_1i_2}|_{U_{i_1i_2}} \in \bigoplus_{1\leqslant i_1<i_2\leqslant M} \Omega^0(U_{i_1i_2}),$$

where

$$\tau_{i_1i_2} = \int_{a_{i_1i_2}}^{z} \frac{\mathrm{d}(\log g_{i_1i_2})}{2\pi\sqrt{-1}} = \frac{1}{2\pi\sqrt{-1}}\big(\log g_{i_1i_2}(z) - \log g_{i_1i_2}(a_{i_1i_2})\big),$$

and $\log g_{i_1i_2}$ is a branch of the log function. Then τ is a preimage of ξ under the homomorphism $D : \bigoplus_{1\leqslant i_1<i_2\leqslant M} \Omega^0(U_{i_1i_2}) \xrightarrow{D} \bigoplus_{1\leqslant i_1<i_2\leqslant M} \mathrm{d}\Omega^0(U_{i_1i_2})$ in diagram (3.21).

It is clear that $\mathrm{d}^1_\Omega(\tau) \in \bigoplus_{1\leqslant i_1<i_2<i_3\leqslant M} \Omega^0(U_{i_1i_2i_3})$ is a Čech 2-cocycle as follows.

$$\mathrm{d}^1_\Omega(\tau) = \bigoplus_{1\leqslant i_1<i_2<i_3\leqslant M} \mathrm{d}^1_\Omega(\tau)_{i_1i_2i_3} \in \bigoplus_{1\leqslant i_1<i_2<i_3\leqslant M} \Omega^0(U_{i_1i_2i_3}),$$

where

$$\begin{aligned}\mathrm{d}^1_\Omega(\tau)_{i_1i_2i_3} &= \tau_{i_1i_2}|_{U_{i_1i_2i_3}} - \tau_{i_1i_3}|_{U_{i_1i_2i_3}} + \tau_{i_2i_3}|_{U_{i_1i_2i_3}} \\ &= \left\{ \frac{1}{2\pi\sqrt{-1}}\big(\log g_{i_1i_2}(z) - \log g_{i_1i_2}(a_{i_1i_2})\big) \right. \\ &\qquad - \frac{1}{2\pi\sqrt{-1}}\big(\log g_{i_1i_3}(z) - \log g_{i_1i_3}(a_{i_1i_3})\big) \\ &\qquad \left. + \frac{1}{2\pi\sqrt{-1}}\big(\log g_{i_2i_3}(z) - \log g_{i_2i_3}(a_{i_2i_3})\big) \right\}\Bigg|_{U_{i_1i_2i_3}}.\end{aligned}$$

Since $g_{i_1i_2} \cdot g_{i_2i_3} \cdot g_{i_3i_1} \equiv 1$ on $U_{i_1i_2i_3}$, $\mathrm{d}^1_\Omega(\tau)_{i_1i_2i_3}$ is a constant function for $z \in U_{i_1i_2i_3}$. Therefore, $\mathrm{d}^1_\Omega(\tau) \in \bigoplus_{1\leqslant i_1<i_2<i_3\leqslant M} \mathbb{C}(U_{i_1i_2i_3})$. Thanks to the fact that $g_{jk}(a_{jk})$ is a constant function defined in U_{ik}, we conclude that $\mathrm{d}^1_\Omega(\tau)$ defines the same two-cocycle as

$$\begin{aligned}\widetilde{\tau} &:= \bigoplus_{1\leqslant i_1<i_2<i_3\leqslant M} \frac{1}{2\pi\sqrt{-1}}\big(\log g_{i_1i_2} - \log g_{i_1i_3} + \log g_{i_2i_3}\big)\big|_{U_{i_1i_2i_3}} \\ &\in \bigoplus_{1\leqslant i_1<i_2<i_3\leqslant M} \mathbb{C}(U_{i_1i_2i_3}).\end{aligned} \tag{3.30}$$

We conclude that $\widetilde{\tau}$ is the image of 1_{W_1} under the map $\check{\delta}^1 \circ \check{\Delta}^0$.

By Proposition §1.1 in [7], the above $\widetilde{\tau}$ corresponds to the first Chern class $c_1(W_1)$ of the divisor W_1. This completes the proof of Theorem 3.7. □

Remark-Definition 3.9. — When W runs through all reduced divisors of X, we can extend the homomorphism $\delta^1 \circ \Delta^0$ from $H^0(X, R^1(W))$ to the vector space of all finite formal sums of divisors on X. We call the image $(\delta^1 \circ \Delta^0)(D)$ of D, a $\mathbb{C}$-linear finite formal sum of divisors on X, the first Chern class of D in De Rham cohomology.

Remark 3.10. — It is easy to verify that the homomorphism Δ^0 in commutative diagram (3.25) is the restriction of the $\mathcal{Q}$-flat class to the subspace $H^0(X, R^1(W))$.

3.4. Property (H) and the $\mathcal{Q}$-flat class of a $\mathbb{C}$-linear finite formal sum of divisors

In this subsection, we will introduce the Property (H) of complex manifolds and the $\mathcal{Q}$-flat classes of holomorphic line bundles and of $\mathbb{C}$-linear finite formal sums of divisors.

Definition 3.11. — *A complex manifold X is said to have Property* (H) *if X is compact and the following equality holds.*

$$\dim H^1(X,\mathbb{C}) = \dim H^0(X,\mathrm{d}\Omega^0) + \dim H^1(X,\mathcal{O}_X). \tag{3.31}$$

Remark 3.12. — Notice that there is a natural long exact sequence of cohomology groups induced by the short exact sequence (3.1) of $\mathbb{C}$-sheaves as follows.

$$\begin{aligned}0 \longrightarrow H^0(X,\mathrm{d}\Omega^0) \longrightarrow H^1(X,\mathbb{C})&\\ \longrightarrow H^1(X,\mathcal{O}_X) \longrightarrow H^1(X,\mathrm{d}\Omega^0) \longrightarrow \cdots&.\end{aligned} \tag{3.32}$$

Hence, in general we only have that $\dim H^1(X,\mathbb{C}) \leqslant \dim H^0(X,\mathrm{d}\Omega^0) + \dim H^1(X,\mathcal{O}_X)$.

Remark 3.13. — Recall that $H^0(X,\mathrm{d}\Omega^0)$ is the vector space consisting of closed holomorphic differential 1-forms on X and when X is a compact Kähler manifold (or more generally of Fujiki class $\mathcal{C}$), each holomorphic 1-form on X is closed (see [4] or [20]). The following Hodge decomposition for $H^1(X,\mathbb{C})$ holds.

$$0 \longrightarrow H^0(X,\Omega^1) \longrightarrow H^1(X,\mathbb{C}) \longrightarrow H^1(X,\mathcal{O}_X) \longrightarrow 0. \tag{3.33}$$

Hence, Kähler manifolds or manifolds of Fujiki class $\mathcal{C}$ have Property (H). More generally, Property (H) is satisfied whenever the Frölicher spectral sequence of X degenerates at E_1.

Proposition 3.14. — *Property* (H) *holds for all smooth, compact complex surfaces.*

Proof. — Let X be a smooth, compact complex surface. Then each holomorphic 1-form on X is closed. Moreover, by Kodaira's results (see [11, 12]), one of the following two cases holds.

(1) $\dim H^0(X,\Omega^1) = q-1$, $\dim H^1(X,\mathcal{O}_X) = q$, $\dim H^1(X,\mathbb{C}) = 2q-1$;
(2) $\dim H^0(X,\Omega^1) = q$, $\dim H^1(X,\mathcal{O}_X) = q$, $\dim H^1(X,\mathbb{C}) = 2q$.

Hence,

$$\begin{aligned}(3.34)\quad \dim H^0(X, \mathrm{d}\Omega^0) + \dim H^1(X, \mathcal{O}_X)\\ &= \dim H^0(X, \Omega^1) + \dim H^1(X, \mathcal{O}_X)\\ &= \dim H^1(X, \mathbb{C}).\end{aligned}$$

We conclude Proposition 3.14. □

Lemma 3.15. — *Property* (H) *is preserved under blow-ups.*

Proof of Lemma 3.15. — Let $f : Y \to X$ be a blow-up. Since X and Y are smooth, $f_*\mathcal{O}_Y = \mathcal{O}_X$ and $R^p f_*\mathcal{O}_Y = 0$ for $p > 0$. Then $H^1(Y, \mathcal{O}_Y) = H^1(X, f_*\mathcal{O}_Y) = H^1(X, \mathcal{O}_X)$. It is easy to verify that $H^1(X, \mathbb{C}) = H^1(Y, \mathbb{C})$.

Next we will prove that $H^0(X, \mathrm{d}\Omega^0_X) = H^0(Y, \mathrm{d}\Omega^0_Y)$. On the one hand, if ω is a closed holomorphic 1-form on X, then $f^*\omega$ is a closed holomorphic 1-form on Y. On the other hand, for any closed holomorphic 1-form τ on Y, its pushforward $f_*\tau$ is a closed holomorphic 1-form on $X\backslash V$ where V is of codimension at least two; hence $f_*\tau$ is a closed holomorphic 1-form on X by extension.

This completes the proof. □

Recall that there is a natural homomorphism i_2 from $H^1(X, \Omega^1)$ to $H^2(X, \mathbb{C})$ and a natural homomorphism j_1 from $H^1(X, \mathrm{d}\Omega^0)$ to $H^1(X, \Omega^1)$. We have the following proposition.

Proposition 3.16 (See [6, Thm. 4.5]). — *Let X be a compact complex manifold. The first Chern class maps factor through the $\mathcal{Q}$-flat class map F as*

$$(3.35)\qquad H^1(X, \mathcal{O}^*) \xrightarrow{F} H^1(X, \mathrm{d}\Omega^0) \xrightarrow{j_1} H^1(X, \Omega^1) \xrightarrow{i_2} H^2(X, \mathbb{C}).$$

That is to say, for each holomorphic line bundle W of X, $(j_1 \circ F)(W)$ is the first Chern class of W in the Dolbeault cohomology group $H^1(X, \Omega^1)$; $(i_2 \circ j_1 \circ F)(W)$ is the first Chern class of W in the De Rham cohomology group $H^2(X, \mathbb{C})$.

If X has Property (H), we have the following lemma.

Lemma 3.17. — *If X has Property* (H)*, then the flat class, the first Chern class in the Dolbeault cohomology and the first Chern class in the De Rham cohomology coincide for each element in $H^0(X, R^1(W))$. In particular, in this case a divisor D of X is homologous to zero if and only if mD is flat for some nonzero integer m.*

Proof of Lemma 3.17. — Thanks to the commutative diagram (3.25), we have the following commutative diagram.

$$\begin{tikzcd} & & & H^0(X,R^1(W)) \arrow[d, "\Delta^0"] \arrow[dr, "\delta^1\circ\Delta^0"] & \\ 0 \arrow[r] & H^0(X,\mathrm{d}\Omega^0) \arrow[r] & H^1(X,\mathbb{C}) \arrow[r] & H^1(X,\Omega^0) \xrightarrow{j} H^1(X,\mathrm{d}\Omega^0) \arrow[r, "\delta^1"] \arrow[d, "j_1"] & H^2(X,\mathbb{C}). \\ & & & H^1(X,\Omega^1) \arrow[ur, "i_2"] & \end{tikzcd} \tag{3.36}$$

Notice that for each element $D \in H^0(X, R^1(W))$ the $\mathcal{Q}$-flat class $F(D) = \Delta^0(D)$; the first Chern class of D in the Dolbeault cohomology is $(j_1 \circ \Delta^0)(D)$; the first Chern class in the De Rham cohomology is $(\delta^1 \circ \Delta^0)(D)$. Since X has Property (H), the homomorphism j is a zero map, and hence the homomorphisms δ^1 and j_1 are injective. Then Lemma 3.17 follows. □

Remark 3.18. — As W runs through all the reduced divisors of X, it is clear that Lemma 3.17 holds for each $\mathbb{C}$-linear finite formal sum of divisors of X. (See Remark-Definition 3.9.)

4. Logarithmic forms

In this section, we will study logarithmic 1-forms.

4.1. Proofs of Theorems 1.9 and 1.12 and some corollaries

Proof of Theorem 1.9. — Recall the bottom horizontal line of long exact sequence (3.24). It is clear the there exists an element $\phi \in H^0(X, \Phi^1(W))$ with residue divisor D if and only if $\Delta^0(D) = 0$. Since $\Delta^0(D)$ is the $\mathcal{Q}$-flat class of D, we conclude Theorem 1.9. □

Proof of Theorem 1.12. — Recall the commutative diagram (3.25). When X has Property (H), the homomorphism j is an injection. Hence $\Delta^0(D) = 0$ if and only if $(\delta^1 \circ \Delta^0)(D) = 0$. By Theorem 3.7 and the Poincaré duality, we conclude Theorem 1.12. □

Proof of Corollary 1.10. — Since

$$D - D' + \sum_{i=1}^{m} a_i \cdot (f_i) = 0, \tag{4.1}$$

the $\mathcal{Q}$-flat class of D is the same as the $\mathcal{Q}$-flat class of D'. By Theorem 1.9, we conclude Corollary 1.10. □

Proof of Corollary 1.13. — This is a direct consequence of Theorem 1.12 and Proposition 3.14. □

Thanks to Remark 3.13 and Lemma 3.17, we further have the following corollaries.

Corollary 4.1 (The theorem of Weil and Kodaira). — *For a Kähler manifold X, there exists a closed logarithmic 1-form with residue divisor D on X if and only if D is homologous to zero.*

Corollary 4.2 ([7, Lem. in §2.2]). — *Given a finite set of points $\{p_\lambda\}$ on compact Riemann surface S and complex numbers $\{a_\lambda\}$ such that $\sum a_\lambda = 0$, there exists a differential of the third kind on S, holomorphic in $S - \{p_\lambda\}$ and has residue a_λ at p_λ.*

Corollary 4.3. — *Suppose S is a connected, smooth element of an ample line bundle over X. Then closed meromorphic 1-forms with singularities on S are locally exact.*

Proof of Corollary 4.3. — This is an easy consequence of the fact that $c_1(S) \neq 0$. □

In the following, we collect some well-known examples on prescribing residues.

Example 4.4 (See [1] *and* [13]*).* — Let X be a generic Hopf surface (or generic Hopf manifold). There are finitely many divisors on X, each of which is associated with a flat line bundle. Then we can prescribe residues on each divisor (see [17]). On the other hand, the second singular cohomology group of X is zero, and hence each divisor is homologous to zero.

Example 4.5 (See [15]*).* — Each type VII surface has at most finitely many curves; each curve is homologous to zero and associated with a flat bundle.

Example 4.6 (See [14]*).* — Let X be an Iwasawa manifold. Then

$$\begin{aligned} &\dim H^0(X, \mathrm{d}\Omega^0) = 2, \quad \dim H^0(X, \Omega^1) = 3, \\ &\dim H^1(X, \mathcal{O}_X) = 2, \quad \dim H^1(X, \mathbb{C}) = 4. \end{aligned} \tag{4.2}$$

X has Property (H) but the Hodge decomposition does not hold (there are non-closed holomorphic 1-forms on X). On the other hand, X is a fibration over an abelian variety T of complex dimension 2; the divisors on X are the pull backs of the divisors on T.

4.2. Proof of the decomposition theorem for logarithmic 1-forms

Proof of Theorem 1.14. — Without loss of generality, we can assume that W is a reduced, normal crossing divisor by Lemma 3.15. In the following, we fix a good cover $\mathcal{U} = \{U_i\}_{i=1}^M$ of X and choose local coordinates $(z_1^i, \ldots, z_n^i)$ on U_i, for $i = 1, \ldots, n$, such that $W \cap U_i = \{z_1 \ldots z_{l_i} = 0\}$ for certain $0 \leqslant l_i \leqslant n$ (when $W \cap U_i = \emptyset$, $l_i = 0$ by convention).

Let us recall the following construction of the residue divisors for logarithmic 1-forms (see [2] or [16]). Let $W = \bigcup_{i=1}^m W_i$ be the irreducible decomposition of W. Let $\iota_i : \widetilde{W_i} \to W_i$ be the normalization of W_i for $i = 1, \ldots, m$. Then $\widetilde{W} := \coprod_{i=1}^m \widetilde{W_i}$ is the normalization of W of which the map we denote by $\iota : \widetilde{W} \to W$. Since W is a subvariety of X, by abuse of notation, we also denote by ι_i the map $\widetilde{W_i} \to X$, $i = 1, \ldots, m$, and ι the map $\widetilde{W} \to X$. Fix a point $x \in X$. Then there is a holomorphic local coordinate system $(z_1, \ldots, z_m)$ in a neighborhood U of x such that $x = (0, \ldots, 0)$ and $W \cap U = \{z_1 \ldots z_l = 0\} \cap U$ where l is an nonegative integer between 0 and n. Without loss of generality, we can assume that $W_i \cap U = \{z_i = 0\} \cap U$ for $i = 1, \ldots, l$ and $W_i \cap U = \emptyset$ for $i = l+1, \ldots, m$. For $\omega \in H^0(X, \Omega^1(\log W))$ we can write

$$\omega = \sum_{i=1}^{l} \frac{1}{2\pi\sqrt{-1}} \frac{\mathrm{d}z_i}{z_i} \wedge \eta_i + \omega' \text{ in } U, \tag{4.3}$$

where $\eta_i \in H^0(U, \mathcal{O}_X)$ for $i = 1, \ldots, l$ and $\omega' \in H^0(U, \Omega^1)$. Put $\mathrm{Res}_{\widetilde{W}_i}(\omega) = \iota_i^*(\eta_i)$ in $\widetilde{W_i} \bigcap \iota^{-1}(U)$ for $i = 1, \ldots, m$. Then $\mathrm{Res}_{\widetilde{W}_i}(\omega)$ is globally well-defined and

$$\mathrm{Res}_{\widetilde{W}_i}(\omega) \in H^0(\widetilde{W}_i, \mathcal{O}_{\widetilde{W}_i}) \text{ for } i = 1, \ldots, m. \tag{4.4}$$

By pushing forward, we have the following short exact sequence of $\mathcal{O}_X$-sheaves on X.

$$0 \longrightarrow \Omega_X^1 \longrightarrow \Omega_X^1(\log W) \xrightarrow{\mathrm{Res}} \iota_* \mathcal{O}_{\widetilde{W}} \longrightarrow 0. \tag{4.5}$$

We next consider the Čech cohomology groups associated with exact sequence (4.5) with respect to $\mathcal{U}$. In the same way as in Setion 3.3, the following commutative diagram holds.

(4.6)
$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{i_1}\Omega^1_X(U_{i_1}) \arrow[r] \arrow[d, "\mathrm{d}^0_\Omega"] & \bigoplus_{i_1}\Omega^1_X(\log W)(U_{i_1}) \arrow[r, "H"] \arrow[d, "\mathrm{d}^0_{\log}"] & \bigoplus_{i_1}\iota_*\mathcal{O}_{\widetilde{W}}(U_{i_1}) \arrow[r] \arrow[d, "\mathrm{d}^0_W"] & 0 \\
0 \arrow[r] & \bigoplus_{i_1<i_2}\Omega^1_X(U_{i_1i_2}) \arrow[r, "G"] \arrow[d, "\mathrm{d}^1_\Omega"] & \bigoplus_{i_1<i_2}\Omega^1_X(\log W)(U_{i_1i_2}) \arrow[r] \arrow[d, "\mathrm{d}^1_{\log}"] & \bigoplus_{i_1<i_2}\iota_*\mathcal{O}_{\widetilde{W}}(U_{i_1i_2}) \arrow[r] \arrow[d, "\mathrm{d}^1_W"] & 0 \\
0 \arrow[r] & \bigoplus_{i_1<i_2<i_3}\Omega^1_X(U_{i_1i_2i_3}) & \cdots & \cdots &
\end{tikzcd}$$

Moreover, we have

(4.7)
$$\begin{tikzcd}
\check{H}^0(\mathcal{U},\Omega^1_X) \arrow[r] \arrow[d, "\cong"] & \check{H}^0(\mathcal{U},\Omega^1_X(\log W)) \arrow[r, "\mathrm{Res}"] \arrow[d, "\cong"] & \check{H}^0(\mathcal{U},\iota_*\mathcal{O}_{\widetilde{W}}) \arrow[r, "\check{\Delta}^0"] \arrow[d, "\cong"] & \check{H}^1(\mathcal{U},\Omega^1_X) \arrow[r] & \cdots \\
H^0(X,\Omega^1_X) \arrow[r] & H^0(X,\Omega^1_X(\log W)) \arrow[r, "\mathrm{Res}"] & H^0(X,\iota_*\mathcal{O}_{\widetilde{W}}) \arrow[r, "\Delta^0"] & H^1(X,\Omega^1_X) \arrow[r] & \cdots
\end{tikzcd}$$

Notice that

$$\begin{aligned}(4.8)\quad H^0(X,\iota_*\mathcal{O}_{\widetilde{W}}) \cong H^0(\widetilde{W},\mathcal{O}_{\widetilde{W}}) \cong H^0(W,\mathcal{O}_W)& \\ \cong \bigoplus_{i=1}^m \mathbb{C}\cdot 1_{W_i} \cong H^0(X,R^1(W)).&\end{aligned}$$

Hence each element $\sigma \in \check{H}^0(\mathcal{U},\iota_*\mathcal{O}_{\widetilde{W}})$ can be represented by a Čech 0-cocycle

$$(4.9)\qquad \sigma = \bigoplus_{1\leqslant i_1\leqslant M}\left(\sum_{k=1}^m a_k\cdot 1_{W_k}\right)\Bigg|_{U_{i_1}} \in \bigoplus_{1\leqslant i_1\leqslant M}\iota_*\mathcal{O}_{\widetilde{W}}(U_{i_1}),$$

where $a_k\in\mathbb{C}$ for $k=1,\dots,m$.

We will show that the homomorphism Δ^0 between $H^0(X,\iota_*\mathcal{O}_{\widetilde{W}})$ and $H^1(X,\Omega^1_X)$ is induced by the first Chern classes as follows.

Claim. — *The map $\Delta^0 : H^0(X,\iota_*\mathcal{O}_{\widetilde{W}}) \to H^1(X,\Omega^1_X)$ is given by*

$$(4.10)\qquad \Delta^0 : \sum_{j=1}^m a_j\cdot W_j \longmapsto \sum_{j=1}^m a_j\cdot c_1(W_j),$$

where $c_1(W_j)$ is the first Chern class of the holomorphic line bundle $[W_j]$ in the Dolbeault cohomology group $H^1(X,\Omega^1_X)$.

Proof of Claim. — Since Δ^0 is a linear map, without loss of generality, it suffices to prove Claim when $a_1 = 1, a_2 = a_3 = \cdots = a_m = 0$. Suppose that W_1 is defined by a holomorphic function f_i in U_i for $i = 1, \dots, M$ and the transition function $g_{ij} = \frac{f_i}{f_j}$ in U_{ij} for $i, j = 1, \dots, M$. Then the element $1 \cdot W_1 \in H^0(X, \iota_* \mathcal{O}_{\widetilde{W}})$ can be represented by a Čech 0-cocycle σ as

$$\sigma = \bigoplus_{1 \leqslant i_1 \leqslant M} (1_{W_1})\big|_{U_{i_1}} \in \bigoplus_{1 \leqslant i_1 \leqslant M} \iota_* \mathcal{O}_{\widetilde{W}}(U_{i_1}). \tag{4.11}$$

A preimage η of σ under H can be taken as

$$\eta = \bigoplus_{1 \leqslant i_1 \leqslant M} \left(\frac{1}{2\pi\sqrt{-1}} \frac{\mathrm{d}f_{i_1}}{f_{i_1}} \right)\Bigg|_{U_{i_1}} \in \bigoplus_{1 \leqslant i_1 \leqslant M} \Omega^1_X(\log W)(U_{i_1}). \tag{4.12}$$

The Čech one-cocycle $\mathrm{d}^0_{\log}(\eta)$ takes the form

$$\begin{aligned} \mathrm{d}^0_{\log}(\eta) = \bigoplus_{1 \leqslant i_1 < i_2 \leqslant M} \left(\frac{1}{2\pi\sqrt{-1}} \frac{\mathrm{d}f_{i_1}}{f_{i_1}} - \frac{1}{2\pi\sqrt{-1}} \frac{\mathrm{d}f_{i_2}}{f_{i_2}} \right)\Bigg|_{U_{i_1 i_2}} \\ \in \bigoplus_{1 \leqslant i_1 < i_2 \leqslant M} \Omega^1_X(\log W)(U_{i_1 i_2}). \end{aligned} \tag{4.13}$$

Since $f_{i_1} = g_{i_1 i_2} f_{i_2}$,

$$\frac{1}{2\pi\sqrt{-1}} \frac{\mathrm{d}f_{i_1}}{f_{i_1}} - \frac{1}{2\pi\sqrt{-1}} \frac{\mathrm{d}f_{i_2}}{f_{i_2}} = \frac{1}{2\pi\sqrt{-1}} \mathrm{d}(\log g_{i_1 i_2}) \in \Omega^1_X(U_{i_1 i_2}). \tag{4.14}$$

Therefore we can lift η to a Čech 1-cocycle ξ as

$$\xi := \bigoplus_{1 \leqslant i_1 < i_2 \leqslant M} \left(\frac{1}{2\pi\sqrt{-1}} \mathrm{d}(\log g_{i_1 i_2}) \right)\Bigg|_{U_{i_1 i_2}} \in \bigoplus_{1 \leqslant i_1 < i_2 \leqslant M} \Omega^1_X(U_{i_1 i_2}). \tag{4.15}$$

By Proposition §1.1 in [7], we conclude that ξ is the first Chern class of W_1 as an $(1,1)$ form. Therefore, the proof of Claim is complete. □

Now we proceed to prove Theorem 1.14. Let $\omega \in H^0(X, \Omega^1_X(\log W))$. Then the first Chern class of $\mathrm{Res}(\omega)$ in the Dolbeault cohomology is trivial. Since X has Property (H), the $\mathcal{Q}$-flat class of $\mathrm{Res}(\omega)$ is trivial by Corollary 3.17. Therefore, there is a closed meromorphic 1-form ω_1 with the residue class $\mathrm{Res}(\omega)$. Since the residue divisor of $\omega - \omega_1$ is zero, $\omega - \omega_1$ is a holomorphic 1-form by (4.7). This completes the proof of Theorem 1.14. □

5. Forms with higher order poles

In this section, we will study the closed meromorphic 1-forms with poles of higher orders. We first refine Hodge and Atiyah's criterion as follows.

Theorem 5.1. — *Let X be a compact complex manifold and W a reduced divisor of X. Suppose that $D \in H^0(X, R^1(W))$ is a $\mathbb{C}$-linear formal sum of divisors of X supported on W. Let k be a nonnegative integer. The following statements are equivalent.*

- *$J_k(F(D))$ is trivial in $H^1(X, \mathrm{d}\Omega^0(kW))$, where $J_k : H^1(X, \mathrm{d}\Omega^0) \to H^1(X, \mathrm{d}\Omega^0(kW))$ is the natural homomorphism associated with the homomorphism of sheaves $j_k : \mathrm{d}\Omega^0 \to \mathrm{d}\Omega^0(kW)$;*
- *there is a closed meromorphic 1-form $\phi \in H^0(X, \Phi((k+1)W))$ with residue divisor D.*

Proof of Theorem 5.1. — By Lemma 3.2, we have the following short exact sequence of sheaves.

$$0 \longrightarrow \mathrm{d}\Omega^0(kW) \xrightarrow{\phi} \Phi^1((k+1)W) \xrightarrow{\psi} R^1(W) \longrightarrow 0. \tag{5.1}$$

By Lemma 3.4 we can compute the sheaf cohomology groups by Čech cohomology groups in the same as the approach in Section 3. In particular, we have the following commutative diagram.

(5.2)

$$\begin{array}{ccccccccc}
0 \longrightarrow & \bigoplus_{i_1} \mathrm{d}\Omega^0(kW)(U_{i_1}) & \longrightarrow & \bigoplus_{i_1} \Phi^1((k+1)W)(U_{i_1}) & \longrightarrow & \bigoplus_{i_1} R^1(W)(U_{i_1}) & \longrightarrow 0 \\
& \big\downarrow {\scriptstyle \mathrm{d}^0_{\mathrm{d}\Omega}} & & \big\downarrow {\scriptstyle \mathrm{d}^0_{\Phi}} & & \big\downarrow {\scriptstyle \mathrm{d}^0_{R}} & \\
0 \longrightarrow & \bigoplus_{i_1<i_2} \mathrm{d}\Omega^0(kW)(U_{i_1 i_2}) & \longrightarrow & \bigoplus_{i_1<i_2} \Phi^1((k+1)W)(U_{i_1 i_2}) & \longrightarrow & \bigoplus_{i_1<i_2} R^1(W)(U_{i_1 i_2}) & \longrightarrow 0 \\
& \big\downarrow {\scriptstyle \mathrm{d}^1_{\mathrm{d}\Omega}} & & \big\downarrow {\scriptstyle \mathrm{d}^1_{\Phi}} & & \big\downarrow {\scriptstyle \mathrm{d}^1_{R}} & \\
0 \longrightarrow & \bigoplus_{i_1<i_2<i_3} \mathrm{d}\Omega^0(kW)(U_{i_1 i_2 i_3}) & & \cdots & & \cdots &
\end{array}$$

Moreover, we also have the following commutative diagrams.

$$\begin{array}{ccc}
\check{H}^0(\mathcal{U}, \mathrm{d}\Omega^0(kW)) & \xrightarrow{\cong} & H^0(X, \mathrm{d}\Omega^0(kW)) \\
\big\downarrow & & \big\downarrow \\
\check{H}^0(\mathcal{U}, \Phi^1((k+1)W)) & \xrightarrow{\cong} & H^0(X, \Phi^1((k+1)W)) \\
\big\downarrow & & \big\downarrow \\
\check{H}^0(\mathcal{U}, R^1(W)) & \xrightarrow{\cong} & H^0(X, R^1(W)) \\
\big\downarrow {\scriptstyle \check{\Delta}^0_k} & & \big\downarrow {\scriptstyle k\Delta^0_k} \\
\check{H}^1(\mathcal{U}, \mathrm{d}\Omega^0(kW)) & \xrightarrow{\cong} & H^1(X, \mathrm{d}\Omega^0(kW)).
\end{array} \tag{5.3}$$

For homomorphism $j_k : \mathrm{d}\Omega^0 \to \mathrm{d}\Omega^0(kW)$, we have a natural homomorphism between the Čech complexes as

$$\begin{tikzcd}
\bigoplus_{i_1} \mathrm{d}\Omega^0(U_{i_1}) \arrow[r] \arrow[d, "\mathrm{d}^0_{\mathrm{d}\Omega}"] & \bigoplus_{i_1} \mathrm{d}\Omega^0(kW)(U_{i_1}) \arrow[d, "\mathrm{d}^0_{\Phi}"] \\
\bigoplus_{i_1<i_2} \mathrm{d}\Omega^0(U_{i_1 i_2}) \arrow[r] \arrow[d, "\mathrm{d}^1_{\mathrm{d}\Omega}"] & \bigoplus_{i_1<i_2} \mathrm{d}\Omega^0(kW)(U_{i_1 i_2}) \arrow[d, "\mathrm{d}^1_{\Phi}"] \\
\bigoplus_{i_1<i_2<i_3} \mathrm{d}\Omega^0(U_{i_1 i_2 i_3}) \arrow[r] & \bigoplus_{i_1<i_2<i_3} \mathrm{d}\Omega^0(kW)(U_{i_1 i_2 i_3}).
\end{tikzcd} \tag{5.4}$$

Hence, the following commutative diagram holds.

$$\begin{tikzcd}
\check{H}^1(\mathcal{U}, \mathrm{d}\Omega^0) \arrow[r, "\check{J}_k"] \arrow[d, "\cong"] & \check{H}^1(\mathcal{U}, \mathrm{d}\Omega^0(kW)) \arrow[d, "\cong"] \\
H^1(X, \mathrm{d}\Omega^0) \arrow[r, "J_k"] & H^1(X, \mathrm{d}\Omega^0(kW)),
\end{tikzcd} \tag{5.5}$$

where $\check{J}_k$ is induced from diagram (5.4). Repeating the diagram chasing, we have that the map Δ^0_k factors through the $\mathcal{Q}$-flat class map F, that is,

$$H^0(X, R^1(W)) \xrightarrow{F} H^1(X, \mathrm{d}\Omega^0) \xrightarrow{J_k} H^1(X, \mathrm{d}\Omega^0(kW)), \tag{5.6}$$

where $J_k \circ F = \Delta^0_k$. Therefore, by the exactness of (5.3) the proof of Theorem 5.1 is complete. □

Proof of Theorem 1.11. — Notice that for each closed meromorphic 1-form $\phi \in H^0(X, \Phi(*))$, there is a reduced divisor W of X and an non-negative integer k such that $\phi \in H^0(X, \Phi(kW))$. Consider the short exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow \Omega^0(kW) \longrightarrow \mathrm{d}\Omega^0(kW) \longrightarrow 0. \tag{5.7}$$

In addition to (5.2), we have the following commutative diagrams.

(5.8)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{i_1} \mathbb{C}(U_{i_1}) & \longrightarrow & \bigoplus_{i_1} \Omega^0(kW)(U_{i_1}) & \longrightarrow & \bigoplus_{i_1} \mathrm{d}\Omega^0(kW)(U_{i_1}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{d}^0_{\mathbb{C}}} & & \downarrow{\scriptstyle \mathrm{d}^0_{\Omega}} & & \downarrow{\scriptstyle \mathrm{d}^0_{\mathrm{d}\Omega}} & & \\
0 & \longrightarrow & \bigoplus_{i_1<i_2} \mathbb{C}(U_{i_1 i_2}) & \longrightarrow & \bigoplus_{i_1<i_2} \Omega^0(kW)(U_{i_1 i_2}) & \rightarrow & \bigoplus_{i_1<i_2} \mathrm{d}\Omega^0(kW)(U_{i_1 i_2}) & \rightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{d}^1_{\mathbb{C}}} & & \downarrow{\scriptstyle \mathrm{d}^1_{\Omega}} & & \downarrow{\scriptstyle \mathrm{d}^1_{\mathrm{d}\Omega}} & & \\
0 & \rightarrow & \bigoplus_{i_1<i_2<i_3} \mathbb{C}(U_{i_1 i_2 i_3}) & \rightarrow & \bigoplus_{\cdots} \Omega^0(kW)(U_{i_1 i_2 i_3}) & \rightarrow & \bigoplus_{\cdots} \mathrm{d}\Omega^0(kW)(U_{i_1 i_2 i_3}) & \rightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{d}^2_{\mathbb{C}}} & & \downarrow{\scriptstyle \mathrm{d}^2_{\Omega}} & & \downarrow{\scriptstyle \mathrm{d}^2_{\mathrm{d}\Omega}} & & \\
 & & \cdots & & \cdots & & \cdots & &
\end{array}$$

and

(5.9)
$$\begin{array}{ccccccc}
\check{H}^1(\mathcal{U}, \Omega^0(kW)) & \longrightarrow & \check{H}^1(\mathcal{U}, \mathrm{d}\Omega^0(kW)) & \xrightarrow{\check{\delta}^1_k} & \check{H}^2(\mathcal{U}, \mathbb{C}) & \longrightarrow & \check{H}^2(\mathcal{U}, \Omega^0(kW)) \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\
H^1(X, \Omega^0(kW)) & \longrightarrow & H^1(X, \mathrm{d}\Omega^0(kW)) & \xrightarrow{\delta^1_k} & H^2(X, \mathbb{C}) & \longrightarrow & H^2(X, \Omega^0(kW)).
\end{array}$$

Combining (5.3) and (5.9), we have a homomorphism $\delta^1_k \circ \Delta^0_k$ from $H^0(X, R^1(W))$ to $H^2(X, \mathbb{C})$ such that the following commutative diagram holds.

(5.10)
$$\begin{tikzcd}
 & H^0(X, R^1(W)) \arrow[d, "\Delta^0_k"] \arrow[dr, "\delta^1_k \circ \Delta^0_k"] & & \\
H^1(X, \Omega^0(kW)) \arrow[r] & H^1(X, \mathrm{d}\Omega^0(kW)) \arrow[r, "\delta^1_k"] & H^2(X, \mathbb{C}) \arrow[r] & H^2(X, \Omega^0(kW)).
\end{tikzcd}$$

By diagram chasing, we can conclude that $(\delta^1_k \circ \Delta^0_k)(D)$ is the first Chern class of D in the De Rham cohomology for each $\mathbb{C}$-linear finite formal sum of divisors D. By Poincaré duality, we conclude Theorem 1.11. □

Next we show that for manifolds with Property (H) a decomposition is possible.

Proof of Theorem 1.15. — Let $\phi \in H^0(X, \Phi(*))$. Choose a reduced divisor W and a nonnegative integer k such that $\phi \in H^0(X, \Phi(kW))$. Thanks to (5.10) and the fact that Δ^0_k factors through $H^1(X, \mathrm{d}\Omega^0)$, we have the following commutative diagram.

$$
\begin{tikzcd}
H^0(X, R^1(W)) \arrow[d, "F"] \arrow[ddr, "c_1"] & \\
H^1(X, \mathrm{d}\Omega^0) \arrow[d, "J_k"] \arrow[dr, "\delta^1"] & \\
H^1(X, \mathrm{d}\Omega^0(kW)) \arrow[r, "\delta^1_k"] & H^2(X, \mathbb{C}).
\end{tikzcd}
\tag{5.11}
$$

Since X has Property (H), δ^1 is injective; hence $F(\mathrm{Res}(\phi)) = 0$. Therefore, we can find a closed logarithmic form $\widetilde{\phi} \in H^0(X, \Phi(W))$ such that $\mathrm{Res}(\phi) = \mathrm{Res}(\widetilde{\phi})$. Then $\psi := \phi - \widetilde{\phi}$ is a locally exact meromorphic 1-form. This completes the proof of Theorem 1.15 □

6. Constructing pluriharmonic functions with mild singularities

In this section, we will investigate Question 1.2. Our construction of pluriharmonic functions is by integrating closed meromorphic 1-forms. Notice that the integration often results in a multi-valued function; the obstructions are twofold, that is, the long period vector(1) and the short period vector(2) (to be defined in the following). In order to get a single-valued function, we construct a conjugate, namely, a closed anti-meromorphic 1-form, for each closed meromorphic 1-form, so that the cancellation of the periods is possible when integrating the sum of the conjugate pair.

In what follows, we always assume that X is a compact algebraic manifold.

6.1. Decorated log pairs, period vectors and conjugate pairs

In this subsection, we will introduce some notions for the purpose of calculating the obstructions explicitly.

(1) We name it the long period vector because it represents the obstruction corresponding to the integration along closed curves on $X\backslash W$.
(2) We name it the short period vector because it represents the obstruction corresponding to the integration along a small circle around W.

Definition 6.1. — *Let X be a compact algebraic manifold and W a reduced divisor of X. We call the pair (X, W) a log pair. We say that a closed meromorphic 1-form or a closed anti-meromorphic 1-form belongs to the log pair (X, W) if it is defined on X with singularities on W.*

Definition 6.2. — *Let (X, W) be a log pair. Assume that W has the irreducible decomposition $W = \bigcup_{j=1}^{l} W_j$. Take a point $p \in X\backslash W$ and a basis $\{\gamma_1, \ldots, \gamma_m\}$ of $H_1(X, \mathbb{C})$ such that γ_i is a smooth Jordan curve based at p and contained in $X\backslash W$ for $1 \leqslant i \leqslant m$.We call the quadruple $\big(X, W, (W_1, \ldots, W_l), (\gamma_1, \ldots, \gamma_m)\big)$ a decorated log pair of the log pair (X, W).*

Definition 6.3. — *Let $G = (X, W)$ be a log pair, Φ a closed meromorphic 1-form belonging to G, and $A = \big(X, W, (W_1, \ldots, W_l), (\gamma_1, \ldots, \gamma_m)\big)$ a decorated log pair of G. We call $(b_1, \ldots, b_m)$ the long period vector of Φ with respect to A if*

$$b_i = \int_{\gamma_i} \Phi \text{ for } 1 \leqslant i \leqslant m.$$

Before defining the short period vector, we state the following lemma.

Lemma 6.4. — *Let (X, W) be a log pair and $W = \bigcup_{i=1}^{l} W_i$ the irreducible decomposition of W. Assume Φ is a closed meromorphic 1-form on X with singularities on W. Let a_1, a_2 be two smooth points of W_i; suppose $D_1, D_2 \subset X$ are two one-dimensional analytic discs intersecting W_i transversally at a_1, a_2, respectively; suppose S_k is a small disc contained in D_k centered at a_k with ∂S_k the counterclockwise oriented circle boundary of S_k for $k = 1, 2$. Then the following integrals are equal.*

$$\int_{\partial S_1} \Phi = \int_{\partial S_2} \Phi.$$

Proof. — Since W_i is irreducible, there is a proper subvariety $\operatorname{Sing} W_i$ of W_i, such that $a_1, a_2 \in W_i^{\mathrm{Reg}} := W_i \backslash \operatorname{Sing} W_i$ and W_i^{Reg} is connected. Hence, we can find a smooth curve $\gamma \subset W_i^{\mathrm{Reg}}$ connecting a_1, a_2, and an open set U of γ in W_i^{Reg} such that $\gamma \subset U \subset\subset W_i^{\mathrm{Reg}}$. By Theorem (5.2) of [8, §4], U has a tubular neighborhood E in X in the sense that

$$\pi : E \longrightarrow U \text{ is a disc bundle; } J : E \hookrightarrow X \text{ is an embedding.}$$

Notice that $J(E)$ is an open set of X and the zero section of E is mapped diffeomorphically to U. Without loss of generality, we assume $J(E) \cap W = U$.

When S_i is small enough, we have that $S_i \subset J(E)$ and ∂S_i is homotopic to the counterclockwise oriented boundary Γ_i of $J(\pi^{-1}(a_i))$ for $i = 1, 2$.

Notice that the homotopies can be chosen away from W; moreover, Γ_1 and Γ_2 are diffeomorphic in the circle bundle (the boundary of E) over γ. Therefore, in a same way as the proof of Lemma B.1, we can find smooth two-chains $\sum_{i=1}^p a_i\sigma_i^1$, $\sum_{j=1}^q b_j\sigma_j^2$ and $\sum_{k=1}^r c_k\sigma_k^3$, such that they are disjoint from W and

$$\partial\left(\sum_{i=1}^p a_i\sigma_i^1\right)=\partial S_1-\Gamma_1;\ \partial\left(\sum_{j=1}^q b_j\sigma_j^2\right)=\partial S_2-\Gamma_2;\ \partial\left(\sum_{k=1}^r c_k\sigma_k^3\right)=\Gamma_1-\Gamma_2.$$

Since Φ is well-defined on $\sigma_i^1, \sigma_j^2, \sigma_k^3$, we can apply the Stokes theorem as follows.

$$0=\int_{\partial\left(\sum_{i=1}^p a_i\sigma_i^1\right)}\mathrm{d}\Phi=\int_{\partial S_1}\Phi-\int_{\Gamma_1}\Phi;$$
$$0=\int_{\partial\left(\sum_{j=1}^q b_j\sigma_j^2\right)}\mathrm{d}\Phi=\int_{\partial S_2}\Phi-\int_{\Gamma_2}\Phi;$$
$$0=\int_{\partial\left(\sum_{k=1}^r c_k\sigma_k^3\right)}\mathrm{d}\Phi=\int_{\Gamma_1}\Phi-\int_{\Gamma_2}\Phi.$$

We conclude Lemma 6.4. □

Definition 6.5. — *Let $G=(X,W)$ be a log pair, Φ a closed meromorphic* 1*-form belonging to G, and $A=\big(X,W,(W_1,\dots,W_l),(\gamma_1,\dots,\gamma_m)\big)$ a decorated log pair of G. We call $(d_1,\dots,d_l)$ the short period vector of Φ with respect to A if*

$$d_j=\int_{\partial S_j}\Phi \ \textit{ for } 1\leqslant i\leqslant l.$$

Here S_j, as in Lemma 6.4, is a small analytic disc intersecting W_j transversally at a smooth point $a\in W_j$; ∂S_j is the counterclockwise oriented circle boundary of S_j. By Lemma 6.4, this definition is independent of the choice of the disc S_j.

Remark 6.6. — The residue divisor $\mathrm{Res}(\Phi)$ of a closed meromorphic 1-form Φ satisfies the following relation.

$$\mathrm{Res}(\Phi)=\sum_{j=1}^l d_jW_j,$$

where d_j is the j^{th} component of the short period vector of Φ.

Remark 6.7. — A log pair $G=(X,W)$ may have many decorated log pairs. Firstly, we can rearrange the order of the irreducible components in the decomposition $W=\bigcup_{j=1}^l W_j$; secondly, we can choose different bases for $H_1(X,\mathbb{C})$. More precisely, let A and A' be two decorated log pair associated with the log pair G; let Φ be a meromorphic 1-form belonging to G. Then we have the following transformation rules for the period vectors with respect to A and A'.

$$\begin{aligned}(b_1^{A'},\dots,b_m^{A'}) &= (b_1^{A},\dots,b_m^{A})\cdot P_A^{A'} + (d_1^{A},\dots,d_l^{A})\cdot Q_A^{A'},\\ (d_1^{A'},\dots,d_l^{A'}) &= (d_1^{A},\dots,d_l^{A})\cdot J_A^{A'}.\end{aligned} \tag{6.1}$$

Here $P_A^{A'}$, $Q_A^{A'}$ and $J_A^{A'}$ are constant matrices depending on A and A' only; $J_A^{A'}$ is an $l\times l$ matrix corresponding to a certain permutation; $P_A^{A'}$ is an $m\times m$ nondegenerate matrix; $Q_A^{A'}$ is an $l\times m$ matrix.

Remark 6.8. — Let Φ_1,Φ_2 belong to a log pair $G=(X,W)$. If Φ_1 and Φ_2 have the same long period vector and short period vector with respect to one decorated log pair of G, then so do they with respect to any decorated log pairs of G by Remark 6.7. Hence we can just say Φ_1,Φ_2 have the same long period vector and short period vector.

DEFINITION 6.9. — *Let $G=(X,W)$ be a log pair. Let Φ (resp. $\widehat{\Phi}$) be a meromorphic 1-form (resp. an anti-meromorphic 1-form) belonging to G. We call $(\Phi,\widehat{\Phi})$ a conjugate pair belonging to G if and only if there is a decorated log pair $A=\big(X,W,(W_1,\dots,W_l),(\gamma_1,\dots,\gamma_m)\big)$ of G such that the following holds.*

$$\begin{cases} b_i=-\widehat{b}_i, & 1\leqslant i\leqslant m,\\ c_j=-\widehat{c}_j, & 1\leqslant j\leqslant l.\end{cases} \tag{6.2}$$

Here $(b_1,\dots,b_m)$ (resp. $(\widehat{b}_1,\dots,\widehat{b}_m)$) is the long period vector of Φ (resp. $\widehat{\Phi}$) with respect to A; $(c_1,\dots,c_l)$ (resp. $(\widehat{c}_1,\dots,\widehat{c}_l)$) is the short period vector of Φ (resp. $\widehat{\Phi}$) with respect to A. In this case, we call $\widehat{\Phi}$ an anti-meromorphic conjugate of Φ and Φ a meromorphic conjugate of $\widehat{\Phi}$. For convenience, we call Φ a conjugate of $\widehat{\Phi}$, and vice versa.

Remark 6.10. — By Remark 6.7, the definition of conjugate pair does not depend on the choice of the decorated log pair A; that is, if $(\Phi,\widehat{\Phi})$ is a conjugate pair, then (6.2) holds for every decorated pair of (X,W).

PROPOSITION 6.11. — *Let $G=(X,W)$ be a log pair. Let $\widehat{\Phi}$ be a closed anti-meromorphic 1-form belonging to G. Then the conjugate of $\widehat{\Phi}$ is unique modulo the differential of a meromorphic function. To be more precise, if $(\Phi_1,\widehat{\Phi})$ and $(\Phi_2,\widehat{\Phi})$ are two conjugate pairs belonging to G, then $\Phi_1-\Phi_2=\mathrm{d}f$ where f is a meromorphic function on X with singularities on W.*

Proof. — By assumption, $\Phi_1 - \Phi_2$ is a closed meromorphic with vanishing long period vector and short period vector (see Remark 6.8). Then by Lemma 6.12 in the following, we draw the conclusion. ☐

LEMMA 6.12 (Uniqueness Lemma). — *Let $G = (X, W)$ be a log pair and Φ a closed meromorphic 1-form belonging to G. Assume the long period vector of Φ and the short period vector of Φ are both zero. Then Φ is the differential of a meromorphic function on X.*

Proof. — Let $A = \big(X, W, (W_1, \dots, W_l), (\tau_1, \dots, \tau_m)\big)$ be a decorated log pair of G. Then the smooth Jordan curves $\tau_1, \dots, \tau_m$ form a basis of the singular homology $H_1(X, \mathbb{C})$, where τ_i is contained in $X \backslash W$ with a common base point $p \in X$ for $1 \leqslant i \leqslant m$. For each point $x \in X \backslash W$, we can find a smooth curve γ_x connecting p and x. Define function $f(x)$ as follows.

$$f(x) := \int_{\gamma_x} \Phi.$$

We shall prove that $f(x)$ is well-defined, or equivalently, the integral is independent of the curve γ_x connecting p and x. It suffices to prove that for any smooth Jordan $\Gamma \subset X \backslash W$ based at p,

$$\int_\Gamma \Phi = 0. \tag{6.3}$$

Since Γ is homologous to $\sum_{i=1}^m a_i \tau_i$, by Lemma B.1 we can find finitely many smooth 2-simplexes $\{\sigma_j\}_{j=1}^J$ such that the following properties hold.

(1) $\sigma_j : \Delta_2 \to X$ is smooth for $1 \leqslant j \leqslant J$. Here $\Delta_2 := \{(x, y) \subset \mathbb{R}^2 | 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1, x + y \leqslant 1\}$ is the standard 2-simplex.

(2) $\partial\big(\sum_{j=1}^J b_j \sigma_j\big) = \Gamma - \sum_{k=1}^m a_k \tau_k$, where $0 \neq b_j \in \mathbb{C}$ for $1 \leqslant j \leqslant J$. (By abuse of notation, we identify the image of Γ with its corresponding 1-cycle; the same for $\tau_k, 1 \leqslant k \leqslant m$.)

(3) There are finitely many 1-simplexes $\{\widetilde{\tau}_l\}_{l=1}^L$ and finitely many 0-simplexes $\{A_n\}_{n=1}^N$ such that

$$\partial\sigma_j = \sum_{l=1}^L c_{jl} \widetilde{\tau}_l \text{ with } c_{jl} \in \{0, 1, -1\}, \qquad \text{for } 1 \leqslant j \leqslant J, 1 \leqslant l \leqslant L;$$

$$\partial\widetilde{\tau}_l = \sum_{n=1}^K \mathrm{d}_{ln} A_n \text{ with } \mathrm{d}_{ln} \in \{0, 1, -1\}, \quad \text{for } 1 \leqslant l \leqslant L, 1 \leqslant n \leqslant N.$$

Notice that $\{\Gamma, \tau_1, \dots, \tau_m\} \subset \{\widetilde{\tau}_l\}_{l=1}^L$ and $p \in \{A_n\}_{n=1}^N$.

(4) The above $\{\sigma_j\}, \{\widetilde{\tau}_l\}$ and $\{A_n\}$ are transversal to W in the sense that: $A_n \notin W$ for $1 \leqslant n \leqslant N$; $\widetilde{\tau}_l \cap W = \varnothing$ for $1 \leqslant l \leqslant L$; $\sigma_j \cap \mathrm{Sing}(W) = \varnothing$ and σ_j intersects W transversally for $1 \leqslant j \leqslant J$.

For $\sigma_j, 1 \leqslant j \leqslant J$, denote by $a_1^j, \dots, a_{i_j}^j \subset \Delta_2$ the intersection points of σ_j with W. Take small circles $S_1^j, \dots, S_{i_j}^j \subset \Delta_2$ around $a_1^j, \dots, a_{i_j}^j$, respectively. Notice that $\sigma_j^*(\Phi)$ is a well-defined closed 1-form on $\widehat{\Delta}_2^j := \Delta_2 \backslash \left(\cup_{q=1}^{j_i} S_q^j\right), 1 \leqslant j \leqslant J$. By Stokes' theorem we have for $1 \leqslant j \leqslant J$

$$\begin{aligned} (6.4) \quad 0 = \int_{\widehat{\Delta}_2^j} \sigma_j^*(\mathrm{d}\Phi) &= \int_{\partial \widehat{\Delta}_2^j} \sigma_j^*(\Phi) \\ &= \sum_{l=1}^{L} \pm c_{jl} \int_{[0,1]} \widetilde{\tau}_l^*(\Phi) + \sum_{q=1}^{i_j} \pm \int_{S_q^j} \sigma_j^*(\Phi). \end{aligned}$$

When S_i^j is small enough, by Lemma 6.4 and the assumption that the short period vector of Φ is zero, we get

$$\sum_{q=1}^{i_j} \pm \int_{S_q^j} \sigma_j^*(\Phi) = 0, \ 1 \leqslant j \leqslant J. \tag{6.5}$$

Since $\partial\left(\sum_{j=1}^{J} b_j \sigma_j\right) = \Gamma - \sum_{k=1}^{m} a_k \tau_k$, we obtain

$$\int_\Gamma \Phi = \sum_{k=1}^{m} a_k \int_{\tau_k} \Phi = 0.$$

The last identity is because the long period vector of Φ is zero.

Therefore, we proved that $f(x)$ is well-defined. It is easy to see that f is a meromorphic function on X with singularities on W. Hence, we conclude Lemma 6.12. $\square$

Remark 6.13. — Lemma 6.12 holds for closed anti-meromorphic 1-forms in a similar way, and hence Proposition 6.11 also holds for conjugates of a closed meromorphic 1-form. That is, if there are two conjugate pairs $(\Phi, \widehat{\Phi}_1)$ and $(\Phi, \widehat{\Phi}_2)$, then $\widehat{\Phi}_1 - \widehat{\Phi}_2 = \mathrm{d}f$ where f is an anti-meromorphic function on X with singularities on W.

6.2. The existence of anti-meromorphic conjugates

In this subsection we will prove a existence theorem for anti-meromorphic conjugates of a closed meromorphic 1-form belonging to a log pair with an extra condition.

Lemma 6.14. — *Let $A = (X, W, (W_1, \dots, W_l), (\gamma_1, \dots, \gamma_m))$ be a decorated log pair of the log pair $G = (X, W)$. Let $\widehat{W}$ be an effective, ample divisor with the support contained in W. Then for any vector $\vec{b} \in \mathbb{C}^m$ and* *the $\mathbb{C}$-linear formal sum of divisors $D \in \bigoplus_{i=1}^{l} \mathbb{C}W_i$ with $c_1(D) = 0$, there is a closed meromorphic 1-form belonging to G with long period vector $\vec{b}$ and residue divisor D.*

Proof. — According to Theorem 1.6, there exists a closed meromorphic 1-form $\Phi \in H^0(X, \Phi^1(W))$ with residue divisor D. In the same way as Proposition 2.10, we can find Ψ, a locally exact meromorphic 1-form, such that $\Psi \in H^0(X, \Phi^1(W))$ and $\Psi+\Phi$ has long period vector $\vec{b}$. This completes the proof of Lemma 6.14. □

Next we will prove the following existence theorem.

Theorem 6.15. — *Let $G = (X, W)$ be a log pair and Φ a closed meromorphic 1-form belonging to G. Suppose there is an effective, ample divisor of X whose support is contained in W. Then there exists a conjugate of Φ belonging to G.*

Proof. — Take a decorated log pair $A = \big(X, W, (W_1, \ldots, W_l), (\gamma_1, \ldots, \gamma_m)\big)$ of G. Denote by $(b_1, \ldots, b_m)$ the long period vector of Φ. Let $\mathrm{Res}(\Phi)$ be the residue divisor of Φ with the form $\mathrm{Res}(\Phi) = \sum_{i=1}^{l} a_i W_i$. Since Φ is a closed meromorphic 1-form, the first Chern class of $\mathrm{Res}(\Phi)$ is zero, that is, $\sum_{i=1}^{l} a_i \cdot c_1(W_i) = 0 \in H^2(X, \mathbb{C})$. Since $\sum_{i=1}^{l}(-\overline{a_i}) \cdot c_1(W_i) = 0 \in H^2(X, \mathbb{C})$, by Lemma 6.14, we can derive a closed meromorphic 1-form Ψ belonging to G, with residue divisor $\sum_{i=1}^{l}(-\overline{a_i}) \cdot W_i$ and long period vector $(-\overline{b_1}, \ldots, -\overline{b_m})$.

Take the complex conjugate $\overline{\Psi}$ of Ψ; that is, if we write $\Psi = \psi_1 + \sqrt{-1}\psi_2$ where ψ_1 and ψ_2 are closed real 1-forms with singularities on W, then $\overline{\Psi} = \psi_1 - \sqrt{-1}\psi_2$. It is easy to verify that $\overline{\Psi}$ is an anti-meromorphic conjugate of Φ. This completes the proof of Theorem 6.15. □

6.3. Constructing pluriharmonic functions with log singularities

In this subsection we will construct pluriharmonic functions with log poles by integrating the sum of conjugate pairs.

Theorem 6.16. — *Let $G = (X, W)$ be a log pair and $(\Phi, \widehat{\Phi})$ a conjugate pair belonging to G. Then the following integral is well-defined on $X \backslash W$ and gives a pluriharmonic function with singularities on W.*

$$h_{\Phi\widehat{\Phi}}(z) := \int_{\gamma_z^p} (\Phi + \widehat{\Phi}). \tag{6.6}$$

Here $p \in X \backslash W$ is a fixed point and γ_z^p is a smooth curve connecting p and $z \in X \backslash W$.

Proof. — Take a decorated log pair $A=\big(X,W,(W_1,\dots,W_l),(\gamma_1,\dots,\gamma_m)\big)$ of G. It suffices to prove that along each Lipschitz loop $\gamma\subset X\backslash W$ the following holds.

$$\int_\gamma(\Phi+\widehat{\Phi})=0. \tag{6.7}$$

In the same way as the proof of Lemma 6.12, we can find complex numbers $\{a_i\}_{i=1}^m$ and $\{b_j\}_{j=1}^l$ so that

$$\int_\gamma(\Phi+\widehat{\Phi})=\sum_{i=1}^m a_i\int_{\gamma_i}(\Phi+\widehat{\Phi})+\sum_{j=1}^l b_j\int_{\partial S_j}(\Phi+\widehat{\Phi}). \tag{6.8}$$

Here ∂S_j is the circle boundary of a small analytic disc intersecting W_j at its smooth point. Notice that the sum of the long period vector (resp. the short period vector) of Φ and the long period vector (resp. the short period vector) of $\widehat{\Phi}$ is zero. Then each term on the right hand side of (6.8) vanishes. We draw the conclusion. □

Definition 6.17. — *We call $h_{\Phi\widehat{\Phi}}$ a pluriharmonic functions coming from the conjugate pair $(\Phi,\widehat{\Phi})$.*

Proof of Theorem 1.16. — Let $A=\big(X,W,(W_1,\dots,W_l),(\gamma_1,\dots,\gamma_m)\big)$ be a decorated log pair of the log pair (X,W). By Theorem 6.15, for each closed meromorphic 1-form Φ with poles on W, there exists a conjugate $\widehat{\Phi}$ of Φ. By Theorem 6.16, we conclude Theorem 1.16. □

Let $G=(X,W)$ be a log pair. Denote by M_G the vector space of meromorphic functions on X with singularities on W; denote by $\widehat{M_G}$ the vector space of anti-meromorphic functions with singularities on W. Define $T_G:=M_G+\widehat{M_G}$. Next we will establish the uniqueness of pluriharmonic functions and estimate the dimension of the vector spaces of pluriharmonic functions coming from conjugate pairs (modulo T_G).

Lemma 6.18. — *Let $G=(X,W)$ be a log pair. Assume $(\Phi_1,\widehat{\Phi}_1)$ and $(\Phi_2,\widehat{\Phi}_2)$ are two conjugate pairs belonging to G. Then $h_{\Phi_1\widehat{\Phi}_1}=h_{\Phi_2\widehat{\Phi}_2}$ (modulo T_G) if and only if Φ_1 and Φ_2 have the same long period vector and the same short period vector.*

Proof. — Assume that Φ_1 and Φ_2 have the same long period vector and the same short period vector. Then so do $\widehat{\Phi}_1$ and $\widehat{\Phi}_2$. By Lemma 6.12, the following functions are well-defined on X.

$$f:=\int_{\gamma_z^p}(\Phi_1-\Phi_2);\quad \bar{g}:=\int_{\gamma_z^p}(\widehat{\Phi}_1-\widehat{\Phi}_2).$$

Since f is meromorphic and $\bar{g}$ is anti-meromorphic, we can thus conclude that $h_{\Phi_1\hat{\Phi}_1} = h_{\Phi_2\hat{\Phi}_2}$ (modulo T_G).

On the other hand if $h_{\Phi_1\hat{\Phi}_1} = h_{\Phi_2\hat{\Phi}_2}$ (modulo T_G), then there are meromorphic function f and anti-meromorphic function $\bar{g}$ on X such that

$$\begin{aligned} &\int_{\gamma_z^p}(\Phi_1+\widehat{\Phi}_1) - \int_{\gamma_z^p}(\Phi_2+\widehat{\Phi}_2) = \bar{g} - f \text{ or equivalently} \\ &\qquad f + \int_{\gamma_z^p}(\Phi_1-\Phi_2) = \bar{g} + \int_{\gamma_z^p}(\widehat{\Phi}_2 - \widehat{\Phi}_1). \end{aligned} \tag{6.9}$$

Notice that the left hand side of (6.9) is holomorphic and the right hand side is anti-holomorphic, away from the singularities of $\Phi_1, \widehat{\Phi}_1, \Phi_2, \widehat{\Phi}_2, f$ and $\bar{g}$. Then both sides are locally constants. Taking differential of the left hand side, we have the following equality on X.

$$\mathrm{d}f + \Phi_1 - \Phi_2 = 0. \tag{6.10}$$

Integrating $\mathrm{d}f + \Phi_1 - \Phi_2$ along γ_i and ∂S_j, we conclude that the long period vector (resp. the short period vector) of Φ_1 and that of Φ_2 are the same with respect to any decorated log pair of G. This completes the proof. □

Theorem 6.19. — *Let $G = (X, W)$ be a log pair. Suppose there is an effective, ample divisor of X whose support is contained in W. Denote by k the dimension of the kernel of the map $\delta^1 \circ \Delta^0 : H^0(X, R(W)) \to H^2(X, \mathbb{C})$. Then the dimension of the vector space of the pluriharmonic functions coming from conjugate pairs with singularities on W (modulo T_G) is $k + \dim H^1(X, \mathbb{C})$.*

Proof. — By Theorem 2.8, the dimension of locally exact meromorphic 1-forms modulo the differentials of meromorphic functions is $\dim H^1(X, \mathbb{C})$. By Theorems 1.5, 1.6 and 1.15, the dimension of the closed meromorphic 1-forms modulo the locally exact meromorphic 1-forms is k. By Theorems 6.15 and 6.16, we can construct pluriharmonic functions for every closed meromorphic 1-form; by Theorem 6.18, these functions are linearly independent modulo T_G. Therefore we conclude Theorem 6.19. □

Remark 6.20. — Denote by V be the vector space of pluriharmonic functions coming from a closed meromorphic 1-form. Then $\dim V/T_G = \infty$. On the other hand, denote by V_2 the vector space of pluriharmonic functions coming from a locally exact meromorphic 1-form. Then $\dim V_2/T_G = \dim H^1(X, \mathbb{C}) < \infty$.

Proof of Theorem 1.17. — Let h be a pluriharmonic function on X of local form (1.1). Then h locally takes the form of

$$h(z) = g_1(z) + g_2(\bar{z}) + \sum_{i=1}^{l} a_i \log |f_i|^2, \tag{6.11}$$

where $a_1, \dots, a_l$ are constants and $g_1, g_2, f_1, \dots, f_l$ are meromorphic functions. Taking differential of h, we have that locally

$$\begin{aligned} \mathrm{d}h &= \partial g_1(z) + \sum_{i=1}^{l} a_i \frac{\partial f_i}{f_i} + \bar{\partial} g_2(\bar{z}) + \sum_{i=1}^{l} a_i \frac{\bar{\partial}\overline{f_i}}{\overline{f_i}}\ ; \\ \Phi &:= \partial g_1(z) + \sum_{i=1}^{l} a_i \frac{\partial f_i}{f_i};\ \ \widehat{\Phi} := \bar{\partial} g_2(\bar{z}) + \sum_{i=1}^{l} a_i \frac{\bar{\partial}\overline{f_i}}{\overline{f_i}}. \end{aligned} \tag{6.12}$$

Notice that Φ is a closed meromorphic 1-form and $\widehat{\Phi}$ is a closed anti-meromorphic 1-form. Moreover, it is clear that $\mathrm{d}h = \Phi + \widehat{\Phi}$ holds globally on X; that is, the definition of Φ and $\widehat{\Phi}$ does not depend on the choice of the local charts. If there is no log term in formula (6.11), then Φ is a locally exact meromorphic 1-form. One can show that h is the integral of the sum of Φ and $\widehat{\Phi}$ in the same way as Theorem 6.16.

Notice that κ by $\kappa(h) = \Phi$ for every $h \in Ph(X)$. Next we will show that κ is injective; that is, for $h_1, h_2 \in Ph(X)$ if $\kappa(h_1) = \kappa(h_2)$, then $h_1 - h_2 \in (K(X) + \overline{K}(X))$. Recall that the singularities of h_i is contained in a divisor of X for $i = 1, 2$. Take an ample, reduced divisor W of X such that the singularities of h_i is contained in W for $i = 1, 2$ and consider the log pair (X, W). Then by Theorem 6.18, we conclude the injectivity of κ.

Finally, we will show that κ is surjective. Let $\Phi \in H^0(X, \Phi^1(*))$. Take an ample, effective, reduced divisor W of X such that the singularities of h_i is contained in W for $i = 1, 2$. Consider the log pair (X, W). Then by Theorem 6.15, we can find a conjugate of Φ. By Theorem 6.16, we conclude the surjectivity.

Similarly, we can show that κ_0 is an isomorphism. Hence we conclude Theorem 1.17. □

Remark 6.21 ([24]). — The log term in the above formula has an interesting explanation related to superfluid vortices in physics. A single vortex has energy growing asymptotically like $\log z_i$ and hence unstable. In low temperature, there will be no free vortices, only clusters of zero total vorticity.

Appendix A. Good Cover Lemma

In this appendix we will present a detailed proof of the following well-known lemma for reader's convenience.

Lemma A.1 (Good cover lemma). — *Let X be a compact complex manifold X and $\mathcal{V}$ an open cover of X. Then, there is a finite open cover $\mathcal{U} := \{U_i\}_{i=1}^M$ of X such that the following properties hold.*

1. *U_i is a Stein space for $i = 1, \ldots, M$;*
2. *Each intersection of U_i is contractible for $i = 1, \ldots, M$, if it is nonempty;*
3. *$\mathcal{U}$ is a refinement of $\mathcal{V}$.*

In order to prove Lemma A.1, we will first prove the following lemma.

Lemma A.2. — *Let X be a complex manifold of complex dimension m. Suppose there are two holomorphic local parametrizations of X as*

$$\Phi_i : V_i \longrightarrow U_i \subset X,\ i = 1, 2, \tag{A.1}$$

where V_i is an open set in $\mathbb{C}^m$ and U_i is biholomorphic to V_i. Moreover, for each point $x = (x_1, \ldots, x_m) \in V_i$ and real number $r > 0$, denote by $B_{i,r}(x)$ the following complex ball in V_i.

$$B_{i,r}(x) := \left\{ (y_1, \ldots, y_m) \in V_i \;\middle|\; \begin{aligned} &|y_1 - x_1|^2 + |y_2 - x_2|^2 \\ &\quad + \cdots + |y_m - x_m|^2 < r^2 \end{aligned} \right\}. \tag{A.2}$$

Then for each point $x \in \Phi_1^{-1}(U_1 \cap U_2)$, there is a positive number R such that for any positive number $r < R$ the complex ball $B_{1,r}(x) \subset \Phi_1^{-1}(U_1 \cap U_2)$ and, moreover, its biholomorphic image $(\Phi_2^{-1} \circ \Phi_1)(B_{1,r}(x))$ is a real convex set in V_2.

Proof. — Fix a point $x \in \Phi_1^{-1}(U_1 \cap U_2)$. Denote by $(z_1, \ldots, z_m)$ the complex coordinates for V_1; denote by $(w_1, \ldots, w_m)$ the complex coordinates for V_2. Without loss of generality, we can assume that the point $x \in \Phi_1^{-1}(U_1 \cap U_2)$ has complex coordinates $(0, \ldots, 0)$ in V_1 and the point $y := (\Phi_2^{-1} \circ \Phi_1)(x)$ has complex coordinates $(0, \ldots, 0)$ in V_2. For convenience, denote by ϕ the restriction of the holomorphic map $\Phi_1^{-1} \circ \Phi_2$ to $\Phi_2^{-1}(U_1 \cap U_2)$, that is,

$$\begin{aligned} &\phi : \Phi_2^{-1}(U_1 \cap U_2) \longrightarrow \Phi_1^{-1}(U_1 \cap U_2), \\ &\qquad (w_1, \ldots, w_m) \longmapsto (z_1, \ldots, z_m) = (\Phi_1^{-1} \circ \Phi_2)(w_1, \ldots, w_m). \end{aligned} \tag{A.3}$$

Note that $\phi(y) = x$. Restricted to a small ball $B_{2,s}(y) \subset \Phi_2^{-1}(U_1 \cap U_2)$, $s > 0$, we have the following Taylor expansion.

$$z_1 = \sum_{i=1}^{n} a_{1i} w_i + O(|w|^2); z_2 = \sum_{i=1}^{n} a_{2i} w_i + O(|w|^2); \dots; \\ z_m = \sum_{i=1}^{n} a_{mi} w_i + O(|w|^2). \tag{A.4}$$

Since ϕ is a biholomorphic map, the matrix $A := (a_{ij})_{i,j=1}^m$ is nondegenerate. Define a real analytic function $F(w_1, \dots, w_m)$ in $B_{2,s}(y)$ as follows.

$$\begin{aligned} F(w_1, \dots, w_m) &= \sum_{i=1}^{m} |z_i(w_1, \dots, w_m)|^2 \\ &= \sum_{i=1}^{m} \left| \sum_{j=1}^{n} a_{ij} w_j + O(|w|^2) \right|^2 \\ &= \sum_{i=1}^{m} \left| \sum_{j=1}^{n} a_{ij} w_j \right|^2 + O(|w|^3) \\ &= (\overline{w}_1, \dots, \overline{w}_m) \cdot \overline{A}^T \cdot A \cdot (w_1, \dots, w_m)^T + O(|w|^3). \end{aligned} \tag{A.5}$$

Take a complex ball $B_{1,\widetilde{R}}(x)$ centered at x such that $B_{1,\widetilde{R}}(x) \subset \phi(B_{2,s}(y))$; then, the set $\phi^{-1}(B_{1,r}(x))$ in V_2 is the same as the set $\{F < r^2\}$ for $0 < r < \widetilde{R}$.

Since $A^T \cdot A$ is a strictly positive definite Hermitian matrix, we can find an $m \times m$ unitary matrix U such that

$$U^T \cdot A^T \cdot A \cdot \overline{U} = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & a_m \end{pmatrix} =: D, \tag{A.6}$$

where a_i is a positive real number for $i = 1, \dots, m$. Take an unitary change of coordinates for $B_{1,\widetilde{R}}(x)$ as

$$(\widetilde{w}_1, \dots, \widetilde{w}_m) = (w_1, \dots, w_m) \cdot U. \tag{A.7}$$

Then F in the new coordinates takes the form of

$$\sum_{i=1}^{m} a_i |\widetilde{w}_i|^2 + O(|\widetilde{w}|^3); \tag{A.8}$$

we denote it by $\widetilde{F}$.

Notice that a unitary transformation of $\mathbb{C}^m$ does not change the shape of domains. Therefore, in order to prove Lemma A.2, it suffices to prove that there exists a positive number R such that $\{\widetilde{F} < r^2\}$ is a real convex domain in V_2 for each $0 < r < R$. Denote the real coordinates of $(\widetilde{w}_1, \dots, \widetilde{w}_m)$ by $(x_1, \dots, x_m, y_1, \dots, y_m)$ where $\widetilde{w}_i = x_i + \sqrt{-1}y_i$ for $i = 1, \dots, m$. Computation of the real Hessian of $\widetilde{F}$ yields that

$$\mathrm{Hess}(\widetilde{F}) = \begin{pmatrix} (\frac{\partial^2 \widetilde{F}}{\partial x_i \partial x_j}) & (\frac{\partial^2 \widetilde{F}}{\partial x_i \partial y_j}) \\ (\frac{\partial^2 \widetilde{F}}{\partial y_i \partial x_j}) & (\frac{\partial^2 \widetilde{F}}{\partial y_i \partial y_j}) \end{pmatrix} = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} + O(|\widetilde{w}|); \tag{A.9}$$

hence the real Hessian of $\widetilde{F}$ is strictly positive in a small neighborhood of y. Therefore, we can choose a positive number R such that $\{\widetilde{F} < r^2\}$ is a real convex domain in V_2 for each $0 < r < R$. □

Now we turn to the proof of Lemma A.1.

Proof of Lemma A.1. — Fix a Riemannian metric g on X. For any points $x, y \in X$, denote by $\mathrm{d}_g(x, y)$ the distance between them with respect to g. Since X is compact, we can have finitely many local parametrizations $\{\phi_i : V_i \to U_i\}_{i=1}^M$ such that the following properties hold.

(1) $V_i = \{z = (z_1, \dots, z_m) \in \mathbb{C}^m \,|\, \mathrm{d}_g(\phi_i(0), \phi_i(z)) < 4\}$ for $1 \leqslant i \leqslant M$.
(2) $U_i \subset X$ and ϕ_i is the biholomorphic map between V_i and U_i for $1 \leqslant i \leqslant M$.
(3) For $1 \leqslant i \leqslant M$, denote by $V_{i,1}$, $V_{i,2}$, $U_{i,1}$ and $U_{i,2}$ the set $\{z \in \mathbb{C}^m \,|\, \mathrm{d}_g(\phi_i(0), \phi_i(z)) < 1\}$, $\{z \in \mathbb{C}^m \,|\, \mathrm{d}_g(\phi_i(0), \phi_i(z)) < 2\}$, $\phi_i(V_{i,1})$ and $\phi_i(V_{i,2})$, respectively. Then, $\{U_{i,1}\}_{i=1}^M$ is an open cover of X.

For convenience, denote by I the index set $\{1, 2, \dots, M\}$. Moreover, similar to formula (A.2), for $1 \leqslant i \leqslant M$, $0 < r < 1$ and each point $x \in V_{i,1}$ with complex coordinates $(x_1, \dots, x_m)$, we denote by $B_{i,r}(x)$ the set $\{(y_1, \dots, y_m) \in V_i \,|\, |y_1 - x_1|^2 + |y_2 - x_2|^2 + \dots + |y_m - x_m|^2 < r^2\}$.

Next, we will construct a special open cover of $V_{i,1}$ for $1 \leqslant i \leqslant M$. Fix i and, for each point $x \in V_{i,1}$, let I_x^i be the index set define by

$$I_x := \{j \in I \,|\, \phi_i(x) \in U_{j,2}\}. \tag{A.10}$$

Then, for each $j \in I_x^i$, we can apply Lemma A.2 and derive a positive number R_j such that for any $0 < r < R_j$ the following properties hold.

- $\phi_i(B_{i,r}(x)) \subset U_{j,2}$;
- $\phi_j^{-1}(\phi_i(B_{i,r}(x)))$ is a real convex set in $V_{j,2}$.

Choose a positive number r_x such that $r_x < \min_{j \in I_x^i}\{R_j\}$, the diameter of $B_{i,r_x}(x)$ with respect to d_g is less than $\frac{1}{3}$, and $\phi_i(B_{i,r}(x)$ is contained in a certain open set in open cover $\mathcal{V}$. Notice that $B_{i,r_x}(x)$ is an open neighborhood of x; moreover, the following properties hold.

($*$) $\phi_i(B_{i,r_x}(x)) \subset U_{j,2}$ for each $j \in I_x^i$,

($\star$) $\phi_j^{-1}(\phi_i(B_{i,r_x}(x)))$ is a real convex set in $V_{j,2}$ for each $j \in I_x^i$.

It is clear that $\left\{B_{i,r_x}(x), x \in V_{i,1}\right\}$ is an open cover of $V_{i,1}$.

Since $\bigcup_{i=1}^M \left\{\phi_i(B_{i,r_x}(x)), x \in V_{i,1}\right\}$ is an open cover of X, we can find a finite subcover Ω of X with the form of

$$\Omega = \bigcup_{i=1}^{M} \left\{\phi_i(B_{i,r_{a_{ij}}}(a_{ij})), a_{ij} \in V_{i,1}, 1 \leqslant j \leqslant N_i\right\}. \tag{A.11}$$

We claim that the cover Ω satisfies the properties required in Lemma A.1. The first and the third properties are clear, for each open set in Ω is biholomorphic to a complex ball in $\mathbb{C}^m$ and is contained in a certain open set in $\mathcal{V}$. Since convext sets are contractible, in order to establish the second property, it suffices to show that each nonempty intersections of open sets in Ω is biholomorphic to a real convex set in $\mathbb{C}^m$.

Let B be an nonempty intersection of open sets in Ω. Without loss of generality, by rearranging the indices, we can assume that B takes the form of

$$B = \bigcap_{i=1}^{M} \bigcap_{j=1}^{l_i} \phi_i(B_{i,r_{a_{ij}}}(a_{ij})), \tag{A.12}$$

where $l_i \leqslant N_i$. Without loss of generality, we can further assume $l_1 \geqslant 1$. Since B is nonempty, by the construction we have $a_{ij} \in V_{1,2}$. Therefore, $1 \in I_{a_{ij}}$ for $1 \leqslant i \leqslant M$ and $1 \leqslant j \leqslant l_i$. By property ($*$) and property ($\star$), we have that $\phi_i(B_{i,r_x}(x)) \subset U_{1,2}$ and $\phi_1^{-1}(\phi_i(B_{i,r_{a_{ij}}}(a_{ij})))$ is a real convex set in $V_{1,2}$ for each for $1 \leqslant i \leqslant M$ and $1 \leqslant j \leqslant l_i$. Since the intersection of convex sets are convex, B is a convex set.

Therefore, we conclude Lemma A.1. □

We also include a proof of the following very good cover lemma for the completeness.

Lemma A.3 (Very good cover). — *Let X be a compact complex manifold, W a normal crossing divisor on X and $\mathcal{V}$ an open cover of X. Then, there is a finite open cover $\mathcal{U} := \{U_i\}_{i=1}^M$ of X such that the following properties hold.*

(1) *U_i is a Stein space for $1 \leqslant i \leqslant M$;*

(2) *Each intersection of U_i, $1 \leqslant i \leqslant M$, is contractible, if it is nonempty;*

(3) *There are only finitely many irreducible components of* $U_i \bigcap W$ *for* $1 \leqslant i \leqslant M$*; moreover, if* W_{ij} *is an irreducible component of* $U_i \cap W$*, then* W_{ij} *is contractible;*
(4) $\mathcal{U}$ *is a refinement of* $\mathcal{V}$*.*

Proof of Lemma A.3. — We first prove the following Claim.

Claim. — *Let* U *be an open set in* $\mathbb{C}^m$*. Suppose that* $W \subset U$ *is a reduced analytic subvariety of codimension one with normal crossing singularities only. For each point* $x \in U$*, there is a Euclidean complex ball* $B_r(x) \subset U$ *such that there are only finitely many irreducible components in the irreducible decomposition of* $W \bigcap B_r(x)$ *and each irreducible component is contractible.*

Proof of Claim. — Since the case $x \in U \backslash W$ is trivial, let $x \in W \bigcap U$; without loss of generality, we assume $x = (0, \ldots, 0) \in U$. By taking the complex ball $B_R := \left\{ z \,\middle|\, |z|^2 < R^2 \right\}$ with radius $R > 0$ small enough, we can assume that $B_R \subset U$ and W is defined in B_R by equation $\prod_{i=1}^{l} f_i = 0$, where $f_1, \ldots, f_l$ are holomorphic functions in B_R such that they are irreducible, pairwise coprime and vanishing at $(0, \ldots, 0)$. Without loss of generality, we assume that $f_1, \ldots, f_l$ take the form of

$$f_i(z) = \sum_{j=1}^{m} a_{ij} z_j + O(|z|^2) \tag{A.13}$$

with $a_{ij} \in \mathbb{C}$ for $1 \leqslant i \leqslant l,\ 1 \leqslant j \leqslant m$ and $z \in B_R$. Since W is a normal crossing divisor, f_i has a nonzero linear part for $1 \leqslant i \leqslant l$.

Define $W_i := \{f_i = 0\} \bigcap B_R$. It is clear that $\bigcup_{i=1}^{l} W_i = W \bigcap B_R$. Hence it suffices to prove that each W_i is contractible when R is small enough for $1 \leqslant i \leqslant l$. In the following, we will prove this for W_1; the proof for others is the same which we will omit.

Without loss of generality, we assume that $a_{11} = -1$ in formula (A.13). By the implicit function theorem, we can solve z_1 in terms of $(z_2, \ldots, z_m)$ as

$$\begin{aligned} z_1 &= h(z_2, \ldots, z_m) \\ &= a_{12} z_2 + a_{13} z_3 + \cdots + a_{1m} z_m + O(|z_2|^2 + \cdots + |z_m|^2). \end{aligned} \tag{A.14}$$

Shrinking the radius R, we have a parametrization $(G, \widetilde{W})$ of W_1 as

$$\begin{aligned} G:\ \widetilde{W} &\longrightarrow W_1 \subset \mathbb{C}^m, \\ (z_2, \ldots, z_m) &\longmapsto (h(z_2, \ldots, z_m), z_2, z_3, \ldots, z_m), \end{aligned} \tag{A.15}$$

where $\widetilde{W}$ is a domain in $\mathbb{C}^{m-1}$ defined by

$$\rho(z_2,\dots,z_m) := \left|h(z_2,\dots,z_m)\right|^2 + \sum_{i=2}^{m} \left|z_i\right|^2 < R^2. \tag{A.16}$$

Since W_1 and $\widetilde{W}$ are biholomorphic, it suffices to show that $\widetilde{W}$ is contractible. Computation yields that

$$\begin{aligned}\rho &= \left|a_{12}z_2 + a_{13}z_3 + \dots + a_{1m}z_m + O(|z|^2)\right|^2 + |z_2|^2 + \dots + |z_m|^2 \\ &= |a_{12}z_2 + a_{13}z_3 + \dots + a_{1m}z_m|^2 + |z_2|^2 + \dots + |z_m|^2 + O(|z|^3).\end{aligned} \tag{A.17}$$

Taking derivatives, we get

$$\begin{aligned}\frac{\partial\rho}{\partial z_2} &= (a_{12}\bar{a}_{12}+1)\bar{z}_2 + a_{12}\bar{a}_{13}\bar{z}_3 + \dots + a_{12}\bar{a}_{1m}\bar{z}_m + O(|z|^2),\\ \frac{\partial\rho}{\partial z_3} &= a_{13}\bar{a}_{12}\bar{z}_2 + (a_{13}\bar{a}_{13}+1)\bar{z}_3 + \dots + a_{13}\bar{a}_{1m}\bar{z}_m + O(|z|^2),\\ &\;\vdots\\ \frac{\partial\rho}{\partial z_m} &= a_{1m}\bar{a}_{12}\bar{z}_2 + a_{1m}\bar{a}_{13}\bar{z}_3 + \dots + (a_{1m}\bar{a}_{1m}+1)\bar{z}_m + O(|z|^2).\end{aligned} \tag{A.18}$$

Multiplying $\frac{\partial\rho}{\partial z_i}$ by z_i, $1 \leqslant i \leqslant m$, and summing up the products, we obtain

$$\sum_{i=1}^{m} \frac{\partial\rho}{\partial z_i} z_i = \left|\sum_{i=1}^{m} a_{1i}z^i\right|^2 + \sum_{i=1}^{m} |z_i|^2 + O(|z|^3). \tag{A.19}$$

Then for $R > 0$ small enough

$$\left|\sum_{i=1}^{m} \frac{\partial\rho}{\partial z_i} z_i\right| > \frac{1}{2}\sum_{i=2}^{m} |z_i|^2 \text{ for } (z_2,\dots,z_m) \in \widetilde{W};$$

hence $\nabla\rho$ is nonzero except at the origin. By Morse theory $\widetilde{W}$ is contractible. This completes the proof of the claim. □

The remainder of the proof is exact the same as the proof of Lemma A.1 except that when choosing $B_{i,r_x}(x)$, we require the following condition in addition

$$\phi_i(B_{i,r_x}(x)) \cap W = \bigcup_{j=1}^{l_i} W_j \tag{A.20}$$

where W_j is irreducible and contractible for $j = 1,\dots,l_i < \infty$. □

Appendix B. Existence of a smooth, transversal two-chain

In this appendix we give a detailed proof of the following lemma.

Lemma B.1. — *Let X be a compact algebraic manifold, $p \in X$ a fixed base point and W a normal crossing divisor. Suppose $\widetilde{\tau}_1, \dots, \widetilde{\tau}_m$ is a basis of the singular homology $H_1(X, \mathbb{C})$, where*

$$\widetilde{\tau}_k : [0,1] \longrightarrow X \text{ is smooth with } \widetilde{\tau}_k(0) = \widetilde{\tau}_k(1) = p \text{ for } 1 \leqslant k \leqslant m.$$

Assume $\widetilde{\tau}$ is also a one-cycle, where $\widetilde{\tau} : [0,1] \to X$ is smooth with $\widetilde{\tau}(0) = \widetilde{\tau}(1) = p$. Moreover we assume that $\widetilde{\tau}([0,1]) \cap W = \varnothing$ and $\widetilde{\tau}_k([0,1]) \cap W = \varnothing$ for $1 \leqslant k \leqslant m$. Then if $\widetilde{\tau}$ is homologous to $\sum_{k=1}^m a_k \widetilde{\tau}_k$ with constants $\{a_k\}_{k=1}^m \subset \mathbb{C}$, we can find finitely many smooth 2-simplexes $\{\sigma_j\}_{j=1}^J$ such that the following properties hold.

1. *$\sigma_j : \Delta_2 \to X$ is smooth for $1 \leqslant j \leqslant J$ where Δ_2 is the standard 2-simplex defined by $\{(x,y) \subset \mathbb{R}^2 | 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1, x+y \leqslant 1\}$.*
2. *$\partial\big(\sum_{j=1}^J b_j \sigma_j\big) = \widetilde{\tau} - \sum_{k=1}^m a_k \widetilde{\tau}_k$, where $0 \neq b_j \in \mathbb{C}$ for $1 \leqslant j \leqslant J$.*
3. *There are finitely many 1-simplexes $\{\tau_l\}_{l=1}^L$ and finitely many 0-simplexes $\{A_n\}_{n=1}^N$ such that*

$$\partial \sigma_j = \sum_{l=1}^{L} c_{jl} \tau_l \text{ with } c_{jl} \in \{0, 1, -1\} \text{ for } 1 \leqslant j \leqslant J, 1 \leqslant l \leqslant L;$$

$$\partial \tau_l = \sum_{n=1}^{K} \mathrm{d}_{ln} A_n \text{ with } \mathrm{d}_{ln} \in \{0, 1, -1\} \text{ for } 1 \leqslant l \leqslant L, 1 \leqslant n \leqslant N.$$

   Notice that $\{\widetilde{\tau}, \widetilde{\tau}_1, \dots, \widetilde{\tau}_m\} \subset \{\tau_j\}_{l=1}^L$ and $p \in \{A_n\}_{n=1}^N$.

4. *The above $\{\sigma_j\}, \{\tau_l\}$ and $\{A_n\}$ are transversal to W in the sense that: $A_n \notin W$ for $1 \leqslant n \leqslant N$ and $\tau_l \cap W = \varnothing$ for $1 \leqslant l \leqslant L$; $\sigma_j \cap \mathrm{Sing}(W) = \varnothing$ and σ_j intersects W transversally for $1 \leqslant j \leqslant J$.*

Proof. — Since $\widetilde{\tau}$ is homologous to $\sum_{k=1}^m a_k \widetilde{\tau}_k$, there are finitely many 2-simplexes $\{\sigma_j\}_{j=1}^J$ such that $\partial\big(\sum_{j=1}^J b_j \sigma_j\big) = \widetilde{\tau} - \sum_{k=1}^m a_k \widetilde{\tau}_k$, where $0 \neq b_j \in \mathbb{C}$ for $1 \leqslant j \leqslant J$.

The idea of the proof is to perturb the simplexes homotopically, thicken the lower dimensional simplexes and extend the perturbation from simplexes with lower dimension to simplexes with higher dimension. Notice that the 0-simplex p and the 1-simplexes $\{\widetilde{\tau}, \widetilde{\tau}_1, \dots, \widetilde{\tau}_m\}$ are fixed throughout the proof. We divide the proof into seven steps as follows.

Step 1: Perturb $\{A_n\}$ so that $A_n \notin W$ for $1 \leqslant n \leqslant N$. — To be more precise, we shall construct homotopies $I_n : [0,1] \to X, 1 \leqslant n \leqslant N$, such that:

(1) I_n is smooth;
(2) $I_n(0) = A_n, I_n(1) = \widetilde{A}_n$ and $\widetilde{A}_n \notin W$.

If $A_n \notin W$, we define I_n to be the identity map. Otherwise we choose a point $\widetilde{A}_n \notin W$ near A_n and draw a smooth curve I_n connecting A_n and $\widetilde{A}_n$.

Step 2: Extend the above homotopies to the complex $\{A_n\}_{n=1}^N \bigcup \{\tau_l\}_{l=1}^L$. — To be more precise, we shall construct homotopies $T_l : [0,1] \times [0,1] \to X$ for $1 \leqslant l \leqslant L$ satisfying the following properties.

(1) T_l is continuous;
(2) $T_l(\,\cdot\,, 0) = \tau_l(\,\cdot\,)$;
(3) $T(0, \cdot) = I_{l_0}(\,\cdot\,)$ and $T(1, \cdot) = I_{l_1}(\,\cdot\,)$, where $l_0, l_1 \in \{1, \dots, N\}$ such that $\tau_l(0) = A_{l_0}$ and $\tau_l(1) = A_{l_1}$.

Firstly we define the homotopies of 1-simplexes $\{\widetilde{\tau}, \widetilde{\tau}_1, \dots, \widetilde{\tau}_m\}$ to be identities. Next we will construct homotopy of T_l for the remaining τ_l as shown in Figure B.1.(A). It is easy to see that the projection from the star-shaped point induces a strong deformation retraction F of the unit square to the union of three intervals $\overline{AD}$, $\overline{CD}$ and $\overline{BC}$ in the following sense.

$$\begin{aligned} F : [0,1] \times \big([0,1] \times [0,1]\big) &\longrightarrow [0,1] \times [0,1], \\ (t, x, y) &\longmapsto (F_1(t,x,y), F_2(t,x,y)), \end{aligned}$$

where $F(0,x,y) = (x,y)$, $F(1,x,y) \subset \overline{AD} \cup \overline{CD} \cup \overline{BC}$ and $F(t,x,y)$ is an identity map when $(x,y) \in \overline{AD} \cup \overline{CD} \cup \overline{BC}$. Denote by $\widetilde{T}_l$ the following continuous map defined on $\overline{AD} \cup \overline{CD} \cup \overline{BC}$.

$$\begin{aligned} &\widetilde{T}_l : \big(\{0\} \times [0,1]\big) \cup \big([0,1] \times \{0\}\big) \cup \big(\{1\} \times [0,1]\big) \longrightarrow X; \\ &\widetilde{T}_l(x,0) = \tau_l(x),\ \widetilde{T}_l(0,y) = I_{l_0}(y) \text{ and } \widetilde{T}_l(1,y) = I_{l_1}(y) \text{ for } x, y \in [0,1]. \end{aligned}$$

Then the desired homotopy T_l can be defined by the following formula.

$$T_l(x,y) = \widetilde{T}_l(F_1(1,x,y), F_2(1,x,y)).$$

Step 3: Smooth $\{\tau_l\}_{l=1}^L$ homotopically with fixed boundary 0-simplexes $\{A_n\}_{n=1}^N$. — To be more precise, we will construct homotopy $T_l : [0,1] \times [0,1]$ for $1 \leqslant l \leqslant L$ satisfying the following properties:

(1) T_l is continuous and $T_l(\,\cdot\,, 1)$ is a smooth map from $[0,1]$ to X;
(2) $T_l(\,\cdot\,, 0) = \tau_l(\,\cdot\,)$; $T(0, \cdot) = \tau_l(0)$ and $T(1, \cdot) = \tau_l(1)$;
(3) $T(x, 1) = \tau_l(0)$ when $x \approx 0$; $T(x, 1) = \tau_l(1)$ when $x \approx 1$.

Firstly we will thicken the boundary as illustrated by Figure B.2.(A). We define homotopy T_l^1 as follows.

$$T_l^1 : [0,1] \times [0,1] \longrightarrow X;$$
$$(x,y) \longmapsto \begin{cases} \tau_l(0) & \text{when } (x,y) \in \text{Region III;} \\ \tau_l(1) & \text{when } (x,y) \in \text{Region II;} \\ \tau_l\left(\frac{2x-y/2}{2-y}\right) & \text{when } (x,y) \in \text{Region I.} \end{cases}$$

Notice that the map $T_l^1(\,\cdot\,,1)$ is smooth in a small neighborhood of two boundary points. Then by Theorem (10.1.2) in [5] and Remark (2) therein, we can find a smooth map $\widehat{\tau}_l$ homotopic to $T_l^1(\,\cdot\,,1)$. Moreover by Remark (1) of Theorem (10.1.2) in [5], $\widehat{\tau}_l$ can be chosen so that $\widehat{\tau}_l(x) = T_l^1(x,1)$ if $x \approx 0$ or $x \approx 1$.

Step 4: Perturb $\{\tau_l\}_{l=1}^L$ homotopically with fixed boundary points so that τ_l is disjoint from W for $1 \leqslant l \leqslant L$. — First we take a stratification of W as $W = W_1 \cup W_2 \cup \cdots \cup W_v$, where W_i is smooth for $1 \leqslant i \leqslant v$ and $\dim W_i < \dim W_{i-1}$ for $2 \leqslant i \leqslant v$. According to the proof of Theorem (10.3.2) in [5], we can find a smooth 1-simplex $\widetilde{\tau}_l^v$ for τ_l such that

(1) $\widetilde{\tau}_l^v$ is transversal to W_v ;
(2) $\widetilde{\tau}_l^v$ is homotopic to τ_l;
(3) $\widetilde{\tau}_l^v$ coincides with τ_l in a small neighborhood of boundary points.

By dimension counting, we have that $\widetilde{\tau}_l^v$ is disjoint from W_v. Repeating the procedure for $W_{v-1}, W_{v-2}, \ldots, W_1$, we end with $\widetilde{\tau}_l^1$ which is disjoint with W. Notice that $\{\widetilde{\tau}, \widetilde{\tau}_1, \ldots, \widetilde{\tau}_m\}$ satisfy the desired property without any perturbation.

Step 5: Extend the obtained homotopies of $\{\tau_l\}_{l=1}^L$ to $\{\sigma_j\}_{j=1}^J$. — After the above steps we have a homotopy T_l for τ_l such that

(1) T_l is continuous;
(2) $T_l(\,\cdot\,,0) = \tau_l(\,\cdot\,)$, $T(x,1) = T(0,1)$ for $x \approx 0$ and $T(x,1) = T(1,1)$ for $x \approx 1$;
(3) $T_l(\,\cdot\,,1)$ is smooth and disjoint from W.

We shall construct homotopy $S_j : \Delta_2 \times [0,1] \to X$ for $1 \leqslant j \leqslant J$ satisfying the following properties.

(1) S_j is continuous.
(2) $S_j(\,\cdot\,,0) = \sigma_j(\,\cdot\,)$.
(3) Suppose that $S_j(t,1-t,0) = \pm\tau_{j_{12}}(t)$, $S_j(0,t,0) = \pm\tau_{j_{01}}(t)$, and $S_j(t,0,0) = \pm\tau_{j_{02}}(t)$ for $j_{01}, j_{02}, j_{12} \in \{1,\ldots,J\}$ and $t \in [0,1]$ (the signs depend on the orientation). Then for $t \in [0,1]$, $S_j(t,1-t,\cdot) =$

$\pm T_{j_{12}}(t,\cdot)$, $S_j(0,t,\cdot)=\pm T_{j_{01}}(t,\cdot)$, and $S_j(t,0,\cdot)=\pm T_{j_{02}}(t,\cdot)$ with the compatible signs.

Similar to Step 2, we first construct a strong deformation retract as illustrated in Figure B.1.(B). It is clear that the projection from the star-shaped point induces a strong deformation retraction F of the triangle based prism to the union of the triangle $\overline{ABC}$ and three parallelograms $\overline{ACC'A'}$, $\overline{CBB'C'}$ and $\overline{ACC'A'}$ in the following sense.

$$\begin{aligned} F:[0,1]\times\big(\Delta_2\times[0,1]\big)&\longrightarrow \Delta_2\times[0,1],\\ (t,x,y,s)&\longmapsto (F_1(t,x,y,s),F_2(t,x,y,s),F_3(t,x,y,s)),\end{aligned}$$

where $F(0,x,y,t)=(x,y,t)$, $F(1,x,y,t)\subset\overline{AD}\cup\overline{CD}\cup\overline{BC}$ and $F(t,x,y,s)$ is an identity map when $(x,y,s)\in\overline{ABC}\cup\overline{ACC'A'}\cup\overline{CBB'C'}\cup\overline{ACC'A'}$.

We construct S_j for σ_j as follows. Denote by $\widetilde{S}_j$ the following continuous map defined on $\overline{AD}\cup\overline{CD}\cup\overline{BC}$.

$$\begin{aligned} &\widetilde{S}_j:\overline{ABC}\cup\overline{ACC'A'}\cup\overline{CBB'C'}\cup\overline{ACC'A'}\longrightarrow X;\\ &\widetilde{S}_j|_{A\bar{B}C}=\sigma_j,\ \widetilde{S}_j|_{ACC'A'}=\pm T_{j_{01}},\ \widetilde{S}_j|_{CBB'C'}=\pm T_{j_{02}}\\ &\text{and } \widetilde{S}_j|_{ACC'A'}=\pm T_{j_{12}}.\end{aligned}$$

Then the desired homotopy S_j is given by

$$S_j(x,y,s)=\widetilde{S}_j(F_1(1,x,y,s),F_2(1,x,y,s),F_3(1,x,y,s)).$$

Step 6: Smooth $\{\sigma_j\}_{j=1}^J$ homotopically with fixed boundary 1-simplexes $\{\tau_l\}_{l=1}^L$. — To be more precise, we will construct homotopy $S_j:\Delta_2\times[0,1]$ for $1\leqslant j\leqslant J$ satisfying the following properties.

(1) S_j is continuous and $S_j(\,\cdot\,,1)$ is a smooth map from Δ_2 to X.
(2) $S_j(\,\cdot\,,0)=\sigma_j(\,\cdot\,)$.
(3) $S_j(x,y,\cdot)=\sigma_j(x,y)$ for $(x,y)\in\big(\{0\}\times[0,1]\big)\cup\big([0,1]\times\{0\}\big)\cup\{(t,1-t)\,|\,0\leqslant t\leqslant 1\}$.

We proceed in a similar way to Step 3. Firstly we thicken the boundary as illustrated by Figure B.2.(B). We can define a homotopy $S_j^1:\Delta_2\times[0,1]\to X$ such that $S_j^1(\,\cdot\,,\cdot\,,1)$ is defined as follows.

$$S_j^1(x,y,1)=\begin{cases}\sigma_j\left(\frac{\frac{1}{\sqrt2+2}y+(1-\frac{1}{\sqrt2+2})x-\frac{1}{\sqrt2+2}}{x+y-\frac{2}{\sqrt2+2}},\frac{\frac{1}{\sqrt2+2}x+(1-\frac{1}{\sqrt2+2})y-\frac{1}{\sqrt2+2}}{x+y-\frac{2}{\sqrt2+2}}\right)&\\ &\text{when }(x,y)\in\text{Region I};\\ \sigma_j\big(2x-\frac{1}{\sqrt2+2},2y-\frac{1}{\sqrt2+2}\big)&\text{when }(x,y)\in\text{Region II};\\ \sigma_j\left(0,\frac{1}{\sqrt2+2}\frac{x-y}{x-\frac{1}{\sqrt2+2}}\right)&\text{when }(x,y)\in\text{Region III};\\ \sigma_j\left(\frac{1}{\sqrt2+2}\frac{y-x}{y-\frac{1}{\sqrt2+2}},0\right)&\text{when }(x,y)\in\text{Region IV}.\end{cases}$$

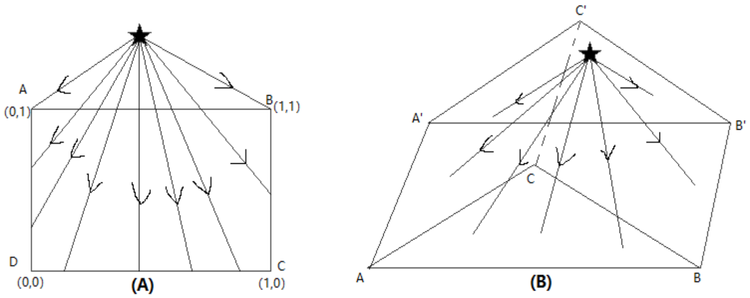

Figure B.1. Extension of the homotopies

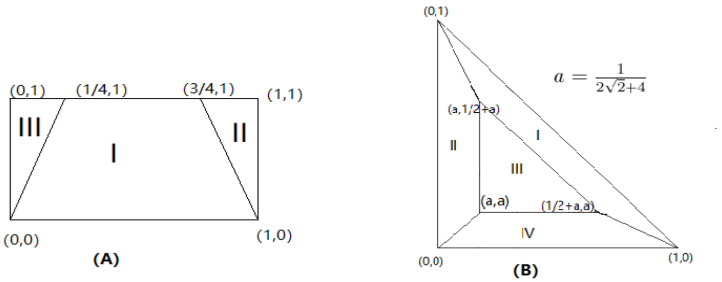

Figure B.2. Thickening the boundary

Notice that after Step 1-5 τ_j is smooth and constant near the boundary $\tau_j(0)$ and $\tau_j(1)$. Then $S^1_j(\cdot,\cdot,1)$ is smooth in a small neighborhood of the boundary. By Theorem (10.1.2) in [5] and Remark (2) of it, we can find a smooth map $\widehat{\sigma}_j$ homotopic to $S^1_j(\cdot,\cdot,1)$. Moreover by Remark (1) of Theorem (10.1.2) in [5], we can assume $\widehat{\sigma}_j$ coincides with $\widetilde{\sigma}_j$ in a small neighborhood of the boundary of Δ_2. Hence we get the desired homotopy of σ_j.

Step 7: Perturb $\{\sigma_j\}_{j=1}^J$ so that σ_j is disjoint from the singularities of W and intersects the smooth part of W transversally. — First we take a stratification of W as $W = W_1 \cup W_2 \cup \cdots \cup W_v$, where W_i is smooth for $i = 1, \ldots, v$ and $\dim W_i < \dim W_{i-1}$ for $i = 2, \ldots, v$.

Next we will find a desired perturbation for σ_l. Notice that a certain neighborhood of the boundary of σ_j is transversal to W_v. Then following the proof of Theorem (10.3.2) in [5], we can find a smooth 2-simplex $\widetilde{\sigma}^v_j$ for σ_j such that $\widetilde{\sigma}^v_j$ is transversal to W_v and homotopic to τ_l with boundary points fixed. Repeating the procedure for $W_{v-1}, W_{v-2}, \ldots, W_1$, we derive the corresponding transversal maps $\widetilde{\sigma}_j^{v-1}, \widetilde{\sigma}_j^{v-2}, \ldots$ and $\widetilde{\sigma}_j^1$ accordingly. By dimension counting, we conclude that the $\widetilde{\sigma}_j^1$ is disjoint from $\bigcup_{i=2}^{v} W_i$ and intersect W_1 transversally in the interior.

It is easy to verify that $\widetilde{\sigma}_j^1$ satisfy all the required property in the lemma. This completes the proof. □

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