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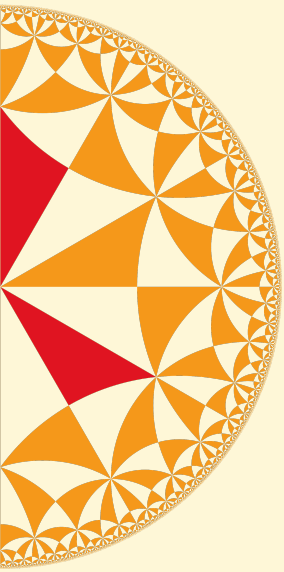

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TÊTE-À-TÊTE TWISTS, MONODROMIES AND REPRESENTATION OF ELEMENTS OF MAPPING CLASS GROUP

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Abstract. — We study monodromies of plane curve singularities and pseudo-periodic homeomorphisms of oriented surfaces with boundary using tête-à-tête graphs and twists. A tête-à-tête twist is a generalisation of the classical Dehn twist. We introduce the class of mixed tête-à-tête graphs and twists, and prove that mixed tête-à-tête twists contain the monodromies of irreducible plane curve singularities. In a sequel paper, the fourth author and B. Sigurdsson have extended this to the reducible case.

Keywords: Monodromy, pseudo-periodic homeomorphism, tête-à-tête graphs and twists.
2010 *Mathematics Subject Classification:* 32S40, 57M50, 32B10, 32S05.
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Résumé. — Nous étudions la monodromie des singularités de courbes planes et les homéomorphisms pseudo-périodiques de surfaces orientées à bord en utilisant graphes et cisaillements tête-à-tête. Un cisaillement tête-à-tête est une généralisation du twist de Dehn classique. Nous introduisons la classes des graphes et cisaillements tête-à-tête mélangé, et démontrons que les monodromies locales de courbes planes irréductibles appartiennent à cette classe. Dans un travail ultérieur le quatrième auteur et B. Sigurdsson ont étendu ce résultat au cas des singularités réductibles.

Contents

1. Introduction

Max Dehn has introduced the so called Dehn twist. More precisely, given an embedded copy α of the circle in the interior an oriented surface S, he has defined a mapping class D_α of the surface S. The mapping class D_α has for every open subset U that contains α a representative δ_U with support in U, that leaves α invariant and whose restriction to α has order 2.

The first author has observed that the geometric monodromy of a Pham singularity $x^a + y^b$ in two complex variables is a "twist" attached in rather similar manner to the Pham graph $Ph_{a,b}$ in the Milnor fiber $F_{a,b}$. More precisely, the graph $\Gamma = Ph_{a,b}$ is the complete bi-coloured graph with a vertices of one colour and b vertices of the other colour and is embedded

into the oriented surface $F_{a,b}$ such that the tête-à-tête property holds with respect to a metric. This property (see Definition 3.5), allows to define a mapping class D_Γ of $F_{a,b}$ with a representative that is the identity away from Γ and whose restriction to Γ has finite order (Definition 5.4). This class is called a tête-à-tête twist and is a generalization of Dehn twists.

A first optimism suggests that all geometric monodromies of plane curve singularities are tête-à-tête twists attached to a graph in the Milnor fiber. This is particularly nice since the tête-à-tête graph is a vanishing spine of the Milnor fibre: is the part of the Milnor fibre that collapses approaching the singular fibre. Then the vanishing spine, together with an additional metric structure codifies the geometric monodromy. However, this is only true for finite order monodromies.

In fact it is natural to explore which mapping classes are tête-à-tête twists. This task is completed in the present paper, where the following result is proved (Corollary 5.11): given any oriented surface Σ with boundary, any mapping class $\phi \in MCG(\Sigma, \partial\Sigma)$ is a tête-à-tête twist if and only if ϕ is boundary-free isotopic to a periodic homeomorphism, and the fractional Dehn twist coefficients (see Definition 4.16) at each of the components of the boundary are positive. In fact a more general result is proven (see Theorem 5.10 and Theorem 5.14): by extending the definitions to that of *signed relative tête-à-tête graphs and twists*, if $\partial^1\Sigma$ is a non-empty union of boundary components, any mapping class $\phi \in MCG(\Sigma, \partial^1\Sigma)$ is a signed relative tête-à-tête twist if and only if ϕ is boundary-free isotopic to a periodic homeomorphism. This results improve results of Graf [4, 5], who proved the analogous realization, but using a class of graphs and twists that strictly contains ours. We also care of representing all periodic boundary-free mapping classes by tête-à-tête twists (Theorem 5.18).

The above optimism on geometric monodromies becomes a Theorem, if one works with graphs that satisfy the mixed tête-à-tête property (Definition 7.8), and with the mapping classes defined by them, which are called mixed tête-à-tête twists and are also introduced in this paper. The proof of this theorem is the conjunction of the results of the second part of this paper and a subsequent paper by the fourth author and B. Sigurdsson [9]. Here we prove that the monodromy of any irreducible plane branch is a mixed tête-à-tête twist (this is a consequence of the more general Theorem 7.20):

We introduce a filtered vanishing spine on a surface, together with a metric structure (which is called a mixed tête-à-tête graph), associate to it a mixed tête-à-tête twist. In the case of irreducible plane curve singularities

we find a mixed tête-à-tête graph on the Milnor fibre, whose associated mixed tête-à-tête twist is the geometric monodromy. In [9], mixed tête-à-tête twists has been characterized as the geometric monodromy for an isolated curve singularity $f : X \to \mathbb{C}$ on a normal surface germ, covering in particular the reducible plane curve case.

The plane curve case suggested the following higher dimensional generalization to the first author. The Pham spine $Ph_{a_0,\cdots,a_n}$, $n > 1$, in the Milnor fiber of $z_0^{a_0} + \cdots + z_n^{a_n}$ has also a tête-à-tête property if one realizes the simplices of the spine as orthogonal spherical simplices. In this case π-walks are broken spherical geodesics . Again the monodromy is the corresponding twist. Taking into account the natural symplectic structure on the Milnor fibre one finds that the Pham spine is a Lagrangian vanishing spine, in the sense explained above. Such a symplectic interpretation allows an optimism about the geometric monodromy of complex hypersurface singularities. This topic needs however further investigation in future.

The gist for the ideas in this paper is contained in the first author preprint [2]. The present paper fully develop the ideas contained there for the surface case and systematically studies mapping classes in terms of tête-à-tête graphs.

We have tried to make the paper as self-contained as possible. The structure is as follows. In Section 2 we introduce the basic definitions on ribbon graphs and thickening surfaces, and set up the notation and terminology for the rest. In Section 3 we define tête-à-tête graphs and prove their basic properties. In Section 4 we recall basic facts on periodic mapping classes, and introduce the fractional Dehn twist coefficients. In Section 5 tête-à-tête twists are introduced, and the characterization of tête-à-tête twists explained above is proved. In Section 6 we recall the facts we need on pseudo-periodic homeomorphisms and set up the notation for the rest of the paper. In Section 7.2 mixed tête-à-tête graphs and twists are introduced, and the realization theorem explained above, which implies that monodromies of irreducible branches are tête-à-tête twists, is proved. In this last section, for pedagogical reasons, we start by analyzing a class of pseudo-periodic homeomorphisms, and constructing a spine on the underlying surface with special properties. This motivates the definition of mixed tête-à-tête graphs and twists. After this previous analysis the proof of main Theorem 7.20 follows.

Acknowledgements

First author thanks Professors Makoto Sakuma and Ichiro Shimada for their invitation to expose about monodromy and tête-à-tête twists at the conference "Branched Coverings, Degenerations and Related Topics, 1916", at the Graduate School of Sciences, Hiroshima University. Together with forth author we thank Professors Yakov Eliashberg, David Nadler and Laura Paul Starkston for their invitation to present higher dimensional symplectic extensions in the conference "Arborealization of singularities of Lagrangian skeleta, 1918", at the American Institut for Mathematics, San Jose, CA.

2. Graphs, spines and regular thickenings

A *graph* Γ is a 1-dimensional finite CW-complex. We denote by $v(\Gamma)$ the *set of vertices* and by $e(\Gamma)$ the *set of edges*. We allow *loops* (edges starting and ending in the same vertex) and also several edges connecting two vertices. For a vertex v we denote by $e(v)$ the set of edges adjacent to v, where an edge e appears twice in case it is a loop joining v with v. The *valency* of a vertex is the cardinality of $e(v)$. Unless we state the contrary we assume that there are no vertices of valency 1.

A *ribbon graph* is a graph Γ such that for every vertex we fix a cyclic order in the set of edges $e(v)$.

A *regular thickening* of a ribbon graph Γ is a piecewise smooth embedding $\Gamma \hookrightarrow \Sigma$ of the graph, as a deformation retract of an oriented surface with boundary, such that the cyclic ordering of the incoming edges at each vertex is induced counterclockwise by the orientation of the surface. The thickening surface (Σ, Γ) is unique only up to orientation preserving homeomorphism of the surface.

Reciprocally, every oriented surface with finite topology and non-empty boundary has a *spine* Γ (i.e. an embedded graph in $\Sigma \setminus \partial\Sigma$ that is a *regular retract* of Σ) with a ribbon graph structure whose regular thickening is (Σ, Γ).

Example 2.1. — Let $K_{p,\,q}$ be the bipartite graph (p, q). The set of vertices is the union of two sets A and B of p and q vertices respectively. The edges are exactly all the possible non-ordered pairs of points one in A and one in B.

Now we fix cyclic orderings in A and B. These give cyclic orderings in the sets of edges adjacent to vertices in B and A respectively.

One can check that the thickening surface has as many boundary components as $\gcd(p,q)$ and genus equal to $\frac{1}{2}[(p-1)(q-1)-\gcd(p,q)+1]$.

In Figure 2.1 we have the example of $K_{2,3}$.

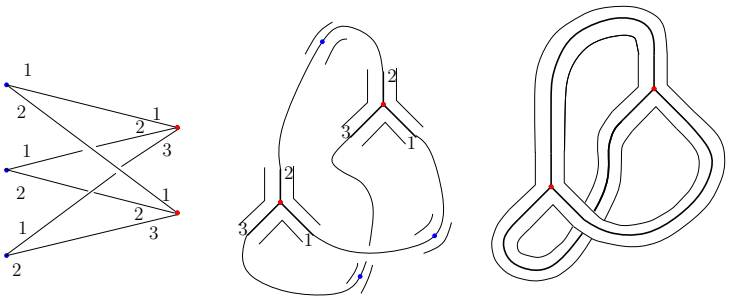


Figure 2.1. Thickening of the graph $K_{2,3}$ in three steps. First we have a planar projection of $K_{2,3}$ where the two subsets of 2 and 3 vertices are vertically ordered in different parallel lines. Then we thicken a neighbourhood of every vertex and finally we glue the pieces. The resulting surface is homeomorphic to the once-punctured torus.

We introduce a generalization of the notion of spine of a surface with boundary Σ, which treats in a special way a certain union of boundary components. Let us start by the corresponding graph theoretic notion.

Definition 2.2. — *Let (Γ, A) be a pair formed by a graph Γ and an oriented subgraph A such that each of its connected components A_i is homeomorphic to the oriented circle $\mathbb{S}^1$. The pair (Γ, A) is a* relative ribbon graph *if for any vertex $v \in A$ the set of incident edges $e(v)$ is endowed with a cyclic ordering $e(v) = \{e_1, \ldots, e_k\}$* compatible *with the orientation of A, which means that*

- *$e_i < e_{i+1}$ and $e_k < e_1$,*
- *the edges belonging to A (that necessarily belong to a single component A_i) are e_1 and e_k,*
- *if we consider a small interval in A_i around v and we parametrize it in the direction induced by the orientation of A_i, then we pass first by e_1 and after by e_k.*

A *regular thickening* of a relative ribbon graph (Γ, A) is a piecewise smooth embedding of pairs $(\Gamma, A) \hookrightarrow (\Sigma, \partial\Sigma)$ into a surface Σ with boundary such that each A_i is sent homeomorphically (orientation reversing) to a component of $\partial\Sigma$, the subgraph $\Gamma \setminus A$ is sent to the interior of Σ, and

the image of Γ is a deformation retract of Σ. The cyclic ordering of the incoming edges at each vertex is induced counterclockwise by the orientation of the surface. The thickening surface (Σ, Γ, A) is unique only up to orientation preserving homeomorphism of the surface.

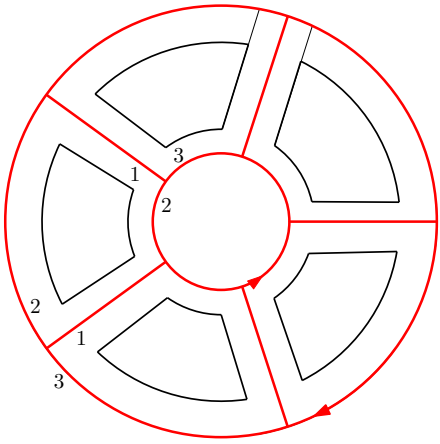


Figure 2.2. Thickening of a relative ribbon graph with two relative components. The arrows in the picture indicate the direction that safe walk takes in the relative components. The numbers indicate the cyclic ordering of the edges adjacent to two different vertices. The surface is a sphere with 7 holes.

Reciprocally: for any pair (Σ, A), given by an oriented surface Σ and a union of some boundary components A, if there is a graph Γ embedded in Σ with $A \subset \Gamma$ such that Γ is a regular retract of Σ, then (Σ, Γ, A) is a thickening of (Γ, A).

Notation 2.3. — From now on the letter I denotes an interval, unless otherwise specified, it denotes the unit interval. Let (Γ, A) be a relative ribbon graph and (Σ, Γ, A) a thickening. There is a connected component of $\Sigma \setminus \Gamma$ for each boundary component C_i of Σ not contained in A, this component is homeomorphic to $C_i \times (0, 1]$. We denote by $\widetilde{\Sigma}_i$ the compactification of $C_i \times (0, 1]$ to $C_i \times I$. We denote by Σ_Γ the surface obtained by cutting Σ along Γ, that is taking the disjoint union of the $\widetilde{\Sigma}_i$. Let

$$g_\Gamma : \Sigma_\Gamma \to \Sigma$$

be the gluing map. We denote by $\widetilde{\Gamma}_i$ the boundary component of the cylinder $\widetilde{\Sigma}_i$ that comes from the graph (that is $g_\Gamma(\widetilde{\Gamma}_i) \subset \Gamma$) and by C_i the one

coming from a boundary component of Σ (that is $g_\Gamma(C_i) \subset \partial\Sigma$). From now on, we take the convention that C_i is identified with $C_i \times \{1\}$ and that $\widetilde{\Gamma}_i$ is identified with $C_i \times \{0\}$. We set $\Sigma_i := g_\Gamma(\widetilde{\Sigma}_i)$ and $\Gamma_i := g_\Gamma(\widetilde{\Gamma}_i)$. Finally we denote $g_\Gamma(C_i)$ also by C_i since $g_\Gamma|_{C_i}$ is bijective. The orientation of Σ induces an orientation on every cylinder $\widetilde{\Sigma}_i$ and on its boundary components.

3. Tête-à-tête graphs

We now consider metric relative ribbon graphs (Γ, A, d). A metric graph is a graph Γ together with lengths $l(e) \in \mathbb{R}$ for every edge $e \in e(\Gamma)$. In an edge e, we take a homogeneous metric that gives e total length $l(e)$. We consider the distance $d(x, y)$ on Γ given by the minimum of the lengths of the paths joining x and y.

Definition 3.1. — *A* walk *in a graph Γ is a continuous mapping*

$$\gamma : I \to \Gamma,$$

from an interval I, possibly infinite, and such that for any $t \in I$ there exists a neighbourhood around t where γ is injective.

The notion of safe walk is central in this paper. We start by a purely graph theoretical definition.

Definition 3.2 (Safe walk). — *Let (Γ, A) be a metric relative ribbon graph. A* safe walk *for a point p in the interior of some edge is a walk $\gamma_p : \mathbb{R}_{\geqslant 0} \to \Gamma$ with $\gamma_p(0) = p$ and such that:*

1. *The absolute value of the speed $|\gamma_p'|$ measured with the metric of Γ is constant and equal to 1. Equivalently, the safe walk is parametrized by arc length, i.e. for any t and any s small enough we have $d(\gamma_p(t), \gamma_p(t+s)) = s$.*
2. *when γ_p gets to a vertex, it continues along the next edge in the given cyclic order.*
3. *If p is in an edge of A, the walk γ_p starts running in the direction prescribed by the orientation of A.*

In this paper, we always work with restrictions of safe walks to intervals. An ℓ-safe walk is the restriction of a safe walk to the interval $[0, \ell]$. If a length is not specified when referring to a safe walk, we will understand that its length is π.

The notion in (2) of *continuing along the next edge in the order* of $e(v)$ is equivalent to the notion of *turning to the right* in every vertex for paths parallel to Γ in any thickening surface (Σ, Γ, A) as it is explained in [2].

Following the first author choice in [2], in the first part of the paper, we adopt the convention of working mainly with π-safe walks. In Section 5.1 we will need safe walks of different lengths, that is why we introduce it in such a generality.

Remark 3.3 (Safe walk via cylinder decomposition.). — Taking a thickening (Σ, Γ, A), the previous definition extends to safe walks γ_p starting also at $p \in v(\Gamma)$ by replacing condition (2) by

(2') the path γ_p admits a lifting $\widetilde{\gamma}_p : \mathbb{R} \to \Sigma_\Gamma$ in the cylinder decomposition of Σ_Γ (see Notation 2.3), which runs in the opposite direction to the one indicated by the orientation induced at the boundary of the cylinder.

Remark 3.4 (Definition of γ_p and ω_p.). — Given a relative ribbon graph (Γ, A) and a thickening (Σ, Γ, A), we make some observations in order to help fixing ideas:

(a) every vertex $v \in v(\Gamma)$ has as many preimages by g_Γ as its valency $e(v)$. These preimages belong to certain $\widetilde{\Gamma}_i \subseteq \widetilde{\Sigma}_i$ for certain cylinders $\widetilde{\Sigma}_i$ which could occasionally be the same. An interior point of an edge not included in A has always two preimages. An interior point of an edge included in A has always one preimage.

(b) for every point $p \in \Gamma$ and every oriented direction from p along Γ compatible with the orientation of A there is a safe walk starting on p following that direction. This safe walk admits a lifting to one of the cylinders $\widetilde{\Sigma}_i$.
In particular:

(b1) For p an interior point of an edge not belonging to A, that is for $p \in \Gamma \setminus (v(\Gamma) \cup A)$, only 2 starting directions for a safe walk are possible, corresponding to the two different preimages of p by g_Γ. We will denote the corresponding safe walks by γ_p and ω_p. If p is at the interior of an edge contained in A only one starting direction for a safe walk at p is possible.

(b2) For a vertex v, not belonging to A there are as many starting directions as edges in $e(v)$, and for any vertex v belonging to A, there are as many starting directions as edges in $e(v)$ minus 1 (the edge in A whose orientation arrives to v does not count).

Definition 3.5 (Tête-à-tête property and tête-à-tête graph). — *Let (Γ, A, d) be a metric relative ribbon graph without univalent vertices. We say that Γ satisfies the* ℓ-tête-à-tête property, *or that Γ is an* ℓ-tête-à-tête graph *if*

- *For any point $p \in \Gamma \setminus (A \cup v(\Gamma))$ the two different ℓ-safe walks starting at p (see Remark 3.4), that we denote by γ_p and ω_p, satisfy $\gamma_p(\ell) = \omega_p(\ell)$.*
- *for a point p in $A \setminus v(\Gamma)$, the end point of the unique ℓ-safe walk starting at p belongs to A.*

If (Σ, Γ, A) is the regular thickening of the graph (Γ, A) where A denotes the corresponding union of boundary components, we say that (Γ, A) gives a relative ℓ-tête-à-tête structure to (Σ, Γ, A) *or that* (Γ, A) *is a* relative ℓ-tête-à-tête graph or spine for (Σ, Γ, A).

If $A = \emptyset$, we call it a pure ℓ-tête-à-tête structure or graph.

Lemma 3.6 (Lemma and Definition). — *For an ℓ-tête-à-tête graph (Γ, A, d), the mapping $\sigma_\Gamma : \Gamma \to \Gamma$ defined by $\sigma_\Gamma(p) = \gamma_p(\ell)$ is a well defined homeomorphism.*

Proof. — Let

$$\sigma : \coprod_i \widetilde{\Gamma}_i \to \coprod_i \widetilde{\Gamma}_i$$

be the homeomorphism which restricts to the metric circle $\widetilde{\Gamma}_i$ to the negative rotation of amplitude ℓ (move each point to a point which is at distance l in the negative sense with respect to the orientation induced as boundary of the cylinder). The tête-à-tête property implies that σ is compatible with the gluing g_Γ at any point which is not the preimage of a vertex. By continuity the compatibility extends to all the points. The mapping σ descends to the mapping σ_Γ. This proves the assertion. □

Remark 3.7. — There is a special and easy case for π-tête-à-tête graphs: when Γ is homeomorphic to $\mathbb{S}^1$. The thickening surface is in this case the cylinder.

If Γ is $\mathbb{S}^1$, then the only possibilities for σ_Γ are the identity or the π rotation (for the homogenous metric). Then Γ has total length of $2\pi/n$ for some $n \in \mathbb{N}$.

Corollary 3.8. — *The homeomorphism σ_Γ has the following properties:*

(1) *it is an isometry,*
(2) *it preserves the cyclic orders of $e(v)$ for every $v \in v(\Gamma)$,*

(3) *it takes vertices of valency $k > 2$ to vertices of the same valency,*
(4) *it has finite order.*

Proof. — Point (1) follows from Lemma 3.6 because σ_Γ is a homeomorphism that is an isometry restricted to the edges. Point (2) follows from the fact that $\gamma_p(t+t') = \gamma_{\gamma_p(t)}(t')$. More precisely, take e' appearing after e in the cyclic order of $e(v)$. Take $p \in e$ and $\gamma_p(t) \in e'$ for t small. Then, it is clear that the next edge of $\sigma_\Gamma(e)$ in the cyclic order of $v(\sigma_\Gamma(v))$ is the edge of $\gamma_p(l+t)$.

Point (3) is immediate since σ_Γ is a homeomorphism.

To see that σ_Γ has finite order when it is not $\mathbb{S}^1$, we observe that σ_Γ induces a permutation between edges and vertices of Γ' and is an isometry. Then, it has finite order. When the graph is homeomorphic to $\mathbb{S}^1$ it is not clear, a priori, that σ_Γ permutes vertices but the result follows from the observation in Remark 3.7. □

Note that condition (2) implies that σ_Γ can be extended to a homeomorphism of a thickening surface. The induced tête-à-tête homeomorphisms that we will define (see Definition 5.4 or 5.16) are certain extensions of it.

COROLLARY 3.9. — *The following assertions hold:*

(1) *If $\sigma_\Gamma|_e = id$ for some edge e, then σ_Γ is the identity.*
(2) *for every $m \in \mathbb{N}$ the homeomorphism σ_Γ^m is also induced by a tête-à-tête graph,*
(3) *If $\sigma_\Gamma^m|e = id$ for some edge e, then σ_Γ^m is the identity.*

Proof. — Given σ_Γ as in (1), since it preserves the cyclic order at every v, then it fixes all the edges adjacent to the vertices of e. Since the graph is connected, this argument extends to the whole graph and the statement follows.

To see (2) and find a tête-à-tête graph for σ_Γ^m one can take the same combinatorial graph Γ with edge lengths equal to the ones of Γ divided by m.

Assertion (3) follows from (1) and (2). □

Example 3.10. — Given a ribbon graph $K_{p,q}$ as in Example 2.1, we endow it with a metric that gives length $\pi/2$ to any edge. Then we have a tête-à-tête graph. Moreover, the homeomorphism σ_Γ has order $\mathrm{lcm}(p,q)$. There are two special orbits, one of p vertices and another one of q vertices.

When dealing with tête-à-tête graphs and σ_Γ, we can assume that we have particularly simple combinatorics as the following lemma shows:

LEMMA 3.11. — *If (Γ, A, d) is a tête-à-tête graph, only modifying the underlying combinatorics (without changing the topological type of Γ), we can ensure we are in one of the following cases:*

(1) *unless Γ is either homeomorphic to $\mathbb{S}^1$ or contractible, all the vertices have valency $\geqslant 3$,*
(2) *there are no edges joining a vertex with itself and there is at most one edge joining two vertices. In this case the restriction $\sigma_\Gamma|_{v(\Gamma)}$ determines σ_Γ.*
(3) *all the edges have the same length,*
(4) *the graph satisfies properties in (2) and (3) simultaneously.*

Proof. — If Γ is either homeomorphic to $\mathbb{S}^1$ or contractible, after Remark 3.7, the proof is trivial.

Let's see the case where Γ is neither homeomorphic to $\mathbb{S}^1$ nor contractible. To get a graph as in (1) we can consider the graph Γ' forgetting the valency-2-vertices of Γ but keeping distances. It is clearly a tête-à-tête graph.

To get a graph as in (2) we consider the graph Γ as in (2). We add as vertices some mid-point-edges $q_1, \ldots, q_m$ to Γ' in order the new graph has no loops and no more than one edge between any pair of vertices. Now, we have to add as vertices any other point p for which $\gamma_p(\pi)$, the end of the safe walk for Γ, is one of these new vertices q_i. Since σ_Γ takes isometrically edges to edges, it will take midpoint edges to midpoint edges of Γ. Then we have to add at the most all the mid point edges of Γ as vertices to reach the desired graph.

Moreover, we note that in a graph as in (2), the image $\sigma_\Gamma(e)$ of an edge e joining v_i and v_j, has to be the only edge joining $\sigma_\Gamma(v_i)$ and $\sigma_\Gamma(v_j)$. Then, $\sigma_\Gamma|_{v(\Gamma)}$ determines σ_Γ.

To find a tête-à-tête graph as in (3), we start with a graph Γ as in (1). The homeomorphism σ_Γ permutes edges. Moreover the tête-à-tête condition says that certain summations of the lengths $\{l(e)\}_{e \in se(\Gamma)}$ are equal to π. We consider only the summations that come from measuring the lengths of the safe walks that start in vertices of Γ, which are a finite number. We collect all these linear equations in the variables $l(e)$ in a system S. We consider the system of equations S' by replacing the independent term π in the equations of S by 1. It is clear that there exist positive rational solutions $l(e)$ of the system S'. Let N be a common denominator. We consider the graph Γ' by subdividing every edge e of Γ into $N \cdot l(e)$ edges of length π/N obtaining the desired graph.

If Γ' does not satisfy properties in (2), replacing N by $2N$ it does. You can also add the middle points of all the edges as vertices and finish as in the proof of (2). □

Remark 3.12. — The Lemma 3.11 shows that, without loss of generality, we can assume that a tête-à-tête graph has all the edges of the same length 1 and safe walks of integer length ℓ. Anyway, we keep the original definition by historical reasons.

A way to obtain relative tête-à-tête graphs from pure ones is the notion of ϵ-blow up.

DEFINITION 3.13 (ϵ-*Blow up of* (Σ, Γ) *at a vertex of* Γ). — *Let* Γ *be a pure* ℓ*-tête-à-tête graph and* (Σ, Γ) *be a thickening surface. Let* v *be a vertex of valency* p*. We consider the oriented real blow up of* Σ *at* v*. We denote by* Σ' *and* Γ' *the transformations of* Σ *and* Γ*. Note that* Σ *has one more boundary component and* Γ' *has changed the vertex* v *by a circle* $A \cong \mathbb{S}^1$ *with* p *edges attached. Away from* v*, we consider the metric in* Γ' *as in* Γ*. We assign the length* 2ϵ *to the new edges in* Γ' *along* $\mathbb{S}^1$ *and redefine the length of the edges corresponding to each* $e \in e(v)$ *by length*$(e) - \epsilon$*. (See Figure 3.1 below). We do this at every vertex on the orbit of* v *by the* σ_Γ *and denote the resulting space by* $Bl_v(\Gamma, \epsilon)$*. We say that it is the result of performing the* ϵ*-blowing up of* Γ *at* v*.*

It is immediate to check that (Γ', A) is a relative tête-à-tête graph and that (Σ', Γ', A) is a thickening.

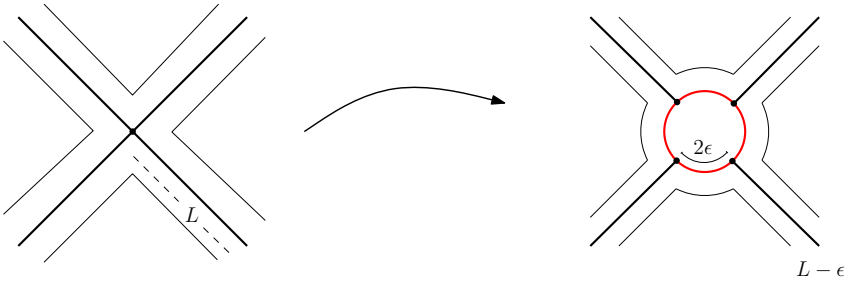


Figure 3.1. Blow-up some vertex v *of valency* 4.

4. Periodic homeomorphisms up to isotopy

In this section, for the reader convenience, we recall some elementary known facts about periodic homeomorphisms both in $MCG(\Sigma)$ and in $MCG(\Sigma, \partial\Sigma)$. We fix notation and conventions for the rest of the work.

Notation 4.1. — Let Σ be a fixed surface with non-empty boundary.

We denote by $[\phi]$ the class of homeomorphisms of ϕ up to isotopy. The group, with the composition, of these equivalence classes is called the Mapping Class Group $MCG(\Sigma)$. Two homeomorphisms of the same class are said to be *boundary-free isotopic* or simply *isotopic*.

A homeomorphism of a surface is *periodic in* $MCG(\Sigma)$ or *periodic up to boundary-free isotopy* if there exists $n \in \mathbb{N}$ such that $[\phi^n] = [id]$.

Let B be a subset of $\partial\Sigma$. We denote by $[\phi]_{B,\, \phi|_B}$ the class of homeomorphisms that are isotopic to ϕ by an isotopy that coincides with $\phi|_B$ at B all along the family. We denote by $MCG(\Sigma, B,\, \phi|_B)$ the set of these classes. If ϕ and ψ are in the same class we say they are *isotopic relative to the action* $\phi|_B$. If the action is the identity on B, we omit the action in the notation and recover the classical notion of *isotopy relative to* B, that means that all the homeomorphisms in the isotopy fix B pointwise. We write these classes simply by $[\phi]_B$. We always consider the case in which B is a union of connected components of $\partial\Sigma$ and denote it by $B = \partial^1\Sigma$. In the case $B = \partial\Sigma$ we will simply write $[\phi]_{\partial,\, \phi|_\partial}$ or $[\phi]_\partial$. Note that in this last case we recover the classical notion of mapping classes fixing pointwise the boundary.

We denote by $MCG^+(\Sigma)$, $MCG^+(\Sigma, B, \phi|_B)$ or $MCG^+(\Sigma, B)$ the corresponding restrictions to homeomorphisms preserving orientation.

Remark 4.2. — Observe that $MCG(\Sigma, \partial, \phi|_\partial)$ is not a group. However, the group $MCG(\Sigma, \partial)$ acts transitive and freely on it.

4.1. Periodic mapping classes in $MCG(\Sigma)$

In this subsection we assume all isotopies are boundary-free. We focus on periodic elements of $MCG^+(\Sigma)$.

A key result, only true in dimension 2 (see [10]), is the following classical theorem:

THEOREM 4.3 (Nielsen's Realization Theorem [7], also see [3, Theorem 7.1]). — *If ϕ^n is isotopic to the identity, then there exists $\widehat{\phi} \in [\phi]$ such that $\widehat{\phi}^n = Id$. Moreover, there exists a metric on Σ such that $\widehat{\phi}$ is an isometry.*

We will use the following well-known fact:

LEMMA 4.4. — *Let $\phi : \Sigma \to \Sigma$ be an orientation-preserving isometry of Σ. Then either the fixed points are isolated and disjoint from the boundary or ϕ is the identity. Moreover, if ϕ is a periodic homeomorphism, then the points with non trivial isotropy for the action generated by ϕ are also isolated and disjoint from the boundary. As a consequence, given a periodic homeomorphism ϕ of a surface that leaves all the boundary components invariant, the restriction to any boundary component has the same order than ϕ.*

Proof. — The second paragraph of the proof of [3, Theorem 6.8 (page 202)], proves that fixed-points of isometries are isolated. That points with non trivial isotropy for the action generated by ϕ are isolated follows from the fact that they are fixed points for some power $\phi^m \neq id$.

The last part of the statement follows by observing that removing points with non-trivial isotropy does not disconnect the surface. □

Notation 4.5. — Let ϕ be a periodic orientation preserving homeomorphism of Σ. We denote by Σ^ϕ the orbit space (which is a surface) and by

$$p : \Sigma \to \Sigma^\phi$$

the quotient mapping. The mapping p is a Galois ramified covering map. The set of points in Σ whose orbit has cardinality strictly smaller than the order of ϕ are called *ramification* points. Its images by p are called *branching* points.

Remark 4.6. — Since the covering map p is Galois, any point at the preimage by p of a branching point is a ramification point.

DEFINITION 4.7. — *Let ϕ be a periodic homeomorphism of Σ that leaves a boundary component $C_i \subseteq \partial\Sigma$ invariant. We cap this boundary component C_i with a disk D^2 obtaining a new surface Σ'. We extend ϕ to a periodic orientation-preserving homeomorphism of Σ' as follows: if θ is the angular and r the radial coordinates for D^2 then we define $\Phi : D^2 \to D^2, (\theta, r) \mapsto (r, \phi(\theta))$. The homeomorphisms Φ and ϕ glue along C_i. We call this extension procedure the* Alexander trick.

Remark 4.8. — Recall that given ϕ and ψ two homeomorphisms of Σ that both leave a spine Γ invariant, if $\phi|_\Gamma$ and $\psi|_\Gamma$ are isotopic, then ϕ and ψ are isotopic. In other words, the isotopy type of the restriction of a homeomorphisms to an invariant spine determines the isotopy type of the homeomorphism of Σ.

LEMMA 4.9. — *Let Σ be a surface with $\partial\Sigma \neq \emptyset$ which is not a disk or a cylinder. Let $\phi : \Sigma \to \Sigma$ be an orientation preserving homeomorphism. Then ϕ is periodic up to isotopy if and only if there exists $\widehat{\phi} \in [\phi]$ such that there exists a spine Γ of Σ which is invariant by $\widehat{\phi}$.*

Proof. — Assume ϕ is periodic up to isotopy. By Nielsen's realization Theorem we can assume that ϕ is periodic. Let Σ^ϕ be the orbit space of ϕ.

The quotient map $p : \Sigma \to \Sigma^\phi$ is a branched covering map whose ramification points are isolated and are contained in the interior of Σ by Lemma 4.4. Pick any spine Γ^ϕ for Σ^ϕ containing all the branch points. The set $\Gamma := p^{-1}(\Gamma^\phi)$ is invariant and it is a spine for being the pre-image of a spine containing the branching locus.

Conversely, assume that an invariant spine Γ exists for some $\widehat{\phi} \in [\phi]$. We consider the spine only with vertices of valency greater than 2. Since $\widehat{\phi}$ leaves the spine invariant, then $\widehat{\phi}$ acts as a permutation on edges and vertices. Then, there is a power of $\widehat{\phi}$, say $\widehat{\phi}^m$ that leaves all the edges and vertices invariant. Thus, $\widehat{\phi}^m|_\Gamma$ is isotopic to the identity, and hence $\widehat{\phi}^m$ too. □

Remark 4.10. — In the theorem above we excluded the cases when Σ is a cylinder or a disk for being trivial. In these cases every homeomorphism is isotopic to a periodic homeomorphism.

Not every spine obtained in the proof of the previous lemma accepts a tête-à-tête structure such that $\sigma_\Gamma = \phi|_\Gamma$ (see Example 5.9). In Theorem 5.10 we will see show how to find one that accepts it.

Notation 4.11. — If a homeomorphism ϕ of Σ leaves a spine Γ invariant, then the homeomorphism lifts to a homeomorphism of Σ_Γ that we denote by $\widetilde{\phi}$.

If ϕ is a periodic homeomorphism of Σ that leaves a boundary component C_i invariant, then $\phi|_{C_i}$ is a periodic homeomorphism of $\mathbb{S}^1$. Then, we can consider the usual Poincare's rotation number $rot(\phi|_{C_i})$ with values in $(0, 1]$.

It is well known that rotation numbers classify periodic homeomorphisms of the circle up to conjugation. As an easy consequence we have the following:

Remark 4.12. — Any orientation preserving periodic homeomorphism of the cylinder which leaves invariant each boundary component is conjugate to a rotation.

4.2. Elements of $MCG(\Sigma, \partial^1\Sigma)$ which are periodic in $MCG(\Sigma)$

Now we take a look at the Mapping Class Group where homeomorphisms and isotopies fix pointwise the union of some boundary components, that we denote by $\partial^1\Sigma \subset \partial\Sigma$ (recall Notation 4.1).

Consider a non-empty union $\partial^1\Sigma$ of the boundary components. We study the elements of $[\phi]_{\partial^1} \in MCG^+(\Sigma, \partial^1\Sigma)$ that are boundary-free isotopic to a periodic one.

If Σ is the disk, it is clear that $MCG^+(\Sigma, \partial\Sigma) \simeq 0$. If Σ is the cylinder, then $MCG^+(\Sigma, \partial\Sigma) \simeq \mathbb{Z}$ and it is generated by the right (or left) Dehn twist along a curve that is parallel to the boundary components. All its elements are boundary-free isotopic to the identity.

Remark 4.13. — In this work we take the convention that *negative* Dehn twists are *right-handed* Dehn twists. See Figure 4.1.

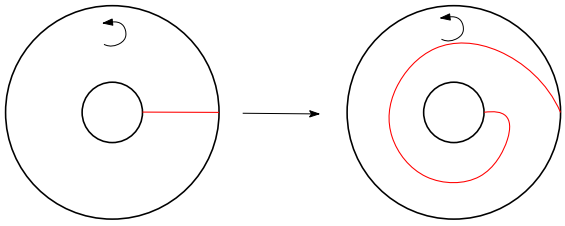


Figure 4.1. We see an oriented annulus with inner radius equal to 1 *and outer radius equal to* 2 *embedded on the plane on the left and the image of the red curve by a right-handed Dehn twist on the right. The little curve on the top of each picture represents the orientation on the annulus. In polar coordinates the transformation is* $(r, \theta) \mapsto (r, \theta + 2\pi(r-1))$.

Let ϕ be an orientation preserving homeomorphism of Σ, fixing pointwise $\partial^1\Sigma$ and boundary-free isotopic to a periodic one $\widehat{\phi}$.

We need the notion of fractional Dehn twist coefficients. The fractional Dehn twist coefficient at a boundary component can be understood as the difference between ϕ and a truly periodic representative at that boundary component. We define fractional rotation number in a slightly different way than the usual one, since it is more convenient for our applications, and allows a self contained exposition for the case that we need.

We start recalling some facts about Dehn twists. If we do not say the contrary, the letter D with a subindex, denotes a negative (right-handed) Dehn twist along some curve that will be clear from the context.

LEMMA 4.14. — *Let Σ be a surface with $r > 0$ boundary components that is not a disk or an cylinder. Then the subgroup of the mapping class group generated by the Dehn twists $D_1, \dots D_r$ along curves parallel to each boundary component is free abelian of rank r.*

Proof. — [3, Lemma 3.17]. □

As a consequence, we find the following result.

LEMMA 4.15. — *Let Σ be a surface that is neither a disk nor a cylinder. Let $\partial^1\Sigma$ be non-empty union of r boundary components of Σ. Let ϕ be an orientation preserving homeomorphism of Σ fixing $\partial^1\Sigma$ pointwise. If ϕ is boundary-free isotopic to the identity then there exist unique integers $n_1, \dots, n_r$ such that we have the equality $[\phi]_{\partial^1} = [D_1^{n_1}]_{\partial^1} \cdot \dots \cdot [D_r^{n_r}]_{\partial^1}$.*

DEFINITION 4.16 (fractional Dehn twist coefficient). — *Let Σ be a surface that is neither a disk nor a cylinder. Let $\partial^1\Sigma$ be non-empty union of r boundary components of Σ. Let $\phi : \Sigma \to \Sigma$ be a homeomorphism fixing pointwise $\partial^1\Sigma$ and boundary-free isotopic to a periodic one. Let $m \in \mathbb{N}$ such that $[\phi^m] = [id]$. Let $t_1, \dots, t_r$ be integers such that $[\phi^m]_{\partial^1} = [D_1^{t_1}]_{\partial^1} \cdot \dots \cdot [D_r^{t_r}]_{\partial^1}$. We define the* fractional Dehn twist coefficient *at C_i by*

$$rot_{\partial^1}(\phi, C_i) := t_i/m.$$

Note that the fractional Dehn twist coefficients do not depend on the number m we choose to compute them or the representative $\phi \in [\phi]_{\partial^1}$.

Now we describe how to compute the fractional Dehn twist coefficient in terms of an invariant spine. First of all, we observe that in the isotopy class fixing $\partial^1\Sigma$ pointwise there is always a representative fixing a spine. Indeed, the following lemma is elementary:

LEMMA 4.17. — *Let ϕ be an orientation preserving homeomorphism of a surface Σ which fixes pointwise a non-empty union $\partial^1\Sigma$ of boundary components, and which is boundary-free isotopic to a periodic homeomorphism $\widehat{\phi}$. Then there there exists a collar U of $\partial^1\Sigma$, a homeomorphism $f : \Sigma \to \overline{\Sigma \setminus U}$, and a homeomorphism ψ of Σ such that: ψ is isotopic relative to $\partial^1\Sigma$ to ϕ and the restriction $\psi|_{\overline{\Sigma\setminus U}}$ is periodic and equal to $f \circ \widehat{\phi} \circ f^{-1}$.*

In particular ψ leaves a spine Γ invariant and $\psi|_\Gamma$ is periodic.

Let Σ be an oriented surface with non-empty boundary that is neither a disk nor a cylinder. Let ϕ be an orientation preserving homeomorphism of Σ that fixes a non-empty union $\partial^1\Sigma$ of components of the boundary and which is boundary-free isotopic to a periodic one. Let A be the union of

the remaining components of the boundary. Suppose that there exists a relative spine (Γ, A) in $(\Sigma, (\Gamma, A))$ which is invariant by ϕ.

We cut Σ along Γ into a disjoint union of cylinders, one for each component C_i of $\partial^1\Sigma$. We use Notations 2.3 and 4.11. We lift the retraction $\Sigma \to \Gamma$ to a retraction $\Sigma_\Gamma \to \widetilde{\Gamma}$ and the homeomorphisms ϕ to a homeomorphism $\widetilde{\phi}$ of Σ_Γ. Let $\frac{p_i}{n}$ be the rotation number of $\widetilde{\phi}|_{\widetilde{\Gamma}_i}$. Choose in the cylinder $\widetilde{\Sigma}_i$ a retraction line L_i from C_i to $\widetilde{\Gamma}_i$. Consider the orientation in $\widetilde{\Sigma}_i$ inherited from the orientation in Σ. We take the classes $[L_i]$ and $[\phi(L_i)]$ in $H_1(\widetilde{\Sigma}_i, \partial\widetilde{\Sigma}_i)$. The class

$$\widetilde{\phi}|^n_{\widetilde{\Sigma}_i}([L_i]) - [L_i]$$

belongs to $H_1(\widetilde{\Sigma}_i)$ since $\widetilde{\phi}|^n_{\widetilde{\Sigma}_i}$ is the identity at the boundary. Let

$$k_i := \left(\widetilde{\phi}|^n_{\widetilde{\Sigma}_i}([L_i]) - [L_i]\right) \cdot [L_i],$$

that is the oriented intersection number of the two homology classes.

LEMMA 4.18. — *We have the equality $rot_{\partial^1}(\phi, C_i) = k_i/n$. In particular k_i does not depend on the chosen spine Γ or even on the representative of $[\phi]_{\partial^1}$.*

Proof. — Note that ϕ^n fixes Γ and that the lifting $\widetilde{\phi^n}|_{\widetilde{\Sigma}_i}$ is isotopic relative to the boundary to the composition of k_i right boundary Dehn twist if k_i is positive (and $-k_i$ left Dehn twists if k_i is negative) around the boundary component C_i. Then $[\phi^n]_{\partial^1} = [D_1]^{k_1} \cdot \cdots \cdot [D_r]^{k_r}$ and the result follows. □

COROLLARY 4.19. — *Let $g, h : \Sigma \to \Sigma$ be two homeomorphisms that fix pointwise a non empty union $\partial^1\Sigma$ of components of the boundary $\partial\Sigma$. Let A be the union of the remaining boundary components. Assume that both preserve a common relative spine (Γ, A) and that they coincide and are periodic at it. Then the equality $rot_\partial(g, C_i) = rot_\partial(h, C_i)$ holds for every i if and only if h and g are isotopic relative to $\partial^1\Sigma$.*

COROLLARY 4.20. — *Let $\phi : \Sigma \to \Sigma$ be a homeomorphism that fixes pointwise a non-empty union $\partial^1\Sigma$ of components of the boundary, and that is isotopic to a periodic homeomorphism $\widehat{\phi}$. Let C_i be a component in $\partial^1\Sigma$. Then, a representative between 0 and 1 of the usual rotation number $rot(\widehat{\phi}|_{C_i})$ equals $|rot_{\partial^1}(\phi, C_i) - \lfloor rot_{\partial^1}(\phi, C_i)\rfloor|$ where $\lfloor x\rfloor$ is the biggest integer less that x.*

Remark 4.21. — We observe that by our convention on Remark 4.13, negative (or equivalently right-handed Dehn twists) produce positive fractional Dehn twist coefficients. This is in accordance with the previous notions of fractional Dehn twist coefficient in the literature.

5. Tête-à-tête twists and periodic classes of $MCG(\Sigma)$.

In this section we introduce tête-à-tête twists, which are mapping classes associated with tête-à-tête graphs, and provide a wide generalization of Dehn twits. In fact we show that all periodic classes of $MCG(\Sigma)$ that leave invariant at least one boundary component is a tête-à-tête twist in two different ways:

- to a tête-à-tête graph we associate an element of $MCG(\Sigma, \partial^1\Sigma)$, that we call the tête-à-tête twist associated with the tête-à-tête graph (see Definition 5.4) in Section 5.1, and prove that all elements of $MCG(\Sigma, \partial\Sigma)$ periodic in $MCG(\Sigma)$, with positive fractional Dehn twits coefficient are tête-à-tête twists. In fact enlarging the definition of tête-à-tête graph to that of the signed tête-à-tête graphs (see Definition 5.2), we can represent all elements of $MCG(\Sigma, \partial\Sigma)$ periodic in $MCG(\Sigma)$, both with positive and negative fractional Dehn twist coefficient. The signed tête-à-tête graphs and twists were also introduced by C. Graf in [4].
- to a tête-à-tête graph we associate a (really) periodic homeomorphisms that we call periodic tête-à-tête twist associeted with the graph (see Definition 5.16) in Section 5.2, and show that any periodic homeomorphism is boundary free isotopic to a periodic tête-à-tête twist.

5.1. Signed tête-à-tête graphs and representatives in $MCG(\Sigma, \partial^1\Sigma)$.

Our objective in this section is to represent homeomorphisms which fix pointwise a union $\partial^1\Sigma$ of components of the boundary, and that are boundary-free isotopic to a periodic one.

We start with the definition of signed tête-à-tête graph that generalizes Definition 3.5.

We will work directly for the class of relative graphs in order to avoid repetitions. The main results are the Theorems 5.10 and 5.18.

Let (Γ, A) be a metric relative ribbon graph. Let (Σ, Γ, A) be a thickening, let

$$g_\Gamma : \Sigma_\Gamma \to \Sigma$$

be the gluing mapping as in Notation 2.3.

We start making an extension of Remark 3.4 adding point (b'):

Remark 5.1 (Definition of γ_p^-, ω_p^-, γ_p^0, γ_p^+ and ω_p^+). —

(b') for every point $p \in \Gamma \setminus v(G)$ and every of the two possible oriented direction from p along Γ, there is a walk starting on p following each of this directions, such that in every vertex v, the walk continues along the *previous* edge in the cyclic order of $e(v)$. We denote by γ_p^- and ω_p^- these walks of length π and speed 1. Each of the oriented directions at p corresponds to a point in $q \in g_\Gamma^{-1}(p)$, which lives in a cylinder $\widetilde{\Sigma}_i$. These walks are the image of the negative sense parametrization of the boundary of $\widetilde{\Sigma}_i$ starting at q.

We denote by γ_p^+ and ω_p^+ the usual safe walks of Definition 3.2 or Remark 3.4 of length π and speed 1. We call γ_p^+ and ω_p^+ the *positive safe walks* and γ_p^- and ω_p^- the *negative safe walks*.

In the case of points in A, since A is oriented, we have also a positive and negative sense for a parametrization. Then, for $p \in A$, we define γ_p^+ (respectively γ_p^-) as the parametrization from p that starts along A in the positive (respectively negative) sense and that when reaching a vertex v takes the next (respectively previous) edge in the order of $e(v)$).

We also define a *safe constant walk* $\gamma_p^0 := p$.

Before stating the next definition, recall that there is a bijection between the set $\mathcal{C}$ of boundary components of Σ not in $\mathcal{A}$ and the cylinders $\widetilde{\Sigma}_i$'s. Given a "sign" mapping $\iota : \mathcal{C} \to \{0, +, -\}$, we denote by $\iota(i)$ the image by ι of the component that corresponds to $\widetilde{\Sigma}_i$ under the bijection.

Definition 5.2 (Signed tête-à-tête property and graph). — *Let (Γ, A) be a metric relative ribbon graph and let (Σ, Γ, A) be a thickening. Let $\mathcal{C}$ denote the set of boundary components of Σ which do not belong to A. Fix a mapping*

$$\iota : \mathcal{C} \to \{0, +, -\}.$$

We say that (Γ, A) satisfies the signed tête-à-tête property for ι *or that (Γ, A, ι) is a* signed relative tête-à-tête graph *if given any point p contained at the interior of an edge the following properties are satisfied:*

(1) *if p does not belong to A and $p \in g_\Gamma(\widetilde{\Sigma}_i) \cap g_\Gamma(\widetilde{\Sigma}_j)$ for some i, j, then we have the equality*

$$\gamma_p^{\iota(i)}(\pi) = \omega_p^{\iota(j)}(\pi).$$

(2) *if p belongs to A and p belongs to $g_\Gamma(\widetilde{\Sigma}_i)$, then the end point $\gamma_p^{\iota(i)}(\pi)$ of the unique signed safe walk starting at p belongs to A.*

Notation 5.3 (Remark and Notation). — Observe that the mapping $\Gamma \setminus v(\Gamma) \to \Gamma$ that sends $p \in \Gamma \setminus v(\Gamma)$ to $\gamma_p^{\iota(i)}(\pi)$ extends to $v(\Gamma)$ and define a homeomorphism of Γ that we denote by $\sigma_{(\Gamma,\,\iota)}$. The proof is as the one of Lemma 3.6.

Definition 5.4 (Definition of (signed) tête-à-tête twist $\phi_{(\Gamma,A,\,\iota)}$).

Let (Γ, A, ι) be a signed relative tête-à-tête graph. For every thickening (Σ, Γ, A) and for every choice of product structure $\widetilde{\Sigma}_i \approx \widetilde{\Gamma}_i \times I$ we consider the homeomorphism

$$\psi_i : \widetilde{\Gamma}_i \times I \longrightarrow \widetilde{\Gamma}_i \times I \tag{5.1}$$

$$(p, s) \mapsto \left(\widetilde{\gamma}_p^{\iota(i)}(s \cdot \pi), s\right)$$

where $\widetilde{\gamma}_p^{\iota(i)}$ is the lifting of the safe walk to $\widetilde{\Gamma}_i$. The homeomorphism ψ_i of the cylinder can be visualized very easily using the universal covering of the cylinder as in Figure 5.1.

These homeomorphisms glue well due to the properties of the signed relative tête-à-tête graph (Γ, ι) and define a homeomorphism of (Σ, Γ, A) that leaves $\partial^1\Sigma$ fixed pointwise and A invariant. We denote by $\phi_{(\Sigma,\,\Gamma,\,A,\,\iota)}$ the resulting homeomorphism of (Σ, A). We call it the induced (signed) tête-à-tête twist*.*

Remark 5.5. — Given two different product structures of the cylinders, the induced tête-à-tête twists are conjugate by a homeomorphism that fixes Γ.

For two different embeddings in Σ, the tête-à-tête twists are conjugate by the same homeomorphism of Σ that relates the two embeddings.

The homeomorphism $\phi_{(\Sigma,\,\Gamma,\,A,\,\iota)}$ leaves Γ invariant and by Lemma 4.18 has obviously the following fractional Dehn twist coefficients (see Figure 5.1):

$$rot_{\partial^1}\left(\phi_{(\Sigma,\,\Gamma,\,A,\,\iota)}, C_i\right) = \iota(i) \cdot \frac{\pi}{length(\widetilde{\Gamma}_i)}. \tag{5.2}$$

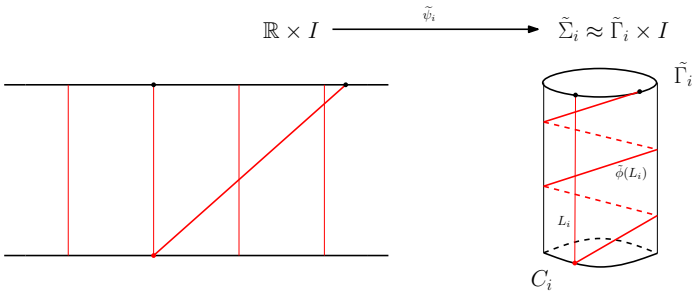


Figure 5.1. Visualization of the lifting $\widetilde{\psi}_i$ of ψ_i to the universal covering of $\widetilde{\Sigma}_i$. You can easily read (5.2) *from it.*

Remark 5.6. — Observe that if $\iota(i) = 0$ for some i, then the homeomorphism ψ_i is the identity and $rot_{\partial^1}(\phi_{(\Gamma, A, \iota)}, C_i) = 0$. In particular, by Corollary 3.9 we have that the signed tête-à-tête twist must be the identity restricted to Γ and hence $rot_{\partial^1}(\phi_{(\Sigma, \Gamma, A, \iota)}, C_j) \in \mathbb{Z}$ for all boundary components. So $\phi_{\Sigma, \Gamma, A, i}$ equals a composition of Dehn twists around curves parallel to the boundary.

Remark 5.7. — If all the signs are positive this notion coincides with the first author's original notion of tête-à-tête twists.

Example 5.8. — The homeomorphisms induced by the tête-à-tête structures on the $K_{p, q}$ ribbon graphs given in Example 3.10 are the monodromies fixing the boundary of the Milnor fibrations associated to the singularities $x^p - y^q = 0$. This is an original construction from [2].

Before going on with the section we note that not every invariant spine admits a metric modeling the corresponding homeomorphism. Indeed, we have the following example.

Example 5.9. — In Figure 5.2 you can find a spine Γ for the genus 1 surface with one boundary components that is invariant by the π-rotation along the vertical axes isotoped to be the identity on both boundary components. In particular, it has fractional Dehn twist coefficient equal to $1/2$ on both boundary components.

This ribbon graph does not admit a tête-à-tête structure that models the given homeomorphism. Let L_1 and L_2 be the lengths of the edges of the graph that meet both cylinders of Σ_Γ. And let L_3 be the other one.

By (5.2), since fractional Dehn twist coefficients are the same on both boundary components, we have that $1/2(L_1 + L_2) = \pi$ and that $1/2(L_1 +$

$L_2 + 2L_3) = \pi$. This is turn would imply the equality $L_3 = 0$ which is not possible since all lengths of edges must be strictly positive.

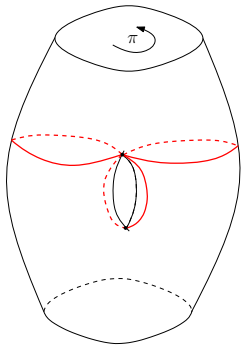


Figure 5.2. A spine for the genus 1 surface with one boundary components that is invariant by the π-rotation along the vertical axes.

In the next theorem we see that every homeomorphism that fixes the boundary pointwise that is boundary-free isotopic to a periodic one, is a signed tête-à-tête homeomorphism up to isotopy fixing the boundary. We state first the non-relative case, which improves [4, Theorem 3.1.1] as we comment on Remark 5.12.

THEOREM 5.10. — *Let Σ be an oriented surface with non-empty boundary. Let ϕ be an orientation preserving homeomorphism fixing pointwise the boundary, and boundary-free isotopic to a periodic one $\widehat{\phi}$. Then,*

(i) *there exists a signed tête-à-tête spine (Γ, ι) embedded in Σ that is invariant by ϕ such that the restriction of ϕ to Γ coincides with $\phi_{(\Sigma,\Gamma,\iota)}$.*
(ii) *The isotopy classes relative to the boundary $[\phi]_\partial$ and $[\phi_{(\Sigma,\Gamma,\iota)}]_\partial$ coincide.*
(iii) *the homeomorphisms ϕ and $\phi_{\Gamma,\iota}$ are conjugate by a homeomorphism that fixes the boundary pointwise, fixes Γ and is isotopic to the identity in $MCG(\Sigma, \partial\Sigma)$.*

COROLLARY 5.11. — *This theorem characterizes the originally defined by the first author tête-à-tête twists as (i.e. non-signed tête-à-tête twists): orientation preserving homeomorphism fixing pointwise $\partial^1\Sigma$ and boundary-free isotopic to a periodic one $\widehat{\phi}$ with strictly positive fractional Dehn twist coefficients.*

Remark 5.12. — We compare the above corollary with [4, Theorem 3.1.1]. In the cited reference, the author proves a similar result but he enlarges the set of *permitted* tête-à-tête graphs, either by allowing vertices of valency 1 or by allowing safe walks of different lengths for each boundary component. A homeomorphism leaving a spine invariant leaves the cylindrical decomposition invariant. Our notion imposes that the homeomorphism consists in rotating all the cylinders at the same speed with respect to the metric, while Graf's notion needs to allow them them rotating at different speeds in the case of no vertices of valency one (or needs vertices of valency 1, (which is equivalent to allow different speeds). Note that Graf's proof can not be adapted to prove our result.

Proof of Theorem 5.10. — We use Notation 2.3, Notation 4.5 and Notation 4.11.

By Lemma 4.17 we can assume that there exist a collar for $\partial\Sigma$ such that $\phi' = \phi|_{\Sigma\setminus U}$ is periodic. Let n be the order of ϕ'. Let $\Sigma^{\phi'}$ be the quotient surface. By abuse the notation we are writing $\Sigma^{\phi'}$ instead of $(\Sigma\setminus U)^{\phi'}$ since they are naturally homeomorphic.

We will construct the invariant graph Γ as in the proof of Lemma 4.9, as the preimage by the quotient map $p : \Sigma \to \Sigma^{\phi'}$ of an appropriate spine $\Gamma^{\phi'}$ for $\Sigma^{\phi'}$. The proof consists in giving a metric for the spine in $\Sigma^{\phi'}$ such that the pullback metric in the corresponding invariant graph in Σ solves the problem.

Let us see what conditions on the lengths of the edges of $\Gamma^{\phi'}$ have to be imposed such that the pullback-metric in Γ defines a tête-à-tête metric adapted to ϕ, that is, so that we have the equality $\gamma_p(\pi) = \phi(p)$ for the safe walks along Γ and such that the given signed tête-à-tête twist $\phi_{\Gamma,\iota}$ has the same fractional Dehn twist coefficients as ϕ.

We define

$$R_i := |rot_{\partial^1}(\phi, C_i)|\,.$$

The tête-à-tête structure of Γ has to satisfy the equality $\gamma_p(\pi) = \phi(p)$ and moreover, the rotation number of $\phi_{\Gamma,\iota}$ at C_i has to be R_i. So, by (5.2), we want that for every i with $R_i \neq 0$ we have

$$rot_{\partial^1}(\phi, C_i) \cdot length(\widetilde{\Gamma}_i) \cdot \iota(i) = \pi, \tag{5.3}$$

If $rot_{\partial}(\phi, C_i)$ equals 0, by the definition of constant safe walk (see the end of Remark 5.1), we obtain no condition.

By the definition of R_i and the fact that both $rot_{\partial^1}(\phi, C_i)$ and $\iota(i)$ have the same sign, this equation becomes:

$$R_i \cdot length(\widetilde{\Gamma}_i) = \pi, \tag{5.4}$$

Moreover, we want the metric on Γ to be invariant by ϕ' so it has to be the pullback of a metric on $\Gamma^{\phi'}$. We denote by $\Sigma^{\phi'}_{\Gamma^{\phi'}}$ the surface obtained by cutting $\Sigma^{\phi'}$ along $\Gamma^{\phi'}$ and consider the gluing map $g_{\Gamma^{\phi'}} : \Sigma^{\phi'}_{\Gamma^{\phi'}} \to \Sigma^{\phi'}$ analogously to Notation 2.3. We consider the lifting of $p : \Sigma \to \Sigma^{\phi'}$ to the cut surfaces and we denote it by $\widetilde{p} : \Sigma_\Gamma \to \Sigma^{\phi'}_{\Gamma^{\phi'}}$. We denote by $\widetilde{p(\Gamma_i)}$ the preimage of $p(\Gamma_i)$ by $g_{\Gamma^{\phi'}}$. Since $\widetilde{p}|_{\widetilde{\Gamma}_i} : \widetilde{\Gamma}_i \to \widetilde{p(\Gamma_i)}$ is a $n : 1$ covering map, we have the equality

$$length(\widetilde{\Gamma}_i) = n \cdot length\left(\widetilde{p(\Gamma_i)}\right). \tag{5.5}$$

Note that one can easily read $length(\widetilde{p(\Gamma_i)})$ looking at the lengths of the edges of $p(\Gamma_i) \subseteq \Gamma^{\phi'}$.

Putting (5.4)-(5.5) together, we have that what we need is that the equality

$$length\left(\widetilde{p(\Gamma_i)}\right) = \frac{\pi}{n \cdot R_i} \tag{5.6}$$

holds for all i with $R_i \neq 0$.

Next, we see that finding a metric spine $\Gamma^{\phi'} \hookrightarrow \Sigma^{\phi'}$ containing all branching points, and whose lengths satisfy (5.6) for every i with $R_i \neq 0$ we prove the theorem by taking $\Gamma = p^{-1}(\Gamma^{\phi'})$ with the pullback metric.

Since $\Gamma^{\phi'}$ contains the branching points, the retraction of $\Sigma^{\phi'}$ to $\Gamma^{\phi'}$ lifts to a retraction of Σ to the preimage $\Gamma := p^{-1}(\Gamma^{\phi'})$. Hence Γ is a spine of Σ.

Then it is clear that the graph $\Gamma := p^{-1}(\Gamma^{\hat{\phi}})$ is a signed tête-à-tête graph for ι defined as $\iota(i) = sign(rot_{\partial^1}(\phi, C_i))$. Let $\phi_{(\Sigma,\,\Gamma,\,\iota)}$ be the induced signed tête-à-tête twist. This tête-à-tête structure on Γ induces by construction a rotation $\widetilde{\phi}_{(\Sigma,\,\Gamma,\,\iota)}|_{\widetilde{\Gamma}_i}$ of rotation number

$$\left| rot_{\partial^1}(\phi, C_i) - \lfloor rot_{\partial^1}(\phi, C_i) \rfloor \right| \text{ in each } \widetilde{\Gamma}_i.$$

It is conjugate to $\widetilde{\phi}|_{\widetilde{\Gamma}_i}$ since they have the same rotation number (recall Corollary 4.20). The orbits of $\widetilde{\phi}|_{\widetilde{\Gamma}_i}$ are the fibres of $\widetilde{p}|_{\widetilde{\Gamma}_i}$. By the choice of lengths, the orbits of the tête-à-tête rotation in $\widetilde{\Gamma}_i$ are also the fibres of $\widetilde{p}|_{\widetilde{\Gamma}_i}$. Conjugation between homeomorphisms of $\mathbb{S}^1$ preserves the cyclic order in $\mathbb{S}^1$ of a point p and its iterations. Then, since $\widetilde{\phi}|_{\widetilde{\Gamma}_i}$ and $\widetilde{\phi}_{(\Sigma,\,\Gamma,\,\iota)}|_{\widetilde{\Gamma}_i}$ are conjugate with the same orbits, they coincide. Then $\phi|_\Gamma$ and $\phi_{(\Sigma,\,\Gamma,\,\iota)}|_\Gamma$ coincide.

By Remark 4.8 we have that ϕ and $\phi_{(\Sigma,\,\Gamma,\,\iota)}$ are isotopic since they coincide on Γ.

We also see that, by the imposed metric, $rot_{\partial^1}(\phi_{(\Sigma,\Gamma,\iota)}, C_i) = rot_{\partial^1}(\phi, C_i)$ (just observe that the signed safe walk corresponding to the cylinder $\widetilde{\Gamma}_i \times [0,1]$ *winds up* R_i times around $\widetilde{\Gamma}_i$). So, since ϕ and $\phi_{(\Sigma,\Gamma,\iota)}$ coincide on a spine with a periodic homeomorphism and have the same fractional Dehn twist coefficients we can conclude by Corollary 4.19 that they are isotopic relative to the boundary.

The restrictions of $\widetilde{\phi}$ and $\widetilde{\phi}_{(\Sigma,\Gamma,\iota)}$ to each invariant cylinder $\widetilde{\Sigma}_i$ are conjugate by a homeomorphism that leaves its boundary fixed pointwise. Then, these conjugation homeomorphisms glue together to a conjugation homeomorphism for ϕ and $\phi_{(\Sigma,\Gamma,\iota)}$ that leaves Γ and $\partial\Sigma$ fixed pointwise. Then, ϕ and $\phi_{(\Sigma,\Gamma,\iota)}$ are also conjugate as required in the statement.

Now we finish the proof finding such an invariant spine Γ.

First we prove the case $g := genus(\Sigma^{\phi'}) \geqslant 1$. To choose a spine in $\Sigma^{\phi'}$, we use a planar representation of $\Sigma^{\phi'}$ as a convex $4g$-gon in $\mathbb{R}^2$ with r disjoint open disks removed from its convex hull. The sides of the $4g$-gon are labelled clockwise like $a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1}$, where edges labelled with the same letter (but different exponent) are identified by an orientation reversing homeomorphism. The number r is the number of boundary components. We number the boundary components $C_i \subseteq \partial\Sigma$, $1 \leqslant i \leqslant r$. We denote by d the arc $a_2 b_2 a_2^{-1} b_2^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1}$. We consider $l_1, \ldots, l_{r-1}$ arcs as in Figure 5.3. We denote by $c_1, \ldots, c_r$ the edges in which a_1^{-1} (and a_1) is subdivided, numbered according to the component $p(C_i)$ they enclose. We consider the spine $\Gamma^{\phi'}$ of $\Sigma^{\phi'}$ given by the union of d, $a_1 b_1 a_1^{-1} b_1^{-1}$ and the l_i's. We construct $\Gamma^{\phi'}$ so that it passes by all the branching points of p.

We denote by D, B_1 and C_i the lengths of d, b_1 and c_i respectively. We will assume all the l_i and b_1 of the same length L. Then the system (5.6) for this case can be expressed as follows:

$$\begin{aligned} 2L + 2C_1 &= \frac{\pi}{n \cdot R_1} \\ 2L + 2C_i &= \frac{\pi}{n \cdot R_i} \qquad \text{for } i = 2, \ldots, r-1 \\ 2L + 2C_r + D &= \frac{\pi}{n \cdot R_r}, \end{aligned} \tag{5.7}$$

which has obviously positive solutions C_i, D after choosing for example $L = \min\{\frac{\pi}{4 \cdot n \cdot R_i}\}$. We assign $length(a_i) = length(b_i) = D/4(g-1)$ for $i > 1$ to get a metric on Γ^{ϕ}. We consider the pullback-metric in Γ. This finishes the case $genus(\Sigma^{\phi'}) \geqslant 1$.

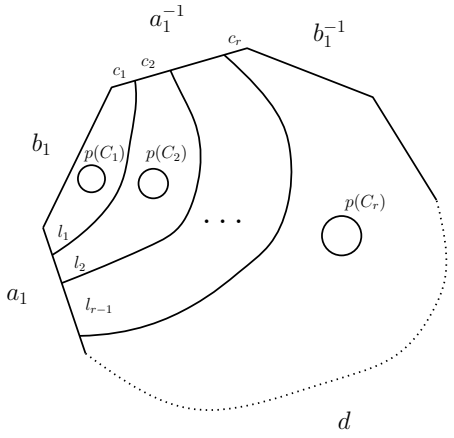


Figure 5.3. Planar representation of the surface Σ^ϕ of genus $\geqslant 1$ and r boundary components $p(C_1), \ldots, p(C_r)$. Drawing of $l_1, \ldots, l_r$ and $c_1, .., c_r$.

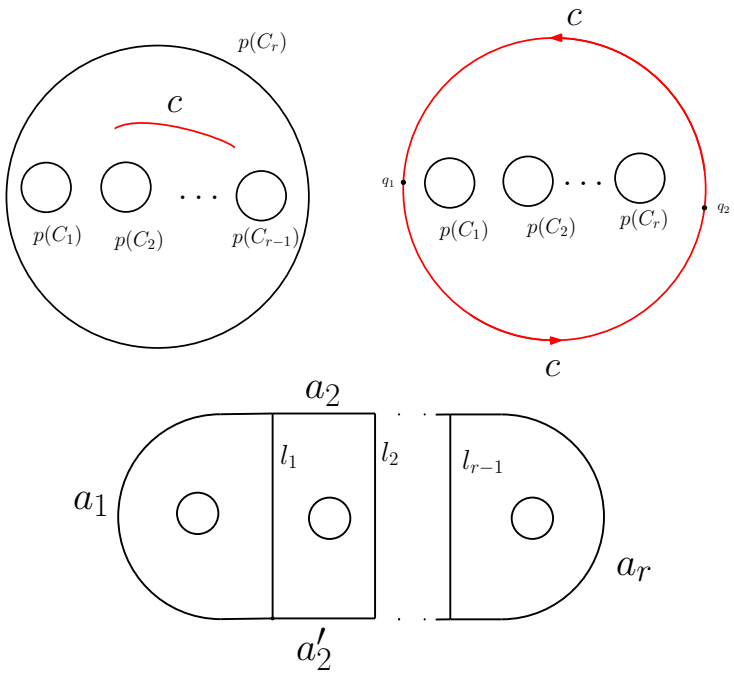


Figure 5.4. In the first two pictures we have the disk with $r-1$ smaller disks removed. In the third one we forget the identification along the exterior and we draw a spine.

For the case $genus(\Sigma^{\phi'}) = 0$ we proceed a bit differently to choose the spine $\Gamma^{\phi'}$. The surface $\Sigma^{\phi'}$ is a disk with $r-1$ smaller disjoint disks removed. We cut the surface along an embedded segment that we call c as we can see in the first image of Figure 5.4. Cutting along c we get another planar representation of $\Sigma^{\phi'}$ as in the second image. The exterior boundary corresponds to cc^{-1}; we call the exterior boundary P and denote by q_1 and q_2 the points in P that come from the two extremes of c. We look at the graph of the third picture in Figure 5.4. We have drawn $r-1$ vertical segments $l_1, \ldots, l_{r-1}$ so that P union with them contains all branch points and is a regular retract of the disk enclosed by P minus the r disks.

The rest of the proof follows by cases on the number of boundary components r and the number of branch points.

If $r = 1$ and there are no branch points or 1 branch point, then Σ is a disk (this follows from the Hurwitz formula) which is not covered by the statement of the theorem. If there are at least 2 branch points, we can get that two branch points lie in q_1 and q_2 so that Γ has no univalent vertices. In this case we set $length(c) = \frac{\pi}{2nR_1}$.

Suppose now $r = 2$. If there are no branch points, then Σ is a cylinder which is not included in the statement of this theorem.

If there is at least 1 branch point we consider two cases, namely $R_1 = R_2$ and $R_1 < R_2$.

In the case $R_1 = R_2$, we choose the graph depicted on the right hand side of Figure 5.5. That is, q_1 and q_2 are exactly $a_1 \cap a_2$. In this case we do not care about the location of the branch point as long as it is contained in the graph. In this case we set $lenght(c) = length(l_1) = \frac{\pi}{2nR_1}$

In the case $R_1 < R_2$, we choose the graph depicted on the left hand side of Figure 5.5 and we choose the branch point to lie on q_1, this way, since q_1 is the only vertex of valency 1, we get that the preimage of this graph by p does not have univalent vertices. We set the lengths, $length(l_1) = \frac{\pi}{nR_1}$ and $length(c) = (\frac{\pi}{nR_2} - \frac{\pi}{nR_1})/2$.

Suppose now $r > 2$.

We are going to assign lengths to every edge in Figure 5.4 and decide how to divide and glue P in order to recover $\Sigma^{\phi'}$. This means that we are going to decide the position of q_1 and q_2 in P, relative to the position of the ends of the l_i's, in order to get a suitable metric spine of the quotient surface $\Sigma^{\phi'}$.

To every vertical interior segment l_j we assign the same length

$$L < \min\left\{\frac{\pi}{2(n \cdot R_i)}\right\}.$$

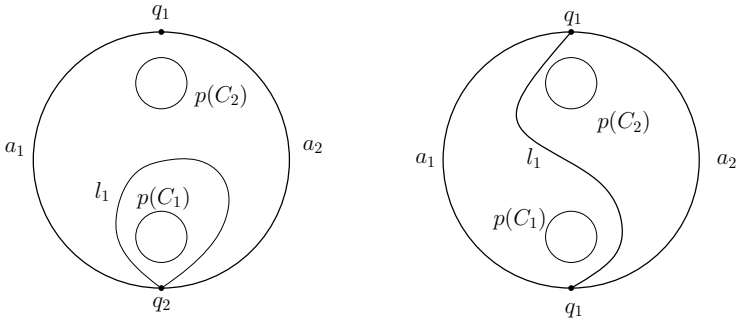


Figure 5.5. On the left, the case $R_1 < R_2$. On the right the case when $R_1 = R_2$.

We look at the segments a_1, a_2, a_2', …, a_{r-1}, a_{r-1}', a_r in which P is divided by the vertical segments (see Figure 5.4) and give lengths A_1, A_2, A_2', …, A_{r-1}, A_{r-1}', A_r. The following system corresponds to (5.6) for this case:

$$\begin{aligned} A_1 + L &= \frac{\pi}{n \cdot R_1} \\ A_i + A_i' + 2L &= \frac{\pi}{n \cdot R_i} \quad \text{for} \quad i = 2, \ldots, r-1 \\ A_r + L &= \frac{\pi}{n \cdot R_r}. \end{aligned} \tag{5.8}$$

It has obviously positive solutions A_i. We choose $A_i = A_i'$ for $i = 2, \ldots, r-1$.

In order this distances can be pullback to the original graph $\Sigma^{\phi'}$, there is one equation left: we have to impose equal length to the two paths c and c^{-1}, or in other words to place q_1 and q_2 dividing P in two segments of equal length.

If ϕ has at least two branching points, we choose q_1 and q_2 to be any two of the branching points. Then, we can choose the metric and the vertical segments so that q_1 and q_2 are the middle points of a_1 and a_r. If we identify the two paths c and c^{-1} joining q_1 and q_2 then we recover $\Sigma^{\phi'}$, and we get a metric graph on it. Then, the preimage by p of the resulting metric graph gives a metric graph. We claim that this graph has no univalent vertices. Indeed, a univalent vertex of this graph has to be the preimage of univalent vertices of the graph below, which are only q_1 and q_2, which are branching points. By Remark 4.6 all their preimages are ramification points and then they are not univalent vertices. The metric induces a tête-à-tête

structure in the graph by construction. Now, we finish the proof as in the case $genus(\Sigma^{\phi'}) \geqslant 1$.

If ϕ has no ramification points or only one, in the previous construction we could obtain univalent vertices at the preimages of q_1 and q_2. So we need to do some changes in the assignations of lengths and in the positions of the vertical segments relative to q_1 and q_2 in order that the extremes q_1 and q_2 of c coincide with vertices of the graph in $\Sigma^{\phi'}$.

Assume that $A_1 \geqslant A_2 \geqslant \cdots \geqslant A_r$.

We choose $q_1 := a_1 \cap a_2$. Let q_2 be the antipodal point (so q_1 and q_2 divides P in two paths of equal length). If q_2 is a vertex we have finished. If it is not, then it is on a segment a'_i, for some $i = 2, \ldots, r_1$. We redefine $A'_i := d(a'_i \cap a'_{i-1}, q_2)$ and $A_i := A_i + d(q_2, a'_i \cap a'_{i+1})$. Now the antipodal point of q_1 is $a_i \cap a_{i+1}$. We redefine $q_2 := a_i \cap a_{i+1}$ and identify orientation reversing the two paths joining q_1 and q_2 to recover Σ^{ϕ}.

Now the pullback of the resulting metric graph has no univalent vertices and gives a tête-à-tête structure since the corresponding system of equations is satisfied. □

Remark 5.13. — Observe that in the case that $genus(\Sigma^{\hat{\phi}}) \neq 0$ or the quotient map by $\widehat{\phi}$ has at least two branching points, we have found a spine Γ of Σ such that for any homeomorphism ψ which fixes pointwise the boundary and is boundary-free isotopic to $\widehat{\phi}$, there is a signed tête-à-tête structure on Γ (that is, a metric and a sign function ι) such that ψ is isotopic relative to the boundary to the corresponding signed tête-à-tête homeomorphism. In other words, there is a *universal* spine which may be endowed of signed tête-à-tête structures representing all boundary fixed isotopy classes of homeomorphisms which are boundary-free isotopic to ϕ.

Now we state the relative case in a simple way, skipping the obvious strengthenings similar to the previous theorem:

THEOREM 5.14. — *Let Σ be an oriented surface with non-empty boundary. Let $\partial^1\Sigma$ be a non empty union of boundary components. Let A be the union of the boundary components not contained in $\partial^1\Sigma$. Let ϕ be an orientation preserving homeomorphism fixing pointwise $\partial^1\Sigma$ and boundary-free isotopic to a periodic one $\widehat{\phi}$. Then, there exists a signed tête-à-tête spine $(\Gamma, A, \iota) \hookrightarrow \Sigma$ such that $\phi_{(\Sigma, \Gamma, A, \iota)}$ is isotopic relative to $\partial^1\Sigma$ to ϕ. Moreover, if ϕ is periodic outside a collar of $\partial^1\Sigma$, we have also that $[\phi]_{\partial, \phi|_\partial} = [\phi_{(\Sigma, \Gamma, A, \iota)}]_{\partial, \phi|_\partial}$.*

Proof. — Apply Alexander's trick (see Definition 4.7) to the boundary components in A in order to obtain a larger surface and a homeomorphism

fixing pointwise the boundary. Construct a signed tête-à-tête graph inducing this homeomorphism like in the proof of Theorem 5.10. We can always get that this signed tête-à-tête graph contains as vertices the centers of the disks added by Alexanders's trick. Now apply an ϵ-blow up (recall Definition 3.13) to these vertices to get the desired signed relative tête-à-tête graph. □

Remark 5.15. — Note that this produces a relative tête-à-tête graph where all the vertices in the boundary have valency at most 3.

5.2. Periodic tête-à-tête twists and periodic representatives in $MCG(\Sigma)$.

In this section we introduce a periodic twist induced by a tête-à-tête graph. We prove Theorem 5.25 as a corollary of the previous section, which shows that periodic tête-à-tête twists give rise to all truly periodic homeomorphisms (leaving at least one boundary component invariant) up to isotopy and conjugacy.

DEFINITION 5.16 (Periodic tête-à-tête twist). — *Replacing the map* (5.1) *in Definition 5.4 by*

$$\begin{aligned} \psi_i : \widetilde{\Gamma}_i \times I &\longrightarrow \widetilde{\Gamma}_i \times I \\ (p, s) &\mapsto \left(\widetilde{\gamma}_p^+(\pi), s\right) \end{aligned}$$

we get a truly periodic homeomorphism that we denote by $\phi^{per}_{\Sigma,\Gamma,A,\iota}$. *We call it the induced* periodic tête-à-tête twist.

Remark 5.17. — For two different product structures of the cylinders, the induced periodic tête-à-tête twists are conjugate by a homeomorphism that fixes Γ.

For two different embeddings in Σ, the periodic tête-à-tête twists are conjugate by the same homeomorphism of Σ that relates the two embeddings.

Note that obviously $\phi^{per}_{\Sigma,\Gamma,A,\iota}$ is boundary-free isotopic to $\phi_{(\Sigma,\Gamma,A,\iota)}$ and coincides along Γ.

Given a periodic homeomorphism ϕ, we can clearly choose a representative ϕ' of $[\phi]$ that leaves $\partial^1\Sigma$ fixed pointwise (by isotoping ϕ near $\partial^1\Sigma$ until it is the identity on it). Then, we can find a signed tête-à-tête graph (Γ, A, ι) embedded in Σ to represent it using Theorems 5.10 and 5.14. Then, we can consider the periodic homeomorphism $\phi^{per}_{\Sigma,\Gamma,A,\iota}$. Note that we can always

get that ϕ' has all its fractional Dehn twist coefficients positive, which means $\iota \equiv +$. Then, we get the following theorem by using Remark 4.12:

THEOREM 5.18. — *Let Σ be a connected surface with non-empty boundary which is not a disk or a cylinder. Let ϕ be an orientation preserving periodic homeomorphism of Σ that leaves (at least) one boundary component invariant. Let A be the set containing all boundary components that are not invariant by ϕ. Then there exists a relative tête-à-tête graph (Γ, A) embedded in (Σ, A), which is invariant by ϕ, such that*

(i) *We have the equality of boundary-free isotopy classes*
$$\left[\phi^{per}_{(\Sigma,\,\Gamma,\,A,\,\iota)}\right]_{A,\,\phi|_A} = [\phi]_{A,\,s\phi|_A}$$
(see Notation 4.1).

(ii) *the homeomorphism ϕ is conjugate to $\phi^{per}_{(\Sigma,\,\Gamma,\,A,\,\iota)}$ by a homeomorphism that fixes Γ.*

Remark 5.19. — Observe that in order to represent all truly periodic homeomophisms we do not need the extension to signed tête-à-tête graphs in Definition 5.2; The first author original definition, given in Definition 3.5 is enough.

The following corollary recovers a known result giving an elementary proof of it. See, for example, [3, Theorem 7.14] and the introduction to section 7.4 in that same reference for a different proof.

COROLLARY 5.20. — *Let Σ be the surface of genus g and r boundary components. There are finitely many conjugacy classes of finite-order mapping classes in $MCG(\Sigma)$.*

Proof. — Observe that there are only a finite amount of spines without vertices of valency 1. By Theorem 5.18, these are enough to model all periodic mapping classes. By Remark 5.17, two distinct embeddings of the same graph produce conjugate periodic homeomorphisms and the result follows. □

6. Pseudo-periodic homeomorphisms.

Definition and conventions

We recall some definitions and fix some conventions on pseudo-periodic homeomorphisms of surfaces with boundary.

DEFINITION 6.1. — *A homeomorphism $\phi : \Sigma \to \Sigma$ is* pseudo-periodic *if it is isotopic to a homeomorphism satisfying that there exists a finite collection of disjoint simple closed curves $\mathcal{C}$ such that*

(1) $\phi(\mathcal{C}) = \mathcal{C}$
(2) $\phi|_{\Sigma\setminus\mathcal{C}}$ *is boundary-free isotopic to a periodic homeomorphism.*

This system of cut curves $\mathcal{C}$ is called a system of cut curves subordinated to ϕ.

The following theorem is due to Nielsen (see [8, Section 15]):

THEOREM 6.2. — *Given a pseudo-periodic homeomorphism ϕ, there exists a minimal system of cut curves $\mathcal{C}$. In particular, none of the connected components of $\Sigma \setminus \mathcal{C}$ is neither a disk nor an annulus. A minimal system of cut curves is unique up to isotopy.*

Remark 6.3 (Quasi-Canonical Form). — Given a system of cut curves $\mathcal{C} = \{\mathcal{C}_i\}$ subordinated to ϕ, it is clear that there exists an isotopic homeomorphism ϕ' that admits annular neighbourhoods $\mathcal{A} = \{\mathcal{A}_i\}$ of the curves in $\mathcal{C}$ with $\mathcal{C}_i \subset \mathcal{A}_i$ such that

(1) $\phi'(\mathcal{A}) = \mathcal{A}$,
(2) the map $\phi'|_{\overline{\Sigma\setminus\mathcal{A}}}$ is periodic.

Moreover, in the case ϕ fixes pointwise some components $\partial^1\Sigma$ of the boundary $\partial\Sigma$, we can always find an isotopic homeomorphism ϕ'' relative to $\partial^1\Sigma$ that coincides with a homeomorphism satisfying (1) and (2) outside a collar neighborhood U of $\partial^1\Sigma$. We may assume that there exists an isotopy connecting ϕ and ϕ'' relative to $\partial^1\Sigma$.

We say ϕ' and ϕ'' are *quasi-canonical forms* for ϕ with respect to the set $\mathcal{C}$ of cut curves.

DEFINITION 6.4 (Canonical Form). — *A quasi-canonical form for a quasi-periodic homeomorphism ϕ with respect to a minimal system of cut curves is called a* canonical form *for ϕ.*

Remark 6.5. — Note that the uniqueness up to isotopy of a minimal system of cut curves $\mathcal{C}$ (see Theorem 6.2) implies that $\phi|_{\mathcal{A}}$ is unique up to conjugacy where $\mathcal{A}$ is the collar neighbourhood of $\mathcal{C}$ in the definition of canonical form.

Notation 6.6. — Let $m, c \in \mathbb{R}$. We denote by $\mathcal{D}_{m,\,c}$ the homeomorphism of $\mathbb{S}^1 \times I$ induced by $(x,t) \mapsto (x + mt + c, t)$ (we are taking $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$). Observe that

$$\mathcal{D}_{m,\,c} \circ \mathcal{D}_{m',\,c'} = \mathcal{D}_{m+m',\,c+c'}, \tag{6.1}$$

$$\mathcal{D}_{m,\,c}^{-1} = \mathcal{D}_{-m,\,-c}. \tag{6.2}$$

In any case, in this work we will always have $m \in \mathbb{Q}$.

LEMMA 6.7 (Linearization. [6, Lemma 2.1]). — *Let $\mathcal{A}_1$ be an annulus and let $\phi : \mathcal{A}_1 \to \mathcal{A}_1$ be a homeomorphism that does not exchange boundary components. Suppose that $\phi|_{\partial\mathcal{A}_1}$ is periodic. Then, after an isotopy of ϕ preserving the action at the boundary, there exists a parametrization $\eta : \mathbb{S}^1 \times I \to \mathcal{A}_1$ such that*

$$\phi = \eta \circ \mathcal{D}_{-m,\,-c} \circ \eta^{-1}$$

for some $m, c \in \mathbb{Q}$.

Remark 6.8. — In the case $\phi|_{\partial\mathcal{A}_1}$ is the identity, we have that

$$\phi = \eta \circ \mathcal{D}_{m,\,0} \circ \eta^{-1}$$

for some $m \in \mathbb{Z}$.

In the case $m = 1$, that is the conjugacy class of $\mathcal{D}_{1,\,0}$, is what is called in the literature negative Dehn twist (compare to convention in Remark 4.13).

The same name of negative Dehn twist is used for the extension of ϕ by the identity to a homeomorphism of a bigger surface.

Remark 6.9. — Note that m and c are completely determined in Lemma 6.7. The parameter c (which only matters modulo $\mathbb{Z}$) equals the rotation number of $\phi|_{\eta(\mathbb{S}^1\times\{0\})}$. The sum $c+m$ modulo $\mathbb{Z}$ equals the rotation number of $\phi|_{\eta(\mathbb{S}^1\times\{1\})}$ measured with the orientation induced by the direct identification with $\mathbb{S}^1 \times \{0\}$ (and not as boundary of the annulus). Moreover, the parameter m is completely determined since $\mathcal{D}_{m,\,c}$ and $\mathcal{D}_{m',\,c}$ with $m \neq m'$ and $m \equiv m' \,mod\, \mathbb{Z}$ are never isotopic relative to the boundary.

DEFINITION 6.10 (Screw number). — *Let $\{\mathcal{A}_i\}_{i=1}^{\alpha}$ be a set of annuli cyclically permuted by a homeomorphism ϕ, i.e. $\phi(\mathcal{A}_i) = \mathcal{A}_{i+1}$ and $\phi(\mathcal{A}_\alpha) = \mathcal{A}_1$. Denote $\mathcal{A} := \bigcup_{i=1}^{\alpha} \mathcal{A}_i$. Assume that $\phi|_{\partial\mathcal{A}}$ is periodic and let n be its order.*

By Remark 6.8, $\phi^n|_{\mathcal{A}_i}$ equals a conjugate to $\mathcal{D}_{e_i,\,0}$ for a certain $e \in \mathbb{Z}$. We define

$$s(\mathcal{A}_i) := -\frac{e}{n}\theta \tag{6.3}$$

where $\theta = \alpha$ if $\phi^\alpha|_{\mathcal{A}_i}$ does not interchange boundary components and $\theta = 2\alpha$ in the other case. We call $s(\mathcal{A}_i)$ the screw number *of ϕ at $\mathcal{A}_i$.*

Let $\mathcal{C} = \{\mathcal{C}_i\}$ be a system of cut curves subordinated to a pseudo-periodic homeomorphism ϕ. Assume ϕ is in quasi-canonical form (see Remark 6.3)

with respect to $\mathcal{C}$. We define the screw numbers $s(\mathcal{C}_i)$ as the screw number of the corresponding annulus.

Remark 6.11. — Let $\{\mathcal{C}_i\}_{i=1}^{\alpha}$ be curves cyclically permuted by a homeomorphism ϕ and that are a subset of a system of cut curves subordinated to a pseudo-periodic homeomorphism ϕ that we assume in quasi-canonical form (see Remark 6.3). It can be checked that these numbers are well defined independently of the choice of $\mathcal{C}$ and $\mathcal{A}$.

Remark 6.12. — Compare Definition 6.10 with [6, p. 4] and with [6, Definition 2.4]. The original definition is due to Nielsen [8, Section 12].

Remark 6.13. — By [6, Corollary 2.2] we have that $\mathcal{D}_{s,\,c}$ has a screw number equal to $-s$. In particular the negative Dehn twist $\mathcal{D}_{1,\,0}$ has screw number -1.

We start with an easy lemma that is important for Theorem 7.20.

LEMMA 6.14. — *Let $\{\mathcal{A}_i\}_{i=1}^{\alpha}$ be a set of annuli and ϕ as in Definition 6.10 permuting them cyclically. Suppose that $\phi|_{\mathcal{A}_1}^{\alpha}$ does not interchange boundary components. Then, after an isotopy of ϕ preserving the action at all the boundary components, there exist coordinates*

$$\eta_i : \mathbb{S}^1 \times I \to \mathcal{A}_i$$

for the annuli in the orbit such that

$$\eta_{j+1}^{-1} \circ \phi \circ \eta_j = \mathcal{D}_{-m/\alpha,\,-c/\alpha}$$

where m and c are associated to $\mathcal{A}_1$ and $\phi^{\alpha}|_{\mathcal{A}_1}$ as in Lemma 6.7.

Proof. — Isotope ϕ if necessary in order to take a parametrization of $\mathcal{A}_1$, associated to $\phi^{\alpha} : \mathcal{A}_1 \to \mathcal{A}_1$ as in Lemma 6.7. Denote this parametrization by $\eta_1 : \mathbb{S}^1 \times I \to \mathcal{A}_1$.

Define recursively $\eta_j := \phi \circ \eta_{j-1} \circ \mathcal{D}_{m/\alpha,\,c/\alpha}$ (see Notation 6.6). Then, we have

$$\eta_{j+1}^{-1} \circ \phi \circ \eta_j = \mathcal{D}_{-m/\alpha,\,-c/\alpha}.$$

Since for every j we have that $\eta_j = \phi^{j-1} \circ \eta_1 \circ \mathcal{D}_{m(j-1)/\alpha,\,c(j-1)/\alpha}$ we have also that

$$\begin{aligned} \eta_1^{-1} \circ \phi \circ \eta_\alpha &= \eta_1^{-1} \circ \phi \circ \phi^{\alpha-1} \circ \eta_1 \circ \mathcal{D}_{m(\alpha-1)/\alpha,\,c(\alpha-1)/\alpha} \\ &= \mathcal{D}_{-m/\alpha,\,-c/\alpha}. \end{aligned}$$

□

Remark 6.15. — In particular, in the setting of this lemma, we have the equality $m = s(\mathcal{A}_1)$ since by (6.1) we have $e = m \cdot n/\alpha$ and in this case we have $\theta = \alpha$ in (6.3).

Moreover, after this proof we can check that $\eta_k^{-1} \circ \phi^\alpha \circ \eta_k = \mathcal{D}_{-s,\,-c}$ to see that the screw number $s = s(\mathcal{A}_i)$ and the parameter c modulo $\mathbb{Z}$ of Lemma 6.14 only depend on the orbit of $\mathcal{A}_i$.

6.1. Gluings and boundary Dehn twists

In this section we introduce some notions and examples that will be used in the rest of the paper. We begin with two easy remarks.

Remark 6.16. — Given a homeomorphism ϕ of a surface Σ with $\partial\Sigma \neq \emptyset$. Let C be a connected component of $\partial\Sigma$. Let A be a cylinder parametrized by $\eta : \mathbb{S}^1 \times I \to A$. We glue A with Σ by an identification $f : \mathbb{S}^1 \times \{0\} \to C$. Then, we can extend ϕ *trivially* along A by defining a homeomorphism $\bar{\phi}$ of $\Sigma \cup_f A$ as $\bar{\phi}|_\Sigma = \phi$ and for any $p \in A$ we set $\phi(p) = \phi(f \circ p_1 \circ \eta^{-1}(p))$ where p_1 is the canonical projection from $\mathbb{S}^1 \times I$ to $\mathbb{S}^1 = \mathbb{S}^1 \times \{0\}$.

Remark 6.17. — Given a homeomorphism ϕ of a surface Σ with $\partial\Sigma \neq \emptyset$. Let C be a connected component of $\partial\Sigma$. Let A be a compact collar neighborhood of C (isomorphic to $I \times C$) in Σ. Let $\eta : \mathbb{S}^1 \times I \to A$ be a parametrization of A, with $\phi(\mathbb{S}^1 \times \{1\}) = C$.

Then, there exists a homeomorphism ϕ' isotopic to ϕ relative to the boundary such that

- the restriction to A satisfies $p_2 \circ \eta^{-1} \circ \phi'|_A \circ \eta(t, x) = p_2 \circ \eta^{-1} \circ \phi'|_A \circ \eta(t', x)$ for all $t, t' \in I$, where $p_2 : \mathbb{S}^1 \times I \to \mathbb{S}^1$ is the canonical projection.

Definition 6.18 (Boundary Dehn twist). — *Let C be a component of $\partial\Sigma$ and let A be a compact collar neighbourhood of C in Σ. Suppose that C has a metric and total length is equal to ℓ. Let $\eta : \mathbb{S}^1 \times I \to A$ be a parametrization of A, such that $\eta|_{\mathbb{S}^1 \times \{1\}} : \mathbb{S}^1 \times \{1\} \to C$ is an isometry. Suppose that $\mathbb{S}^1$ has the metric induced from taking $\mathbb{S}^1 = \mathbb{R}/\ell\mathbb{Z}$ with $\ell \in \mathbb{R}_{>0}$ and the standard metric on $\mathbb{R}$. A* boundary Dehn twist *of length $r \in \mathbb{R}_{>0}$ along C is a homeomorphism $\mathcal{D}_r^\eta(C)$ of Σ such that:*

1. *it is the identity outside A*
2. *the restriction of $\mathcal{D}_r^\eta(C)$ to A in the coordinates given by η is given by $(x, t) \mapsto (x + r \cdot t, t)$.*

The isotopy type of $\mathcal{D}_r^\eta(C)$ by isotopies fixing the action on $\partial\Sigma$ does not depend on the parametrization η. When we write just $\mathcal{D}_r(C)$, it means that we are considering a boundary Dehn twist with respect to some *parametrization η.*

Example 6.19. — We can restate the $\ell-$tête-à-tête property in terms of boundary Dehn twists of length ℓ as follows. Let (Σ, Γ) be a thickening surface of a metric ribbon graph Γ. Let $g_\Gamma : \Sigma_\Gamma \to \Sigma$ be the gluing map. Consider the pull back metric on $g^{-1}(\Gamma)$. Denote by $\mathcal{D}_{\underline{\ell}}$ the composition of the boundary Dehn twists $\mathcal{D}_\ell$ along each $\widetilde{\Gamma}_j \subset \widetilde{\Gamma}$. Then (Σ, Γ) holds the ℓ-tête-à-tête property if and only if $\mathcal{D}_{\underline{\ell}}$ is compatible with the gluing g_Γ. We see from this that the lenghts of $\widetilde{\Gamma}_j$ are in $\ell\mathbb{Q}_+$.

7. Mixed tête-à-tête graphs and twists and pseudo-periodic homeomorphisms.

In this section we introduce the notion of mixed tête-à-tête graphs and the homeomorphisms that they induce which we call mixed tête-à-tête twists. Mixed tête-à-tête graphs and twists are generalizations of tête-à-tête graphs and twists, but they are able to model pseudo-periodic homeomorphisms. Our main result is a realization theorem for a certain class of homeomorphisms whose isotopy classes are representable by a tête-à-tête twists (see Theorem 7.20). The class contains the monodromies of arbitrary plane branches, generalizing the construction for branches with 2 Puiseux pairs in [2].

Before introducing the definition of mixed tête-à-tête graphs and twists, we analyze in Section 7.1 the structure of the class of pseudo-periodic homeomorphisms refrerred above, and show how to codify them using metric spines. We think this section makes the reading easier, but strictly speaking the reader could skip it now, read the definition of mixed tête-à-tête graphs and twists in Section 7.2 and Section 7.3, read the statement of Theorem 7.20 in Section 7.4, and come back to Section 7.1 for its proof. In a sequel article by the third author and B. Sigurdsson [9] it is proved a general realization theorem for pseudo-periodic mapping classes in terms of the mixed tête-à-tête graphs defined here.

7.1. A restricted type of pseudo-periodic homeomorphisms

In this section we work with a restricted type of pseudo-periodic homeomorphisms and give a natural construction of an embedded metric filtered graph that is a retract of the surface and that codifies the homeomorphism up to isotopy relative to the boundary. It is important, however, to notice

that, unlike in the case of the previous sections, the graph is not left invariant by the homeomorphism (in fact this would force the homeomorphism to be boundary-free periodic). This study will lead to the more general definition of mixed tête-à-tête graph and the induced tête-à-tête twist developed in the next sections.

We start giving the hypothesis of our homeomorphism.

Let ϕ be a pseudo-periodic homeomorphism of a surface Σ with $\partial\Sigma \neq \emptyset$. Let $\mathcal{C}$ be a system of cut curves for ϕ as in Definition 6.1. Let $G(\phi, \Sigma)$ be a graph constructed as follows:

(1) It has a vertex for each connected component of $\Sigma \setminus \mathcal{C}$.
(2) There are as many edges joining two vertices as curves in $\mathcal{C}$ intersect the two surfaces corresponding to those vertices.

7.1.1. Assumptions on ϕ:

(1) the graph $G(\phi, \Sigma)$ is a tree and
(2) the screw numbers are all non-positive.
(3) We assume that
   (3') it leaves at least one boundary component pointwise fixed,
   (3") the fractional Dehn twist coefficients along at least one of these fixed-boundary components is positive (we extend in the obvious way the notion of fractional Dehn twist coefficients in Definition 4.16 and Lemma 4.18 to pseudo-periodic homeomorphisms by considering the restriction of the homeomorphism to the connected component in $\Sigma \setminus \mathcal{A}$ that contains C). Recall that $\mathcal{A}$ is a collection of pairwise disjoint annuli (Remark 6.3).

We denote by $\partial^1\Sigma$ the union of some, at least one, connected components of $\partial\Sigma$ contained in a single connected component of $\Sigma \setminus \mathcal{C}$ and that are fixed pointwise by ϕ and have positive fractional Dehn twist coefficient.

We will obtain a metric spine codifying $[\phi]_{\partial\Sigma,\, \phi|_{\partial\Sigma}}$ (recall Notation 4.1).

Remark 7.1. — Observe that $G(\phi, \Sigma)$ being a tree implies that for any invariant orbit of annuli associated to the system of cut curves, we are under the hypothesis of Lemma 6.14.

Notation 7.2. — Figure 7.1 may help. We assume ϕ is in the quasi-canonical form of Remark 6.3 with respect to $\partial^1\Sigma$. We denote by $\widehat{\Sigma}$ the closure of $\Sigma \setminus \mathcal{A}$ in Σ.

Let $v(G(\phi, \Sigma))$ be the set of vertices of $G(\phi, \Sigma)$. We choose as root of $G(\phi, \Sigma)$ the vertex $v \in v(G(\phi, \Sigma))$ corresponding to the connected component of $\Sigma \setminus \mathcal{A}$ that contains $\partial^1\Sigma$. We say that $G(\phi, \Sigma)$ is rooted at v.

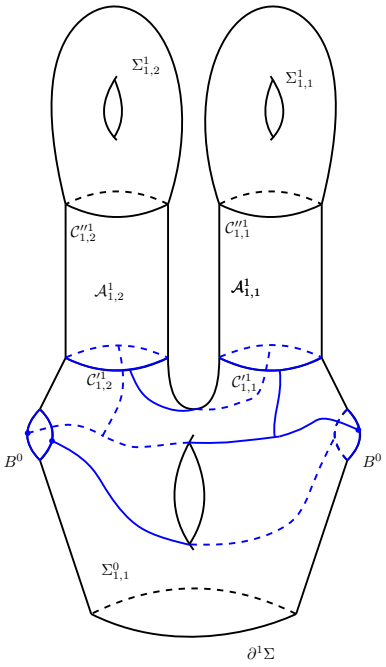


Figure 7.1. This figure helps understand Notation 7.2. It is the surface of genus 3 and 3 boundary components. Suppose that the notation is induced by an homeomorphism that rotates by π around the z-axis composed by some Dehn twist around the annuli $\mathcal{A}$ and isotoped to be the identity on the bottom boundary. We also include in blue the relative spine $(\Gamma^{\leqslant 0}, B^{\leqslant 0})$ of the first step in the construction of an associated metric relative spine. In particular B^0 consists of the boundary components in Σ^0 not in $\partial^1\Sigma$. It can be checked that this relative spine admits a relative tête-à-tête structure that induces $\phi|_{\Sigma^0_{1,1}}$.

Since $\widehat{\phi}$ permutes the surfaces in $\widehat{\Sigma}$, it induces a permutation of the set $v(G(\phi,\Sigma))$ which we denote by σ_ϕ.

We begin filtering the set $v(G(\phi,\Sigma))$:

(1) Denote the vertex chosen as the root by $v^0_{1,1}$. Let $V^0 := \{v^0_{1,1}\}$.
(2) Let $d : G(\phi,\Sigma) \to \mathbb{Z}_{\geqslant 0}$ be the distance function to V^0, that is, $d(v)$ is the number of edges of the bamboo in $G(\phi,\Sigma)$ that joins v with V^0. This bamboo is unique because $G(\phi,\Sigma)$ is a tree. Let $V^i := d^{-1}(i)$. Observe that the permutation σ_ϕ leaves the set V^i invariant.

There is a labelling of V^i induced by the orbits of σ_ϕ: suppose it has β_i different orbits. For each $j = 1, \dots, \beta_i$, we label the vertices in that orbit by $v^i_{j,k}$ with $k = 1, \dots, \alpha_j$ so that $\sigma_\phi(v^i_{j,k}) = v^i_{j,k+1}$ and $\sigma_\phi(v^i_{j,\alpha_j}) = v^i_{j,1}$.

We accordingly give names to distinct parts of the surface Σ (see Figure 7.3 for an example):

- Denote by $\Sigma^i_{j,k}$ the surface in $\widehat{\Sigma}$ corresponding to the vertex $v^i_{j,k}$.
- Denote by Σ^i the union of the surfaces corresponding to the vertices in V^i. Note that Σ^0 equals $\Sigma^0_{1,1}$.
- Denote by $\mathcal{A}^{i+1}_{j,k}$ the only annulus in $\mathcal{A}$ that intersects both $\Sigma^{i+1}_{j,k}$ and Σ^i. Observe that if there were more than one such annulus, $G(\phi, \Sigma)$ would not be a tree.
- Denote by $\mathcal{C}'^{i+1}_{j,k}$ the boundary component of $\mathcal{A}^{i+1}_{j,k}$ that lies in $\partial\Sigma^i$ and by $\mathcal{C}''^{i+1}_{j,k}$ the other boundary component.
- Denote $\mathcal{A}^{i+1}$ the union of all the annuli that intersect Σ^{i+1} and Σ^i and define analogously $\mathcal{C}'^{i+1}$ and $\mathcal{C}''^{i+1}$.
- We also define recursively

$$\Sigma^{\leqslant 0} := \Sigma^0, \quad \Sigma^{\leqslant i+1} := \Sigma^{\leqslant i} \cup \mathcal{A}^{i+1} \cup \Sigma^{i+1}.$$

We recall that α^i_j is the smallest positive number such that

$$\phi^{\alpha^i_j}\left(\mathcal{A}^{i+1}_{j,k}\right) = \mathcal{A}^{i+1}_{j,k},$$

and in consequence the least such that

$$\phi^{\alpha^i_j}\left(\mathcal{C}'^{i+1}_{j,k}\right) = \mathcal{C}'^{i+1}_{j,k} \quad \text{and} \quad \phi^{\alpha^i_j}\left(\mathcal{C}''^{i+1}_{j,k}\right) = \mathcal{C}''^{i+1}_{j,k}.$$

7.1.2. Construction of the metric relative spines for $\Sigma^{\leqslant i}$

We start now recursively building an embedded metric spine $(\Gamma^{\leqslant i}, B^{\leqslant i})$ for $\Sigma^{\leqslant i}$ codifying $\phi_{\Sigma^{\leqslant i}}$. The way in which the metric spine codifies the automorphism will become clear along the construction The starting point is the following:

- By Theorem 5.18, there exists a π-relative tête-à-tête graph (Λ^0, B^0) embedded in Σ^0
  - it induces $\phi|_{\Sigma^0}$ relative to $\partial^1\Sigma$, that is $[\phi|_{\Sigma^0}]_{\partial^1\Sigma} = [\phi_{\Lambda^0}]_{\partial^1\Sigma}$
  - it contains the boundary components of Σ_0 that intersect with $\mathcal{A}^1$ and any other boundary component of Σ^0 different from ∂^1, that is $B^0 = \partial\Sigma^0 \setminus \partial^1\Sigma$.
  - $\phi|_{B^0} = \phi_{\Lambda^0}|_{B^0}$, (in particular $\phi|_{B^0}$ is an isometry).

– All the vertices in B^0 have valency 3 (see Remark 5.15).

To construct the metric relative spine $(\Gamma^{\leqslant i+1}, B^{\leqslant i+1})$ from $(\Gamma^{\leqslant i}, B^{\leqslant i})$ we will use the following:

- parametrizations $\{\eta^{i+1}_{j,k}\}$ of the annuli $\{\mathcal{A}^{i+1}_{j,k}\}$ as in Lemma 6.14 for each $j = 1, \ldots, \beta_{i+1}$ and for each $k = 1, \ldots, \alpha_j$. For using this lemma we might have to isotope $\phi|_{\mathcal{A}^{i+1}_{j,k}}$ by an isotopy that preserves the action at the boundary, so in particular, it does not change ϕ outside $\mathcal{A}$. We choose the parametrizations such that $\eta^{i+1}_{j,k}(\mathbb{S}^1 \times \{0\}) = \mathcal{C}'^{i+1}_{j,k}$.
- a metric relative spine $(\Lambda^{i+1}_{j,k}, B^{i+1}_{j,k})$ for $\Sigma^{i+1}_{j,k}$ invariant by $\phi^{\alpha^i_j}|_{\Sigma^{i+1}_{j,k}}$ for each $j = 1, \ldots, \beta_{i+1}$, and for each $k = 1, \ldots, \alpha^i_j - 1$ such that
  (i) $B^{i+1}_{j,k}$ contains all the boundary components of $\Sigma^{i+1}_{j,k}$ except $\mathcal{C}''^{i+1}_{j,k}$ (which is invariant by $\phi^{\alpha^i_j}|_{\Sigma^{i+1}_{j,k}}$). This may include boundary components from the surface Σ not in $\partial^1\Sigma$, not only boundary components with non-empty intersection with $\mathcal{A}^{i+1}$,
  (ii) all the vertices in $B^{i+1}_{j,k}$ have valency 3 (the important ones are the ones in $\mathcal{C}'^{i+1}$).

  Note that condition (1) of the *assumptions on* ϕ (page 2687) implies that the surface obtained by cutting $\Sigma^{i+1}_{j,k}$ along $\Lambda^{i+1}_{j,k}$ is a unique cylinder $\widetilde{\Sigma}^{i+1}_{j,k}$. Let

$$g_{i+1}|_{\widetilde{\Sigma}^{i+1}_{j,k}} : \widetilde{\Sigma}^{i+1}_{j,k} \to \Sigma^{i+1}_{j,k}$$

  be the gluing mapping.
- mappings $r^{i+1}_{j,k}$:

$$r^{i+1}_{j,k} : \widetilde{\Lambda}^{i+1}_{j,k} \times I \to \Sigma^{i+1}_{j,k}, \tag{7.1}$$

  composition of the gluing map $g_{i+1}|_{\widetilde{\Sigma}^{i+1}_{j,k}}$ with a product structure for the cylinder $\widetilde{\Sigma}^{i+1}_{j,k}$

$$\widetilde{r}^{i+1}_{j,k} : \widetilde{\Lambda}^{i+1}_{j,k} \times I \to \widetilde{\Sigma}^{i+1}_{j,k}, \tag{7.2}$$

  which is invariant by the lifting of $\phi^{\alpha_j}|_{\Sigma^{i+1}_{j,k}}$ to $\widetilde{\Sigma}^{i+1}_{j,k}$.

  We choose $r^{i+1}_{j,1}(x,1)$ such that:
  (iii) $r^{i+1}_{j,k}(x,1)$ is not a vertex of $\Lambda^{i+1}_{j,1}$ whenever $\eta^{i+1}_{j,k}(x,0)$ is a vertex of $\Gamma^{\leqslant i}$

The reader can now have a look at the final construction of the spine in (a)-(e) in page 2692, in order to have an impression of the final construction.

We first explain more details about how to get $(\Lambda^{i+1}_{j,k}, B^{i+1}_{j,k})$ and the $r^{i+1}_{j,k}$ satisfying (i)-(iii). Afterwards we will show how we choose the right metric.

To construct $\Lambda^{i+1}_{j,k}$, we first find $\Lambda^{i+1}_{j,1}$ and $r^{i+1}_{j,1}$ and then define $\Lambda^{i+1}_{j,k+1} := \phi^k(\Lambda^{i+1}_{j,1})$.

To find the spine $(\Lambda^{i+1}_{j,1}, B^{i+1}_{j,1})$ we consider the quotient map by the action of ϕ^{α_j} in $\Sigma^{i+1}_{j,1}$ as in the proof of Theorem 5.10 or Theorem 5.18. We choose a relative spine of the quotient surface such that:

- it contains the image of all points whose isotropy subgroup by the action of the group generated by $\phi^{\alpha^i_j}|_{\Sigma^{i+1}_{j,k}}$ is non-trivial,
- it contains all the boundary components except the image by the quotient map of $\mathcal{C}''^{i+1}$,
- it has only vertices of valency 3 along the boundary.

Then we denote by $\Lambda^{i+1}_{j,k}$ its preimage by the quotient map which is a relative spine satisfying (i)-(ii). Moreover, we can lift any regular retraction (or product structure in the cylinder) in the quotient and find a product structure for the cylinder $\widetilde{\Sigma}^{i+1}_{j,1}$

$$\widetilde{r}^{i+1}_{j,1} : \widetilde{\Lambda}^{i+1}_{j,1} \times I \to \widetilde{\Sigma}^{i+1}_{j,1}, \tag{7.3}$$

that is invariant by the lifting of $\phi^{\alpha_j}|_{\Sigma^{i+1}_{1,j}}$ to $\widetilde{\Sigma}^{i+1}_{j,1}$. To get (iii) we choose carefully the retraction in the quotient such that $\widetilde{r}^{i+1}_{j,1}(x,1)$ does not correspond to a vertex of $\Lambda^{i+1}_{j,1}$ whenever $\widetilde{r}^{i+1}_{j,1}(x,0)$ is a point preimage either of a vertex of $\Gamma^{\leqslant i}$ or the image of a vertex of $\Gamma^{\leqslant i}$ by any power of ϕ.

Now we choose the right metric.

The metric of the graph $\Gamma^{\leqslant i}$ assigns a metric on $\mathcal{C}'^{i+1}_{j,k}$ for every j,k. We use the natural identification from $\mathcal{C}'^{i+1}_{j,k}$ to $\mathcal{C}''^{i+1}_{j,k}$ by $\eta^{i+1}_{j,k}$ (i.e. $\eta^{i+1}_{j,k}(x,0) \mapsto \eta^{i+1}_{j,k}(x,1)$) to put a metric on $\mathcal{C}''^{i+1}_{j,k}$.

Note that from the 2 boundary components of $\widetilde{\Sigma}^{i+1}_{j,1}$, one comes from cutting $\Lambda^{i+1}_{j,1}$ and the other is $\mathcal{C}''^{i+1}_{j,1}$. Now, for each $j = 1, \dots, \beta_{i+1}$ we put a metric on $\widetilde{\Lambda}^{i+1}_{j,1}$ by pull back the metric in $\mathcal{C}''^{i+1}_{j,1}$ with the mapping given by $\widetilde{r}^{i+1}_{j,1}(x,1) \mapsto \widetilde{r}^{i+1}_{j,1}(x,0)$.

Let $g_{i+1}|_{\widetilde{\Lambda}^{i+1}_{j,1}} : \widetilde{\Lambda}^{i+1}_{j,1} \to \Sigma^{i+1}_{j,1}$ be the restriction of the gluing map. The metric on $\widetilde{\Lambda}^{i+1}_{j,1}$ is compatible with the gluing because $\phi^{\alpha_j}|_{\mathcal{C}''^{i+1}_{j,1}}$ is an isometry and ϕ^{α_j} respects retraction lines.

We define the metric on $\Lambda^{i+1}_{j,k+1} = \phi^k(\Lambda^{i+1}_{j,1})$ to be the pullback metric on $\Lambda^{i+1}_{j,1}$.

We denote by (Λ^{i+1}, B^{i+1}) the union of the graphs $(\Lambda^{i+1}_{j,k}, B^{i+1}_{j,k})$ for all j,k. Note that the metric that we have put on makes $\phi|_{\Lambda^{i+1}}$ an isometry.

We build $\Gamma^{\leqslant i+1}$ starting with $(\Gamma^{\leqslant i}, B^{\leqslant i}) \hookrightarrow \Sigma^{\leqslant i} \subseteq \Sigma^{\leqslant i+1}$ and doing following:

(a) We remove $\mathcal{C}'^{i+1}_{j,k}$ from $\Gamma^{\leqslant i}$ for every j and k.
(b) For every edge e in $\Gamma^{\leqslant i} \setminus \mathcal{C}'^{i+1}$ containing a vertex in $\mathcal{C}'^{i+1}_{j,k}$, if L is its length in $\Gamma^{\leqslant i}$, then, we redefine its metric to $L - \epsilon$ (to simplify we take ϵ smaller than the lengths of every edge).
(c) We add the embedded segments $\eta^{i+1}_{j,k}(I \times \{x\})$ for all x such that $\eta^{i+1}_{j,k}(0,x)$ is a vertex of B^i (that is, a vertex in $\mathcal{C}'^{i+1}$. We set the length of each of this segments to be $\epsilon/2$.
(d) We add the embedded segments $r^{i+1}_{j,k}(I \times \{x\})$ that concatenate with the ones added in the previous step. We set the length of each of this segments to be $\epsilon/2$.
(e) We add Λ^{i+1} with the metric we defined previously.

We set $B^{\leqslant i+1} := B^{\leqslant i} \cup B^{i+1} \setminus \mathcal{C}'^{i+1}$.

This is obviously a relative metric spine for $\Sigma^{\leqslant i+1}$.

We observe that

$$\left(\Gamma^{\leqslant i+1}_{\Lambda^{i+1}}, B^{\leqslant i+1}_{\Lambda^{i+1}}\right) \text{ is isometric to } \left(\Gamma^{\leqslant i}, B^{\leqslant i}\right).$$

This will allow us to codify all the information of the $(\Gamma^{\leqslant i}, B^{\leqslant i})$ in the metric spine $(\Gamma^{\leqslant d}, B^{\leqslant d})$ (for d maximal). We will define a filtration to keep the recursive steps in the construction we have just explained.

7.1.3. Recovering $[\phi]_{\partial^1}$ from the metric relative spines for $(\Sigma^{\leqslant i}, B^{\leqslant i})$

Now we explain in which sense the sequence of metric spines $(\Gamma^{\leqslant i}, B^{\leqslant i})$ that we have constructed, together with some extra numerical information, codify the automorphisms

$$[\phi_{\Sigma^{\leqslant i}}]_{B^{\leqslant i}, \phi|_{B^{\leqslant i}}}$$

for all i.

For $i = 0$ it is immediate and we don't need extra information because $(\Gamma^{\leqslant 0}, B^{\leqslant 0})$ is a π-tête-à-tête graph codifying it in the sense of Theorem 5.18.

Assume that we know how to recover the automorphisms up to a certain i, let us see how to recover the automorphism for $i+1$.

Firstly, we make some observations about the original ϕ in quasi-canonical form as in (6.3).

We define the homeomorphism $\bar{\phi}_i$ as the homeomorphism of the

$$\left(\Sigma^{\leqslant i+1}\right)_{\Lambda^{i+1}}$$

that coincides with $\phi|_{\Sigma^{\leqslant i}}$ in $\Sigma^{\leqslant i}$ and that extends *trivially* to the remaining cylinders as in Remark 6.16 using the parametrizations $\eta^{i+1}_{j,k}$ and $\widetilde{r}^{i+1}_{j,k}$ for every j,k successively.
Recall Notation 6.6 and for every j,k consider the Dehn twists

$$\eta^{i+1}_{j,k} \circ \mathcal{D}_{-s_j/\alpha_j,\,0} \circ \left(\eta^{i+1}_{j,k}\right)^{-1}$$

along the annuli $\mathcal{A}^{i+1}_{j,k}$ with s_j the screw number of ϕ on at $\mathcal{A}^{i+1}_{j,k}$. We extend the composition of all this Dehn twists to $\Sigma^{\leqslant i+1}_{\Lambda^{i+1}}$ in the following way: we extend by the identity to $\Sigma^{\leqslant i}$ and we extend to the cylinders $\Sigma^{i+1}_{\Lambda^{i+1}}$ as in Remark 6.16 using the parametrizations $\widetilde{r}^{i+1}_{j,k}$. We denote by

$$\mathcal{D}_i : \Sigma^{\leqslant i+1}_{\Lambda^{i+1}} \to \Sigma^{\leqslant i+1}_{\Lambda^{i+1}}$$

the extension that we just have constructed.

Observe that $\mathcal{D}_i \circ \bar{\phi}_i$ is compatible with the gluing g_{i+1} because its restriction to $\Sigma^{\leqslant i} \cup \mathcal{A}^{i+1}$ coincides with $\phi|_{\Sigma^{\leqslant i} \cup \mathcal{A}^{i+1}}$ up to isotopy fixing $\mathcal{C}''^{i+1}$. Then, it induces a mapping in $\Sigma^{\leqslant i+1}$ that coincides with $\phi|_{\Sigma^{\leqslant i+1}}$ up to isotopy fixing the boundary.

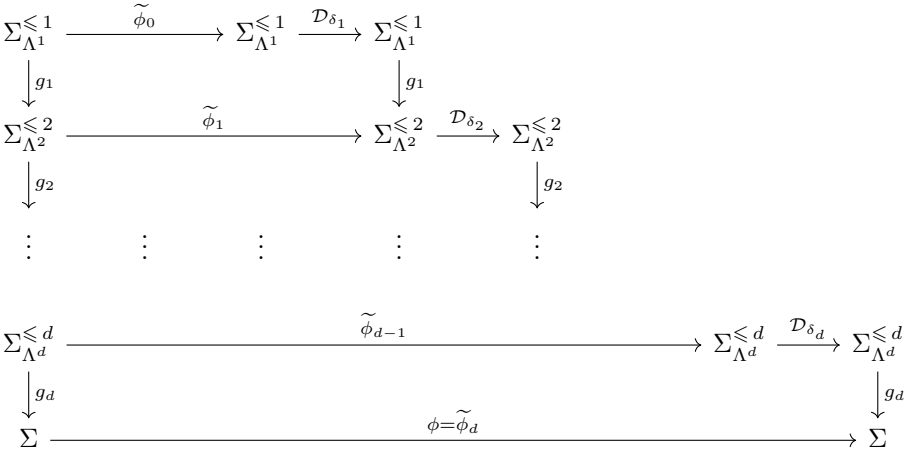
Now, we come back to the graph $(\Gamma^{\leqslant i+1}, B^{\leqslant i+1})$. Assume we know $\phi|_{\Sigma^{\leqslant i}}$ up to isotopy fixing the action at the boundary $B^{\leqslant i}$. We extend trivially $\phi|_{\Sigma^{\leqslant i}}$ using parametrizations of the collars of $\Sigma^{\leqslant i+1}_{\Lambda^{i+1}} \setminus \Sigma^{\leqslant i}$ for which the segments of $\Gamma^{\leqslant i+1}$ inside are retraction lines of the parametrizations (use Remark 6.16). We call it $\widetilde{\phi}_i$. It is a homeomorphism of $\Sigma^{\leqslant i+1}|_{\Lambda^{i+1}}$. Let ℓ^{i+1}_j be the length of $\widetilde{\Lambda}^{i+1}_{j,k}$ for any k, that is, it coincides with the original $\mathit{length}(\mathcal{C}''^{i+1}_{j,1})$. Define

$$\delta^{i+1}_j := -s_j/\alpha_j \cdot \ell^{i+1}_j \tag{7.4}$$

with s_j the screw number of ϕ on $\mathcal{A}^{i+1}_{j,k}$. Let $\mathcal{D}_{\delta_{i+1}}$ be the composition of the boundary Dehn twists of length δ^{i+1}_j along each $\widetilde{\Lambda}^{i+1}_{j,k}$ (see Definition 6.18). Then, it is clear, by the previous observations about ϕ, that $[\widetilde{\phi}_i]_{\partial,\,\widetilde{\phi}_i|_\partial}$ is compatible with the gluing g_{i+1} and that the homeomorphism it induces in $\Sigma^{\leqslant i+1}$ coincides with $\phi|_{\Sigma^{\leqslant i+1}}$ up to isotopy fixing the *action at the* boundary.

Below we see the diagram which shows how $[\phi]_{\partial^1}$ is obtained from the metric graphs $(\Gamma^{\leqslant i}, B^{\leqslant i})$ and the numbers δ^i_j.

(7.5)



We have obtained the following:

PROPOSITION 7.3. — *The collection of metric spines* $(\Gamma^{\leqslant i}, B^{\leqslant i})$ *for* $i = 0, \ldots, d$ *together with the numbers* δ^i_j *obtained in* (7.4) *determine* $[\phi]_{\partial\Sigma,\, \phi|_{\partial\Sigma}}$.

7.1.4. Filtration on Γ^d

We can codify the information of the collection the metric relative spines $(\Gamma^{\leqslant i}, B^{\leqslant i})$ by a filtration on the last spine $(\Gamma^{\leqslant d}, B^{\leqslant d})$ as follows. Define

$$\Gamma^i := \bigcup_{\ell \geqslant i} \left(\Lambda^\ell \setminus \mathcal{C}'^{\ell+1}\right) \cup \bigcup_{\ell \geqslant i\, j,\, k} \eta^{\ell+1}_{j,\,k} \left(I \times \{x\}\right) \cup \bigcup_{\ell \geqslant i\, j,\, k} r^{\ell+1}_{j,\,k} \left(I \times \{x\}\right),$$

$$A^i := B^{\leqslant d} \cap \Gamma^i$$

where x runs exactly as in the construction in the steps (c) and (d).

In this way we obtain a filtered relative spine

$$\left(\Gamma^{\leqslant d}, B^{\leqslant d}\right) = \left(\Gamma^0, A^0\right) \supset \left(\Gamma^1, A^1\right) \supset \cdots \supset \left(\Gamma^d, A^d\right).$$

PROPOSITION 7.4. — *The metric filtered graph defined above, together with the numbers* δ^i_j *in* (7.4)*, determine* $[\phi]_{\partial\Sigma,\, \phi|_{\partial\Sigma}}$.

Proof. — This follows from Proposition 7.3 since $\Gamma^0_{\Gamma^{i+1}}$ is isometric to $\Gamma^{\leqslant i}$. □

The properties of this filtered metric relative spine and the numbers δ_i, which can be summarized in the diagram 7.5, can be restated in terms of mixed tête-à-tête graphs that we introduce in the next section.

7.2. Mixed tête-à-tête graphs.

Now we introduce the definition of mixed tête-à-tête graphs and twists, inspired by the constructions of the previous section.

Let $(\Gamma^\bullet, A^\bullet)$ be a decreasing filtration on a connected relative metric ribbon graph (Γ, A). That is

$$(\Gamma, A) = \left(\Gamma^0, A^0\right) \supset \left(\Gamma^1, A^1\right) \supset \cdots \supset \left(\Gamma^d, A^d\right)$$

where $\supset$ between pairs means $\Gamma^i \supset \Gamma^{i+1}$ and $A^i \supset A^{i+1}$, and where (Γ^i, A^i) is a (possibly disconnected) relative metric ribbon graph for each $i = 0, \ldots, d$. We say that d is the depth of the filtration $\Gamma^\bullet$. We assume each Γ^i does not have univalent vertices and is a subgraph of Γ in the usual terminology in Graph Theory. We observe that since each (Γ^i, A^i) is a relative metric ribbon graph, we have that $A^i \setminus A^{i+1}$ is a disjoint union of connected components homeomorphic to $\mathbb{S}^1$.

For each $i = 0, \ldots, d$, let

$$\delta_i : \Gamma^i \to \mathbb{R}_{\geqslant 0}$$

be a locally constant map (so it is a map constant on each connected component). We put the restriction that $\delta_0(\Gamma^0) > 0$. We denote the collection of all these maps by $\delta_\bullet$.

Given $p \in \Gamma$, we define c_p as the largest natural number such that $p \in \Gamma^{c_p}$.

DEFINITION 7.5 (Mixed safe walk). — *Let $(\Gamma^\bullet, A^\bullet)$ be a filtered relative metric ribbon graph. Let $p \in \Gamma \setminus (A \cup v(\Gamma))$. We define a* mixed safe walk γ_p *starting at p as a concatenation of paths defined iteratively by the following properties*

(i) *γ_p^0 is a safe walk of length $\delta_0(p)$ starting at $p_0^\gamma := p$. Let $p_1^\gamma := \gamma^0(\delta_0)$ be its endpoint.*

(ii) *Suppose that γ_p^{i-1} is defined and let p_i^γ be its endpoint.*
- *If $i > c_p$ or $p_i^\gamma \notin \Gamma^i$ we stop the algorithm.*
- *If $i \leqslant c_p$ and $p_i^\gamma \in \Gamma^i$ then define $\gamma_p^i : [0, \delta_i(p_i)] \to \Gamma^i$ to be a safe walk of length $\delta_i(p_i^\gamma)$ starting at p_i^γ and going in the same direction as γ_p^{i-1}.*

(iii) *Repeat step* (ii) *until algorithm stops.*

Finally, define $\gamma_p := \gamma_p^k \star \cdots \star \gamma_p^0$, *that is, the mixed safe walk starting at* p *is the concatenation of all the safe walks defined in the inductive process above.*

As in the pure case, there are two safe walks starting at each point on $\Gamma \setminus (A \cup v(\Gamma))$. We denote them by γ_p and ω_p.

Definition 7.6 (Boundary mixed safe walk). — *Let* $(\Gamma^\bullet, A^\bullet)$ *be a filtered relative metric ribbon graph and let* $p \in A$. *We define a* boundary mixed safe walk b_p *starting at* p *as a concatenation of a collection of paths defined iteratively by the following properties*

(i) $b_{p_0}^0$ *is a boundary safe walk of length* $\delta_0(p)$ *starting at* $p_0 := p$ *and going in the direction indicated by* A *(as in the relative tête-à-tête case). Let* $p_1 := b_p^0(\delta_0)$ *be its endpoint.*

(ii) *Suppose that* $b_{p_{i-1}}^{i-1}$ *is defined and let* p_i *be its endpoint.*
- *If* $i > c_p$ *or* $p_i \notin \Gamma^i$ *we stop the algorithm.*
- *If* $i \leqslant c(p)$ *and* $p_i \in \Gamma^i$ *then define* $b_{p_i}^i : [0, \delta_i(p_i)] \to \Gamma^i$ *to be a safe walk of length* $\delta_i(p_i)$ *starting at* p_i *and going in the same direction as* $b_{p_{i-1}}^{i-1}$.

(iii) *Repeat step* (ii) *until algorithm stops.*

Finally, define $b_p := b_{p_k}^k \star \cdots \star b_{p_0}^0$, *that is, the boundary mixed safe walk starting at* p *is the concatenation of all the safe walks defined in the inductive process.*

Notation 7.7. — We call the number k in Definition 7.5 (resp. Definition 7.6), the *order* of the mixed safe walk (resp. boundary mixed safe walk) and denote it by $o(\gamma_p)$ (resp. $o(b_p)$).

We denote by $l(\gamma_p)$ the *length* of the mixed safe walk γ_p which is the sum $\sum_{j=0}^{o(\gamma_p)} \delta_j(p_j^\gamma)$ of the lengths of all the walks involved. We consider the analogous definition $l(b_p)$.

As in the pure case, two mixed safe walks starting at $p \in \Gamma \setminus v(\Gamma)$ exist. We denote by ω_p the mixed safe walk that starts at p but in the opposite direction to the starting direction of γ_p.

Observe that since the safe walk $b_{p_0}^0$ is completely determined by p, for a point in A there exists only one boundary safe walk.

Now we define the relative mixed tête-à-tête property.

Definition 7.8 (Relative mixed tête-à-tête property). — *Let* $(\Gamma^\bullet, A^\bullet)$ *be a filtered relative metric ribbon graph and let* $\delta_\bullet$ *be a set of locally constant mappings* $\delta_k : \Gamma^k \to \mathbb{R}_{\geqslant 0}$. *We say that* $(\Gamma^\bullet, A^\bullet, \delta_\bullet)$ *satisfies the*

relative mixed tête-à-tête property or that it is a relative mixed tête-à-tête graph if for every $p \in \Gamma - (v(\Gamma) \cup A)$

(i) *The endpoints of* γ_p *and* ω_p *coincide.*

(ii) $c_{\gamma_p(l(\gamma_p))} = c_p$

and for every $p \in A$*, we have that*

(iii) $b_p(l(b_p)) \in A^{c_p}$

As a consequence of the two previous definitions we have:

LEMMA 7.9. — *Let* $(\Gamma^\bullet, A^\bullet, \delta_\bullet)$ *be a mixed relative tête-à-tête graph, then*

(a) $o(\omega_p) = o(\gamma_p) = c_p$

(b) $l(\gamma_p) = l(\omega_p)$ *for every* $p \in \Gamma \backslash v(\Gamma)$.

Proof.

(a) By Definition 7.5 (ii), we have that $o(\gamma_p) \leqslant c_p$ for all $p \in \Gamma \setminus v(\Gamma)$. Suppose that for some p, we have that $o(\gamma_p) = k < c_p$. This means, that while constructing the mixed safe walk we stopped after constructing the path γ_p^k either because $k > c_p$ which contradicts the supposition, or because the endpoint p_k^γ of γ_p^k is not in Γ^k which contradicts that $c_p = c_{\gamma_p(l(\gamma_p))}$. This proves the equality $o(\gamma_p) = c_p$. In order to prove $o(\omega_p) = c_p$ use the equality $\gamma_p(l(\gamma_p)) = \omega_p(l(\omega_p))$ and repeat the same argument.

(b) Let q be the endpoint of γ_p and ω_p. Since the image of the safe walks $\gamma_p^{c_p}$ and $\omega_p^{c_p}$ lies on the same connected component of Γ^{c_p} we have that their starting points $p_{c_p}^\gamma$ and $p_{c_p}^\omega$ also lie on that same connected component. Therefore $\delta_{c_p}(p_{c_p}^\gamma) = \delta_{c_p}(p_{c_p}^\omega)$.

Suppose now that p_i^γ and p_i^ω lie on the same connected component of Γ^i (and so $\delta_i(p_i^\gamma) = \delta_i(p_i^\omega)$). Then the image of the safe walks γ_p^{i-1} and ω_p^{i-1} lies on the same connected component of Γ^{i-1} and we have that their starting points p_{i-1}^γ and p_{i-1}^ω also lie on that same connected component. So $\delta_{i-1}(p_{i-1}^\gamma) = \delta_{i-1}(p_{i-1}^\omega)$.

We conclude that $\delta_j(p_j^\gamma) = \delta_j(p_j^\omega)$ for all $j = 0, \ldots, d$ which concludes the proof. □

Remark 7.10. — Note that for mixed tête-à-tête graphs it is not true that $p \mapsto \gamma_p(\delta(p))$ gives a continuous mapping from Γ to Γ. A example of this phenomena is given in Figure 7.5 and the discussion at the end of Example 7.23 commenting on it.

The reader can check that the filtered graph associated with the pseudo-periodic homeomorphism ϕ in the previous section is a mixed relative tête-à-tête graph.

7.3. Mixed tête-à-tête twists.

Let $(\Gamma^\bullet, A^\bullet, \delta_\bullet)$ be a a mixed tête-à-tête graph and let (Σ, Γ, A) be a thickening surface of (Γ, A). Let $\partial^1\Sigma$ be the union of the boundary components of Σ not contained in A. In this section we define a pseudo-periodic homeomorphism $\phi_{(\Sigma,\,\Gamma^\bullet,\,\delta)}$ of Σ associated to $(\Gamma^\bullet, A^\bullet, \delta_\bullet)$. This homeomorphism is well defined up to isotopy fixing $\partial^1\Sigma$ and relative to the action on A.

For the sake of simplicity in notation we assume that $A^\bullet = \emptyset$ during the construction. The general case is analogous.

Notation 7.11. — Let $g_{\Gamma^i} : \Sigma_{\Gamma^i} \to \Sigma$ be the gluing map as in Notation 2.3. Let Γ_{Γ^i} be the preimage of Γ by g_{Γ^i}. We also denote by g_{Γ^i} its restriction $g_{\Gamma^i} : \Gamma_{\Gamma^i} \to \Gamma$. The union of the boundary components of Σ_{Γ^i} that come from Γ^i is denoted by $\widetilde{\Gamma}^i$. Observe that a single connected component of Γ^i might produce more than one boundary component in Σ_{Γ^i}.

It's clear that g_{Γ^i} factorizes as follows:

$$\Sigma_{\Gamma^i} \to \Sigma_{\Gamma^{i+1}} \to \ldots \to \Sigma.$$

We denote these mappings by $g_j : \Sigma_{\Gamma^j} \to \Sigma_{\Gamma^{j+1}}$ for $j = 0, \ldots, d-1$ and also their restrictions $g_j : \Gamma_{\Gamma^j} \to \Gamma_{\Gamma^{j+1}}$.

Remark 7.12. — Observe that by Definition 7.8, each connected component of the relative metric ribbon graph $(\Gamma_{\Gamma^1}, \widetilde{\Gamma}^1) \hookrightarrow \Sigma_{\Gamma^1}$ has the relative tête-à-tête property for safe walks of length $\delta_0(\Gamma)$.

Let $\phi_{\Gamma,\,0} : \Sigma_{\Gamma^1} \to \Sigma_{\Gamma^1}$ be the induced tête-à-tête twist fixing each boundary component that is not in $\widetilde{\Gamma}^1$ as in Definition 5.4. It is the homeomorphism induced by the relative tête-à-tête property of each connected component of $(\Gamma_{\Gamma^1}, \widetilde{\Gamma}^1)$ for some choice of product structures on $(\Sigma_{\Gamma^1})_{\Gamma_{\Gamma^1}}$.

Also according to Definition 5.4, observe that since we do not specify anything, we assume that the sign ι is constant $+1$.

Now we continue to define inductively the homeomorphism $\phi_{(\Sigma,\,\Gamma^\bullet,\,\delta)}$.

Notation 7.13. — Let

$$\mathcal{D}_{\delta_i} : \Sigma_{\Gamma^i} \to \Sigma_{\Gamma^i}$$

be the homeomorphism consisting of the composition of all the boundary Dehn twists $\mathcal{D}_{\delta_i(g_{\Gamma^i}(C))}(C)$ for all components in $\widetilde{\Gamma}^i$. Recall Definition 6.18.

Lemma 7.14. — *The homeomorphism*

$$\widetilde{\phi}_{\Gamma,\,1} := \mathcal{D}_{\delta_1} \circ \phi_{\Gamma,\,0} : \Sigma_{\Gamma^1} \to \Sigma_{\Gamma^1}$$

is compatible with the gluing $g_1 : \Sigma_{\Gamma^1} \to \Sigma_{\Gamma^2}$.

Proof. — We use the notation introduced in Definition 7.5.

Since g_1 only identifies points in $\widetilde{\Gamma}^1$, we must show that if x, y are different points in $\widetilde{\Gamma}^1$ such that $g_1(x) = g_1(y) \in \Gamma^1$, then $g_1(\widetilde{\phi}_{\Gamma,1}(x)) = g_1(\widetilde{\phi}_{\Gamma,1}(y))$.

So let $x, y \in \widetilde{\Gamma}^1$ be such that $g_1(x) = g_1(y) = p$. In particular, $c_p = 1$ and we have that $\gamma_p = \gamma_p^1 \star \gamma_p^0$ and $\omega_p = \omega_p^1 \star \omega_p^0$ by Lemma 7.9(a). So the mixed safe walks end in a connected component of Γ^1. Denote by $\widehat{p}$ their endpoint. By Definition 7.8(ii) we have that $c_{\widehat{p}} = 1$.

First observe that $\phi_{\Gamma,0}(x) = b_x(\delta_0(p))$ where $b_x : [0, \delta_0(p)] \to \Gamma_{\Gamma^1}$ is the boundary safe walk of length $\delta_0(p)$ given by the relative tête-à-tête structure on $(\Gamma_{\Gamma^1}, \widetilde{\Gamma}^1)$. Analogously $\phi_{\Gamma,0}(y) = b_y(\delta_0(y))$. So we have

$$g_1\left(\phi_{\Gamma,0}(x)\right) = \gamma_p\left(\delta_0(p)\right)$$

and

$$g_1\left(\phi_{\Gamma,0}(y)\right) = \omega_p\left(\delta_0(p)\right).$$

It is clear that

$$\left(\mathcal{D}_{\delta_1}(\phi_{\Gamma,0}(x))\right) = \widetilde{\gamma}_p^0\big(\delta_1(\phi_{\Gamma,0}(x)) + \delta_0(p)\big)$$

with $\widetilde{\gamma}_p$ the safe walk in $\widetilde{\Gamma}^1$. It is also clear that $\widetilde{\gamma}_p$ is the actual lifting of γ_p along Γ. Then,

$$\begin{aligned} g_1\left(\mathcal{D}_{\delta_1} \circ \phi_{\Gamma,0}(x)\right) &= \gamma_p\big(\delta_1\left(\phi_{\Gamma,0}(x)\right) + \delta_0(p)\big) \\ &= \gamma_p\Big(\delta_1\left(\gamma_p^0(\delta_0(p))\right) + \delta_0(p)\Big) = \gamma_p\left(l\left(\gamma_p\right)\right) \end{aligned}$$

and analogously $g_1(\mathcal{D}_{\delta_1} \circ \phi_{\Gamma,0}(y)) = \omega_p(l(\omega_p))$.

By property (i) of a mixed tête-à-tête graph, we can conclude. □

Now, we consider the homeomorphism induced by $\widetilde{\phi}_{\Gamma,1}$ and we denote it by

$$\phi_{\Gamma,1} : \Sigma_{\Gamma^2} \to \Sigma_{\Gamma^2}.$$

The same argument applies inductively to prove that each map

$$\widetilde{\phi}_{\Gamma,i} := \mathcal{D}_{\delta_i} \circ \phi_{\Gamma,i-1} : \Sigma_{\Gamma^i} \to \Sigma_{\Gamma^i} \tag{7.6}$$

is compatible with the gluing g_i and hence it induces a homeomorphism

$$\phi_{\Gamma,i} : \Sigma_{\Gamma^{i+1}} \to \Sigma_{\Gamma^{i+1}}. \tag{7.7}$$

In the end we get a map

$$\phi_{(\Sigma,\Gamma^\bullet,\delta)} := \phi_{\Gamma,d} : \Sigma \to \Sigma \tag{7.8}$$

which we call the *mixed tête-à-tête twist* induced by $(\Gamma^\bullet, \delta_\bullet)$ and the chosen embedding $\Gamma \hookrightarrow \Sigma$.

Notation 7.15. — We can extend the notation introduced before by defining $\phi_{\Gamma,-1} := id$ and $\widetilde{\phi}_{\Gamma,0} := \mathcal{D}_{\delta_0} \circ \phi_{\Gamma,-1} = \mathcal{D}_{\delta_0}$. Then we can restate Remark 7.12 by saying that $\widetilde{\phi}_{\Gamma,0}$ is compatible with the gluing g_0 and induces the homeomorphism $\phi_{\Gamma,0}$.

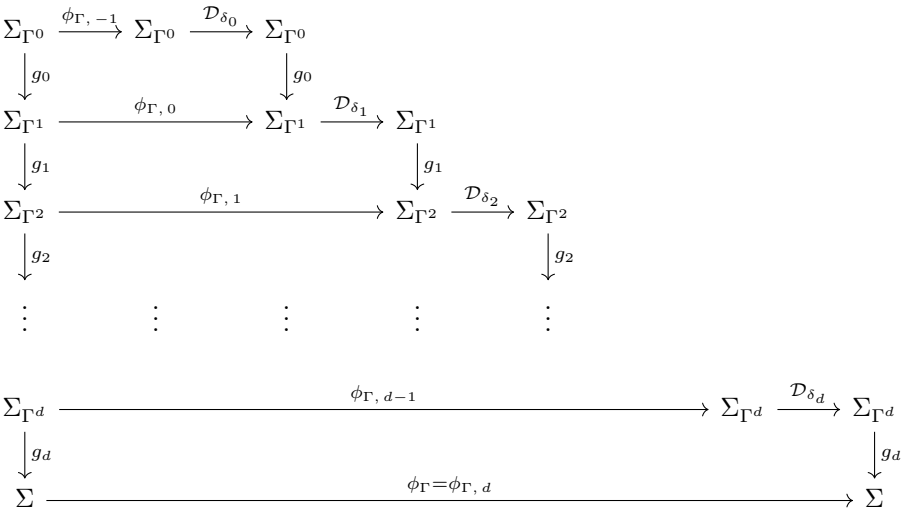
Remark 7.16. — After the description of the construction of the mixed tête-à-tête twist above and the diagram 7.5 and 7.9, we observe that satisfying (i) and (ii) of the mixed tête-à-tête property in Definition 7.8 is equivalent to satisfying:

I') For all $i = 0, \ldots, d-1$, the homeomorphism $\widetilde{\phi}_{\Gamma,i} = \mathcal{D}_{\delta_i} \circ \phi_{\Gamma,i-1}$ is compatible with the gluing g_i, that is,

$$g_i(x) = g_i(y) \Rightarrow g_i\left(\widetilde{\phi}_{\Gamma,i}(x)\right) = g_i\left(\widetilde{\phi}_{\Gamma,i}(y)\right).$$

Below we see the diagram which shows the construction of ϕ_Γ.

(7.9)



We prove the following:

THEOREM 7.17. — *The homeomorphism $\phi_{\Gamma,i}$ is pseudo-periodic for all $i = 0, \ldots, d$. In particular, ϕ_Γ is pseudo-periodic.*

Proof. — The mapping $\phi_{\Gamma,0}$ is periodic. Assume $\phi_{\Gamma,i-1}$ is pseudo-periodic. Let's see so is $\phi_{\Gamma,i}$. Choose a collar neighbourhood $\mathcal{U}^i$ of $\widetilde{\Gamma}^i$, invariant by $\phi_{\Gamma,i-1}$. We denote by $\mathcal{U}^i_{j,1}, \ldots, \mathcal{U}^i_{j,\alpha_j}$ any set of its connected annular

components permuted by $\phi_{\Gamma,i-1}$ such that $\phi_{\Gamma,i-1}(\mathcal{U}^i_{j,k}) = \mathcal{U}^i_{j,k+1}$. Similarly to Remark 6.17 we can assume that, up to isotopy fixing the action on the boundary, the homeomorphism $\phi_{\Gamma,i-1}$ satisfies that for some parametrizations $\eta^i_{j,k} : \mathbb{S}^1 \times I \to \mathcal{U}^i_{j,k}$ we have that

$$p_2 \circ \left(\eta^i_{j,k+1}\right)^{-1} \circ \phi_{\Gamma,i-1}|_{\mathcal{U}^i_{j,k}} \circ \eta^i_{j,k}(x,t) = p_2 \circ \left(\eta^i_{j,k+1}\right)^{-1} \circ \phi_{\Gamma,i-1}|_{\mathcal{U}^i_{j,k}} \circ \eta^i_{j,k}(x,t')$$

for every $t, t' \in I$ where $p_2 : \mathbb{S}^1 \times I \to \mathbb{S}^1$ is the canonical projection.

Now we consider $\widetilde{\phi}_{\Gamma,i} := \mathcal{D}_{\delta_i} \circ \phi_{\Gamma,i-1}$ as in (7.6). The curves

$$\mathcal{C}^i_{j,k} = \eta^i_{j,k}\left(\mathbb{S}^1 \times \{0\}\right)$$

are invariant by $\widetilde{\phi}_{\Gamma,i}$. These curves separate Σ_{Γ^i} in two pieces: $\mathcal{U}^i$ and its complementary that we call $\mathcal{B}$. After quotienting by g_i, we get a $\phi_{\Gamma,i}$-invariant piece that is $g_i(\mathcal{B}) \approx \mathcal{B}$ and another one that is $g_i(\mathcal{U}^i)$. The restriction of $\phi_{\Gamma,i}$ to $g_i(\mathcal{B})$ is conjugate to $\phi_{\Gamma,i-1}$ and then pseudo-periodic. The restriction to $g_i(\mathcal{U}^i)$ has an invariant spine that is $g_i(\widetilde{\Gamma}^i)$ and then, by Lemma 4.9, it is boundary-free isotopic to a periodic one. Then, we have seen that $\phi_{\Gamma,i}$ is pseudo-periodic. □

Remark 7.18. — The screw number associated to the orbit of an invariant annuli at the curve $\mathcal{C}^i_{j,k}$ that appears in the previous proof is

$$s\left(\mathcal{C}^i_{j,k}\right) = -\sum_k \delta^1_{j,k}/l\left(\widetilde{\Gamma}^1_{j,1}\right). \tag{7.10}$$

We check this by giving a more elaborated construction that the one in the proof of Theorem 7.17 from which the computation of the screw number follows easily.

Let $\widetilde{\Gamma}^i_{j,1}, \ldots, \widetilde{\Gamma}^i_{j,\alpha_j}$ be a set of boundary components of Σ_{Γ^i} contained in $\widetilde{\Gamma}^i$ and cyclically permuted by $\phi_{\Gamma,i-1}$. Let $\mathcal{A}^i_{j,k}$ annular neighbourhoods of them as in the previous proof such that

$$\phi_{\Gamma,i-1}\left(\mathcal{U}^i_{j,k}\right) = \mathcal{U}^i_{j,k+1}.$$

Since $\phi^{\alpha}_{\Gamma,i-1}|_{\mathcal{U}^i_{j,1}}$ is periodic, we can choose coordinates/parametrization $r^i_{j,1}$ from $\mathbb{S}^1 \times [0,1]$ (with $\mathbb{S}^1 \approx \mathbb{R}/\mathbb{Z}$ of total length 1) to $\mathcal{U}^i_{j,1}$ with respect to which $\phi^{\alpha}_{\Gamma,i-1}|_{\mathcal{U}^i_{j,1}}$ is a rotation of the annulus, that is

$$\left(r^i_{j,1}\right)^{-1} \circ \phi^{\alpha_j}_{\Gamma,i-1} \circ r^i_{j,1}(x,t) = (x + \tau_{i-1}, t),$$

where τ_{i-1} is the rotation number $rot(\phi^{\alpha_j}_{\Gamma,\,i-1}|_{\widetilde{\Gamma}^i_{j,\,i}})$ (note that τ_{i-1} depends on the considered orbit of annuli, not only on i). Assume

$$\widetilde{\Gamma}^i_{j,\,k} = r^i_{j,\,k}\left(\mathbb{S}^1 \times \{1\}\right).$$

Let $l(\widetilde{\Gamma}^i_{j,\,k})$ be the length of $\widetilde{\Gamma}^i_{j,\,k}$. We assume without loss of generality that $r^i_{j,\,k}|_{\mathbb{S}^1\times\{1\}}$ is a homotethy of ratio $l(\widetilde{\Gamma}^i_{j,\,k})$ onto $\widetilde{\Gamma}^i_{j,\,k}$.

Define

$$r^i_{j,\,k} := \phi^{k-1}_{\Gamma,\,0} \circ r^i_{j,\,1} \text{ for } k = 2, \ldots, \alpha_j - 1.$$

Define now

$$\mathcal{A}^i_{j,\,k} := r^i_{j,\,k}\left(\mathbb{S}^1 \times [0, 1/2]\right) \text{ and } \mathcal{C}^i_{j,\,k} := r^i_{j,\,k}(\mathbb{S}^1 \times \{1/4\})$$

(note that this system of curves $\mathcal{C}^i_{j,\,k}$ is different from that of the previous proof but isotopic to it). We choose a representative of the boundary Dehn twists $\mathcal{D}_{\delta_i}$ (defined up to isotopy fixing the boundary) performing *all the twisting* in the annuli $\mathcal{A}^i_{j,\,k}$. More precisely we can assume $\mathcal{D}_{\delta_i}$ is expressed on $\mathcal{U}^i_{j,\,k}$ in the coordinates $r^i_{j,\,k}$ as follows:

$$\left(r^i_{j,\,k+1}\right)^{-1} \circ \mathcal{D}_{\delta_i} \circ r^i_{j,\,k}(x,t) := \begin{cases} \left(x + 2 \cdot t \cdot \delta^i_{j,\,k}/l\left(\widetilde{\Gamma}^i_{j,\,k}\right), t\right) & \text{if } 0 \leqslant t \leqslant 1/2 \\ \left(x + \delta^i_{j,\,k}/l\left(\widetilde{\Gamma}^i_{j,\,k}\right), t\right) & \text{if } 1/2 \leqslant t \leqslant 1 \end{cases}$$

where we denote by $\delta^i_{j,\,k} := \delta_i(g_i(\widetilde{\Gamma}^i_{j,\,k}))$ with $g_i : \Sigma_{\Gamma^i} \to \Sigma_{\Gamma^{i+1}}$ the gluing function.

Then, we have that $\widetilde{\phi}_{\Gamma,i} := \mathcal{D}_{\delta_i} \circ \phi_{\Gamma,\,i-1}$ satisfies that

$$\left(r^i_{j,\,1}\right)^{-1} \circ \widetilde{\phi}^{\alpha_j}_{\Gamma,\,i} \circ r^i_{j,\,1}(x,t) = \begin{cases} \left(x + \tau_{i-1} + 2 \cdot t \cdot \sum_k \delta^i_{j,\,k}/l\left(\widetilde{\Gamma}^i_{j,\,1}\right), t\right) & \text{if } 0 \leqslant t \leqslant 1/2 \\ \left(x + \tau_{i-1} + \sum_k \delta^i_{j,\,k}/l\left(\widetilde{\Gamma}^i_{j,\,1}\right), t\right) & \text{if } 1/2 \leqslant t \leqslant 1 \end{cases} \tag{7.11}$$

where

$$\tau_{i-1} = rot\left(\phi^{\alpha_j}_{\Gamma,\,i-1}\big|_{\widetilde{\Gamma}^i_{j,\,i}}\right) \text{ is the rotation number of } \phi^{\alpha_j}_{\Gamma,\,i-1} \text{ at } \widetilde{\Gamma}^i_{j,\,1}.$$

Recall that this rotation number is computed orienting $\widetilde{\Gamma}^i_{j,\,1}$ with the orientation induced by relative safe walks, that is, the opposite as the orientation that it inherits as boundary component of Σ_{Γ^i}.

From the expression in (7.11) we see that the restriction of $\widetilde{\phi}^{\alpha_j}_{\Gamma,\,i}$ to $\mathcal{A}^i_{j,\,k}$ is conjugated to $\mathcal{D}_{s,\tau_{i-1}}$ (note that we have to reparametrize $t = t/2$ in order

to get the expression of $\mathcal{D}_{s,\tau_{i-1}}$ which is defined in $\mathbb{S}^1\times[0,1]$) where s is the opposite of the expression in (7.10). Applying Definition 6.10, we get that the screw number at $\mathcal{A}^i_{j,k}$ are given exactly by by the expression (7.10).

By Definition 6.10 (see also Remark 6.11) this computes the screw number at $\mathcal{C}^i_{j,k}$ since the mapping $\widetilde{\phi}_{\Gamma,i}$, after quotienting by g_i, goes down to a homeomorphism that is periodic restricted to $g_i(\mathcal{U}^i_j\setminus\mathcal{A}^i_j)$ where $\mathcal{U}^i_j=\cup_k\mathcal{U}^i_{j,k}$ and $\mathcal{A}^i_j=\cup_k\mathcal{A}^i_{j,k}$, and to a pseudo-periodic homeomorphism of $\Sigma_{\Gamma^i}\setminus\mathcal{U}^i\subseteq\Sigma_{\Gamma^{i+1}}$ in a quasi-canonical form (see Remark 6.3).

Remark 7.19. — Given a relative mixed tête-à-tête graph $(\Gamma^\bullet,A^\bullet,\delta_\bullet)$ and given two different embeddings of (Γ,A) into the surface Σ, the induced mixed tête-à-tête twists are conjugated by the same homeomorphism that relates the two embeddings.

7.4. A realization theorem

It is straightforward that the filtered metric graph described at the end of Section 7.1 is a mixed relative tête-à-tête graph with the given δ's. The construction given in that section proves the following:

THEOREM 7.20. — *Let ϕ be a pseudo-periodic homeomorphism satisfying assumptions* (1)-(3) *at the beginning of Section 7.1. Let $\partial^1\Sigma$ be a union of some, at least one, connected components of $\partial\Sigma$ that:*

- *are fixed pointwise by ϕ,*
- *are contained in a single connected component of $\Sigma\setminus\mathcal{C}$ and*
- *have positive fractional Dehn twist coefficient.*

Then, there exists a mixed tête-à-tête graph $(\Gamma^\bullet,A^\bullet,\delta_\bullet)$ with $A^0=\partial\Sigma\setminus\partial^1\Sigma$ and an embedding $\Gamma\hookrightarrow\Sigma$ such that:

$$\left[\phi_{(\Sigma,\Gamma)}\right]_{\partial\Sigma,\,\phi|_{\partial\Sigma}}=[\phi]_{\partial\Sigma,\,\phi|_{\partial\Sigma}}.$$

And we have as a corollary the case of the monodromy of plane branches:

COROLLARY 7.21. — *Let $f:\mathbb{C}^2\to\mathbb{C}$ be an irreducible polynomial with an isolated singularity at 0. Let Σ be the corresponding Milnor fiber and let $h:\Sigma\to\Sigma$ be a representative of the monodromy that fixes $\partial\Sigma$ pointwise. Then, there exists a mixed tête-à-tête graph $(\Gamma^\bullet,\delta_\bullet)$ (with no relative boundaries) embedded in Σ such that $[\phi_{(\Sigma,\Gamma)}]_{\partial\Sigma}=[h]_{\partial\Sigma}$. Notice that in the subsequent paper of the fourth author with B. Sigurdsson* [9] *this has been generalized to any function $f:X\to\mathbb{C}$ defined on a normal surface singularity X.*

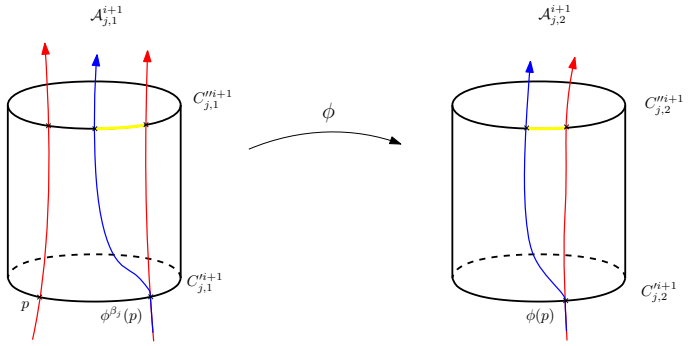


Figure 7.2. In red, the edges of $\widetilde{\Gamma}^{\leqslant i+1}$ *passing by a point* $p \in \Sigma^{\leqslant i} \cap \mathcal{A}_1$, *by* $\phi(p)$ *and by* $\phi^{\beta_{i+1}}(p)$. *In black, the image by* $\phi_{\Gamma^{i+1}}$ *or* $\widehat{\phi}^{i+1}$ *of the first one, and the image through the point* $\phi^{\beta}(p)$ *of an edge through the point* $\phi^{\beta-1}(p)$. *In yellow, a curve in* $\mathcal{C}''_1$ *and* $\mathcal{C}''_2$ *that has length* $-s \cdot length(\mathcal{C}''_1)$ *and* $-s/\alpha \cdot length(\mathcal{C}''_1)$ *respectively.*

Proof. — In [1] it is given a description of the Milnor fiber and the monodromy, which, in particular shows that:

(0) h is pseudo-periodic,
(1) $G(h, \Sigma)$ is a tree,
(2) it has all screw numbers negative,
(3) the Milnor fiber has 1 boundary component and the fractional Dehn twist coefficient with respect to it is positive.

hence we conclude by Theorem 7.20. □

Remark 7.22. — We could broaden the definition of mixed tête-à-tête twist allowing $\delta_i : \Gamma^i \to \mathbb{R}$. In this way, we allow turning in the other direction along the separating annuli. We would find a *signed* mixed tête-à-tête twist that would model pseudo-periodic homeomorphisms with positive screw numbers as well.

Example 7.23. — Let Σ be the surface of which is symmetrically embedded in $\mathbb{R}^3$ as in Figure 7.4 with its boundary component being the unit circle in the xy-plane. We follow the notation of the figure.

Consider the rotation of π radians around the z-axis. By the symmetric embedding of the surface, it leaves the surface invariant. Isotope the rotation so that it is the identity on $z \leqslant 0$ and its restriction to Σ has fractional Dehn twist coefficient $1/2$ at the only boundary component. We denote the homeomorphism after the isotopy by R_π.

Let D_i be a full positive Dehn twist on $\mathcal{A}^1_{1,k}$, $k = 1, 2$. We define the homeomorphism

$$\phi := D_2 \circ D_1^{-2} \circ R_\pi|_\Sigma \tag{7.12}$$

It is clear that ϕ is a pseudo-periodic homeomorphism and it is already in quasi-canonical form of Remark 6.3. We consider the decomposition of Σ adapted to ϕ as in Figure 7.3. In particular we have a single annuli orbit $\mathcal{A}^1$ and $s(\mathcal{A}^1) = -1$.

One can follow the proof of Theorem 7.20 to find a mixed tête-à-tête graph embedded in Σ that models ϕ. In Figure 7.3 you can find the abstract mixed tête-à-tête graph for $\delta_0 = \pi$, $\delta_1 = \pi/18$ which determines the homeomorphism only up to conjugacy (see Remark 7.19). In Figure 7.4 you can find the embedded graph one obtains following the proof of Theorem 7.20 which determines also the isotopy classes $[\phi]_{\partial, \phi|_\partial}$ or $[\phi]$.

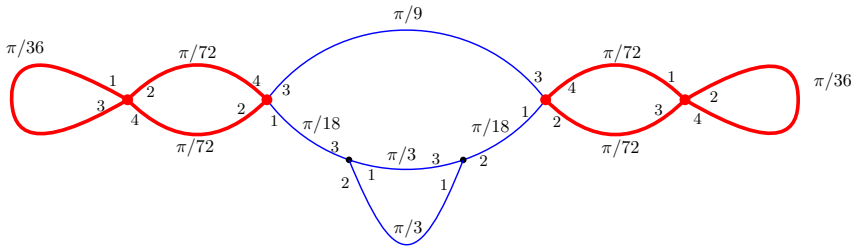


Figure 7.3. Mixed tête-à-tête graph with Γ_1 in red.

In Figure 7.5 we can see a modified graph which is still a mixed tête-à-tête graph for the same values of the functions δ. This serves to exemplify the phenomenon announced in Remark 7.10. Pick one of the two newly created vertices on the component of the component of the red graph on the right. Pick two sequences of points converging to this vertex, one in the red part of the graph and the other in the blue part. A direct inspection shows that the end point of the mixed safe walks of the points in the red graph tend to one of the two new vertices on the left component of the red graph; while the end points of the mixed safe walks of the points in the blue graph tend to the other vertex.

Note that in the key example in Section 7.1, the adjacency graph of the connected pieces of the subsurfaces where the restriction of the homeomorphims is periodic, is a tree. This was assumtion (1) in that section. In the following example we show a pseudo-periodic homeomorphism not satifying this restriction, that is a mixed tête-à-tête twist. As we have remarked at

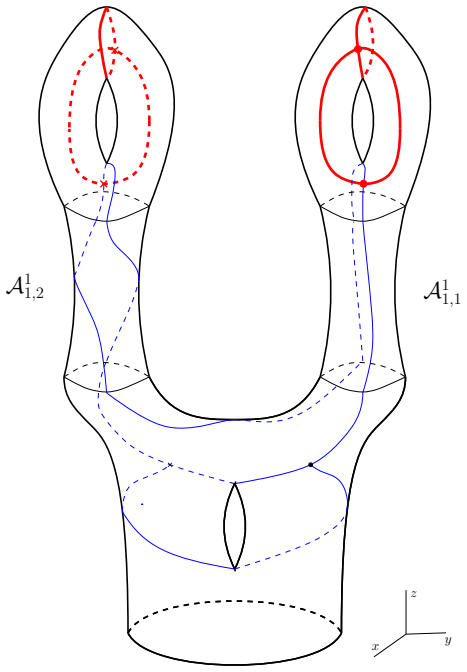


Figure 7.4. We can see the surface Σ of genus 3 and 1 boundary component. In red and blue we see the embedded graph Γ and in red we see Γ^1.

the introduction, in the sequel article [9], it is proved that this restriction is superfluous, and that mixed tête-à-tête twist is a much more general class.

Example 7.24. — This is an example of mixed tête-à-tête twist that does not satisfy the assumption (1) in page 2687 at the beginning of the section.

It is induced by a mixed tête-à-tête graph with a filtration $\Gamma \supset \Gamma^1$. The thickening of Γ is a surface of genus 3 and 1 boundary component.

We obtain the graph Γ as follows. We take the relative tête-à-tête graph of the first figure in Figure 7.6 that induces the rotation of the disk with 3 more boundary components. This will correspond to Γ_{Γ^1}. We also consider a graph $K_{3,3}$ with edges of length $\pi/12$. Note that the thickening of $K_{3,3}$ is a torus with 3 boundary components and that this metric $K_{3,3}$ is a tête-à-tête graph for $\pi/6$.

Now, we identify the three relative components of Γ_{Γ^1} with $\widetilde{K}_{3,3}$ which is obtained by cutting the graph $K_{3,3}$ along itself (and consists in three $\mathbb{S}^1$). After the identification we quotient by the quotient map $\widetilde{K}_{3,3} \to K_{3,3}$ to

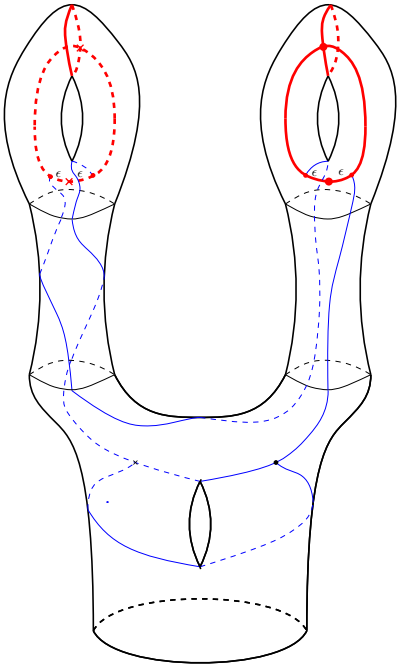


Figure 7.5. We can see the surface Σ of genus 3 and 1 boundary component. In red and blue we see the a modified version of the previously constructed graph Γ. The blue segments arriving at the red graph have been displaced ϵ from their original intersection point creating 4 new vertices. The total lengths of the graphs remain invariant.

obtain a graph Γ as the third one in Figure 7.6. It is a mixed tête-à-tête graph for $\delta_0 = \pi$ and $\delta_1 = \pi/6$ with the embedding of $K_{3,3}$ as the level 1 subgraph Γ^1.

It can be easily checked that the induced homeomorphism, that we denote by ϕ, has restriction to $\Sigma^1_{1,1}$ isotopic to the rotation of Figure 7.6, restriction to $\Sigma^2_{1,1}$ permuting the boundary components in the same way and the screw number of the annuli equals -1.

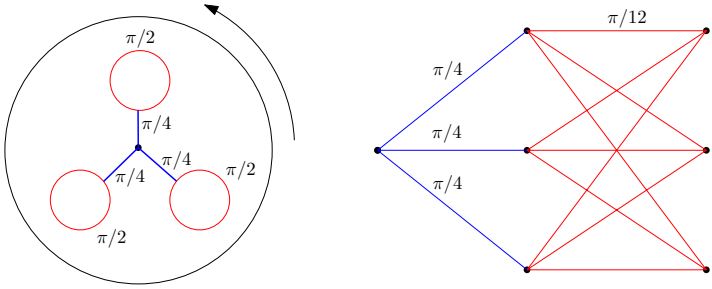


Figure 7.6. On the left we see Σ_{Γ^1}, in blue and red we see Γ_{Γ^1} and in red we see $\widetilde{\Gamma}^1$. On the right, we see a projection on the plane of the mixed tête-à-tête graph for $\delta_0 = \pi$ and $\delta_1 = \pi/6$ with Γ^1 in red (note that Γ^1 is $K_{3,3}$ with each edge of length $\pi/12$. If no cyclic order is indicated, then consider the counterclockwise ordering induced by the orientation of the plane.

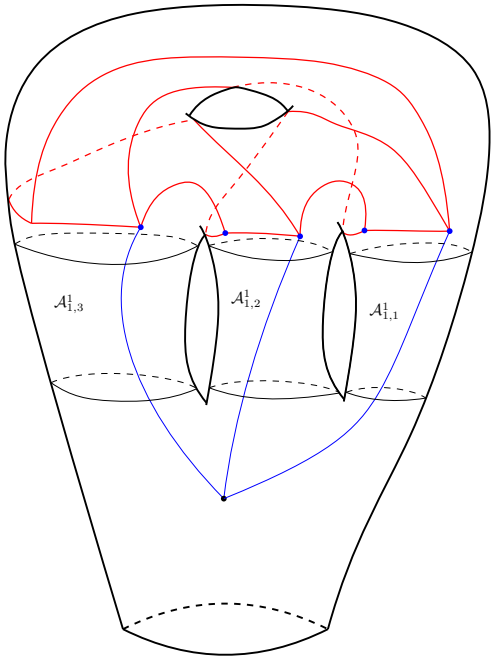


Figure 7.7. The decomposition of Σ adapted to ϕ. The embedding of the mixed tête-à-tête graph for $\delta_0 = \pi$ and $\delta_1 = \pi/6$ with Γ^1 in red.

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