ANNALES DE L'INSTITUT FOURIER

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ABELIANIZATION OF SOME GROUPS OF INTERVAL EXCHANGES

by Octave LACOURTE

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ABSTRACT. — Let IET be the group of permutations of $[0, 1[$ which are continuous outside a finite set, right-continuous and piecewise translations. The abelianization homomorphism of IET was described by Arnoux–Fathi and Sah. It gives an isomorphism between the abelianization of IET and the second exterior power of the reals over the rationals.

Let Γ be a subgroup of $\mathbb{R}/\mathbb{Z}$ and let $\widetilde{\Gamma}$ be its preimage in $\mathbb{R}$. We denote by IET(Γ) the subgroup of IET consisting of all elements continuous outside Γ. We establish an explicit isomorphism between its abelianization and the second skew-symmetric power of $\widetilde{\Gamma}$ over $\mathbb{Z}$. This group often has non-trivial 2-torsion.

Then, we define IET$^{\bowtie}$ as the group of all interval exchange transformations with flips. For every subgroup IET$^{\bowtie}(\Gamma)$ we establish an explicit isomorphism between its abelianization and an explicit 2-elementary abelian group.

RÉSUMÉ. — Soit IET le groupe des permutations de $[0, 1[$ qui sont continues en dehors d'un ensemble fini, continues à droite et qui sont des translations par morceaux. Arnoux–Fathi et Sah ont établi un isomorphisme entre l'abélianisé d'IET et la seconde puissance extérieure des réels sur les rationnels.

Soit Γ un sous-groupe de $\mathbb{R}/\mathbb{Z}$ et $\widetilde{\Gamma}$ sa pré-image. On note par IET(Γ) le sous-groupe d'IET composé de l'ensemble des éléments continus en dehors de Γ. On établit un isomorphisme explicite entre l'abélianisé d'IET(Γ) et la seconde puissance antisymétrique de $\widetilde{\Gamma}$ sur $\mathbb{Z}$. Ce groupe a souvent de la 2-torsion.

Puis nous définissons IET$^{\bowtie}$ comme le groupe de tous les échanges d'intervalles avec renversements. Pour tout sous-groupe IET$^{\bowtie}(\Gamma)$, on établit un isomorphisme explicite entre son abélianisé et un groupe abélien 2-élémentaire explicite.

1. Introduction

Let X be the right-open and left-closed interval $[0, 1[$. We denote by $\mathfrak{S}(X)$ the group of all permutations of X and let $\mathfrak{S}_{\text{fin}}$ be its subgroup consisting of all finitely supported elements. For every group G we denote by $D(G)$ its derived subgroup and by $G_{\text{ab}} = G/D(G)$ its abelianization.

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We define IET (Interval Exchange Transformations) as the subgroup of $\mathfrak{S}(X)$ consisting of all elements that are continuous outside a finite set, right-continuous and that are piecewise a translation. For every subgroup Γ of $\mathbb{R}/\mathbb{Z}$ we define IET(Γ) as the subgroup of IET consisting of all elements f that are continuous outside the preimage $\widetilde{\Gamma}$ of Γ in $\mathbb{R}$.

We also consider an overgroup of IET and IET(Γ), where we allow flips, that is, we replace "translation" with "isometry". There is a difficulty in defining this group, namely that, properly speaking, it cannot be defined as a group acting on $[0,1[$. For this we define $\widehat{\text{IET}^{\bowtie}}$ as the subgroup of $\mathfrak{S}(X)$ consisting of all elements that are continuous outside a finite set and piecewise isometric. Let $\widehat{\text{IET}^{\bowtie}}(\Gamma)$ be its subgroup consisting of all elements f continuous outside $\widetilde{\Gamma}$. We define the group of interval exchanges with flips, denoted by $\text{IET}^{\bowtie}$, as the image of $\widehat{\text{IET}^{\bowtie}}$ in $\mathfrak{S}(X)/\mathfrak{S}_{\text{fin}}(X)$. Similarly let $\text{IET}^{\bowtie}(\Gamma)$ be the image of $\widehat{\text{IET}^{\bowtie}}(\Gamma)$ in $\mathfrak{S}(X)/\mathfrak{S}_{\text{fin}}(X)$.

The group IET is a subgroup of $\widehat{\text{IET}^{\bowtie}}$. Also it is naturally isomorphic to its image in $\text{IET}^{\bowtie}$ thus we freely see IET as a subgroup of $\text{IET}^{\bowtie}$.

Most of the existing research on interval exchanges since the 70s pertains to classical dynamics, mostly studying the dynamical and ergodic properties of a single element of IET (see the survey of Viana [11]). The study of IET as a group started with the identification of its abelianization by Sah [10] and Arnoux–Fathi [1] (see below), and was much later revived by work of Dahmani–Fujiwara–Guirardel [3, 4] and C. Novak [9] in the late 2000s.

In comparison the group $\text{IET}^{\bowtie}$ is considerably less known. P. Arnoux proved in [2] that it is a simple group. Recently I. Liousse and N. Guelman [6] proved that $\text{IET}^{\bowtie}$ is uniformly perfect.

For every $\mathbb{Z}$-module A, we denote by $\ominus\bigwedge_{\mathbb{Z}}^2 A$ the second skew-symmetric power. It is the quotient of the second tensor power $\bigotimes_{\mathbb{Z}}^2 A$ by the $\mathbb{Z}$-submodule generated by the set $\{x \otimes y + y \otimes x \mid x, y \in A\}$. We denote by $a \wedge b$ the projection of $a \otimes b$ in $\ominus\bigwedge_{\mathbb{Z}}^2 A$; the set of all these elements generates $\ominus\bigwedge_{\mathbb{Z}}^2 A$. There exists a canonical surjective group homomorphism from $\ominus\bigwedge_{\mathbb{Z}}^2 A$ into the second exterior power $\bigwedge_{\mathbb{Z}}^2 A$, which is injective if $A = 2A$. In general, its kernel is a 2-elementary abelian group, which is not trivial in general. For instance, if $A \simeq \mathbb{Z}^d$, then $\bigwedge_{\mathbb{Z}}^2 A$ is isomorphic to $\mathbb{Z}^{d(d-1)/2}$, while this kernel is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^d$. In the case where $A = \mathbb{R}$ we recall that every map is $\mathbb{Z}$-bilinear if and only if it is $\mathbb{Q}$-bilinear, thus $\bigwedge_{\mathbb{Q}}^2 \mathbb{R} = \bigwedge_{\mathbb{Z}}^2 \mathbb{R}$.

We denote by λ the Lebesgue measure on $\mathbb{R}$. We recall the result of Arnoux–Fathi and Sah [1, 10] about the identification of the abelianization of IET:

THEOREM (Arnoux–Fathi–Sah [1, 10]). — *The map:*

$$\varphi: \begin{array}{ccc} \mathrm{IET} & \longrightarrow & \bigwedge^2_{\mathbb{Z}} \mathbb{R} \\ f & \longmapsto & [\sum_{a \in \mathbb{R}} a \otimes \lambda((f - \mathrm{Id})^{-1}(\{a\}))] \end{array}$$

is a surjective group homomorphism whose kernel is the derived subgroup $D(\mathrm{IET})$. *It is called the* SAF-*homomorphism. It induces an isomorphism between* $\mathrm{IET}_{\mathrm{ab}}$ *and the second exterior power of the reals over the rationals.*

In this article we identify $\mathrm{IET}(\Gamma)_{\mathrm{ab}}$ and $\mathrm{IET}^{\bowtie}(\Gamma)_{\mathrm{ab}}$ for every subgroup Γ of $\mathbb{R}/\mathbb{Z}$. We denote by $\widetilde{\Gamma}$ the preimage of Γ in $\mathbb{R}$.

When Γ if finite, both $\mathrm{IET}(\Gamma)$ and $\mathrm{IET}^{\bowtie}(\Gamma)$ are finite Coxeter groups and their abelianization can be described easily (see Section 2.1). So from now on, we always assume that Γ is a dense subgroup of $\mathbb{R}/\mathbb{Z}$.

In this case, we show that the derived subgroups $D(\mathrm{IET}(\Gamma))$ and $D(\mathrm{IET}^{\bowtie}(\Gamma))$ are simple (see Section 2.5). They are respectively the smallest normal subgroup of $\mathrm{IET}(\Gamma)$ and $\mathrm{IET}^{\bowtie}(\Gamma)$. Then we see $\mathrm{IET}(\Gamma)$ and $\mathrm{IET}^{\bowtie}(\Gamma)$ as topological full group of the groupoids of germs of their action on a Stone space. Then by the work of Nekrashevych [8], we obtain the simplicity of their smallest normal subgroup.

We denote by φ_Γ the restriction of the SAF-homomorphism φ to $\mathrm{IET}(\Gamma)$ for every subgroup Γ of $\mathbb{R}/\mathbb{Z}$. It turns out that φ_Γ may fail to give the abelianization of $\mathrm{IET}(\Gamma)$ because not every element of order 2 of $\mathrm{IET}(\Gamma)$ is necessarily in $D(\mathrm{IET}(\Gamma))$. In fact the restriction of φ_Γ is a surjective group homomorphism onto $2^{\ominus}\bigwedge^2_{\mathbb{Z}} \widetilde{\Gamma}$ and here the main theorem is:

THEOREM 1.1. — *There exists a surjective group homomorphism* $\varepsilon_\Gamma:$ $\mathrm{IET}(\Gamma) \to {}^{\ominus}\bigwedge^2_{\mathbb{Z}} \widetilde{\Gamma}$ *whose kernel is the derived subgroup* $D(\mathrm{IET}(\Gamma))$. *It induces an isomorphism between* $\mathrm{IET}(\Gamma)_{\mathrm{ab}}$ *and the second skew-symmetric power* ${}^{\ominus}\bigwedge^2_{\mathbb{Z}} \widetilde{\Gamma}$.

Remark 1.2. — One can notice that if $2\widetilde{\Gamma} = \widetilde{\Gamma}$ then $\varepsilon_\Gamma = \varphi_\Gamma$. Actually, in this case, the proof of Arnoux–Fathi–Sah works with immediate changes.

The novelty occurs when $\widetilde{\Gamma} \neq 2\widetilde{\Gamma}$. In this case 2-torsion appears in ${}^{\ominus}\bigwedge^2_{\mathbb{Z}} \widetilde{\Gamma}$ with in particular the subset $\{a \wedge a \mid a \in \widetilde{\Gamma}\}$.

Before constructing ε_Γ it is important to understand the kernel of φ_Γ. For this we introduce some elements of IET and $\mathrm{IET}(\Gamma)$.

A *restricted rotation of type* (a, b) is an element r of IET such that there exist consecutive intervals I and J of length a and b respectively with $\sup(I) = \inf(J)$, where r is the translation by b on I and the translation by $-a$ on J (and the identity outside $I \cup J$). The two intervals I and J are called the *intervals associated* with r. A Γ-*restricted rotation* is a

restricted rotation in IET(Γ). A tuple of restricted rotations is *balanced* if the number of factors of type (a, b) is equal to the one of type (b, a). A product of restricted rotations is balanced if it can be written as the product of a balanced tuple of restricted rotations.

Remark 1.3. — Let r be a restricted rotation of type (a, b) and I and J be the two consecutive intervals associated with r. Then if we only look at the interval $I \cup J$ and identify its endpoints, we obtain that r is the actual rotation of angle the length of J. This is why we called these elements "restricted rotation".

Y. Vorobets proves in [12] the following:

THEOREM (Y. Vorobets [12]). — *The kernel* $\mathrm{Ker}(\varphi)$ *of the* SAF-*homomorphism is generated by the set of all balanced products of restricted rotations and it is also generated by the set of all elements of order* 2.

Here we prove that this result also holds in restriction to IET(Γ):

LEMMA 1.4. — *The set of all balanced products of* Γ-*restricted rotations is a generating subset of* $\mathrm{Ker}(\varphi_\Gamma)$.

A corollary can be proved by following the ideas of Y.Vorobets. We introduce some elements of order 2. An IET-*transposition* of type a is an element of IET that swaps two nonoverlapping intervals of length a while fixing the rest of $[0, 1[$ and an IET(Γ)-*transposition* is an IET-transposition in IET(Γ).

COROLLARY 1.5. — *The set of all* IET(Γ)-*transpositions is a generating subset of the kernel* $\mathrm{Ker}(\varphi_\Gamma)$.

After this, in Section 5, we define the group homomorphism ε_Γ in the same spirit as the signature on finite permutation groups. For every $n \in \mathbb{N}$ and σ in the finite permutation group $\mathfrak{S}_n$, a pair $(x, y) \in \{1, 2, \ldots, n\}$ is an inversion of σ if $x < y$ and $\sigma(x) > \sigma(y)$. The signature of σ is equal to the counting measure of the set of all inversions of σ.

We keep the same definition for an inversion of an element of IET(Γ) and explicit how to measure this set. We define A_Γ as the Boolean algebra of subsets of $[0, 1[$ generated by the set of intervals $\{[a, b[\mid a, b \in \widetilde{\Gamma}\}$. Thanks to the Lebesgue measure λ on $\mathbb{R}$ we obtain a Boolean algebra measure w_Γ for $A_\Gamma \otimes A_\Gamma$ in the second tensor power $\bigotimes_{\mathbb{Z}}^2 \widetilde{\Gamma}$. We show that the set of inversions of an element of IET(Γ) is an union of rectangles inside $A_\Gamma \otimes A_\Gamma$. We define ε_Γ as the projection on $\ominus\bigwedge_{\mathbb{Z}}^2 \widetilde{\Gamma}$ of the measure of the set of all inversions. We show that ε_Γ is a surjective group homomorphism.

We establish the equality $-2\varepsilon_\Gamma = \varphi_\Gamma$ and manage to prove that $\mathrm{Ker}(\varepsilon_\Gamma)$ is equal to the derived subgroup $D(\mathrm{IET}(\Gamma))$.

In Section 6 we treat the case of $\mathrm{IET}^{\bowtie}(\Gamma)$. *A reflection map of type a is* an element of $\widehat{\mathrm{IET}^{\bowtie}}$ that reverses a right-open and left-closed subinterval of $[0, 1[$ of length a. A *reflection of type a* is the image of a reflection map of type a in $\mathrm{IET}^{\bowtie}$. A Γ-reflection is a reflection in $\mathrm{IET}^{\bowtie}(\Gamma)$. We know that the group $\mathrm{IET}^{\bowtie}(\Gamma)$ is generated by the set of reflections (see Proposition 2.9) so its abelianization is a 2-group. We are no longer able to measure the set of inversions but we can measure the union of this set with its symmetric. By projecting on $\bigotimes_{\mathbb{Z}}^2 \widetilde{\Gamma}/2(\bigotimes_{\mathbb{Z}}^2 \widetilde{\Gamma})$ we obtain a group homomorphism $\varepsilon_\Gamma^{\bowtie}$ whose image is $\langle \{a \otimes a \ [\mathrm{mod}\ 2] \mid a \in \widetilde{\Gamma}\} \rangle$. We also prove that its kernel is generated by all reflections of type inside $\widetilde{\Gamma} \smallsetminus (2\widetilde{\Gamma})$. Thus for $a \in \widetilde{\Gamma} \smallsetminus 2\widetilde{\Gamma}$ every restricted rotation of type (a, a) is in the kernel and by the work on $\mathrm{IET}(\Gamma)$ we do not expect it to be in $D(\mathrm{IET}^{\bowtie}(\Gamma))$. We notice that such an element is conjugate to a reflections of type $2a$. In fact we prove:

PROPOSITION 1.6. — *Let Ω_Γ be the conjugate closure of the group generated by the set of all Γ-reflections of type 2ℓ with $\ell \in \widetilde{\Gamma} \smallsetminus 2\widetilde{\Gamma}$. Then:*

$$\mathrm{Ker}(\varepsilon_\Gamma^{\bowtie}) = D(\mathrm{IET}^{\bowtie}(\Gamma))\Omega_\Gamma$$

Then we construct a group homomorphism

$$\psi_\Gamma : \mathrm{IET}^{\bowtie}(\Gamma) \to {}^{\ominus}\bigwedge_{\mathbb{Z}}^2 \widetilde{\Gamma}/2\left({}^{\ominus}\bigwedge_{\mathbb{Z}}^2 \widetilde{\Gamma}\right)$$

which uses ε_Γ, called the positive contribution of $\mathrm{IET}^{\bowtie}(\Gamma)$. Morally we approximate every element $f \in \mathrm{IET}^{\bowtie}(\Gamma)$ by elements in $\mathrm{IET}(\Gamma)$ and we use the group homomorphism ε_Γ on them. This can also be seen as approximate triangles by rectangles in the set of inversions. The construction is strongly dependent on a totally ordered set given by the fact that $\widetilde{\Gamma}$ is ultrasimplicially ordered (see Section 2.4). For every $a \in \widetilde{\Gamma} \smallsetminus 2\widetilde{\Gamma}$, this group homomorphism sends restricted rotation of type (a, a) on a nontrivial element. We also prove that the image of its restriction to $\mathrm{Ker}(\varepsilon_\Gamma^{\bowtie})$ is $\langle \{\ell \wedge \ell \ [\mathrm{mod}\ 2] \mid \ell \in \widetilde{\Gamma}\} \rangle$.

In order to obtain the following theorem we will need to notice that groups of exponent 2 are also $\mathbb{F}_2$-vector spaces.

THEOREM 1.7. — *For every subgroup Γ of $\mathbb{R}/\mathbb{Z}$ we have the following isomorphism:*

$$\mathrm{IET}^{\bowtie}(\Gamma)_{\mathrm{ab}} \simeq \langle \{a \otimes a \ [\mathrm{mod}\ 2] \mid a \in \widetilde{\Gamma}\} \rangle \times \langle \{\ell \wedge \ell \ [\mathrm{mod}\ 2] \mid \ell \in \widetilde{\Gamma}\} \rangle,$$

where the left term of the product is in $\bigotimes_{\mathbb{Z}}^2 \widetilde{\Gamma}/(2\bigotimes_{\mathbb{Z}}^2 \widetilde{\Gamma})$ and the right one is in ${}^{\ominus}\bigwedge_{\mathbb{Z}}^2 \widetilde{\Gamma}/(2{}^{\ominus}\bigwedge_{\mathbb{Z}}^2 \widetilde{\Gamma})$.

The inclusion of $\mathrm{IET}(\Gamma)$ in $\mathrm{IET}^{\bowtie}(\Gamma)$ induces a natural group homomorphism ι from $\mathrm{IET}(\Gamma)_{\mathrm{ab}}/(2\,\mathrm{IET}(\Gamma)_{\mathrm{ab}})$ to $\mathrm{IET}^{\bowtie}(\Gamma)_{\mathrm{ab}}$. We show that ι is injective for every dense subgroup Γ of $\mathbb{R}/\mathbb{Z}$. The image of ι is $\langle\{p\otimes q + q\otimes p\ [\mathrm{mod}\ 2]\mid p,q\in\widetilde{\Gamma}\}\rangle\times\langle\{l\wedge l\ [\mathrm{mod}\ 2]\mid l\in\widetilde{\Gamma}\}\rangle$; it is also isomorphic to $^{\ominus}\bigwedge_{\mathbb{Z}}^{2}\widetilde{\Gamma}/(2^{\ominus}\bigwedge_{\mathbb{Z}}^{2}\widetilde{\Gamma})$. We deduce that ι is surjective if and only if $\Gamma = 2\Gamma$. In the case where Γ if finitely generated we can precise the dimension of $\mathrm{IET}^{\bowtie}(\Gamma)_{\mathrm{ab}}$ and $\mathrm{Im}(\iota)$ as $\mathbb{F}_2$-vector spaces. The group $\widetilde{\Gamma}$ is also finitely generated and we denote by d its rank. Then $\mathrm{IET}^{\bowtie}(\Gamma)_{\mathrm{ab}}$ has dimension $\frac{d(d+3)}{2}$ and $\mathrm{Im}(\iota)$ has dimension $\frac{d(d+1)}{2}$.

Acknowledgments

2. Preliminaries

The aim of this section is to introduce some notation and gather known results used in the next sections. Let Γ be a subgroup of $\mathbb{R}/\mathbb{Z}$ and let $\widetilde{\Gamma}$ be its preimage in $\mathbb{R}$.

We recall that $\mathfrak{S}_n$ is the permutation group of n elements. For every group G we denote by G_{ab} the abelianization of G, which is the quotient of G by its derived subgroup.

For every function f we define the *support of f* by $\mathrm{Supp}(f) := \{x \in [0,1[\, :\, f(x)\neq x\}$.

For any real interval I let I° be its interior in $\mathbb{R}$. If I is equal to $[0,t[$ we agree that its interior I° is equal to $]0,t[$. We denote by $\mathrm{Itv}(\Gamma)$ the set of all intervals $[a,b[$ with $a,b\in\widetilde{\Gamma}$ and $0\leqslant a<b\leqslant 1$. We denote by λ the Lebesgue measure on $\mathbb{R}$.

2.1. When Γ is finite

We assume that Γ is a finite subgroup of $\mathbb{R}/\mathbb{Z}$. Then there exists $n\in\mathbb{N}_{\geqslant 1}$ such that $\widetilde{\Gamma}$ is equal to $\frac{1}{n}\mathbb{Z}$. We deduce that for every $1\leqslant i\leqslant n$, every

element of $\widehat{\mathrm{IET}^{\bowtie}(\Gamma)}$ is continuous on the interval $[\frac{i-1}{n}, \frac{i}{n}[$ up to a finitely supported permutation.

Then the group $\mathrm{IET}(\Gamma)$ is naturally isomorphic to the finite permutation group $\mathfrak{S}_n$. It is a Coxeter group of type A_{n-1} so its abelianization is $\{1\}$ if $n = 1$ and it is $\mathbb{Z}/2\mathbb{Z}$ if $n > 1$.

The group $\mathrm{IET}^{\bowtie}(\Gamma)$ is isomorphic to the signed symmetric group $\mathbb{Z}/2\mathbb{Z} \wr \mathfrak{S}_n$. It is a Coxeter group of type B_n so its abelianization is $\mathbb{Z}/2\mathbb{Z}$ for $n = 1$ and it is $(\mathbb{Z}/2\mathbb{Z})^2$ if $n > 1$.

2.2. Descriptions of an element of IET

In order to describe an element f of IET we use partitions into right-open and left-closed intervals of $[0, 1[$. Let $\mathcal{P}$ be such a partition.

DEFINITION 2.1. — *If f is continuous on the interior of every interval of $\mathcal{P}$ then $\mathcal{P}$ is called* a partition into intervals associated with f. *The set of all partitions into intervals associated with f is denoted by Π_f. If every interval I of $\mathcal{P}$ is in $\mathrm{Itv}(\Gamma)$ then $\mathcal{P}$ is said to be a Γ-partition. We denote by $f(\mathcal{P})$ the partition into intervals of $[0, 1[$ composed of all right-open and left-closed intervals whose interior is the image by f of the interior of an interval in $\mathcal{P}$. It is called* the arrival partition of f associated with $\mathcal{P}$.

Remark 2.2. — For every $f \in \mathrm{IET}$, the set Π_f has a minimal element for the refinement. This element is also the unique partition that has a minimal number of interval.

One can notice that for every $f \in \mathrm{IET}$ there is an equivalence between f belongs to $\mathrm{IET}(\Gamma)$ and the existence of a Γ-partition inside Π_f. From now on every partition is assumed to be finite.

There are descriptions of an element of IET that are more combinatorial. They are useful to understand the SAF-homomorphism and further ε_Γ:

DEFINITION 2.3. — *Let f be an element of IET and let $\mathcal{P} \in \Pi_f$; let n be the number of intervals of $\mathcal{P}$. For $1 \leqslant i \leqslant n$ let I_i (resp. J_i) be the consecutive intervals composing $\mathcal{P}$ (resp. $f(\mathcal{P})$). Let α be the n-tuple such that α_i is the length of I_i for every $1 \leqslant i \leqslant n$. There exists $\tau \in \mathfrak{S}_n$ such that $f(I_i^\circ) = J_{\tau(i)}^\circ$. Then we say that (α, τ) is* a combinatorial description *of f. If each component of α is in $\widetilde{\Gamma}$ we say that (α, τ) is a Γ-combinatorial description of f. The partition $\mathcal{P}$ can also be called* the partition associated with (α, τ).

PROPOSITION 2.4. — *Let $f \in$ IET and let (α, τ) be a combinatorial description of f; let n be the length of α. Then we have:*

$$\varphi(f) = \sum_{j=1}^{n} \left(\sum_{\substack{i: \\ \tau(i) < \tau(j)}} \alpha_i - \sum_{i<j} \alpha_i \right) \wedge \alpha_j$$

Proof. — Let $f \in$ IET and (α, τ) be a combinatorial description of f. Let n be the length of α and let $\{I_1, I_2, \ldots, I_n\}$ be the partition associated with (α, τ). For each j we denote by $v(j)$ the value of $f - \mathrm{Id}$ on I_j. Thus we deduce that $\varphi(f) = \sum_{j=1}^{n} v(j) \wedge \lambda(I_j) = \sum_{j=1}^{n} v(j) \wedge \alpha_j$. Also we know that $v(j) = \sum_{\tau(i)<\tau(j)} \alpha_i - \sum_{i<j} \alpha_i$ (see Figure 2.1 below), and this gives the conclusion.   □
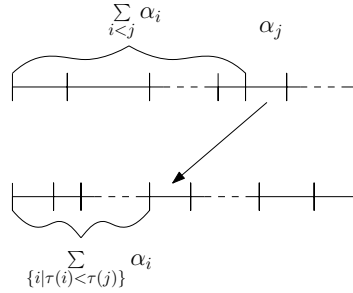

Figure 2.1. Illustration for the value of the SAF-homomorphism in Proposition 2.4

2.3. Three important families

There are two families of elements of IET appearing in this work and one more for $\mathrm{IET}^{\bowtie}$.

DEFINITION 2.5. — *Let $a \in [0, \frac{1}{2}[$ and f be in IET. We call f an IET-transposition of type a if it swaps two subintervals of $[0, 1[$ of length a with non-overlapping support while fixing the rest of $[0, 1[$. If in addition f belongs to IET(Γ) then f is called an IET(Γ)-transposition of type a.*

Also it is natural to consider the identity as an IET(Γ)-transposition. We precise it whenever it is needed.

One can notice that, for f an IET-transposition of type a, asking f to be in IET(Γ) implies that a is in $\widetilde{\Gamma} \cap [0, 1[$ but the converse is false.

These elements are called *interval swap maps* by Y. Vorobets in [12]. They are elements of order 2 and generate $D(\text{IET})$. More precisely, we describe below (Figure 2.2) how such an element can be seen in $D(\text{IET})$. For f an $\text{IET}(\Gamma)$-transposition of type a, this suggests to ask if a belongs to $2\widetilde{\Gamma}$ or not, before concluding that f is in $D(\text{IET}(\Gamma))$ or not.
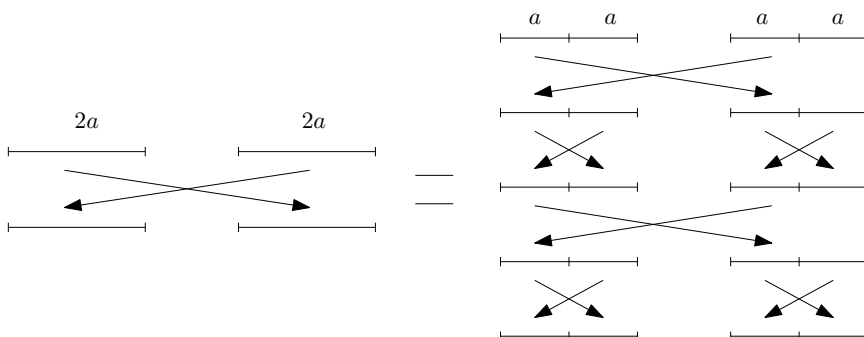

Figure 2.2. *Illustration of the property that an* IET*-transposition of type* $2a$ *is a commutator of an* IET*-transposition of type* a *with a product of two* IET*-transpositions of type* a.

A *restricted rotation of type* (a, b) is an element r of IET such that there exist consecutive intervals I and J of length a and b respectively with $\sup(I) = \inf(J)$, where r is the translation by b on I and the translation by $-a$ on J (and the identity outside $I \cup J$). The two intervals I and J are called the intervals associated with r.

DEFINITION 2.6. — *Let* $a, b \in [0, 1[$ *with* $0 \leqslant a + b \leqslant 1$ *and* f *be in* IET. *We call* f *an* IET-*restricted rotation of type* (a, b) *if there exist consecutive intervals* I *and* J *of length* a *and* b *respectively with* $\sup(I) = \inf(J)$, *where* f *is the translation by* b *on* I *and the translation by* $-a$ *on* J *(and the identity outside* $I \cup J$). *The two intervals* I *and* J *are called the intervals associated with* f. *If in addition* f *belongs to* IET(Γ) *then* f *is called an* Γ-*restricted rotation of type* (a, b).

Again one can notice that for f an IET-restricted rotation of type (a, b), if f is in IET(Γ) this implies that a and b are in $\widetilde{\Gamma} \cap [0, 1[$, but the converse is false.

The case $a = b$ coincide with an IET-transposition of type a.

The family of restricted rotations generates IET and we have more precision on the type of restricted rotations in the decomposition, by the following fact due to Vorobets in [12]:

LEMMA 2.7 (Vorobets). — *Any transformation f in IET is a product of IET-restricted rotations. Furthermore for every $\mathcal{P} \in \Pi_f$ there exists a decomposition of f into IET-restricted rotations with type inside the set of length of the intervals in $\mathcal{P}$.*

In particular if we start with an element of IET(Γ) we can find a decomposition into Γ-restricted rotations that satisfies the condition about the type of restricted rotations.

The group IET can be seen as a subgroup of IET$^{\bowtie}$; thus the notion of restricted rotation still makes sense in IET$^{\bowtie}$. We now add a family that occurs only in IET$^{\bowtie}$.

DEFINITION 2.8. — *For every right-open and left-closed subinterval I of $[0, 1[$ we define the I-reflection map as the element of $\widehat{\text{IET}^{\bowtie}}$ that reverses I° while fixing the rest of $[0, 1[$. The type of an I-reflection map is the length of I. We define the I-reflection as the image of the I-reflection map in IET$^{\bowtie}$. The type of an I-reflection is the length of I. A Γ-reflection is an I-reflection for some $I \in \text{Itv}(\Gamma)$.*

The set of all reflections is a generating subset of IET$^{\bowtie}$. This follows from Proposition 6.10 and the fact that the restricted rotation, whose intervals associated are I and J, is the product of the I-reflection, the J-reflection and the $(I \sqcup J)$-reflection. More precisely, with this proposition and Lemma 2.7 we obtain the following:

PROPOSITION 2.9. — *The set of all Γ-reflections is a generating subset of IET$^{\bowtie}(\Gamma)$.*

We make explicit some elements of the conjugacy class of a reflection.

PROPOSITION 2.10. — *Two reflections r, s which have the same type are conjugate in IET$^{\bowtie}(\Gamma)$.*

PROPOSITION 2.11. — *Let $a \in [0, \frac{1}{2}[$. A reflection of type $2a$ is conjugate to a restricted rotation of type (a, a).*

Proof. — Let r be such a reflection and let I be the interval reversed by r. By cutting I into two intervals of same size I_1 and I_2 we obtain that r is conjugate to the restricted rotation that permutes I_1 and I_2, by the I_1-reflection. $\square$

2.4. Ultrasimplicial groups

Let us introduce some classical terminology from the theory of ordered abelian groups. An ordered abelian group is an abelian group endowed with an invariant partial ordering.

For any subgroup H of $\mathbb{R}$ we denote by $H_+ := \{x \in H \mid x \geqslant 0\}$ the positive cone of H. A difficulty is that H_+ is not, in general, finitely generated as a subsemigroup. For every subset B of $\mathbb{R}$ we denote by $\mathrm{Vect}_{\mathbb{N}}(B)$ the subsemigroup generated by B.

A subsemigroup of an abelian group is *simplicial* if it is generated, as a subsemigroup, by a finite $\mathbb{Z}$-independent subset, and it is *ultrasimplicial* if it is the direct limit of an upward directed set of simplicial subsemigroups. An ordered abelian group is *simplicially ordered* if its positive cone is simplicial and is *ultrasimplicially ordered* if its positive cone is ultrasimplicial. The next theorem is proved by G.A. Elliott in [5]:

THEOREM 2.12. — *Every totally ordered abelian group is ultrasimplicially ordered.*

We deduce the following corollary for the totally ordered abelian group $\widetilde{\Gamma}$:

COROLLARY 2.13. — *There exist an upward directed set $(M_\Gamma, \leqslant)$ and a finite $\mathbb{Z}$-linearly independent subset S_a of $\widetilde{\Gamma}$ for every $a \in M_\Gamma$ such that for every $a, b \in M_\Gamma$ with $a \leqslant b$ we have $\mathrm{Vect}_{\mathbb{N}}(S_a) \subset \mathrm{Vect}_{\mathbb{N}}(S_b)$ and $\widetilde{\Gamma}_+$ is equal to the direct limit $\varinjlim \mathrm{Vect}_{\mathbb{N}}(S_a)$.*

We will use the Theorem 2.12 in the form of the following corollary which specifies the finite rank case:

COROLLARY 2.14. — *Let H be a subgroup of $\mathbb{R}$ which is free abelian of finite rank d. Then there exists a sequence $(B_n)_{n \in \mathbb{N}}$ of $\mathbb{Z}$-basis of H such that for each n we have $\mathrm{Vect}_{\mathbb{N}}(B_n) \subset \mathrm{Vect}_{\mathbb{N}}(B_{n+1})$ and H_+ is equal to the increasing union $\bigcup_n \mathrm{Vect}_{\mathbb{N}}(B_n)$. Furthermore for every $k \in \mathbb{N}$ and $L_1, L_2, \ldots, L_k \in H_+$ there exists a basis $\{\ell_1, \ell_2, \ldots, \ell_d\} \subset H_+$ of H such that for every $1 \leqslant i \leqslant k$ the element L_i is a linear combination of $\ell_1, \ell_2, \ldots, \ell_d$ with coefficients in $\mathbb{N}$.*

2.5. About the derived subgroup

We prove here that the derived subgroups $D(\mathrm{IET}(\Gamma))$ and $D(\mathrm{IET}^{\bowtie}(\Gamma))$ are simple. There are two steps in the proof. The first one is to give a

sufficient condition for a group G to obtain that its derived subgroup $D(G)$ is contained in every nontrivial normal subgroup of G; this is used by Arnoux in [2] to show that $\text{IET}^{\bowtie}$ is simple.

The second step is to remark that $\text{IET}(\Gamma)$ and $\text{IET}^{\bowtie}(\Gamma)$ are subgroups of the group of homeomorphism of a Stone space Y; that is a compact totally disconnected space. We show that they satsify some conditions that allows us to use the work of Nekrashevych [8] which gives us a normal subgroup of $\text{IET}(\Gamma)$ and one of $\text{IET}^{\bowtie}(\Gamma)$ which are simple and contained in all normal subgroups.

DEFINITION 2.15. — *For every $\varepsilon > 0$ we denote by $\mathcal{F}_\varepsilon$ the union of the set of all Γ-restricted rotations of type (a, b) with $a + b \leqslant \varepsilon$, with the set of all $\text{IET}(\Gamma)$-transpositions of type a with $2a \leqslant \varepsilon$. We denote by $\mathcal{F}_\varepsilon^{\bowtie}$ the union of the set $\mathcal{F}_\varepsilon$ with the set of all Γ-reflections of type a with $a \leqslant \varepsilon$.*

LEMMA 2.16. — *For every $\varepsilon > 0$, any element f in $\text{IET}(\Gamma)$ (resp. $\text{IET}^{\bowtie}(\Gamma)$) can be written as a finite product of element in $\mathcal{F}_\varepsilon$ (resp. $\mathcal{F}_\varepsilon^{\bowtie}$).*

Proof. — Let $\varepsilon > 0$ and by density of $\widetilde{\Gamma}$ in $\mathbb{R}$, let $w \in \widetilde{\Gamma}_+$ such that $w < \varepsilon$.

If f is an $\text{IET}(\Gamma)$-transposition of type a let I and J be the two non-overlapping intervals swapped by f. There exist $k \in \mathbb{N}$ and $u \in \widetilde{\Gamma}_+$ such that $u < w$ and $a = kw + u$. Then I and J can be cut into $k + 1$ consecutive intervals $I_1, I_2, \ldots, I_{k+1}$ and $J_1, J_2, \ldots, J_{k+1}$ respectively; the first k intervals with length w and the last one with length u. Then f is equal to the product $s_1 s_2 \ldots s_{k+1}$ where s_i is the $\text{IET}(\Gamma)$-transposition that swaps I_i with J_i.

If f is an I-reflection of type a with $a = kw + u$ as in the previous case. Then we consider $\{I_1, I_2, \ldots, I_{k+1}\}$ and $\{J_1, J_2, \ldots, J_{k+1}\}$ two partitions into consecutive intervals of I such that the length of I_i and J_j is w for every $1 \leqslant i \leqslant k$ and $2 \leqslant j \leqslant k + 1$ and the length of I_{k+1} and J_1 is u. Then f is equal to the product $r_1 r_2 \ldots r_{k+1} s_1 s_2 \ldots s_{k+1}$ where r_i is the I_i-reflection and s_i is the $\text{IET}(\Gamma)$-transposition that swaps I_i with J_{k+1-i}.

If f is a Γ-restricted rotation of type (a, b). We can assume $a \geqslant b$ as the other case is similar. Then f is the product of an $\text{IET}(\Gamma)$-transposition of type b with a restricted rotation of type $(a - b, b)$ (see Figure 2.3). Then by iterating this operation and thanks to the first case we deduce the result.

By Lemma 2.7 and Proposition 2.9 we obtain the result for any f. $\square$
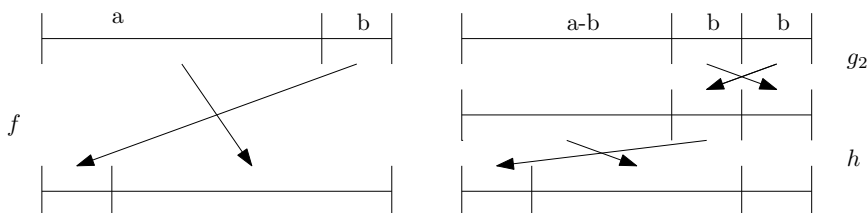
Figure 2.3. Illustration of the decomposition of a restricted rotation in Lemma 2.16.

PROPOSITION   2.17. — *The   derived   subgroup $D(\mathrm{IET}(\Gamma))$ (resp. $D(\mathrm{IET}^{\bowtie}(\Gamma)))$ is contained in every nontrivial normal subgroup of $\mathrm{IET}(\Gamma)$ (resp. $\mathrm{IET}^{\bowtie}(\Gamma))$.*

Proof. — We only do the case of $\mathrm{IET}(\Gamma)$. In $\mathrm{IET}^{\bowtie}(\Gamma)$ one should be careful about having equalities up to finitely supported permutations, but the proof is similar. For every $\varepsilon > 0$, we recall that $\mathcal{F}_\varepsilon$ is a generating subset of $\mathrm{IET}(\Gamma)$. Then the set of all commutators of two elements in $\mathcal{F}_\varepsilon$ is a generating subset of $D(\mathrm{IET}(\Gamma))$.

Let N be a normal subgroup of $\mathrm{IET}(\Gamma)$ and $f \in N \smallsetminus \{\mathrm{Id}\}$. Then there exists a subinterval I of $[0,1[$ such that f is continuous on I and $f(I) \cap I = \varnothing$. Let δ be the length of I and let $\varepsilon = \frac{\delta}{3}$. Let $g_1, g_2 \in \mathcal{F}_\varepsilon$ and let $A := \mathrm{Supp}(g_1) \cup \mathrm{Supp}(g_2)$. There exists $h \in \mathrm{IET}(\Gamma)$ such that $h(A) \subset I$ because the measure of A is at most 2ε which is less than δ. Then the element $f' := h^{-1} \circ f \circ h$ is an element of N and $f'(A) \cap A = \varnothing$. This implies that the support of g_1 and the one of $f'g_2f'^{-1}$ do not overlap; thus these elements commute and we obtain that $[g_1, g_2] = [g_1, [g_2, f']]$. Also N is normal in $\mathrm{IET}(\Gamma)$, hence as $f' \in N$ we deduce that $[g_2, f'] \in N$, thus $[g_1, g_2] \in N$. We deduce that every commutator of elements in $\mathcal{F}_\varepsilon$ is in N so $D(\mathrm{IET}(\Gamma))$ is a subgroup of N. □

DEFINITION 2.18. — *Let X be a Stone space and G be a subgroup of $\mathrm{Homeo}(X)$. The group G is a topological full group if for every $n \in \mathbb{N}$ and for all $(Y_1, Y_2, \ldots, Y_n)$ and $(Z_1, Z_2, \ldots, Z_n)$ finite partitions into clopen subsets of X and every $g_i \in G$ such that $g_i(Y_i) = Z_i$, the element g that satisfies $g = g_i$ on X_i is an element of G.*

One can notice that $\mathrm{IET}(\Gamma)$ and $\mathrm{IET}^{\bowtie}(\Gamma)$ act minimally on the space Y obtained from $\mathbb{R}/\mathbb{Z}$ by replacing every point $x \in \Gamma$ by two points x_- and x_+. The space Y is a Stone space without isolated points. The fact that $\mathrm{IET}(\Gamma)$ and $\mathrm{IET}^{\bowtie}(\Gamma)$ are both topological full group is immediate. We can use the following theorem due to V. Nekrashevych in [8]:

THEOREM 2.19 (Nekrashevych). — *Let X be a infinite Stone space and G be a subgroup of $\mathrm{Homeo}(X)$ such that G is a topological full group and acts minimally on X. Then the subgroup $\mathcal{A}(G)$ of G generated by the set of all elements of order 3, is simple and contained in every nontrivial normal subgroup of G.*

Thanks to this theorem and Proposition 2.17 we deduce that $\mathcal{A}(\mathrm{IET}(\Gamma)) = D(\mathrm{IET}(\Gamma))$ and $\mathcal{A}(\mathrm{IET}^{\bowtie}(\Gamma)) = D(\mathrm{IET}^{\bowtie}(\Gamma))$ and both of them are simple.

3. Behaviour of the composition of two IET-transpositions

Understanding compositions of $\mathrm{IET}(\Gamma)$-transpositions is important to describe the abelianization of $\mathrm{IET}(\Gamma)$. The next lemma relates finite order elements with IET-transpositions (see Definition 2.5). In [12], Y. Vorobets proves it in the case $\Gamma = \mathbb{R}/\mathbb{Z}$ and result from the theorem of decomposition into minimals and periodics components of Arnoux [2]. The general case follows from this case:

LEMMA 3.1. — *Let f be a finite order element of $\mathrm{IET}(\Gamma)$. Then there exist $n \in \mathbb{N}$ and $\mathcal{P} := \{I_1, \ldots, I_n\} \in \Pi_f$ and $\sigma \in \mathfrak{S}_n$ such that $f(I_i) = I_{\sigma(i)}$, moreover f is a product of $\mathrm{IET}(\Gamma)$-transpositions.*

The aim is to show that the product of two IET-transpositions has finite order. It is not a trivial question because we can build such examples of product of order n for every $n \in \mathbb{N}$ (see Proposition 3.8). The result will be used in Section 5. For this we use the notion of reversible maps and some of their properties proved by N. Guelman and I. Liousse in [7].

DEFINITION 3.2. — *Let h, k be two elements of IET. We said that h reverses k if $hkh^{-1} = k^{-1}$.*

A direct consequence of the definition is:

PROPOSITION 3.3. — *Let $h, k \in \mathrm{IET}$ such that h reverses k. Then for every $p \in \mathbb{Z}$ both h and h^{-1} reverse k^p. Thus $k^p h = hk^{-p}$.*

For every $f \in \mathrm{IET}$ we denote by $\mathrm{Per}(f)$ the set of periodic points of f and by $\mathrm{Fix}(f)$ the set of fixed points of f.

LEMMA 3.4. — *Let $h, k \in \mathrm{IET}$ such that h reverses k. Then:*
  (1) *The set $\mathrm{Per}(k)$ is invariant by h;*
  (2) *The set $\mathrm{Fix}(h)$ is a subset of $\mathrm{Per}(k)$.*

Proof.

(1) Let $x \in \mathrm{Per}(k)$ and let $p \in \mathbb{Z}$ such that $k^p(x) = x$. By Proposition 3.3 we have $k^p(h(x)) = hk^p(x) = h(x)$, thus $h(x) \in \mathrm{Per}(k)$.

(2) By contradiction let $x \in \mathrm{Fix}(h) \smallsetminus \mathrm{Per}(k)$. As h is in IET we deduce that there exists $c \in [0,1[$ such that the whole interval $[x,c[\subset \mathrm{Fix}(h)$. Also x is not a periodic point of k thus the point x is a right cluster point of the k-orbit of x. This means that there exists a sequence $(p_n)_n$ of positive integers such that $k^{p_n}(x) > x$ and $\lim_{n \to \infty} k^{p_n}(x) = x$. Hence there exists an integer $p > 0$ such that $k^p(x) \in [x,c[$ thus it is a fixed point of h. Then $k^p(x) = h(k^p(x)) = k^{-p}(h(x)) = k^{-p}(x)$ and we deduce that $x \in \mathrm{Per}(k)$ which is a contradiction. $\qquad\square$

COROLLARY 3.5. — *Let f and g be two* IET-*transpositions. Then f and g reverse gf and for every $x \notin \mathrm{Per}(gf)$ the $\langle f, g \rangle$-orbit of x is contained in $[0,1[\smallsetminus \mathrm{Per}(gf) \subset \mathrm{Supp}(f) \cap \mathrm{Supp}(g)$.*

LEMMA 3.6. — *Let f and g be two* IET-*transpositions. Then gf has finite order.*

Proof. — Let I, J, A, B be the intervals such that $\sup(I) \leqslant \inf(J)$ and f swaps I with J while fixing the rest of $[0,1[$ and $\sup(A) \leqslant \inf(B)$ and g swaps A with B while fixing the rest of $[0,1[$. Let $\alpha, \beta \in [0,1[$ such that $J = I + \alpha$ and $B = A + \beta$. A crucial property is that either $J \cap A$ or $I \cap B$ is the empty set.

By contradiction let x be a point that is not in $\mathrm{Per}(gf)$. By Corollary 3.5 we know that the $\langle f, g \rangle$-orbit of x is included in:

$$\mathrm{Supp}(f) \cap \mathrm{Supp}(g) = (I \cap A) \sqcup (I \cap B) \sqcup (J \cap A) \sqcup (J \cap B)$$

We distinguish three cases:

(1) If the $\langle f, g \rangle$-orbit of x does not intersect $(I \cap A) \sqcup (J \cap B)$ then it is included in $(I \cap B) \cap (J \cap A)$ and as one of them is empty it is included either in I or J which is impossible.

(2) Now assume that the $\langle f, g \rangle$-orbit of x intersects $I \cap A$. Up to change x by a point of its gf-orbit we can assume that $x \in I \cap A$. Then we deduce that $f(x) \in (J \cap A) \sqcup (J \cap B)$.

   (a) If $J \cap A = \varnothing$ then $f(x) \in J \cap B$ thus $gf(x) \in I \cap A$ hence for every $n \in \mathbb{N}$ we have $(gf)^n(x) \in I \cap A$. Then $(gf)^n(x) = x + n(\alpha - \beta)$ hence x has to be inside $\mathrm{Per}(gf)$ which is a contradiction.

(b) If $J \cap A \neq \varnothing$ then $I \cap B = \varnothing$ thus $g(x) \in J \cap B$ and we deduce that $fg(x) \in I \cap A$. Then for ever $n \in \mathbb{N}$ we obtain that $(fg)^n(x) = x + n(\beta - \alpha)$ hence $x \in \mathrm{Per}(fg)$. Also $\mathrm{Per}(fg) = \mathrm{Per}(g^{-1}f^{-1}) = \mathrm{Per}(gf)$ thus $x \in \mathrm{Per}(gf)$ which is a contradiction.

(3) Let now assume that the $\langle f, g \rangle$-orbit of x intersects $J \cap B$ and does not intersect $I \cap A$. Up to change x by a point of its gf-orbit we can assume that $x \in J \cap B$. Then $f(x) \in I \cap B$, in particular $I \cap B \neq \varnothing$ thus $J \cap A = \varnothing$ and we deduce that $g(x) \in I \cap A$ which is a contradiction. $\qquad \square$

Remark 3.7. — In the case where $\alpha = \beta$ every point of $[0, 1[$ has a gf-orbit of at most cardinal three. Thus gf is at most of order 6.

PROPOSITION 3.8. — *For every n in $\mathbb{N}_{\geqslant 1}$ there exist two IET-transpositions f and g such that the product gf has order n.*

Proof. — We distinguish the case where n is even or odd. In both cases we illustrate the proof with Figure 3.1. The case $n = 1$ is given by the equality $f^2 = \mathrm{Id}$ for any IET-transposition f. Let $n \in \mathbb{N}_{\geqslant 1}$.

Let I and J be two consecutive intervals of the same length $\ell \in [0, \frac{1}{2}]$ and let g be the IET-transposition that swaps I and J. Let $A_1, A_2, \ldots, A_{n-1}$ and C be consecutive intervals of length $\frac{\ell}{n}$ such that the left endpoint of A_1 is the left endpoint of I (hence the right endpoint of C is the right endpoint of I). Let D and $B_1, B_2, \ldots, B_{n-1}$ be consecutive intervals of length $\frac{\ell}{n}$ such that the right endpoint of B_{n-1} is the right endpoint of J (hence the left endpoint of D is the left endpoint of J). Let f be the IET-transposition that swaps A_i and B_i for every $1 \leqslant i \leqslant n - 1$. Hence by definition we get $g(A_1) = D$, $g(A_i) = B_{i-1}$ for every $2 \leqslant i \leqslant n - 1$ and $g(C) = B_{n-1}$. So the composition gf is equal to the permutation $(A_1 \ A_2 \ \ldots \ A_{n-1} \ C \ B_{n-1} \ B_{n-2} \ \ldots \ B_1 \ D)$. Thus gf has order $2n$.

It remains the case of order $2n - 1$. Let I, J and K be three consecutive intervals with I and J of length $\ell \in [0, \frac{1}{3}[$ and K of length $\ell' \in]\frac{\ell}{n}, \ell[$. Let g be the IET-transposition that swaps I and J. We define $A_1, A_2, \ldots, A_{n-1}$ consecutive intervals of length $\frac{\ell}{n}$ such that the right endpoint of A_{n-1} is the right endpoint of I. We define also D and $B_1, B_2, \ldots, B_{n-1}$ consecutive intervals of length $\frac{\ell}{n}$ such that the left endpoint of B_{n-1} is the left endpoint of K. Let f be the IET-transposition that swaps A_i and B_i for every $1 \leqslant i \leqslant n - 1$. One can check that the product gf in this case is the permutation $(A_1 \ A_2 \ \ldots \ A_{n-1} \ B_{n-1} \ B_{n-2} \ \ldots \ B_1 \ D)$ so gf has order $2n - 1$. $\qquad \square$
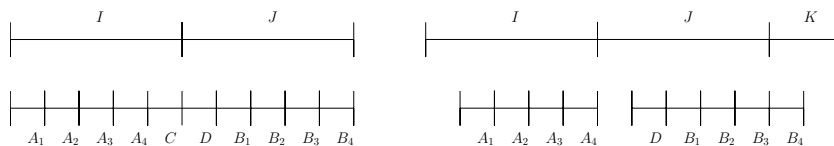
Figure 3.1. Illustration of Proposition 3.8 with $n = 5$. Left: *For the case "fg has order $2n$". Right: For the case "fg has order $2n - 1$".*

4. A generating subset for $\mathrm{Ker}(\varphi_\Gamma)$

Let Γ be a dense subgroup of $\mathbb{R}/\mathbb{Z}$. We follow the idea of Y. Vorobets in [12] and introduce the notion of balanced product of restricted rotations. The aim is to show Lemma 1.4. We recall that φ_Γ is the restriction of the SAF-homomorphism to IET(Γ); and that $\widetilde{\Gamma}_+$ is the positive cone of $\widetilde{\Gamma}$.

4.1. Balanced product of Γ-restricted rotations

DEFINITION 4.1. — *Let $n \in \mathbb{N}$ and let r_i be a restricted rotation for every $1 \leqslant i \leqslant n$. For every $a, b \in \widetilde{\Gamma}_+$ let $n_{a,b}$ be the number of restricted rotation r_i of type (a, b). The tuple $(r_1, r_2, \ldots, r_n)$ is said to be a balanced tuple of restricted rotations if $n_{a,b} = n_{b,a}$ for every $a, b \in \widetilde{\Gamma}_+$. We say that a product of restricted rotations is a balanced product of restricted rotations if it can be written as a product of a balanced tuple of restricted rotations.*

Example 4.2. — Let a be an element in $\widetilde{\Gamma}_+$ with $a \leqslant \frac{1}{2}$. Any Γ-restricted rotation of type (a, a) is a balanced product of restricted rotations. These elements are also IET(Γ)-transposition which swapped two consecutive intervals of same length a.

Example 4.3. — Let $a, b \in \widetilde{\Gamma}_+$ with $a + b \leqslant 1$. If h is a Γ-restricted rotation of type (a, b) then h^{-1} is a Γ-restricted rotation of type (b, a). Thus every element of $D(\mathrm{IET}(\Gamma))$ is a balanced product of Γ-restricted rotations.

In order to get the decomposition of the elements of $\mathrm{Ker}(\varphi_\Gamma)$, we need to know the freeness of some family of $^\ominus\!\bigwedge^2_\mathbb{Z} V$.

LEMMA 4.4. — *Suppose V is a subgroup of $\mathbb{R}$ (then it is a $\mathbb{Z}$-module). Let k in $\mathbb{N}_{\geqslant 1}$ and $v_1, v_2, \ldots v_k$ elements of V which are $\mathbb{Z}$-linearly independent. Then the wedge products $v_i \wedge v_j$ for $1 \leqslant i < j \leqslant k$ are $\mathbb{Z}$-linearly independent in $^\ominus\!\bigwedge^2_\mathbb{Z} V$.*

Proof. — Let $v_1, v_2, \ldots, v_k$ in V which are $\mathbb{Z}$-linearly independent. It is sufficient to prove the lemma for $V = \mathbb{R}$ because being $\mathbb{Z}$-linearly independent in ${}^{\ominus}\bigwedge_{\mathbb{Z}}^2 \mathbb{R}$ implies being $\mathbb{Z}$-linearly independent in ${}^{\ominus}\bigwedge_{\mathbb{Z}}^2 V$. We know that ${}^{\ominus}\bigwedge_{\mathbb{Z}}^2 \mathbb{R}$ is isomorphic to $\bigwedge_{\mathbb{Z}}^2 \mathbb{R}$. Let us assume that $v_1, v_2, \ldots, v_k$ are in $\mathbb{R}$. Then being $\mathbb{Z}$-linearly independent is the same that being $\mathbb{Q}$-linearly independent. Indeed if there exist $p_1, p_2, \ldots, p_k$ in $\mathbb{Z}$ and $q_1, q_2, \ldots, q_k$ in $\mathbb{N}_{\geqslant 1}$ such that $\sum_{i=1}^{k} \frac{p_i}{q_i} v_i = 0$ then $\sum_{i=1}^{k} (\prod_{\substack{j=1 \\ j \neq i}} q_j p_i) v_i = 0$ is an equality in $\mathbb{Z}$. Thus for each $i \in \{1, 2, \ldots, k\}$ we have $\prod_{\substack{j=1 \\ j \neq i}} q_j p_i = 0$. Or $q_j \neq 0$ for every j then $p_i = 0$ for every i.

The $\mathbb{Q}$-vector space generated by all the v_i for $1 \leqslant i \leqslant k$ is isomorphic to $\mathbb{Q}^k$. We can complete the $\mathbb{Q}$-linearly independent set $\{v_1, v_2, \ldots, v_k\}$ in a basis S of $\mathbb{R}$ seen as a $\mathbb{Q}$-vector space. Thus in $\bigwedge_{\mathbb{Z}}^2 \mathbb{R}$ the elements $v_i \wedge v_j$ for $1 \leqslant i < j \leqslant k$ are $\mathbb{Q}$-linearly independent so they are $\mathbb{Z}$-linearly independent in ${}^{\ominus}\bigwedge_{\mathbb{Z}}^2 \mathbb{R}$ and this gives the conclusion. $\square$

LEMMA 4.5. — *Any transformation f in $\operatorname{Ker}(\varphi_\Gamma)$ can be represented as a balanced product of Γ-restricted rotations.*

Proof. — Let $f \in \operatorname{Ker}(\varphi_\Gamma)$. This is trivial if $f = \mathrm{id}$; assume otherwise. Let (μ, σ) be a Γ-combinatorial description of f, let $k \in \mathbb{N}$ and $\{I_1, I_2, \ldots, I_k\}$ be the partition into intervals associated to (μ, σ) (we have $k \geqslant 2$ as f is not the identity). We recall that μ_i is the length of I_i for every $1 \leqslant i \leqslant k$.

We treat the case where Γ is finitely generated. Then $\widetilde{\Gamma}$ is also finitely generated and we denote by d its rank. By Corollary 2.14 there exist $\ell_1, \ell_2, \ldots, \ell_d$ in $\widetilde{\Gamma}_+$ such that $\mathcal{L} := \{\ell_1, \ell_2, \ldots, \ell_d\}$ is a basis of $\widetilde{\Gamma}$ and such that μ_i is a linear combination of $\ell_1, \ell_2, \ldots, \ell_d$ with non-negative integer coefficients for every $1 \leqslant i \leqslant k$. Then I_i can be partitioned into smaller intervals with length in $\mathcal{L}$ for every $1 \leqslant i \leqslant k$. We obtain a partition associated with f whose intervals have length in $\mathcal{L}$. By Lemma 2.7 there exist $n \in \mathbb{N}$ and a restricted rotation f_i of type (a_i, b_i) with $a_i, b_i \in \mathcal{L}$ for $1 \leqslant i \leqslant n$ such that $f = f_1 f_2 \ldots f_n$. For any $i, j \in \{1, 2, \ldots, d\}$ let s_{ij} be the number of Γ-restricted rotation of type (ℓ_i, ℓ_j) in the tuple $(f_1, f_2, \ldots, f_n)$. As $\varphi_\Gamma(f_i) = \ell_j \wedge \ell_i - \ell_i \wedge \ell_j = 2\ell_j \wedge \ell_i$, we obtain that:

$$\varphi_\Gamma(f) = \sum_{i=1}^{d} \sum_{j=1}^{d} 2s_{ij}(\ell_j \wedge \ell_i) = \sum_{1 \leqslant i < j \leqslant d} 2(s_{ij} - s_{ji})(\ell_j \wedge \ell_i)$$

We know that $\{\ell_1, \ell_2, \ldots, \ell_d\}$ is a basis of $\widetilde{\Gamma}$ thus by Lemma 4.4 we obtain that $\{\ell_j \wedge \ell_i\}_{1 \leqslant i < j \leqslant d}$ is a free family of ${}^{\ominus}\bigwedge_{\mathbb{Z}}^2 \widetilde{\Gamma}$. Then the assumption $\varphi_\Gamma(f) = 0$ implies $s_{ij} = s_{ji}$ for every $1 \leqslant i < j \leqslant d$. This means that the product of Γ-restricted rotations $f_1 f_2 \ldots f_n$ is balanced.

We do not assume Γ finitely generated any more. Hence we only know that $\varphi_\Gamma(f) = \sum_{j=1}^k \left(\sum_{\substack{i:\\ \sigma(i)<\sigma(j)}} \mu_i - \sum_{i<j} \mu_i \right) \wedge \mu_j = 0$ in $\ominus\bigwedge_{\mathbb{Z}}^2 \widetilde{\Gamma}$ (see Proposition 2.4). We denote by $\overline{\varphi_\Gamma(f)} := \sum_{j=1}^k \left(\sum_{\substack{i:\\ \sigma(i)<\sigma(j)}} \mu_i - \sum_{i<j} \mu_i \right) \otimes \mu_j$. It is a representative of $\varphi_\Gamma(f)$ in $\bigotimes_{\mathbb{Z}}^2 \widetilde{\Gamma}$. Then there exist a finite set J and $x_j, y_j \in \widetilde{\Gamma}$ for every $j \in J$, such that $\overline{\varphi_\Gamma(f)} = \sum_{j\in J} x_j \otimes y_j + y_j \otimes x_j$. We denote by $\widetilde{A}$ the subgroup of $\widetilde{\Gamma}$ generated by $\{\mu_i\}_{1\leqslant i\leqslant k} \cup \{x_j, y_j\}_{j\in J}$. Then $\widetilde{A}$ is a finitely generated subgroup of $\mathbb{R}$ which contains $\mathbb{Z}$. Its image A in $\mathbb{R}/\mathbb{Z}$ is a finitely generated subgroup of $\mathbb{R}/\mathbb{Z}$. Also we know that f is in IET(A) and (μ, σ) is also a A-combinatorial description of f and $\overline{\varphi_\Gamma(f)}$ is an element of $\bigotimes_{\mathbb{Z}}^2 \widetilde{A}$. Thus in $\ominus\bigwedge_{\mathbb{Z}}^2 \widetilde{A}$ we have:

$$
\begin{aligned}
\varphi_A(f) &= \sum_{j=1}^k \left(\sum_{\substack{i:\\ \sigma(i)<\sigma(j)}} \mu_i - \sum_{i<j} \mu_i \right) \wedge \mu_j \\
&= [\overline{\varphi_\Gamma(f)}]_{\ominus\bigwedge_{\mathbb{Z}}^2 \widetilde{A}} \\
&= \left[\sum_{j\in J} x_j \otimes y_j + y_j \otimes x_j \right]_{\ominus\bigwedge_{\mathbb{Z}}^2 \widetilde{A}} \\
&= 0
\end{aligned}
$$

Then we can applied the previous case and conclude that f is a balanced product of A-restricted rotations, thus a balanced product of Γ-restricted rotations. $\qquad\square$

4.2. Ker(φ_Γ) is generated by IET(Γ)-transpositions

The work of Y.Vorobets [12] done for IET can be adapted to show the next two lemmas:

LEMMA 4.6. — *Let f and g be two Γ-restricted rotations. If they have the same type then $f^{-1}g$ is a product of IET(Γ)-transpositions.*

LEMMA 4.7. — *Let f be a Γ-restricted rotation and g be any transformation in IET(Γ). Then the commutator $[f, g]$ is a product of IET(Γ)-transpositions.*

These lemmas with Lemma 2.16 give us the next theorem. It is proved by Y.Vorobets in the case $\Gamma = \mathbb{R}/\mathbb{Z}$ in [12], and as the proof is the same we only provide here a sketch.

THEOREM 4.8. — *Every balanced product of Γ-restricted rotations can be written as a product of IET(Γ)-transpositions.*

Proof of Theorem 4.8 (sketched). — Let $(f_1, f_2, \ldots, f_n)$ be a balanced tuple of restricted rotations. The proof is by strong induction on the length n of the tuple. If $n = 1$ then (f_1) is a balanced tuple of Γ-restricted rotations, thus f_1 is a Γ-restricted rotation of type (a, a) with $a \in \widetilde{\Gamma}$ so it is also an IET(Γ)-transposition.

For the general case, let (a, b) be the type of f_1. If $a = b$ then f_1 is an IET(Γ)-transposition and $(f_2, f_3, \ldots, f_n)$ is a balanced tuple of restricted rotations. By the induction assumption we obtain the result. If $a \neq b$ then there exists $k \in \{2, \ldots, n\}$ such that f_k is a Γ-restricted rotation of type (b, a). Let $g_1 = f_2 \ldots f_{k-1}$ or $g_1 = \mathrm{Id}$ if $k = 2$. Let $g_2 = f_{k+1} \ldots f_n$ or $g_2 = \mathrm{Id}$ if $k = n$. Then we can write

$$f_1 f_2 \ldots f_n = f_1 g_1 f_k g_2 = (f_1 f_k)(f_k^{-1} g_1 f_k g_1^{-1})(g_1 g_2)$$

Hence, the induction assumption and Lemmas 4.6 and 4.7 give the result.

$\square$

COROLLARY 4.9. — *The kernel $\mathrm{Ker}(\varphi_\Gamma)$ is generated by the set of all IET(Γ)-transpositions.*

5. Description of the abelianization of IET(Γ)

In this section we construct a group homomorphism $\varepsilon_\Gamma : \mathrm{IET}(\Gamma) \to {}^{\ominus}\bigwedge_{\mathbb{Z}}^2 \widetilde{\Gamma}$ whose kernel is the derived subgroup $D(\mathrm{IET}(\Gamma))$.

5.1. Boolean measures

In finite permutation groups there is a natural signature. One way to describe the signature is as follows: the signature of a finite permutation f, viewed in $\mathbb{Z}/2\mathbb{Z}$ is the number modulo 2 of pairs (x, y) such that $x < y$ and $f(x) > f(y)$. In our context where $f \in \mathrm{IET}(\Gamma)$, while this set is infinite, the idea is to measure it in a meaningful sense.

DEFINITION 5.1. — *Let A be a Boolean algebra and G be an abelian group. Let $\mu : A \to G$ be a finitely additive map: $\forall\, U, V \in A$ disjoint, $\mu(U \sqcup V) = \mu(U) + \mu(V)$. Such a μ is called a* Boolean algebra measure *for A in G.*

Notation 5.2. — We recall that $\mathrm{Itv}(\Gamma)$ is the set of all intervals $[a, b[$ with a and b in $\widetilde{\Gamma}$ and $0 \leqslant a < b \leqslant 1$. Let A_Γ be the Boolean algebra of subsets of $[0, 1[$ generated by $\mathrm{Itv}(\Gamma)$. Then A_Γ is a Boolean subalgebra of $\{0, 1\}^{[0,1[}$. By noting λ the Lebesgue measure on $[0, 1[$ we get that λ is a Boolean measure for A_Γ in $\widetilde{\Gamma}$.

It might be useful to notice that for k in $\mathbb{N}_{\geqslant 1}$ and every $I_1, I_2, \ldots, I_k$ intervals in $\mathrm{Itv}(\Gamma)$, the intersection $\bigcap_i I_i$ is still an element of $\mathrm{Itv}(\Gamma)$. Moreover for every I in $\mathrm{Itv}(\Gamma)$, the complement of I is the disjoint union of two elements of $\mathrm{Itv}(\Gamma)$. Thus any Boolean combination of elements of $\mathrm{Itv}(\Gamma)$ is a finite disjoint union of such elements.

PROPOSITION 5.3. — *Let X and Y be two sets, let A be a Boolean subalgebra of $\{0, 1\}^X$ and let B be a Boolean subalgebra of $\{0, 1\}^Y$. Let G and H be two abelian groups, let $\mu : A \to G$ be a Boolean algebra measure for A in G and $\nu : B \to H$ be a Boolean algebra measure for B in H. Let $C := A \otimes B$ be the Boolean algebra product (generated by subsets of the form $a \times b$ with a in A and b in B). Then there exists a unique Boolean algebra measure $\omega : C \to G \otimes H$ for C in $G \otimes H$ such that for every a in A and b in B we have $\omega(a \times b) = \mu(a) \otimes \nu(b)$.*

Proof. — Let ω_1 and ω_2 be two such Boolean algebra measures, thus they are equal on every $a \times b$ for $a \in A$ and $b \in B$. Let c be an element of C, then there exist k in $\mathbb{N}$ and $a_1, \ldots, a_k$ in A and $b_1, \ldots, b_k$ in B such that $c = \bigsqcup_{i=1}^{k} a_i \times b_i$. So $\omega_1(c) = \sum_{i=1}^{k} \omega_1(a_i \times b_i) = \sum_{i=1}^{k} \omega_2(a_i \times b_i) = \omega_2(c)$. Thus $\omega_1 = \omega_2$ and the unicity is proved.

It is sufficient to prove the existence for every finite Boolean subalgebra of C. Indeed if we assume that for every D finite Boolean subalgebra of C there exists a Boolean algebra measure m_D for D in $G \otimes H$ such that $m_D(a \times b) = \mu(a) \otimes \nu(b)$ for every a in A and b in B with $a \times b$ in D. Let c be an element of C. Then $\{0_C, c, \neg c, 1_C\}$ is a finite Boolean subalgebra of C non-trivial. Moreover if c is in $D_1 \cap D_2$ where D_1 and D_2 are two finite Boolean subalgebras of C then by noting D the Boolean subalgebra generated by D_1 and D_2 we get that D is a finite Boolean subalgebra of C containing c. Thus $m_D|_{D_1}$ is a Boolean measure for D_1 in $G \otimes H$ which satisfies $m_D|_{D_1}(a \times b) = \mu(a) \otimes \nu(b)$ for every a in A and b in B with $a \times b$ in D_1. By unicity we get $m_D|_{D_1} = m_{D_1}$ and the same argument gives $m_D|_{D_2} = m_{D_2}$ thus $m_{D_1}(c) = m_D(c) = m_{D_2}(c)$. So by putting $\omega(c) = m_D(c)$, the map ω is well-defined. also if we take two disjoint elements c and c' in C. Then by taking any finite Boolean subalgebra D of C which contains c and c' we get $m_D(c + c') = m_D(c) + m_D(c') = \omega(c) + \omega(c')$ and

the value does not depend on D. Thus ω is the wanted Boolean algebra measure.

Let now D be a finite Boolean subalgebra of C. Then there exist $k, \ell \in \mathbb{N}$ and $a_1, \ldots, a_k \in A$ and $b_1, \ldots, b_\ell \in B$ such that every $d \in D$ is a Boolean combination of $a_i \times b_j$. Then let D' be the finite Boolean algebra generated by all the $a_i \times b_j$ with $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant \ell$. Let U be the finite Boolean subalgebra of A generated by all a_i and let V be the finite Boolean subalgebra of B generated by all b_j. Then U and V are atomic. Let $u_1, \ldots, u_n$ be the atoms of U and $v_1, \ldots, v_m$ be the atoms of V. Hence D' is atomic with atoms given by $u_i \times v_j$ for every $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant \ell$. Then for each element d in D' there exists a unique $J_d \subset \{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\}$ such that $d = \bigsqcup_{(i,j) \in J_d} u_i \otimes v_j$. Hence the map $m_{D'}$ defined by $m_{D'}(d) = m_{D'}(\bigsqcup_{(i,j) \in J_d} u_i \times v_j) = \sum_{(i,j) \in J_d} \mu(u_i) \otimes \nu(v_j)$ is well-defined, finitely additive and satisfies $m_{D'}(a \times b) = \mu(a) \otimes \nu(b)$ for every $a \in A$, $b \in B$ such that $a \times b \in D'$. □

Notation 5.4. — By applying the previous proposition with $X = Y = [0, 1[$ and $A = B = A_\Gamma$ and $\mu = \nu = \lambda$, there exists a unique Boolean algebra measure $\omega_\Gamma : A_\Gamma \otimes A_\Gamma \to \bigotimes_\mathbb{Z}^2 \widetilde{\Gamma}$ such that for every a, b, c and d in $\widetilde{\Gamma}_+$ with $a < b \leqslant 1$ and $c < d \leqslant 1$ we have $\omega_\Gamma([a, b[\times [c, d[) = (b-a) \otimes (d-c)$.

We need to check some IET(Γ)-invariance for the measure ω_Γ. For this we define an action of IET(Γ) on $[0, 1[\times [0, 1[$ by $f.(x, y) = (f(x), f(y))$. Hence for every P in A_Γ we have $f.P$ in A_Γ, this gives us a new Boolean algebra measure $f_* \omega_\Gamma$.

PROPOSITION 5.5. — *For every f in* IET(Γ) *and every P in $A_\Gamma \otimes A_\Gamma$ we have $f.P := \{(f(x), f(y)) \mid (x, y) \in P\}$ in $A_\Gamma \otimes A_\Gamma$. Furthermore we have $f_* \omega_\Gamma = \omega_\Gamma$.*

Proof. — Let $f \in$ IET(Γ) and (μ, σ) be a Γ-combinatorial description of f and let $\{I_1, \ldots, I_n\}$ be the partition into intervals associated. Let P be an element of $A_\Gamma \otimes A_\Gamma$. There exist m in $\mathbb{N}$ and pairwise disjoint elements $p_1, p_2, \ldots, p_m$ of Itv(Γ) $\times$ Itv(Γ) such that $P = \bigsqcup_{i=1}^k p_i$. As f is a permutation of $[0, 1[$ we get $f. \bigsqcup_{i=1}^m p_i = \bigsqcup_{i=1}^m f.p_i$, so it is enough to show that $f.p$ belongs to $A_\Gamma \otimes A_\Gamma$. For $i \in \{1, \ldots, k\}$, let $a_i, b_i, c_i, d_i \in \widetilde{\Gamma}$ such that $p_i = [a_i, b_i[\times [c_i, d_i[$. Then $f.p_i = \bigsqcup_{(k,l)} f([a_i, b_i[\cap I_k) \times f([c_i, d_i[\cap I_l)$ which is a finite disjoint union of elements of Itv(Γ) $\times$ Itv(Γ) because a_i, b_i, c_i, d_i and extremities of I_l are in $\widetilde{\Gamma}$ for each $1 \leqslant l \leqslant n$. In conclusion $f.p_i$ is in $A_\Gamma \otimes A_\Gamma$ thus $f.P$ is in $A_\Gamma \otimes A_\Gamma$.

Also f is piecewise a translation and λ is the Lebesgue measure, so for any J in $\mathrm{Itv}(\Gamma)$ we have $\lambda(J) = \lambda(f(J))$. Thus:

$$
\begin{aligned}
\omega_\Gamma(f.p_i) &= \sum_{(k,l)} \omega_\Gamma(f([a_i, b_i[\cap I_k) \times f([c_i, d_i[\cap I_l)) \\
&= \sum_{(k,l)} \lambda(f([a_i, b_i[\cap I_k)) \otimes \lambda(f([c_i, d_i[\cap I_l)) \\
&= \left(\sum_k \lambda([a_i, b_i[\cap I_k)\right) \otimes \left(\sum_l \lambda([c_i, d_i[\cap I_l)\right) \\
&= \lambda([a_i, b_i[) \otimes \lambda([c_i, d_i[) \\
&= \omega_\Gamma(p_i)
\end{aligned}
$$

This gives us $\omega_\Gamma(f.P) = \sum_{i=1}^m \omega_\Gamma(f.p_i) = \sum_{i=1}^m \omega_\Gamma(p_i) = \omega_\Gamma(P)$. Hence $\omega_\Gamma = f_*^{-1}\omega_\Gamma$. As f^{-1} is also in $\mathrm{IET}(\Gamma)$ we can do the same to deduce $\omega_\Gamma = f_*\omega_\Gamma$. $\qquad\square$

5.2. Creation of a signature

DEFINITION 5.6. — *Let f be a transformation in $\mathrm{IET}(\Gamma)$. Every pair (x, y) in $[0, 1[\times [0, 1[$ such that $x < y$ and $f(x) > f(y)$ is called an* inversion *of f. We denote by $\mathcal{E}_f$ the set of all inversions of f.*

PROPOSITION 5.7. — *Let f be a transformation in $\mathrm{IET}(\Gamma)$ and (μ, τ) be a combinatorial description of f. We have $\mathcal{E}_f = \bigsqcup_{j=1}^n \bigsqcup_{\substack{i<j \\ \tau(i)>\tau(j)}} I_i \times I_j$.*

Proof. — Let (μ, τ) be a Γ-combinatorial description of f and let $\{I_1, I_2, \ldots, I_k\}$ be the partition into intervals associated. Let (x_0, y_0) be an element of $\mathcal{E}_f$. Then there exist $i, j \in \{1, 2, \ldots, k\}$ such that $x_0 \in I_i$ and $y_0 \in I_j$. As $x_0 < y_0$ we have $i \leqslant j$. Furthermore if $i = j$ then as f is an isometry which preserves the order on I_i we get $f(x_0) < f(y_0)$ which is a contradiction, we deduce that $i < j$. By definition of f we have $f(I_i)$, $f(I_j) \in \mathrm{Itv}(\Gamma)$ and they are disjoint. Thus as $f(x_0) > f(y_0)$ we deduce that for every $x \in I_i$ and $y \in I_j$ we have $x < y$ and $f(x) > f(y)$, so $I_i \times I_j \subset \mathcal{E}_f$. Also, this implies $\sigma(i) > \sigma(j)$ and we deduce that $\bigsqcup_{j=1}^n \bigsqcup_{\substack{i<j \\ \tau(i)>\tau(j)}} I_i \times I_j = \mathcal{E}_f$. $\qquad\square$

COROLLARY 5.8. — *For every $f \in \mathrm{IET}(\Gamma)$ we have $\mathcal{E}_f \in A_\Gamma \otimes A_\Gamma$.*

We denote by p the projection from $\bigotimes_{\mathbb{Z}}^2 \widetilde{\Gamma}$ into $\ominus\bigwedge_{\mathbb{Z}}^2 \widetilde{\Gamma}$.

DEFINITION 5.9. — *The signature for* IET(Γ) *is the following map:*

$$\varepsilon_\Gamma : \begin{array}{ccc} \text{IET}(\Gamma) & \longrightarrow & \ominus\bigwedge_{\mathbb{Z}}^2 \widetilde{\Gamma} \\ f & \longmapsto & p \circ \omega_\Gamma(\mathcal{E}_f) \end{array}$$

PROPOSITION 5.10. — *For every A and B in A_Γ we have:*

$$p \circ \omega_\Gamma(A \times B) = -p \circ \omega_\Gamma(B \times A)$$

Proof. — Let $A, B \in A_\Gamma$ then:

$$\begin{aligned} p \circ \omega_\Gamma(A \times B) &= \lambda(A) \wedge \lambda(B) \\ &= -\lambda(B) \wedge \lambda(A) \\ &= -p \circ \omega_\Gamma(B \times A) \qquad\qquad \square \end{aligned}$$

THEOREM 5.11. — *The map ε_Γ is a group homomorphism.*

Proof. — Let $f, g \in \text{IET}(\Gamma)$. We denote by s the symmetry of axis $y = x$. We remark that every element I of $A_\Gamma \otimes A_\Gamma$ satisfies $s(I) \in A_\Gamma \otimes A_\Gamma$. Then $\{(x,y) \mid x < y, \ g(x) > g(y), \ fg(x) < fg(y)\} = \mathcal{E}_g \cap sg^{-1}(\mathcal{E}_f)$ is an element of $A_\Gamma \otimes A_\Gamma$. We also notice that $\{(x,y) \mid x < y, \ g(x) < g(y), \ fg(x) > fg(y)\} = \mathcal{E}_{fog} \cap \mathcal{E}_g^c$ and $\{(x,y) \mid x < y, \ g(x) > g(y), \ fg(x) > fg(y)\} = \mathcal{E}_{fog} \cap \mathcal{E}_g$ are two elements of $A_\Gamma \otimes A_\Gamma$.

For more clarity we do some calculus first. By Proposition 5.5 and Proposition 5.10 we get:

$$\begin{aligned} &- p \circ \omega_\Gamma(\{(x,y) \mid x < y, \ g(x) > g(y), \ fg(x) < fg(y)\}) \\ &\quad = -p \circ \omega_\Gamma(\{(g(x), g(y)) \mid x < y, \ g(x) > g(y), \ fg(x) < fg(y)\}) \\ &\quad = p \circ \omega_\Gamma(\{(g(y), g(x)) \mid x < y, \ g(x) > g(y), \ fg(x) < fg(y)\}) \\ &\quad = p \circ \omega_\Gamma(\{(u,v) \mid g^{-1}(u) > g^{-1}(v), \ u < v, \ f(u) > f(v)\}) \end{aligned}$$

and

$$\begin{aligned} &p \circ \omega_\Gamma(\{(x,y) \mid x < y, \ g(x) < g(y), \ fg(x) > fg(y)\}) \\ &\quad = p \circ \omega_\Gamma(\{(g(x), g(y)) \mid x < y, \ g(x) < g(y), \ fg(x) > fg(y)\}) \\ &\quad = p \circ \omega_\Gamma(\{(u,v) \mid g^{-1}(u) < g^{-1}(v), \ u < v, \ f(u) > f(v)\}) \end{aligned}$$

In addition:

$$p \circ \omega_\Gamma(\{(u,v) \mid g^{-1}(u) > g^{-1}(v),\ u < v,\ f(u) > f(v)\})$$
$$+ p \circ \omega_\Gamma(\{(u,v) \mid g^{-1}(u) < g^{-1}(v),\ u < v,\ f(u) > f(v)\})$$
$$= p \circ \omega_\Gamma(\mathcal{E}_f)$$
$$= \varepsilon_\Gamma(f)$$

Hence by adding and remove the same quantity at the fourth equality we obtain:

$$\varepsilon_\Gamma(f \circ g) = p \circ \omega_\Gamma(\{(x,y) \mid x < y,\ fg(x) > fg(y)\}$$
$$= p \circ \omega_\Gamma(\{(x,y) \mid x < y,\ g(x) > g(y),\ fg(x) > fg(y)\}$$
$$\sqcup \{(x,y) \mid x < y,\ g(x) < g(y),\ fg(x) > fg(y)\})$$
$$= p \circ \omega_\Gamma(\{(x,y) \mid x < y,\ g(x) > g(y),\ fg(x) > fg(y)\})$$
$$+ p \circ \omega_\Gamma(\{(x,y) \mid x < y,\ g(x) < g(y),\ fg(x) > fg(y)\})$$
$$= p \circ \omega_\Gamma(\{(x,y) \mid x < y,\ g(x) > g(y),\ fg(x) > fg(y)\})$$
$$+ p \circ \omega_\Gamma(\{(x,y) \mid x < y,\ g(x) > g(y),\ fg(x) < fg(y)\})$$
$$- p \circ \omega_\Gamma(\{(x,y) \mid x < y,\ g(x) > g(y),\ fg(x) < fg(y)\})$$
$$+ p \circ \omega_\Gamma(\{(x,y) \mid x < y,\ g(x) < g(y),\ fg(x) > fg(y)\})$$
$$= p \circ \omega_\Gamma(\mathcal{E}_g) + p \circ \omega_\Gamma(\mathcal{E}_f)$$
$$= \varepsilon_\Gamma(g) + \varepsilon_\Gamma(f)$$

In conclusion, ε_Γ is additive thus it is a group homomorphism. $\square$

PROPOSITION 5.12. — *Let $a \in \widetilde{\Gamma}_+$ with $a \leqslant \frac{1}{2}$ and f be an IET(Γ)-transposition of type a. Then $\varepsilon_\Gamma(f) = a \wedge a$.*

Proof. — Let $u, v, b \in \widetilde{\Gamma}$ such that $((u,a,b,a,v),(2\ 4))$ is a Γ-combinatorial description of f (see Figure 5.1). Let $\{I_1, \ldots, I_5\}$ be the partition into intervals associated. We already proved in 5.7 that it is sufficient to check if a pair $(x,y) \in I_i \times I_j$ is in $\mathcal{E}_f$ to know that $I_i \times I_j$ is in $\mathcal{E}_f$. We also have $I_i \times I_j \notin \mathcal{E}_f$ if $j \leqslant i$. Thus one can look at the graph of f to find that $\mathcal{E}_f$ is equal to the tiling space on Figure 5.1. We deduce that $\varepsilon_\Gamma(f) = a \wedge b + a \wedge a + b \wedge a = a \wedge a$. $\square$

Thanks to Proposition 5.7 we can calculate the value of ε_Γ for every $f \in$ IET(Γ):
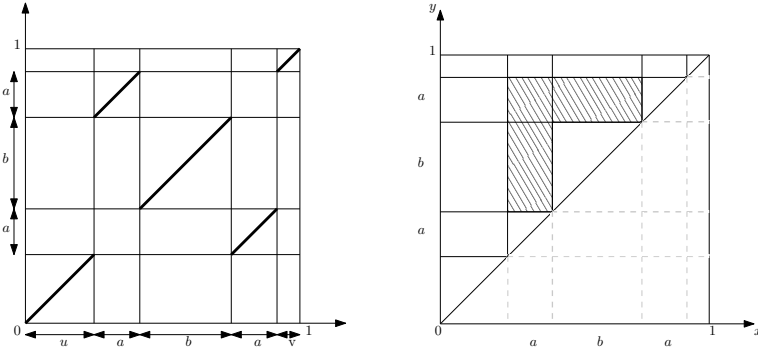
Figure 5.1. Illustration of how to calculate the value of ε_Γ on an IET(Γ)-transposition f in Proposition 5.12. Left: The graph of f. Right: The set $\mathcal{E}_f$.

PROPOSITION 5.13. — Let $f \in$ IET(Γ) and (α, τ) be a Γ-combinatorial description of f. Let n be the length of α. Then

$$\varepsilon_\Gamma(f) = \sum_{j=1}^{n} \sum_{\substack{i<j \\ \tau(i)>\tau(j)}} \alpha_i \wedge \alpha_j$$

PROPOSITION 5.14. — The group homomorphism ε_Γ is surjective.

Proof. — Let $a, b \in \widetilde{\Gamma}$. We assume that $0 \leqslant a, b < 1$ and $0 \leqslant a+b \leqslant 1$. Let r be the IET(Γ) restricted rotation of type (a, b), whose intervals associated are $[0, a[$ and $[a, a+b[$. Then we obtain that $\varepsilon_\Gamma(r) = a \wedge b$.

In the general case, let $w \in \widetilde{\Gamma}$ with $0 \leqslant w \leqslant \frac{1}{2}$. Then there exist $k, \ell \in \mathbb{Z}$ and $a', b' \in \widetilde{\Gamma}$ with $0 \leqslant a', b' < w$ such that $a = kw + a'$ and $b = \ell w + b'$. Then $a \wedge b = k\ell w \wedge w + kw \wedge b' + \ell a' \wedge w + a' \wedge b'$. By the previous case and as ε_Γ is a group homomorphism, we deduce that $a \wedge b$ is in Im(ε_Γ). $\quad\square$

5.3. Description of Ker(ε_Γ)

The aim of this part is to conclude that Ker(ε_Γ) $= D($IET(Γ)) and the induced morphism IET(Γ)$_{\mathrm{ab}} \to {}^\ominus\bigwedge_{\mathbb{Z}}^2 \widetilde{\Gamma}$ is an isomorphism. We recall that φ_Γ is the restriction of the SAF-homomorphism φ to IET(Γ).

LEMMA 5.15. — We have $2\varepsilon_\Gamma = -\varphi_\Gamma$.

Proof. — Let $f \in \mathrm{IET}(\Gamma)$ and (α, τ) be a Γ-combinatorial description of f. Let n be the length of α. Thanks to Propositions 2.4 and 5.13 we have:

$$\varphi_\Gamma(f) = \sum_{j=1}^{n} \left(\sum_{\substack{i \\ \tau(i) < \tau(j)}} \alpha_i - \sum_{i < j} \alpha_i \right) \wedge \alpha_j$$

$$= \sum_{j=1}^{n} \sum_{\substack{i > j \\ \tau(i) < \tau(j)}} \alpha_i \wedge \alpha_j + \sum_{j=1}^{n} \left(\sum_{\substack{i < j \\ \tau(i) < \tau(j)}} \alpha_i - \sum_{i < j} \alpha_i \right) \wedge \alpha_j$$

$$= \sum_{i=1}^{n} \sum_{\substack{j < i \\ \tau(j) > \tau(i)}} \alpha_i \wedge \alpha_j - \sum_{j=1}^{n} \sum_{\substack{i < j \\ \tau(i) > \tau(j)}} \alpha_i \wedge \alpha_j$$

$$= -2 \sum_{i=1}^{n} \sum_{\substack{j < i \\ \tau(j) > \tau(i)}} \alpha_j \wedge \alpha_i$$

$$= -2\varepsilon_\Gamma(f) \qquad \qquad \square$$

COROLLARY 5.16. — *We have the inclusion* $\mathrm{Ker}(\varepsilon_\Gamma) \subset \mathrm{Ker}(\varphi_\Gamma)$.

By Corollary 4.9 we know that $\mathrm{Ker}(\varphi_\Gamma)$ is generated by the set of all $\mathrm{IET}(\Gamma)$-transpositions. Thus it is natural to look at these elements who are also in $\mathrm{Ker}(\varepsilon_\Gamma)$. If σ is an $\mathrm{IET}(\Gamma)$-transposition of type a in $\mathrm{Ker}(\varepsilon_\Gamma)$ then we have the equality $a \wedge a = 0$. We want to prove that σ is in $D(\mathrm{IET}(\Gamma))$ if and only if $a \in 2\widetilde{\Gamma}$.

We denote by $S^2\widetilde{\Gamma}$ the second symmetric power of $\widetilde{\Gamma}$ and we denote by $a \odot a$ image of $a \otimes a$ in $S^2\widetilde{\Gamma}$.

For every group G and every $w \in G$ we use the notation $w \ [\mathrm{mod}\ 2]$ for the image of w in $G/2G$.

PROPOSITION 5.17. — *Let* $a \in \widetilde{\Gamma}$, *if* $a \wedge a \ [\mathrm{mod}\ 2] = 0$ *then* a *belongs to* $2\widetilde{\Gamma}$.

Proof. — For every group G, the group $^\ominus\!\bigwedge_\mathbb{Z}^2 G / 2(^\ominus\!\bigwedge_\mathbb{Z}^2 G)$ is naturally isomorphic to the second symmetric power $S^2 G / 2(S^2 G)$. This comes from the fact that these groups satisfy the following universal property: for every group G and every abelian elementary 2-group We denote i the natural inclusion of $G \times G$ into $S^2 G / 2(S^2 G)$. For every bilinear symmetric group homomorphism $b : G \times G \to A$ there exists a unique group homomorphism $f : S^2 G / 2(S^2 G) \to A$ such that for every $g, h \in G$ we have $b(g, h) = f(i(g, h))$.

Let $a \in \widetilde{\Gamma}$ with $a \neq 0$ (because we already have $0 = 2 \times 0$). We denote by $a \odot a$ the image of $a \otimes a$ in $S^2 G / 2(S^2 G)$, and we assume that $a \odot a \pmod 2 = 0$. The projection $\widetilde{\Gamma} \to \widetilde{\Gamma}/2\widetilde{\Gamma}$ gives rise to a morphism $\zeta : S^2 \widetilde{\Gamma} \to S^2(\widetilde{\Gamma}/2\widetilde{\Gamma})$. As $2(S^2 \widetilde{\Gamma}) \subset \mathrm{Ker}(\zeta)$ we obtain a morphism $\zeta' : S^2 \widetilde{\Gamma}/2S^2 \widetilde{\Gamma} \to S^2(\widetilde{\Gamma}/2\widetilde{\Gamma})$. Hence if $a \notin 2\widetilde{\Gamma}$ then $\zeta(a \odot a) \neq 0$ thus $\zeta'(a \odot a \pmod 2) \neq 0$ which is a contradiction with the assumption. In conclusion $a \in 2\widetilde{\Gamma}$. $\qquad\square$

COROLLARY 5.18. — *Every* IET(Γ)-*transposition* f *in* $\mathrm{Ker}(\varepsilon_\Gamma)$ *is in* $D(\mathrm{IET}(\Gamma))$.

Proof. — Let $a \in \widetilde{\Gamma}_+$, with $a \leqslant \frac{1}{2}$, be the type of f and let $u, v \in \widetilde{\Gamma}_+$ such that $I_1 = [u, u + a[$ and $I_2 = [v, v + a[$ are the two intervals swapped by f. From $f \in \mathrm{Ker}(\varepsilon)$ we deduce that $\varepsilon_\Gamma(f) = a \wedge a = 0$. Hence $a \wedge a \pmod 2 = 0$ in $^\ominus\bigwedge_{\mathbb{Z}}^2 \widetilde{\Gamma}/2(^\ominus\bigwedge_{\mathbb{Z}}^2 \widetilde{\Gamma})$. Then by Proposition 5.17 there exists $b \in \widetilde{\Gamma}$ such that $a = 2b$. Thus if we define g as the unique IET(Γ)-transposition of type b that swaps $[u, u + b[$ and $[v, v + b[$ and h as the unique IET(Γ) that swaps $[u, u + b[$ with $[u + b, u + a[$ and $[v, v + b[$ with $[v + b, v + a[$. Then $f = ghgh$ (see Figure 2.2) and $g^2 = h^2 = \mathrm{Id}$, hence $f \in D(\mathrm{IET}(\Gamma))$. $\qquad\square$

In order to show that $\mathrm{Ker}(\varepsilon_\Gamma) = D(\mathrm{IET}(\Gamma))$ we prove that any element $f \in \mathrm{Ker}(\varepsilon_\Gamma)$ can be written as $f = \sigma h$ where $h \in D(\mathrm{IET}(\Gamma))$ and σ is an IET(Γ)-transposition. This concludes because we just show that an IET(Γ)-transposition which is also in $\mathrm{Ker}(\varepsilon_\Gamma)$ is in $D(\mathrm{IET}(\Gamma))$. We begin by a particular case of a product of IET(Γ)-transpositions with pairwise disjoint support. The aim will be to reduce the general case to this one. We recall that the identity is considered as an IET(Γ)-transposition.

LEMMA 5.19. — *Let* $k \in \mathbb{N}$ *and* $\tau_1, \tau_2, \ldots, \tau_k$ *be* IET(Γ)-*transpositions with pairwise disjoint support. Then* $\tau_1 \tau_2 \ldots \tau_k = \sigma h$ *where* σ *is an* IET(Γ)-*transposition and* h *is an element of* $D(\mathrm{IET}(\Gamma))$. *Moreover the support of* h *and* σ *do not overlap and are included in the union of the supports of the* τ_i.

Proof. — By induction it is enough to show the result in the case $k = 2$ Let respectively a_1 and a_2 be the type of τ_1 and τ_2. As their support do not overlap we know that τ_1 and τ_2 commute. Hence we can assume $a_1 \geqslant a_2$ without loss of generality. If $a_1 = a_2$ then there exists $f \in \mathrm{IET}(\Gamma)$ such that $\tau_1 = f\tau_2 f^{-1}$. Thus $\tau_1 \tau_2$ is in $D(\mathrm{IET}(\Gamma))$. If $a_1 > a_2$ let $u, v \in \widetilde{\Gamma}$ such that $[u, u + a_1[$ and $[v, v + a_1[$ are the intervals swapped by τ_1. Let g and h be IET(Γ)-transpositions such that g swaps the intervals $[u, u + a_2[$ and $[v, v + a_2[$ and h swaps the intervals $[u + a_2, u + a_1[$ and $[v + a_2, v + a_1[$. Thus $\tau_1 = hg$. Moreover g and τ_2 are two IET(Γ)-transpositions with same type and non-overlapping support. Then by the previous case, the product

$g\tau_2$ is in $D(\text{IET}(\Gamma))$ and its support does not intersect the support of h. Then $f = h(g\tau_2)$ is the wanted decomposition. $\qquad\square$

We can now treat the case of finite order elements:

LEMMA 5.20. — *If $f \in \text{IET}(\Gamma)$ has finite order then there exist an IET(Γ)-transposition σ and $h \in D(\text{IET}(\Gamma))$ such that the support of σ and h are inside the support of f and $f = \sigma h$.*

Proof. — Let $f \in \text{IET}(\Gamma)$ as in the statement. By Lemma 3.1, there exist $n \in \mathbb{N}$ and $\mathcal{P} = \{I_1, I_2, \ldots, I_n\} \in \Pi_f$ and $\sigma \in \mathfrak{S}_n$ such that $f(I_i) = I_{\sigma(i)}$. Let $k \in \mathbb{N}$ and $\sigma = c_1 c_2 \ldots c_k$ be the disjoint cycle decomposition for σ. Let f_i be the element of $\text{IET}(\Gamma)$ that is equal to f on I_j for every $j \in \text{Supp}(c_i)$ while fixing the rest of $[0, 1[$. Then $f = f_1 \ldots f_k$ and f_i commutes with f_j for every $1 \leqslant i \neq j \leqslant k$. Then if the statement is true for every f_i we can write $f_i = \tau_i h_i$ with τ_i an IET(Γ)-transposition and $h \in D(\text{IET}(\Gamma))$, both of them with support inside the support of f_i. Then $f = f_1 \ldots f_k = \tau_1 h_1 \ldots \tau_k h_k = \tau_1 \ldots \tau_k h_1 \ldots h_k$ because the support of τ_i does not overlap with the support of τ_j of h_j for every $1 \leqslant j \leqslant k$ and $j \neq i$. We conclude with Lemma 5.19 applied to $\tau_1 \ldots \tau_k$.
Let c by a cycle of length $n \geqslant 2$ and let $I_1, I_2, \ldots, I_n$ be non-overlapping intervals of $\text{Itv}(\Gamma)$ of same length. Let $f \in \text{IET}(\Gamma)$ be the element that permutes the set $\{I_1, I_2, \ldots, I_n\}$ by c. Then if $c \in D(\mathfrak{S}_n) = \mathfrak{A}_n$ we deduce that $f \in D(\text{IET}(\Gamma))$. If $c \notin D(\mathfrak{S}_n)$ then let g be the unique IET(Γ)-transposition that swaps I_1 with $I_{c(1)}$ (we notice that the support of g is included in the support of f). By previous case, $gf \in D(\text{IET}(\Gamma))$ and we conclude that $f = g(gf)$ is a wanted decomposition. $\qquad\square$

COROLLARY 5.21. — *Let τ and τ' be two IET(Γ)-transpositions. There exist an IET(Γ)-transposition σ and $h \in D(\text{IET}(\Gamma))$ such that $\tau\tau' = \sigma h$ and the support of σ and h are included in the union of the support of τ and the one of τ'.*

Proof. — By Lemma 3.6 we deduce that $f := \tau\tau'$ has finite order. Hence by Lemma 5.20 we obtain the result. $\qquad\square$

LEMMA 5.22. — *Let k in $\mathbb{N}$ and $\tau_1, \tau_2, \ldots, \tau_k$ be some IET(Γ)-transpositions. Then there exist an IET(Γ)-transposition σ and $h \in D(\text{IET}(\Gamma))$ such that $\tau_1 \tau_2 \ldots \tau_k = \sigma h$.*

Proof. — The proof is by induction on k. The initialisation $k = 1$ is immediate. The case $k = 2$ is Corollary 5.21. Now if we assume the result for $k \geqslant 2$ let $\tau_1, \tau_2, \ldots, \tau_k, \tau_{k+1}$ be IET(Γ)-transpositions. Then by assumption, applied to $\tau_2 \tau_3 \ldots \tau_{k+1}$, there exist an IET($\Gamma$)-transposition σ and $h \in$

$D(\mathrm{IET}(\Gamma))$ such that $\tau_2\tau_3\ldots\tau_{k+1} = \sigma h$. Hence $\tau_1\tau_2\ldots\tau_{k+1} = \tau_1\sigma h$. By using the case $k = 2$ we deduce that there exist an $\mathrm{IET}(\Gamma)$-transposition σ' and $h' \in D(\mathrm{IET}(\Gamma))$ such that $\tau_1\sigma = \sigma'h'$. Thus $\tau_1\tau_2\ldots\tau_{k+1} = \sigma'g$, with $g = h'h \in D(\mathrm{IET}(\Gamma))$, which is a wanted decomposition. $\square$

Finally we can prove the main theorem of the section:

THEOREM 5.23. — *We have the equality* $\mathrm{Ker}(\varepsilon_\Gamma) = D(\mathrm{IET}(\Gamma))$, *and the induced morphism* $\mathrm{IET}(\Gamma)_{\mathrm{ab}} \to {}^\ominus\bigwedge^2_{\mathbb{Z}}\widetilde{\Gamma}$ *is an isomorphism.*

Proof. — The inclusion from right to left is immediate. For the other inclusion let $f \in \mathrm{Ker}(\varepsilon_\Gamma)$. By Corollary 5.16 we know that $f \in \mathrm{Ker}(\varphi_\Gamma)$, then by Corollary 4.9 there exists $k \in \mathbb{N}$ such that f is equal to the product $\tau_1\tau_2\ldots\tau_k$ where τ_i is an $\mathrm{IET}(\Gamma)$-transposition. By Lemma 5.22 there exist an $\mathrm{IET}(\Gamma)$-transposition σ and $h \in D(\mathrm{IET}(\Gamma))$ such that $f = \sigma h$. Then $\varepsilon_\Gamma(f) = \varepsilon_\Gamma(\sigma)\varepsilon_\Gamma(h) = \varepsilon_\Gamma(\sigma) = 0$. By Corollary 5.18 we deduce $\sigma \in D(\mathrm{IET}(\Gamma))$. Hence $f = \sigma h \in D(\mathrm{IET}(\Gamma))$.

We deduce that the induced group homomorphism $\varepsilon'_\Gamma : \mathrm{IET}(\Gamma)_{\mathrm{ab}} \to {}^\ominus\bigwedge^2_{\mathbb{Z}}\widetilde{\Gamma}$ is injective. Furthermore ε_Γ is surjective by Proposition 5.14 thus ε'_Γ is surjective and we conclude that ε'_Γ is an isomorphism. $\square$

6. Abelianization of $\mathrm{IET}^{\bowtie}(\Gamma)$

In order to identify $\mathrm{IET}^{\bowtie}(\Gamma)_{\mathrm{ab}}$ we construct two group homomorphisms. One is an analogue of the signature homomorphism ε_Γ constructed in Section 5. Its kernel will be slightly larger than $D(\mathrm{IET}^{\bowtie}(\Gamma))$; in fact, it will miss some reflections of a certain type. In order to tackle this issue we notice that, in some cases, a reflection is a conjugate of a restricted rotation and thus we will try to use the group homomorphism ε_Γ. To this we need to affect for every element of $\mathrm{IET}^{\bowtie}(\Gamma)$ an element of $\mathrm{IET}(\Gamma)$.

We recall that the group IET can be seen as a subgroup of $\mathrm{IET}^{\bowtie}$. Thus restricted rotations in $\mathrm{IET}^{\bowtie}$ are well-defined.

The next proposition implies that a lot of Γ-reflections are in $D(\mathrm{IET}^{\bowtie}(\Gamma))$ with a sufficient condition on their type (see Definition 2.8 for the definition of reflection and type of a reflection). It is a consequence of Proposition 2.11 and Theorem 5.23:

PROPOSITION 6.1. — *Every* Γ*-reflection of type* $\ell \in 4\widetilde{\Gamma}$ *belongs to* $D(\mathrm{IET}^{\bowtie}(\Gamma))$

6.1. Partitions associated and positive substitute

6.1.1. Partitions

Partitions of $[0, 1[$ into finite intervals will be really useful there but the definition of $\mathrm{IET}^{\bowtie}(\Gamma)$ imposes to be more careful and to take them up to a finitely supported permutation.

DEFINITION 6.2. — *Let $\mathcal{P}$ be a finite partition of $[0, 1[$ into right-open and left-closed intervals and let $\widehat{f} \in \widehat{\mathrm{IET}^{\bowtie}}$. The partition $\mathcal{P}$ is called a partition associated with $\widehat{f}$ if $\widehat{f}$ is continuous on every $I^{\circ} \in \mathcal{P}$. Also $\mathcal{P}$ is an essential partition associated with $\widehat{f}$ if there exists a finitely supported permutation σ such that $\mathcal{P}$ is a partition associated with $\sigma\widehat{f}$. We denote by $\widehat{f}(\mathcal{P})$ the partition into intervals of $[0, 1[$ composed of all right-open and left-closed intervals whose interior is the image by $\widehat{f}$ of the interior of an interval in $\mathcal{P}$ up to a finite number of points; it is called the arrival partition of $\widehat{f}$ associated to $\mathcal{P}$.*

We can now define partitions associated with an element of $\mathrm{IET}^{\bowtie}$:

DEFINITION 6.3. — *Let $\mathcal{P}$ be a finite partition of $[0, 1[$ into right-open and left-closed intervals. Let $f \in \mathrm{IET}^{\bowtie}$ and let $\widehat{f}$ be a representative of f in $\widehat{\mathrm{IET}^{\bowtie}}$. The partition $\mathcal{P}$ is called a partition associated with f if $\mathcal{P}$ is an essential partition associated to $\widehat{f}$. We denote by $f(\mathcal{P})$ the partition $\widehat{f}(\mathcal{P})$; it is called the arrival partition of f associated to $\mathcal{P}$.*

Remark 6.4. — For every $f \in \mathrm{IET}^{\bowtie}$, the set Π_f has a minimal element for the refinement. This element is also the unique partition that has a minimal number of interval.

We recall that a partition $\mathcal{P}$ is a Γ-partition if for every $I \in \mathcal{P}$ we have $I \in \mathrm{Itv}(\Gamma)$. Thus for every $f \in \mathrm{IET}^{\bowtie}$ we have $f \in \mathrm{IET}^{\bowtie}(\Gamma)$ if and only if there exists a Γ-partition associated with f.

Sometimes we will need to be more precise on where the length of intervals are. We recall that λ denote the Lebesgue measure:

DEFINITION 6.5. — *Let S be a finite subset of $\mathbb{R}$, we say that a subinterval I of $\mathbb{R}$ is a S-interval if $\lambda(I) \in S$.*

In order to have more rigidity we also associate a partition to a tuple:

DEFINITION 6.6. — *Let $n \in \mathbb{N}$ and $f_1, f_2, \ldots, f_n \in \mathrm{IET}^{\bowtie}(\Gamma)$. Let S be a finite subset of $\mathbb{R}$ and let $\mathcal{P}$ be a partition into intervals of $[0, 1[$, we said that $\mathcal{P}$ is a partition into S-intervals associated with $(f_1, f_2, \ldots, f_n)$ if:*

   (1) $\mathcal{P}$ is a partition into S-intervals associated with f_1,
   (2) For every $2 \leqslant i \leqslant n - 1$, $f_i f_{i-1} \ldots f_1(\mathcal{P})$ is a partition into S-intervals associated with f_{i+1}.

Remark 6.7. — Let S and T be two finite subsets of $\mathbb{R}$ and let $\mathcal{P}$ be a partition into S-intervals associated with $(f_1, f_2, \ldots, f_n)$. Then any refinement of $\mathcal{P}$ into T-intervals is a partition into T-intervals associated with $(f_1, f_2, \ldots, f_n)$.

We also want to talk about order-preserving and order-reversing for elements in $\mathrm{IET}^{\bowtie}$.

DEFINITION 6.8. — Let $f \in \mathrm{IET}^{\bowtie}$ and $\mathcal{P} \in \Pi_f$. Let $I \in \mathcal{P}$, we say that f is order-preserving on I (resp. order-reversing on I) if there exists a representative of f in $\widehat{\mathrm{IET}^{\bowtie}}$ that is order-preserving on I (resp. order-reversing on I).

Thanks to this when we have a partition $\mathcal{P}$ associated with an element $f \in \mathrm{IET}^{\bowtie}$ we can always say that f is either order-preserving or order-reversing on every interval of $\mathcal{P}$.

6.1.2. Positive substitute

We introduce the notion of positive substitute to affect an element of IET for every element of $\mathrm{IET}^{\bowtie}$. This substitute depends on a partition associated with f.

DEFINITION 6.9. — Let $\widehat{f} \in \widehat{\mathrm{IET}^{\bowtie}}$ and $\mathcal{P}$ be an essential partition associated with f. Let f be its image in $\mathrm{IET}^{\bowtie}$. Then there exists a unique $f_{\mathcal{P}}^{+} \in \mathrm{IET}$ such that $\mathcal{P}$ is a partition associated with $f_{\mathcal{P}}^{+}$ and such that for every $I \in \mathcal{P}$ we have $f_{\mathcal{P}}^{+}(I) = \widehat{f}(I)$ up to a finite number of points. This element is called the positive $\mathcal{P}$-substitute of $\widehat{f}$ and f.

This element exists because there exists a finitely supported permutation σ such that $\mathcal{P}$ is a partition associated with $\sigma \widehat{f}$ and we define $f_{\mathcal{P}}^{+}$ as the element of IET such that $f_{\mathcal{P}}^{+}(I) = \sigma f(I)$ for every $I \in \mathcal{P}$. The unicity come from the right continuity of $f_{\mathcal{P}}^{+}$. The dependance on the partition is really important as we can see on Figures 6.1 and 6.2.

This gives us a decomposition for every element of $\widehat{\mathrm{IET}^{\bowtie}}$.

PROPOSITION 6.10. — Let $\widehat{f} \in \widehat{\mathrm{IET}^{\bowtie}}$ and $\mathcal{P}$ be an essential partition associated with $\widehat{f}$. Then there exist a unique finitely supported permutation σ and a product r of I-reflection maps with $I \in \widehat{f}(\mathcal{P})$ such that $\widehat{f} = \sigma r f_{\mathcal{P}}^{+}$.
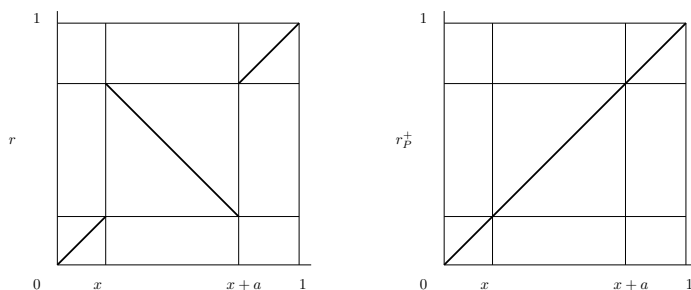
Figure 6.1. *Positive substitute for a reflection in the case* $S = \{x, a, 1 - (x + a)\}$.
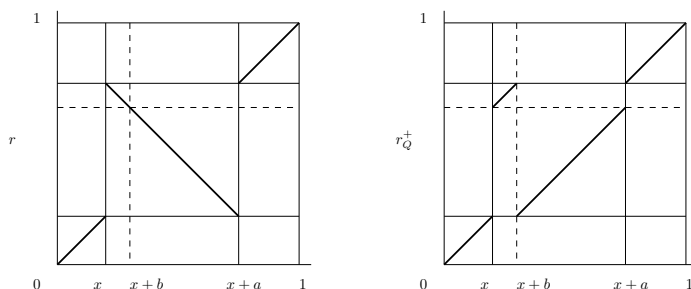

Figure 6.2. *Positive substitute for a reflection in the case* $S = \{x, b, a - b, 1 - (x + a)\}$.

With some conditions we deduce how positive substitute behaves with composition.

PROPOSITION 6.11. — *Let* $n \in \mathbb{N}$ *and* $f_1, f_2, \ldots, f_n \in \text{IET}^{\bowtie}(\Gamma)$. *Let* $\mathcal{P}$ *be a partition into* S-*intervals associated with* $(f_1, f_2, \ldots, f_n)$. *Let* $g_1 = (f_1)^+_{\mathcal{P}}$ *and* $g_i = (f_i f_{i-1} \ldots f_1)^+_{f_{i-1} \ldots f_1(\mathcal{P})}$ *for every* $2 \leqslant i \leqslant n$. *Then*

$$(f_n f_{n-1} \ldots f_1)^+_{\mathcal{P}} = g_n g_{n-1} \ldots g_1.$$

Proof. — For this proof we denote by r_I the I-reflection for every subinterval I of $[0, 1[$.

By iteration it is sufficient to show the result for $n = 2$. Let $f, g \in \text{IET}^{\bowtie}(\Gamma)$ and $\mathcal{P}$ be a partition into S-intervals associated with (f, g). Let $n \in \mathbb{N}$ such that $\mathcal{P} = \{I_1, I_2, \ldots, I_n\}$ and $f(\mathcal{P}) = \{J_1, J_2, \ldots, J_n\}$ and $f^+_{\mathcal{P}}(I_i) = J_i$ for $1 \leqslant i \leqslant n$. Let $1 \leqslant i \leqslant n$, we notice that $r_{J_i} \circ f^+_{\mathcal{P}}|_{I_i} \circ r_{I_i} = f^+_{\mathcal{P}}|_{I_i}$. There are 4 cases:

(1) If f is order-preserving on I_i and g is order-preserving on J_i then $g \circ f$ is order-preserving on I_i so:

$$(g \circ f)_{\mathcal{P}}^+|_{I_i} = (g \circ f)|_{I_i} = g|_{J_i} \circ f|_{I_i} = g_{f(\mathcal{P})}^+|_{J_i} \circ f_{\mathcal{P}}^+|_{I_i}$$

(2) If f is order-preserving on I_i and g is order-reversing on J_i then $g \circ f$ is order-reversing on I_i so:

$$(g \circ f)_{\mathcal{P}}^+|_{I_i} = (g \circ f)_{I_i} \circ r_{I_i} = g|_{J_i} \circ f|_{I_i} \circ r_{I_i}$$
$$= g_{f(\mathcal{P})}^+|_{J_i} \circ r_{J_i} \circ f_{\mathcal{P}}^+|_{I_i} \circ r_{I_i} = g_{f(\mathcal{P})}^+|_{J_i} \circ f_{\mathcal{P}}^+|_{I_i}$$

(3) If f is order-reversing on I_i and g is order-preserving on J_i then $g \circ f$ is order-reversing on I_i so:

$$(g \circ f)_{\mathcal{P}}^+|_{I_i} = (g \circ f)_{I_i} \circ r_{I_i} = g|_{J_i} \circ f|_{I_i} \circ r_{I_i}$$
$$= g_{f(\mathcal{P})}^+|_{J_i} \circ f_{\mathcal{P}}^+|_{I_i} \circ r_{I_i} \circ r_{I_i} = g_{f(\mathcal{P})}^+|_{J_i} \circ f_{\mathcal{P}}^+|_{I_i}$$

(4) If f is order-reversing on I_i and g is order-reversing on J_i then $g \circ f$ is order-preserving on I_i so:

$$(g \circ f)_{\mathcal{P}}^+|_{I_i} = (g \circ f)|_{I_i} = g|_{J_i} \circ f|_{I_i}$$
$$= g_{f(\mathcal{P})}^+|_{J_i} \circ r_{J_i} \circ f_{\mathcal{P}}^+|_{I_i} \circ r_{I_i} = g_{f(\mathcal{P})}^+|_{J_i} \circ f_{\mathcal{P}}^+|_{I_i} \qquad \square$$

6.2. Analogue of the signature

6.2.1. Balanced product of reflections

We give here a first description of $D(\mathrm{IET}^{\bowtie})$. It is inspired of the work done for IET with balanced product restricted rotations.

DEFINITION 6.12. — *Let $n \in \mathbb{N}$ and $r_1, r_2, \ldots, r_n$ be some Γ-reflections. For every $\ell \in \widetilde{\Gamma}_+$ let n_ℓ be the number of Γ-reflections of type ℓ among these elements. The tuple $(r_1, r_2, \ldots, r_n)$ is a* balanced tuple of Γ-reflections *if 2 divides n_ℓ for every $\ell \in \widetilde{\Gamma}_+$. We say that a product of Γ-reflections is a a* balanced product of Γ-reflections *if it can be written as a product of a balanced tuple of Γ-reflections.*

LEMMA 6.13. — *The set of all balanced products of Γ-reflections is a generating subset of $D(\mathrm{IET}^{\bowtie}(\Gamma))$.*

Proof. — As any element of $\mathrm{IET}^{\bowtie}(\Gamma)$ is a finite product of Γ-reflections (see Proposition 2.9) and as a reflection has order 2 we deduce that every element of $D(\mathrm{IET}^{\bowtie}(\Gamma))$ is a balanced product of reflections.

Let r and s be two reflections with the same type. By Proposition 2.10 we deduce that they are conjugate. As a reflection has order 2, the product rs is a commutator.

Let $n \in \mathbb{N}$ and $r_1, r_2, \ldots, r_n$ be Γ-reflections such that $r_1 r_2 \ldots r_n$ is a balanced product of reflections. Then n is even and up to compose with an element of $D(\mathrm{IET}^{\bowtie})$ we can assume that r_{2i-1} and r_{2i} have the same type for every $1 \leqslant i \leqslant \frac{n}{2}$. Thus by the previous case we deduce that $r_1 r_2 \ldots r_n$ is in $D(\mathrm{IET}^{\bowtie})$. $\qquad\square$

By Proposition 2.9 and Lemma 6.13 we deduce the following:

COROLLARY 6.14. — *Let $f \in \mathrm{IET}^{\bowtie}(\Gamma)$ then $f^2 \in D(\mathrm{IET}^{\bowtie}(\Gamma))$.*

PROPOSITION 6.15. — *Any balanced product of Γ-restricted rotations with type in $\{(a,b) \in (\widetilde{\Gamma}_+)^2 \mid a \neq b\}$ is a balanced product of Γ-reflections.*

Proof. — Let $a, b \in \widetilde{\Gamma}_+$. Let r be a restricted rotation of type (a,b) an s be a restricted rotation of type (b,a). Let I and J be the two consecutive intervals associated with r. Then r is the composition of the I-reflection, the J-reflection and the $I \cup J$-reflection thus a product of a reflections of type a with one of type b and one of type $a+b$. The same is true for s thus we obtain that rs is the product of two reflections of type a, two of type b and two of type $a+b$; so rs is a balanced product of Γ-reflections. $\qquad\square$

6.2.2. The group homomorphism

Here we start with the work done in Section 5. We remark that if we denote $A'_{\mathbb{R}}$ the Boolean algebra of subsets of $[0,1[$ generated by the set of all intervals $[a,b]$, $A_{\mathbb{R}}$ the one generated by the set of all intervals $[a,b[$ and A_{fin} the one generated by all the singletons $\{x\}$ then $A_{\mathbb{R}}$ is isomorphic to $A^*_{\mathbb{R}} := A'_{\mathbb{R}}/A_{\mathrm{fin}}$. This is why we do not make a difference between A_{Γ} and its image in $A^*_{\mathbb{R}}$.

The notion of inversions as defined in Definition 5.6 is no longer relevant because for every reflection r we have $\mathcal{E}_r \notin A_{\Gamma} \otimes A_{\Gamma}$; indeed $\mathcal{E}_r$ is a triangle. We need to be more precise:

DEFINITION 6.16. — *For every element $f \in \widehat{\mathrm{IET}^{\bowtie}}(\Gamma)$ we define:*
  (1) $\mathcal{E}_{f,1} := \{(x,y) \mid x < y, \ f(x) > f(y)\}$, *the set of all inversions of type 1 of f,*
  (2) $\mathcal{E}_{f,2} := \{(x,y) \mid y < x, \ f(y) > f(x)\}$, *the set of all inversions of type 2 of f,*
  (3) $\mathcal{E}^{\bowtie}_f := \mathcal{E}_{f,1} \cup \mathcal{E}_{f,2}$, *the set of all inversions of f.*

From now on we will write $\mathcal{E}_f^{\bowtie} = \mathcal{E}_f$.

Example 6.17.
  (1) If we consider a Γ-restricted rotation r of type (a, b) (with $a, b \in \widetilde{\Gamma}_+$), we obtain $w_\Gamma(\mathcal{E}_r) = a \otimes b + b \otimes a$ (see Figure 6.3).
  (2) Let $I \in \mathrm{Itv}(\Gamma)$ be a subinterval of $[0, 1[$ of length a. Let r be the I-reflection. We have $w_\Gamma(\mathcal{E}_r) = a \otimes a$ (see Figure 6.3).
  (3) Let τ a finitely supported permutation. Then $\mathcal{E}_\tau$ is a union of singletons thus $\mathcal{E}_\tau = \varnothing \in A_\Gamma \otimes A_\Gamma$ and $w_\Gamma(\mathcal{E}_\tau) = w_\Gamma(\varnothing) = 0$.
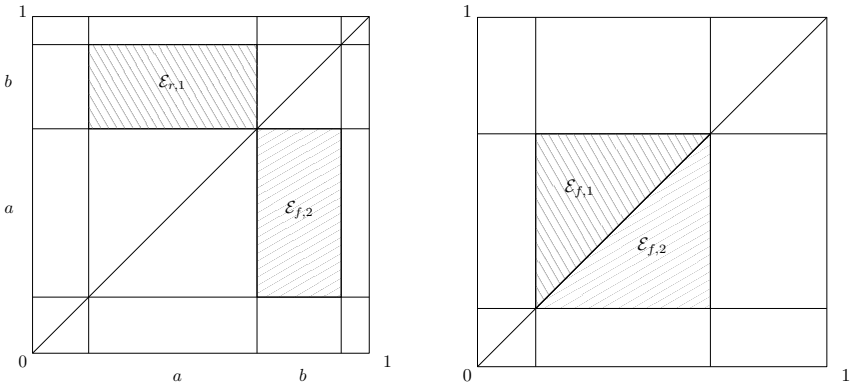

Figure 6.3. Left: *Set of inversions for a restricted rotation.* Right: *Set of inversions for a I-reflection*

PROPOSITION 6.18. — *For every $f \in \widehat{\mathrm{IET}^{\bowtie}}(\Gamma)$ we have $\mathcal{E}_f \in A_\Gamma \otimes A_\Gamma$.*

Proof. — Let $f \in \widehat{\mathrm{IET}^{\bowtie}}$ and let $\mathcal{P}$ be a partition associated with f. Let $f_\mathcal{P}^+$ be the positive $\mathcal{P}$-substitute of f. Let $\mathcal{J} \subset \mathcal{P}$ be the subset of intervals where f is order-reversing. By seeing A_Γ as its image in $A_\mathbb{R}^*$ then we deduce that $\mathcal{E}_f = \mathcal{E}_{f_\mathcal{P}^+} \cup \bigcup_{I \in \mathcal{J}} I \times I$ is an element of $A_\Gamma \otimes A_\Gamma$. $\qquad\square$

Then this set can be measured with the same measure ω_Γ used in Section 5. Here we expect a 2-group for the image of our group homomorphism. We denote by π the projection of $\bigotimes_\mathbb{Z}^2 \widetilde{\Gamma}$ onto $\bigotimes_\mathbb{Z}^2 \widetilde{\Gamma}/2(\bigotimes_\mathbb{Z}^2 \widetilde{\Gamma})$ and for every $a \in \widetilde{\Gamma}$ we denote $\pi(a \otimes a)$ by $a \otimes a$ [mod 2].

DEFINITION 6.19. — *We define the signature for $\widehat{\mathrm{IET}^{\bowtie}}(\Gamma)$ as the map:*

$$\widehat{\varepsilon_\Gamma^{\bowtie}} : \quad \widehat{\mathrm{IET}^{\bowtie}}(\Gamma) \quad \longrightarrow \quad \bigotimes_\mathbb{Z}^2 \widetilde{\Gamma}/2(\bigotimes_\mathbb{Z}^2 \widetilde{\Gamma})$$
$$f \quad \longrightarrow \quad w_\Gamma(\mathcal{E}_f) \text{ [mod 2]}$$

For more clarity we explicit some equalities used to show that $\widehat{\varepsilon_\Gamma^{\bowtie}}$ is a group homomorphism.

LEMMA 6.20. — Let $f, g \in \widehat{\mathrm{IET}^{\bowtie}}(\Gamma)$. We have the following equalities:

(1) $\mathcal{E}_{f \circ g} \cup (\mathcal{E}_g \cap g^{-1}(\mathcal{E}_f)) = \mathcal{E}_g \cup g^{-1}(\mathcal{E}_f)$,

(2) $\mathcal{E}_{f \circ g} \cap \mathcal{E}_g \cap g^{-1}(\mathcal{E}_f) = \varnothing$.

Remark 6.21. — We notice that $\mathcal{E}_g \cap g^{-1}(\mathcal{E}_f)$ is an element of $A_\Gamma \otimes A_\Gamma$.

Proof.

(1) We proceed by double inclusions.
From left to right, we know that $\mathcal{E}_g \cap g^{-1}(\mathcal{E}_f) \subset \mathcal{E}_g \cup g^{-1}(\mathcal{E}_f)$ hence it is sufficient to show the inclusion $\mathcal{E}_{f \circ g} \subset \mathcal{E}_g \cup g^{-1}(\mathcal{E}_f)$. Let (x, y) be in $\mathcal{E}_{f \circ g}$. We can assume that $x < y$, the case $x > y$ is similar. Then we deduce that $f(g(x)) > f(g(y))$. We have two cases, if $g(x) > g(y)$ then $(x, y) \in \mathcal{E}_g$ else we have $g(x) < g(y)$ and $f(g(x)) > f(g(y))$ thus $(x, y) = g^{-1}(g(x), g(y)) \in g^{-1}\mathcal{E}_f$.
From right to left, let $(x, y) \in \mathcal{E}_g \cup g^{-1}(\mathcal{E}_f)$. We can assume that $x < y$ the case $x > y$ is similar. If $(x, y) \in \mathcal{E}_g \cap g^{-1}(\mathcal{E}_f)$ then it is done. We have two cases, if $(x, y) \in \mathcal{E}_g$ and $(x, y) \notin g^{-1}(\mathcal{E}_f)$ then as $x < y$ we have $g(x) > g(y)$ and $f(g(x)) > f(g(y))$ thus $(x, y) \in \mathcal{E}_{f \circ g}$. If $(x, y) \in g^{-1}(\mathcal{E}_f)$ and $(x, y) \notin \mathcal{E}_g$ then as $x < y$ we have $g(x) < g(y)$ and $f(g(x)) > f(g(y))$ thus $(x, y) \in \mathcal{E}_{f \circ g}$.

(2) By contradiction let us assume that there exists $(x, y) \in \mathcal{E}_{f \circ g} \cap \mathcal{E}_g \cap g^{-1}(\mathcal{E}_f)$. We can assume that $x < y$, the case $x > y$ is similar. As $x < y$ and $(x, y) \in \mathcal{E}_{f \circ g} \cap \mathcal{E}_g$ we know that $g(x) > g(y)$ and $f(g(x)) > f(g(y))$. However $g.(x, y) = (g(x), g(y)) \in \mathcal{E}_f$ thus as $g(x) > g(y)$ we have $f(g(x)) < f(g(y))$ which is a contradiction. $\square$

THEOREM 6.22. — The signature $\widehat{\varepsilon_\Gamma^{\bowtie}}$ is a group homomorphism.

Proof. — Let $f, g \in \widehat{\mathrm{IET}^{\bowtie}}(\Gamma)$. Thanks to Lemma 6.20 we have:

$$\widehat{\varepsilon_\Gamma^{\bowtie}}(f \circ g) = \omega_\Gamma(\mathcal{E}_{f \circ g}) \ [\mathrm{mod}\ 2]$$

$$= \omega_\Gamma(\mathcal{E}_{f \circ g}) + \omega_\Gamma(\mathcal{E}_g \cap g^{-1}(\mathcal{E}_f)) - \omega_\Gamma(\mathcal{E}_g \cap g^{-1}(\mathcal{E}_f)) \ [\mathrm{mod}\ 2]$$

$$= \omega_\Gamma(\mathcal{E}_{f \circ g} \sqcup (\mathcal{E}_g \cap g^{-1}(\mathcal{E}_f))) - \omega_\Gamma(\mathcal{E}_g \cap g^{-1}(\mathcal{E}_f)) \ [\mathrm{mod}\ 2]$$

$$= \omega_\Gamma(\mathcal{E}_g \cup g^{-1}(\mathcal{E}_f)) - \omega_\Gamma(\mathcal{E}_g \cap g^{-1}(\mathcal{E}_f)) \ [\mathrm{mod}\ 2]$$

$$= \omega_\Gamma(\mathcal{E}_g) + \omega_\Gamma(g^{-1}(\mathcal{E}_f)) - 2\omega_\Gamma(\mathcal{E}_g \cap g^{-1}(\mathcal{E}_f)) \ [\mathrm{mod}\ 2]$$

$$= \omega_\Gamma(\mathcal{E}_g) + \omega_\Gamma(g^{-1}(\mathcal{E}_f)) \ [\mathrm{mod}\ 2]$$

$$= \widehat{\varepsilon_\Gamma^{\bowtie}}(g) + \widehat{\varepsilon_\Gamma^{\bowtie}}(f) \qquad \square$$

We notice that every finitely supported permutation of support in $\widetilde{\Gamma}$ is in $\mathrm{Ker}(\widehat{\varepsilon_\Gamma^{\bowtie}})$.

COROLLARY 6.23. — *There exists a nontrivial group homomorphism* $\varepsilon_\Gamma^{\bowtie} : \mathrm{IET}^{\bowtie}(\Gamma) \to \bigotimes_{\mathbb{Z}}^2 \widetilde{\Gamma}/2(\bigotimes_{\mathbb{Z}}^2 \widetilde{\Gamma})$.

Example 6.24. — We give the value of $\varepsilon_\Gamma^{\bowtie}$ for two kinds of elements:
(1) Let r be a Γ-reflection of type a then $\varepsilon_\Gamma^{\bowtie}(r) = a \otimes a \ [\mathrm{mod}\ 2]$,
(2) Let s be a Γ-restricted rotation of type (p, q) then:

$$\varepsilon_\Gamma^{\bowtie}(s) = p \otimes q + q \otimes p \ [\mathrm{mod}\ 2]$$

As $\mathrm{IET}^{\bowtie}$ is generated by reflections, we deduce the image of $\varepsilon_\Gamma^{\bowtie}$:

COROLLARY 6.25. — *We have the following isomorphism:*

$$\mathrm{IET}^{\bowtie}(\Gamma)/\mathrm{Ker}(\varepsilon_\Gamma^{\bowtie}) \simeq \mathrm{Im}(\varepsilon_\Gamma^{\bowtie}) = \langle \{a \otimes a \ [\mathrm{mod}\ 2] \mid a \in \widetilde{\Gamma}\}\rangle,$$

where $\langle \{a \otimes a \ [\mathrm{mod}\ 2] \mid a \in \widetilde{\Gamma}\}\rangle$ *is a subgroup of* $\bigotimes_{\mathbb{Z}}^2 \widetilde{\Gamma}/2(\bigotimes_{\mathbb{Z}}^2 \widetilde{\Gamma})$.

6.2.3. Description of $\mathrm{Ker}(\varepsilon_\Gamma^{\bowtie})$

As $\bigotimes_{\mathbb{Z}}^2 \widetilde{\Gamma}/2(\bigotimes_{\mathbb{Z}}^2 \widetilde{\Gamma})$ is an abelian group we know that $D(\mathrm{IET}^{\bowtie}(\Gamma))$ is included in $\mathrm{Ker}(\varepsilon_\Gamma^{\bowtie})$. We will see later that the other inclusion is false in general.

Notation 6.26. — We denote by Ω_Γ the conjugate closure of the group generated by the set of all Γ-reflections of type 2ℓ with $\ell \in \widetilde{\Gamma} \smallsetminus 2\widetilde{\Gamma}$ (the closure inside $\mathrm{IET}^{\bowtie}(\Gamma)$).

The inclusion $\Omega_\Gamma \subset \mathrm{Ker}(\varepsilon_\Gamma^{\bowtie})$ is immediate. By 2.11 we know that every restricted rotation of type (ℓ, ℓ), with $\ell \in \widetilde{\Gamma}$, is in Ω_Γ.

The aim here is to show the equality $\mathrm{Ker}(\varepsilon_\Gamma^{\bowtie}) = D(\mathrm{IET}^{\bowtie}(\Gamma))\Omega_\Gamma$.

We begin by proving the result in the specific case where Γ has finite rank. We will reduce the general case to this one.

LEMMA 6.27. — *Let Γ be a finitely generated subgroup of $\mathbb{R}/\mathbb{Z}$. Then for every $f \in \mathrm{Ker}(\varepsilon_\Gamma^{\bowtie})$ there exist $\delta \in D(\mathrm{IET}^{\bowtie}(\Gamma))$ and $h \in \Omega_\Gamma$ such that $f = \delta h$.*

Proof. — As Γ is finitely generated, we know that $\widetilde{\Gamma}$ is finitely generated. Let d be the rank of $\widetilde{\Gamma}$.

Let $f \in \mathrm{Ker}(\varepsilon_\Gamma^{\bowtie})$. Let $n \in \mathbb{N}$ and $\mathcal{P} := \{I_1, I_2, \ldots, I_n\}$ be a partition into Γ-intervals associated with f. We denote by L_i the length of I_i for every $1 \leqslant i \leqslant n$. By Corollary 2.14 there exists $B := \{\ell_1, \ell_2, \ldots, \ell_d\}$ a basis of $\widetilde{\Gamma}$

with elements in $\widetilde{\Gamma}_+$ such that $L_i \in \mathrm{Vect}_{\mathbb{N}}(B)$ for every $1 \leqslant i \leqslant n$. Hence we can cut each I_i into smaller intervals with length in B. This operation gives us a new partition $\mathcal{Q} := \{J_1, J_2, \ldots, J_k\}$, with $k \in \mathbb{N}$, into Γ-intervals associated with f.

For every $1 \leqslant i \leqslant k$ we define r_i as the J_i-reflection if f is order-reversing on J_i else we put $r_i = \mathrm{Id}$. Let g be the product $fr_1r_2\ldots r_k$; it is an element of $\mathrm{IET}(\Gamma)$ and $\mathcal{Q}$ is a partition into Γ-intervals associated with g. By Lemma 2.7 g can be written as a finite product of Γ-restricted rotations with type inside $\{(l_p, l_q) \mid p, q \in \{1, 2, \ldots, d\}\}$. Thanks to an element of $D(\mathrm{IET}^{\bowtie}(\Gamma))$ we can organize this product to put all Γ-restricted rotations of type (l_p, l_p) together (with $1 \leqslant p \leqslant d$; they are elements in Ω_Γ): there exist $w_1 \in D(\mathrm{IET}^{\bowtie}(\Gamma))$ and $h \in \Omega_\Gamma$ and $m \in \mathbb{N}$ and $s_1, s_2, \ldots, s_m$ some Γ-restricted rotations with type inside $\{(l_p, l_q) \mid p, q \in \{1, 2, \ldots, d\}, \ p \neq q\}$ such that $g = w_1 h s_1 s_2 \ldots s_m$. Then $f = w_1 h s_1 s_2 \ldots s_m r_k r_{k-1} \ldots r_1$.

We define $u_p := \mathrm{Card}\{i \in \{1, 2, \ldots, k\} \mid r_i \neq \mathrm{Id}, \ \mathrm{type}(r_i) = l_p\}$. Let $v_{p,p} = 0$ for every $1 \leqslant p \leqslant d$ and let $v_{p,q} := \mathrm{Card}\{j \in \{1, 2, \ldots, m\} \mid \mathrm{type}(s_j) = (l_p, l_q)\}$ for every $1 \leqslant p \neq q \leqslant d$. Then we have:

$$\varepsilon_\Gamma^{\bowtie}(f) = \sum_{i=1}^{k} \varepsilon_\Gamma^{\bowtie}(r_i) + \sum_{j=1}^{m} \varepsilon_\Gamma^{\bowtie}(s_j)$$

$$= \sum_{p=1}^{d} u_p l_p \otimes l_p + \sum_{p=1}^{d}\sum_{q=1}^{d} v_{p,q}(l_p \otimes l_q + l_q \otimes l_p) \ [\mathrm{mod} \ 2]$$

$$= 0$$

We notice that $\sum_{p=1}^{d}\sum_{q=1}^{d} v_{p,q}(l_p \otimes l_q + l_q \otimes l_p) = \sum_{p=1}^{d}\sum_{q=1}^{d}(v_{p,q} + v_{q,p})l_p \otimes l_q$. Furthermore B is a basis of $\widetilde{\Gamma}$ so $\{l_p \otimes l_q\}_{1 \leqslant p,q \leqslant d}$ is a basis of $\bigotimes_{\mathbb{Z}}^{2} \widetilde{\Gamma}$, thus we deduce that 2 divides u_p for every $1 \leqslant p \leqslant d$ and 2 divides $v_{p,q} + v_{q,p}$ for every $1 \leqslant p, q \leqslant d$.

We obtain that $r_1 r_2 \ldots r_k$ is a balanced product of Γ-reflections hence by Lemma 6.13 it is an element of $D(\mathrm{IET}^{\bowtie}(\Gamma))$ denoted w_2. We also deduce that the product $s_1 s_2 \ldots s_m$ is a balanced product of Γ-restricted rotations with type inside $\{(a, b) \mid a \neq b \in \widetilde{\Gamma}_+\}$. Hence by Proposition 6.15 we obtain that it is also an element of $D(\mathrm{IET}^{\bowtie}(\Gamma))$, denoted w_3.

Finally we have $f = w_1 h w_3 w_2 = \delta h$ with $\delta \in D(\mathrm{IET}^{\bowtie}(\Gamma))$ and $h \in \Omega_\Gamma$. $\qquad\square$

The next lemma gives an inclusion used to conclude in the general case:

LEMMA 6.28. — *For all Γ, A subgroups of $\mathbb{R}/\mathbb{Z}$ such that $A \subset \Gamma$ we have:*

$$D(\mathrm{IET}^{\bowtie}(A))\Omega_A \subset D(\mathrm{IET}^{\bowtie}(\Gamma))\Omega_\Gamma$$

Proof. — The inclusion $D(\mathrm{IET}^{\bowtie}(A)) \subset D(\mathrm{IET}^{\bowtie}(\Gamma))$ is immediate. It is sufficient to show that $\Omega_A \subset D(\mathrm{IET}^{\bowtie}(\Gamma))\Omega_\Gamma$. Let $\widetilde{A}$ be the preimage of A in $\mathbb{R}$.

Let f be an element of Ω_A. Then there exist $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n \in \widetilde{A} \smallsetminus 2\widetilde{A}$ and $w_1, w_2, \ldots, w_n$ some Γ-reflections such that the type of w_i is $2a_i$ and there exist $g_1, g_2, \ldots, g_n \in \mathrm{IET}^{\bowtie}(A)$ such that $f = \prod_{i=1}^n g_i w_i g_i^{-1}$.

Let $U := \{i \in \{1, 2, \ldots, n\} \mid a_i \in \widetilde{\Gamma} \smallsetminus 2\widetilde{\Gamma}\}$. By definition we have $\{g_i w_i g_i^{-1} \mid i \in U\} \subset \Omega_\Gamma$. Take $V := \{1, 2, \ldots, n\} \smallsetminus U$. As A is a subgroup of Γ we deduce that $V = \{j \in \{1, 2, \ldots, n\} \mid a_j \in 2\widetilde{\Gamma}\}$. Thus for every $j \in V$ the type of w_j is in $4\widetilde{\Gamma}$, then by Proposition 6.1 we deduce that w_j and $g_j w_j g_j^{-1}$ belong to $D(\mathrm{IET}^{\bowtie}(\Gamma))$. We know that there exists $h \in D(\mathrm{IET}^{\bowtie}(\Gamma))$ such that:

$$f = h \prod_{j \in V} g_j w_j g_j^{-1} \prod_{i \in U} g_i w_i g_i^{-1}$$

Then $f \in D(\mathrm{IET}^{\bowtie}(\Gamma))\Omega_\Gamma$. $\square$

We can prove the theorem for the general case:

THEOREM 6.29. — *For any dense subgroup Γ of $\mathbb{R}/\mathbb{Z}$ we have:*

$$\mathrm{Ker}(\varepsilon_\Gamma^{\bowtie}) = D(\mathrm{IET}^{\bowtie}(\Gamma))\Omega_\Gamma$$

Proof. — The inclusion from right to left is already proved.

Let $f \in \mathrm{Ker}(\varepsilon_\Gamma^{\bowtie})$, let $n \in \mathbb{N}$ and $\mathcal{P} := \{I_1, I_2, \ldots, I_n\}$ be a partition into Γ-intervals associated with f. We denote by L_i the length of I_i for every $1 \leqslant i \leqslant n$. As $\varepsilon_\Gamma^{\bowtie}(f) = 0$ we know that there exist $k \in \mathbb{N}$ and $a_1, a_2 \ldots, a_k, b_1, b_2, \ldots, b_k \in \widetilde{\Gamma}$ such that $\omega_\Gamma(\mathcal{E}_f) = 2 \sum_{i=1}^k a_i \otimes b_i$ inside $\bigotimes_{\mathbb{Z}}^2 \widetilde{\Gamma}$.

Let $\widetilde{A}$ be the subgroup of $\mathbb{R}$ generated by $\{L_i\}_{i=1\ldots n} \cup \{1\} \cup \{a_i, b_i\}_{i=1\ldots k}$. Then $\widetilde{A}$ contains $\mathbb{Z}$, is finitely generated and is a subgroup of $\widetilde{\Gamma}$. Let A be the image of $\widetilde{A}$ in $\mathbb{R}/\mathbb{Z}$. The partition $\mathcal{P}$ is also a partition into A-intervals associated with f thus f belongs to $\mathrm{IET}^{\bowtie}(A)$. Furthermore we

have $\omega_A(\mathcal{E}_f) = \omega_\Gamma(\mathcal{E}_f)$. Hence:

$$\varepsilon_A^{\bowtie}(f) = [\omega_A(\mathcal{E}_f)]_{\bigotimes_{\mathbb{Z}}^2 A/2(\bigotimes_{\mathbb{Z}}^2 A)}$$

$$= [\omega_\Gamma(\mathcal{E}_f)]_{\bigotimes_{\mathbb{Z}}^2 A/2(\bigotimes_{\mathbb{Z}}^2 A)}$$

$$= \left[2 \sum_{i=1}^{k} a_i \otimes b_i \right]_{\bigotimes_{\mathbb{Z}}^2 A/2(\bigotimes_{\mathbb{Z}}^2 A)}$$

$$= 0$$

By Lemma 6.27 we deduce that $f \in D(\text{IET}^{\bowtie}(A))\Omega_A$ and by Lemma 6.28 we deduce that $f \in D(\text{IET}^{\bowtie}(\Gamma))\Omega_\Gamma$. □

6.3. The positive contribution

We know that every Γ-reflection of type 2ℓ with $\ell \in \widetilde{\Gamma} \smallsetminus 2\widetilde{\Gamma}$ is conjugated to a Γ-restricted rotation of type (ℓ, ℓ). Also this is an element of $\text{IET}^+(\Gamma)$ which is not send on the trivial element by the morphism ε_Γ. We use the notion of positive substitute in order to use the group homomorphism ε_Γ to send such a reflection on a nontrivial element. For this we need to use the upward directed set M_Γ and the $\mathbb{Z}$-linearly independent finite subset S_a of $\widetilde{\Gamma}_+$ for every $a \in M_\Gamma$ given by Corollary 2.13. We recall that for every $a, b \in M_\Gamma$ with $a \leqslant b$ we have $\text{Vect}_\mathbb{N}(S_a) \subset \text{Vect}_\mathbb{N}(S_b)$ and $\widetilde{\Gamma}_+$ is equal to the direct limit $\varinjlim \text{Vect}_\mathbb{N}(S_a)$.

For every $a \in M_\Gamma$, we construct a map depending on S_a. They are not group homomorphisms but satisfy the group homomorphism property on some products thanks to Proposition 6.11.

6.3.1. Some subsets of $\text{IET}^{\bowtie}(\Gamma)$

Let S be a finite set of $\widetilde{\Gamma}_+$. We denote by G_S the set of all f in $\text{IET}^{\bowtie}(\Gamma)$ such that there exists a partition $\mathcal{P}$ into S-intervals associated with f. We remark that G_S is not a group in general. We want to know how these sets and $\text{IET}^{\bowtie}(\Gamma)$ are linked.

PROPOSITION 6.30. — *Let S and T be two finite subsets of $\widetilde{\Gamma}_+$. If $S \subset \text{Vect}_\mathbb{N}(T)$ then $G_S \subset G_T$. More precisely for every partition $\mathcal{P}$ into S-intervals of $[0, 1[$ there exists a refinement $\mathcal{Q}$ of $\mathcal{P}$ which is a partition into T-intervals of $[0, 1[$.*

Proof. — Let $f \in G_S$ and let $\mathcal{P}$ be a partition into S-intervals associated with f. As $S \subset \text{Vect}_{\mathbb{N}}(T)$ each interval $I \in P$ can be subdivided with intervals of length in T. After subdividing this way the intervals of $\mathcal{P}$, we obtain a refinement $\mathcal{Q}$ of $\mathcal{P}$. We notice that $\mathcal{Q}$ is a partition into T-intervals associated with f so $f \in G_T$. □

As $\widetilde{\Gamma}_+$ is the direct limit of the $\text{Vect}_{\mathbb{N}}(S_a)$ we deduce the next proposition:

PROPOSITION 6.31. — *For every finite subset S of $\widetilde{\Gamma}_+$, there exists $a \in M_\Gamma$ such that for every $b \geqslant a$ we have $S \subset \text{Vect}_{\mathbb{N}}(S_b)$.*

From Propositions 6.30 and 6.31 we obtain:

COROLLARY 6.32. — *Let $f \in \text{IET}^{\bowtie}(\Gamma)$. There exists $a \in M_\Gamma$ such that for every $b \geqslant a$ we have $f \in G_{S_b}$.*

We also want to check that if we take any product of elements in $\text{IET}^{\bowtie}(\Gamma)$ there will be a moment where we have a partition associated to the tuple of these elements.

PROPOSITION 6.33. — *Let $k \in \mathbb{N}$ and $f_1, f_2, \ldots, f_k \in \text{IET}^{\bowtie}(\Gamma)$. There exists $a \in M_\Gamma$ such that for every $b \geqslant a$ there exists $\mathcal{P}_b$ a partition into S_b-intervals associated with $(f_1, f_2, \ldots, f_k)$.*

Proof. — Thanks to Proposition 6.30 and Remark 6.7 we deduce that it is sufficient to find only one $a \in M_\Gamma$ such that there exists a partition into S_a-intervals associated with $(f_1, f_2, \ldots, f_k)$.

By Corollary 6.32 there exist $a_i \in M_\Gamma$ and $\mathcal{P}_i$ a partition into S_{a_i} intervals associated with f_i for every $1 \leqslant i \leqslant k$. We denote by V_i the set of all the endpoints of the intervals in $\mathcal{P}_i$. Let $V = V_1 \cup f_1^{-1}(V_2) \cup \cdots \cup (f_{k-1} \ldots f_1)^{-1}(V_k)$. We know that V is finite thus there exist $m \in \mathbb{N}$ and $v_0, v_1, \ldots, v_m \in \widetilde{\Gamma}_+$ such that $V = \{v_i\}_{i=0\ldots m}$. Up to change the order we can assume that $v_0 = 0 < v_1 < v_2 < \cdots < v_{m-1} < v_m = 1$. Let I_j be the interval $[v_{j-1}, v_j[$ and ℓ_j be the length of I_j for every $1 \leqslant j \leqslant m$. Let $T = \{\ell_j\}_{j=1\ldots m}$ and $\mathcal{P} = \{I_j\}_{j=1\ldots m}$.

We have $V_1 \subset V$ then $\mathcal{P}$ is a refinement of $\mathcal{P}_1$ so $\mathcal{P}$ is a partition into T-intervals associated with f_1. Similarly for every $2 \leqslant i \leqslant k$ we know that $V_i \subset f_{i-1} \ldots f_1(V)$ thus $f_{i-1} \ldots f_1(\mathcal{P})$ is a refinement of $\mathcal{P}_i$ so $f_{i-1} \ldots f_1(\mathcal{P})$ is a partition into T-intervals associated with f_i. Hence $\mathcal{P}$ is a partition into T-intervals associated with $(f_1, f_2, \ldots, f_k)$.

Thanks to Proposition 6.31 there exists $a \in M_\Gamma$ such that for every $b \geqslant a$ we have $T \subset \text{Vect}_{\mathbb{N}}(S_b)$ and by Proposition 6.30 and Remark 6.7 there exists a refinement $\mathcal{Q}$ of $\mathcal{P}$ which is a partition into S_a-intervals associated with $(f_1, f_2, \ldots, f_k)$. □

6.3.2. The S-map

Let S be a finite subset of $\widetilde{\Gamma}_+$ for all this subsection. We assume that S is free inside $\mathrm{Vect}_{\mathbb{Z}}(S)$. For every $a, b \in \widetilde{\Gamma}$, we use the notation $[a \wedge b]_{2 \ominus \bigwedge^2_{\mathbb{Z}} \widetilde{\Gamma}} = a \wedge b \,[\mathrm{mod}\ 2]$.

Let $f \in G_S$ and $\mathcal{P}$ be a S-partition associated with f. We show that the value $\varepsilon_\Gamma(f_{\mathcal{P}}^+)\,[\mathrm{mod}\ 2]$ does not depend on $\mathcal{P}$, where ε_Γ is the group homomorphism define in 5.9.

PROPOSITION 6.34. — Let $f \in \mathrm{IET}^{\bowtie}(\Gamma)$ and let $\mathcal{P}, \mathcal{Q}$ be two partitions into S-intervals associated with f. Then $\varepsilon_\Gamma(f_{\mathcal{P}}^+)\,[\mathrm{mod}\ 2] = \varepsilon_\Gamma(f_{\mathcal{Q}}^+)\,[\mathrm{mod}\ 2]$.

Proof. — For this proof we denote by r_I the I-reflection for every subinterval I of $[0, 1[$ and we recall that λ is the Lebesgue measure. First we reduce the case to a simpler one.

Let $f \in \mathrm{IET}^{\bowtie}(\Gamma)$ and $\mathcal{P}, \mathcal{Q}$ be two partitions into S-intervals associated with f. Let $n, k \in \mathbb{N}$ such that $\mathcal{P} = \{I_1, I_2, \ldots, I_n\}$ and $\mathcal{Q} = \{J_1, J_2, \ldots, J_k\}$. Up to change the index we can assume that the I_i are consecutive intervals and so are the J_i. We denote by $\mathcal{M}$ the unique partition into Γ-intervals associated with f that has the minimal number of intervals. Let m be this number and $\mathcal{M} := \{M_1, M_2, \ldots, M_m\}$ where the M_i are consecutive intervals. As $\mathcal{P}, \mathcal{Q}$ are also partitions into Γ-intervals we know that they are refinements of M.

Let $1 \leqslant i \leqslant m$ and let $n_0 = k_0 = 0$. There exist $n_1 < n_2 < \cdots < n_m, k_1 < k_2 < \cdots < k_m \in \mathbb{N}$ such that $M_i = \bigsqcup_{j=n_{i-1}+1}^{n_i} I_j = \bigsqcup_{j=k_{i-1}+1}^{k_i} J_j$. Hence $\lambda(M_i) = \sum_{j=n_{i-1}+1}^{n_i} \lambda(I_i) = \sum_{j=k_{i-1}+1}^{k_i} \lambda(J_j)$. For every $s \in S$ we define $a_s = \mathrm{Card}(\{j \in \{n_{i-1}+1, \ldots, n_i\} \mid \lambda(I_j) = s\}$ and $b_s = \mathrm{Card}(\{j \in \{k_{i-1}+1, \ldots, k_i\} \mid \lambda(J_j) = s\})$. Then $\sum_{s \in S} a_s s = \sum_{s \in S} b_s s$. As S is free in $\mathrm{Vect}_{\mathbb{Z}}(S)$ we deduce that $a_s = b_s$ for every $s \in S$. This implies that $n_i = k_i$ for every $1 \leqslant i \leqslant n$ and $n = k$. We also get the existence of a permutation σ_i of the set $\{n_{i-1}+1, \ldots, n_i\}$ such that $\lambda(I_j) = \lambda(J_{\sigma_i(j)})$ for every $j \in \{n_{i-1}+1, \ldots, n_i\}$.

We deduce that it is sufficient to show the result when only one permutation σ_i is a transposition $(a\ a+1)$ with $n_{i-1}+1 \leqslant a < n_i$.

Let i_0 in $\{1, 2, \ldots, m\}$ and $a \in \{n_{i_0-1}+1, \ldots, n_{i_0}\}$. We assume now that σ_{i_0} is the transposition $(a\ a+1)$ and $\sigma_i = \mathrm{Id}$ for every $i \neq i_0$. Then $I_j = J_j$ for every $j \in \{1, 2, \ldots, n\} \smallsetminus \{a, a+1\}$. We also have $I_a \cup I_{a+1} = J_a \cup J_{a+1}$ and $\lambda(I_{a+1}) = \lambda(J_a)$. In the case where f is order-preserving on M_{i_0} then $f_{\mathcal{P}}^+ = f_{\mathcal{Q}}^+$. In the case where f is order-reversing on M_{i_0} we deduce that

$(f_{\mathcal{Q}}^+)^{-1} \circ f_{\mathcal{P}}^+$ is equal to the square of the restricted rotation whose intervals associated are I_a and I_{a+1} (see Figure 6.4).

In both cases we deduce $\varepsilon_\Gamma(f_P^+)$ [mod 2] $= \varepsilon_\Gamma(f_Q^+)$ [mod 2]. $\qquad\square$
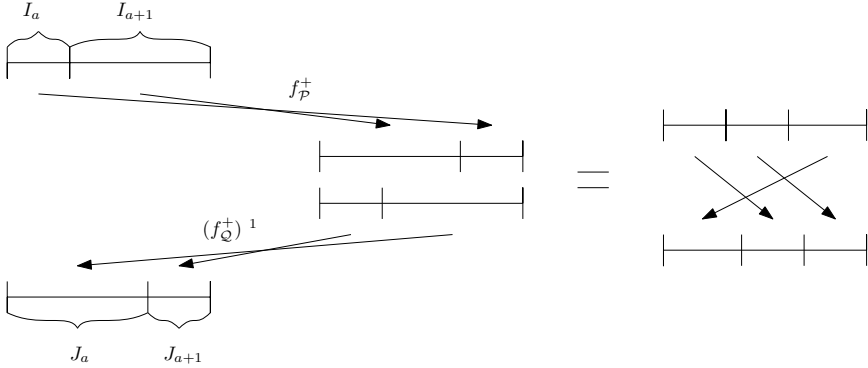

Figure 6.4. Illustration of the fact that $(f_{\mathcal{Q}}^+)^{-1} \circ f_{\mathcal{P}}^+$ is equal to the square of a restricted rotation in Proposition 6.34

DEFINITION 6.35. — *We define the S-map as:*

$$\psi_S : \ \mathrm{IET}^{\bowtie}(\Gamma) \ \longrightarrow \ {}^{\ominus}\!\bigwedge\nolimits_{\mathbb{Z}}^2 \widetilde{\Gamma}/2^{\ominus}\!\bigwedge\nolimits_{\mathbb{Z}}^2 \widetilde{\Gamma}$$

$$f \ \longmapsto \ \begin{cases} \varepsilon_\Gamma(f_{\mathcal{P}}^+) \ [\mathrm{mod}\ 2] & f \in G_S, \\ 0 & f \notin G_S. \end{cases}$$

Where $\mathcal{P}$ is a partition into S-intervals associated with f.

Thanks to Proposition 6.34, the map ψ_S is well-defined. We check that ψ_S satisfies the morphism condition when we have the existence of a partition in S-intervals associated with a tuple.

PROPOSITION 6.36. — *Let $n \in \mathbb{N}$ and $f_1, f_2, \ldots, f_n \in \mathrm{IET}^{\bowtie}(\Gamma)$. If there exists $\mathcal{P}$ a partition into S-intervals associated with $(f_1, f_2, \ldots, f_n)$ then:*

$$\psi_S(f_n f_{n-1} \ldots f_1) = \sum_{i=1}^n \psi_S(f_i)$$

Proof. — Let $n \in \mathbb{N}$ and $f_1, f_2, \ldots, f_n \in \mathrm{IET}^{\bowtie}(\Gamma)$. The case $n = 1$ is trivial thus we assume $n \geqslant 2$. Let $\mathcal{P}$ be a partition into S-intervals associated with $(f_1, f_2, \ldots, f_n)$. Let $g_1 = (f_1)_{\mathcal{P}}^+$ and for every $2 \leqslant i \leqslant n$ let

$g_i = (f_i)^+_{f_{i-1}\ldots f_1(\mathcal{P})}$. By Proposition 6.11 we know that $(f_n f_{n-1} \ldots f_1)^+_{\mathcal{P}} = g_n g_{n-1} \ldots g_1$, then:

$$\psi_S(f_n f_{n-1} \ldots f_1) = \varepsilon_\Gamma((f_n f_{n-1} \ldots f_1)^+_{\mathcal{P}}) \ [\text{mod } 2]$$
$$= \varepsilon_\Gamma(g_n g_{n-1} \ldots g_1) \ [\text{mod } 2]$$

Also ε_Γ is a group homomorphism thus:

$$\psi_S(f_n f_{n-1} \ldots f_1) = \sum_{i=1}^n \varepsilon_\Gamma(g_i) \ [\text{mod } 2] = \sum_{i=1}^n \psi_S(f_i) \qquad \square$$

6.3.3. The group homomorphism

We define the group of germs at infinity of functions from M_Γ to $^\ominus\!\bigwedge^2_\mathbb{Z} \widetilde{\Gamma}/2^\ominus\!\bigwedge^2_\mathbb{Z} \widetilde{\Gamma}$ as the quotient: $\Xi_\Gamma = (^\ominus\!\bigwedge^2_\mathbb{Z} \widetilde{\Gamma}/2^\ominus\!\bigwedge^2_\mathbb{Z} \widetilde{\Gamma})^{M_\Gamma} / \text{Net}_0$, where Net_0 is the normal subgroup of eventually zero functions defined by:

$$\text{Net}_0 := \{(w_a)_{a \in M_\Gamma} \mid \exists a \in M_\Gamma \ \forall b \in M_\Gamma, \ b \geqslant a \Rightarrow w_b = 0\}$$

We are now able to define a new group homomorphism from $\text{IET}^{\bowtie}(\Gamma)$ to Ξ_Γ.

DEFINITION 6.37. — *We define the* positive contribution *of* $\text{IET}^{\bowtie}(\Gamma)$ *as the next map:*

$$\begin{array}{rccc}
\psi_\Gamma : & \text{IET}^{\bowtie}(\Gamma) & \longrightarrow & \Xi_\Gamma \\
& f & \longmapsto & [(\psi_{S_a}(f))_{a \in M_\Gamma}]
\end{array}$$

PROPOSITION 6.38. — *The map* ψ_Γ *is a group homomorphism.*

Proof. — Let $f, g \in \text{IET}^{\bowtie}(\Gamma)$. By Proposition 6.33, there exists $a_0 \in M_\Gamma$ such that for every $b \geqslant a_0$, there exists a partition $\mathcal{P}_b$ into S_b-intervals associated with (f, g). Then by Proposition 6.36 we deduce that for every $b \geqslant a_0$ we have $\psi_{S_b}(g \circ f) = \psi_{S_b}(g) + \psi_{S_b}(f)$.

We denote by $w = (w_a)_{a \in M_\Gamma}$ the element of $(^\ominus\!\bigwedge^2_\mathbb{Z} \widetilde{\Gamma}/2^\ominus\!\bigwedge^2_\mathbb{Z} \widetilde{\Gamma})^{M_\Gamma}$ defined by $w_a = \psi_{S_a}(g \circ f) - \psi_{S_a}(g) - \psi_{S_a}(f)$ for every $a \in M_\Gamma$. We remark that $w_b = 0$ for every $b \geqslant a_0$ thus $w \in \text{Net}_0$. Then we have $(\psi_{S_a}(g \circ f))_a = (\psi_{S_a}(g))_a + (\psi_{S_a}(f))_a - w$. So $\psi_\Gamma(g \circ f) = \psi_\Gamma(g) + \psi_\Gamma(f)$. $\qquad \square$

The following lemma gives the value of an element in $\text{IET}(\Gamma)$. The proof is immediate from the definition of ψ_{S_a}.

LEMMA 6.39. — *For every* $f \in \text{IET}(\Gamma)$, *there exists* $a_0 \in M_\Gamma$ *such that for every* $b \in M_\Gamma$ *with* $b \geqslant a_0$ *we have we have* $\psi_{S_b}(f) = \varepsilon_\Gamma(f) \ [\text{mod } 2]$. *In particular* $\psi_\Gamma(f) = [(\varepsilon_\Gamma(f) \ [\text{mod } 2])_{a \in M_\Gamma}]$.

Remark 6.40. — In fact the definition of Ψ_{S_a} gives us that for every $a \in M_\Gamma$ there exists $a' \in M_\Gamma$ with $a' \geqslant a$ such that for every $b \geqslant a'$ we have $\psi_{S_b}(f) = \varepsilon_\Gamma(f)$ [mod 2]. This is a sufficient condition to obtain the wanted equality in Ξ_Γ. It is also a necessary condition. Indeed, let us assume that there exists $a_0 \in M_\Gamma$ such that for every $b \in M_\Gamma$ with $b \geqslant a_0$ we have $\psi_{S_b}(f) = \varepsilon_\Gamma(f)$ [mod 2]. Then thanks to the property of M_Γ to be an upward directed set we deduce that for every $a \in M_\Gamma$ there exists $c_a \in M_\Gamma$ such that $c_a \geqslant a$ and $c_a \geqslant a_0$. Thus for every $b \in M_\Gamma$ such that $b \geqslant c_a$ we have $\psi_{S_b}(f) = \varepsilon_\Gamma(f)$ [mod 2].

The next proposition show that the set we used to define Ω_Γ is not sent to the trivial element by ψ_Γ; thus this is not a subset of $D(\mathrm{IET}^{\bowtie}(\Gamma))$.

PROPOSITION 6.41. — *Let* $\ell \in \widetilde{\Gamma} \smallsetminus 2\widetilde{\Gamma}$ *and* r *be a* Γ*-reflection of type* 2ℓ. *Then* $\psi_\Gamma(r) = [(\ell \wedge \ell \text{ [mod 2]})_{a \in M_\Gamma}] \neq 0$.

Proof. — We know that such a Γ-reflection is conjugate to a Γ-restricted rotation of type (ℓ, ℓ). We denote by s this Γ-restricted rotation. We have $\varepsilon_\Gamma(s) = \ell \wedge \ell$ [mod 2]. Then by Lemma 6.39 we have $\psi_\Gamma(r) = \psi_\Gamma(s) = [(\ell \wedge \ell \text{ [mod 2]})_a]$. Thus if $\psi_\Gamma(r) = 0$ then there exists $(w_a)_{a \in M_\Gamma} \in \mathrm{Net}_0$ such that $(\ell \wedge \ell \text{ [mod 2]})_a + (w_a)_a = 0$. We deduce that $\ell \wedge \ell$ [mod 2] $= 0$. Or this implies $\ell \in 2\widetilde{\Gamma}$ and this is a contradiction. Hence $\psi_\Gamma(r) \neq 0$. □

6.4. Description of IET$^{\bowtie}(\Gamma)_{\mathrm{ab}}$

With both morphisms $\varepsilon_\Gamma^{\bowtie}$ and ψ_Γ we are now able to describe $D(\mathrm{IET}^{\bowtie}(\Gamma))$. We recall that Ω_Γ is the conjugate closure of the group generated by the set of all Γ-reflections of type 2ℓ with $\ell \in \widetilde{\Gamma} \smallsetminus 2\widetilde{\Gamma}$.

LEMMA 6.42. — *We have the inclusion:*

$$\Omega_\Gamma \cap \mathrm{Ker}(\psi_\Gamma) \subset D(\mathrm{IET}^{\bowtie}(\Gamma))$$

Proof. — Let $w \in \Omega_\Gamma \cap \mathrm{Ker}(\psi_\Gamma)$. There exist $n \in \mathbb{N}$ and $g_1, g_2, \ldots, g_n \in \mathrm{IET}^{\bowtie}(\Gamma)$ and $r_1, r_2, \ldots, r_n$ some Γ-reflections with type inside $\widetilde{\Gamma} \smallsetminus 2\widetilde{\Gamma}$ such that $w = \prod_{i=1}^{n} g_i r_i g_i^{-1}$. We know that every Γ-reflection is conjugate to a Γ-restricted rotation so there exist $h_1, h_2, \ldots, h_n \in \mathrm{IET}^{\bowtie}(\Gamma)$ and $s_1, s_2, \ldots, s_n$ some Γ-restricted rotations such that $s_i = h_i r_i h_i^{-1}$ for every $1 \leqslant i \leqslant n$. Then $w = \prod_{i=1}^{n} g_i h_i^{-1} s_i h_i g_i^{-1}$. By Lemma 6.13 it is sufficient to show that w is a balanced product of Γ-reflections.

As for every $1 \leqslant i \leqslant n$ we have g_i and g_i^{-1} which appear the same number of time in w (the same is true for h and h^{-1}) we deduce that it is sufficient to show that $w' := \prod_{i=1}^n s_i$ is a balanced product of Γ-reflections. By Lemma 6.13 it is enough to show that $w' \in D(\mathrm{IET}^{\bowtie}(\Gamma))$.

We notice that $\psi_\Gamma(w) = \psi_\Gamma(w')$ thus by the assumption we deduce that $\psi_\Gamma(w') = 0$. Hence by Lemma 6.39 we deduce that $\varepsilon_\Gamma(w')$ [mod 2] $= 0$. This equality stands in $\ominus\bigwedge_{\mathbb{Z}}^2 \widetilde{\Gamma}/2\ominus\bigwedge_{\mathbb{Z}}^2 \widetilde{\Gamma}$. There exists $k \in \mathbb{N}$ and for every $1 \leqslant j \leqslant k$ there exist $a_j, b_j \in \widetilde{\Gamma}_+$ with $a_j + b_j < 1$ and $n_j \in \mathbb{N}$ and $\nu_j \in \{-1, 1\}$ such that $\varepsilon_\Gamma(w') + \sum_{j=1}^k 2n_j\nu_j a_j \wedge b_j = 0$. For every $1 \leqslant j \leqslant k$ let γ_j be a Γ-restricted rotation of type (a_j, b_j). Then the element $w' \prod_{j=1}^k (\gamma_j^{n_j\nu_j})^2$ is in $\mathrm{IET}(\Gamma)$ and satisfies:

$$\varepsilon_\Gamma\left(w' \prod_{j=1}^k (\gamma_j^{n_j\nu_j})^2\right) = \varepsilon_\Gamma(w') + \sum_{j=1}^k 2n_j\nu_j a_j \wedge b_j = 0$$

By Theorem 5.23 the element $w' \prod_{j=1}^k (\gamma_j^{n_j\nu_j})^2$ is in $D(\mathrm{IET}(\Gamma)) \subset D(\mathrm{IET}^{\bowtie}(\Gamma))$. By Corollary 6.14 we know that $\prod_{j=1}^k (\gamma_j^{n_j\nu_j})^2$ is in $D(\mathrm{IET}^{\bowtie}(\Gamma))$. Then we deduce that $w' \in D(\mathrm{IET}^{\bowtie}(\Gamma))$. Hence w is in $D(\mathrm{IET}^{\bowtie}(\Gamma))$. $\qquad\square$

THEOREM 6.43. — *We have $D(\mathrm{IET}^{\bowtie}(\Gamma)) = \mathrm{Ker}(\varepsilon_\Gamma^{\bowtie}) \cap \mathrm{Ker}(\psi_\Gamma)$.*

Proof. — The inclusion from left to right is trivial.

Let $f \in \mathrm{Ker}(\varepsilon_\Gamma^{\bowtie}) \cap \mathrm{Ker}(\psi_\Gamma)$. By Theorem 6.29 there exist $g \in D(\mathrm{IET}^{\bowtie}(\Gamma))$ and $h \in \Omega_\Gamma$ such that $f = gh$. We deduce that $\psi_\Gamma(h) = \psi_\Gamma(f) = 0$ so $h \in \mathrm{Ker}(\psi_\Gamma) \cap \Omega_\Gamma$. By Lemma 6.42 we obtain that $h \in D(\mathrm{IET}^{\bowtie}(\Gamma))$, thus $f \in D(\mathrm{IET}^{\bowtie}(\Gamma))$. $\qquad\square$

COROLLARY 6.44. — *We have $D(\mathrm{IET}^{\bowtie}(\Gamma)) = \mathrm{Ker}(\psi_\Gamma|_{\mathrm{Ker}(\varepsilon_\Gamma^{\bowtie})})$.*

LEMMA 6.45. — *The quotient $\mathrm{Ker}(\varepsilon_\Gamma^{\bowtie})/D(\mathrm{IET}^{\bowtie}(\Gamma))$ is isomorphic to the subgroup $\langle\{\ell \wedge \ell$ [mod 2] $\mid \ell \in \widetilde{\Gamma}\}\rangle$ of $\ominus\bigwedge_{\mathbb{Z}}^2 \widetilde{\Gamma}/2\ominus\bigwedge_{\mathbb{Z}}^2 \widetilde{\Gamma}$.*

Proof. — By Corollary 6.44 we have:

$$\mathrm{Ker}(\varepsilon_\Gamma^{\bowtie})/D(\mathrm{IET}^{\bowtie}(\Gamma)) \simeq \mathrm{Im}(\psi_\Gamma|_{\mathrm{Ker}(\varepsilon_\Gamma^{\bowtie})}) = \psi_\Gamma(\mathrm{Ker}(\varepsilon_\Gamma^{\bowtie}))$$

By Theorem 6.29 and as ψ_Γ is a group homomorphism we have the equality $\psi_\Gamma(\mathrm{Ker}(\varepsilon_\Gamma^{\bowtie})) = \psi_\Gamma(\Omega_\Gamma)$. Furthermore Ω_Γ is the normal closure of the group generated by all Γ-reflections of type 2ℓ with $\ell \in \widetilde{\Gamma} \setminus 2\widetilde{\Gamma}$. Hence we deduce that:

$$\psi_\Gamma(\Omega_\Gamma) = \langle\{[(\ell\wedge\ell \text{ [mod 2]})_{n\in\mathbb{N}}] \mid \ell \in \widetilde{\Gamma}\setminus 2\widetilde{\Gamma}\rangle = \langle\{[(\ell\wedge\ell \text{ [mod 2]})_{n\in\mathbb{N}}] \mid \ell \in \widetilde{\Gamma}\rangle$$

Thus $\psi_\Gamma(\mathrm{Ker}(\varepsilon_\Gamma^{\bowtie})) \simeq \langle\{\ell \wedge \ell$ [mod 2] $\mid \ell \in \widetilde{\Gamma}\}\rangle$. $\qquad\square$

THEOREM 6.46. — *We have the following group isomorphisms:*

$$\mathrm{IET}^{\bowtie}(\Gamma)_{\mathrm{ab}} \simeq \mathrm{Im}(\varepsilon_\Gamma^{\bowtie}) \times \mathrm{Ker}(\varepsilon_\Gamma^{\bowtie})/D(\mathrm{IET}^{\bowtie}(\Gamma))$$

$$\simeq \langle\{a \otimes a \;[\mathrm{mod}\;2] \mid a \in \widetilde{\Gamma}\}\rangle \times \langle\{\ell \wedge \ell \;[\mathrm{mod}\;2] \mid \ell \in \widetilde{\Gamma}\}\rangle,$$

where the left term of the product is in $\bigotimes^2_{\mathbb{Z}} \widetilde{\Gamma}/(2\bigotimes^2_{\mathbb{Z}} \widetilde{\Gamma})$ *and the right one is in* ${}^{\ominus}\bigwedge^2_{\mathbb{Z}} \widetilde{\Gamma}/(2{}^{\ominus}\bigwedge^2_{\mathbb{Z}} \widetilde{\Gamma})$.

Proof. — The second isomorphism is given by Corollary 6.25 and Lemma 6.45.

For the first isomorphism we recall that we have the following exact sequence:

$$1 \to \mathrm{Ker}(\varepsilon_\Gamma^{\bowtie})/D(\mathrm{IET}^{\bowtie}(\Gamma)) \to \mathrm{IET}^{\bowtie}(\Gamma)_{\mathrm{ab}} \to \mathrm{IET}^{\bowtie}(\Gamma)/\mathrm{Ker}(\varepsilon_\Gamma^{\bowtie}) \to 1$$

Each group in this exact sequence has exponent 2. Then they are also $\mathbb{F}_2$-vectorial spaces. We deduce that this exact sequence is an exact sequence of $\mathbb{F}_2$-vectorial spaces, thus it splits and gives the result. $\qquad\square$

Remark 6.47. — If $\widetilde{\Gamma}$ has dimension d then $\langle\{a \otimes a \;[\mathrm{mod}\;2] \mid a \in \widetilde{\Gamma}\}\rangle$ has dimension $\frac{d(d+1)}{2}$, as $\mathbb{F}_2$-vector space, and $\langle\{\ell \wedge \ell \;[\mathrm{mod}\;2] \mid \ell \in \widetilde{\Gamma}\}\rangle$ has dimension d, as $\mathbb{F}_2$-vector space, so $\mathrm{IET}^{\bowtie}(\Gamma)_{\mathrm{ab}}$ has dimension $\frac{d(d+3)}{2}$ over $\mathbb{F}_2$.

Remark 6.48. — The inclusion of $\mathrm{IET}(\Gamma)$ in $\mathrm{IET}^{\bowtie}(\Gamma)$ induces a group morphism ι from $\mathrm{IET}(\Gamma)_{\mathrm{ab}}/2(\mathrm{IET}(\Gamma)_{\mathrm{ab}})$ to $\mathrm{IET}^{\bowtie}(\Gamma)_{\mathrm{ab}}$. By Lemma 2.7 we know that $\mathrm{IET}(\Gamma)$ is generated by Γ-restricted rotations thus we deduce that the image of ι is the subgroup $\langle\{p \otimes q + q \otimes p \;[\mathrm{mod}\;2] \mid p, q \in \widetilde{\Gamma}\}\rangle \times \langle\{a \wedge a \;[\mathrm{mod}\;2]\}\rangle$ of $\bigotimes^2_{\mathbb{Z}} \widetilde{\Gamma}/(2\bigotimes^2_{\mathbb{Z}} \widetilde{\Gamma}) \times {}^{\ominus}\bigwedge^2_{\mathbb{Z}} \widetilde{\Gamma}/(2{}^{\ominus}\bigwedge^2_{\mathbb{Z}} \widetilde{\Gamma})$. This is isomorphic to ${}^{\ominus}\bigwedge^2_{\mathbb{Z}} \widetilde{\Gamma}/(2{}^{\ominus}\bigwedge^2_{\mathbb{Z}} \widetilde{\Gamma})$ and if $\widetilde{\Gamma}$ has dimension d then its dimension is $\frac{d(d+1)}{2}$ as $\mathbb{F}_2$-vector space. In this case ι is not surjective and its cokernel has dimension d over $\mathbb{F}_2$. In the case where $\widetilde{\Gamma}$ has infinite dimension over $\mathbb{Z}$ we deduce that $\mathrm{Im}(\iota)$ also has infinite dimension over $\mathbb{F}_2$. By Proposition 5.17 we deduce that the group $\langle\{p \otimes q + q \otimes p \;[\mathrm{mod}\;2] \mid p, q \in \widetilde{\Gamma}\}\rangle$ is equal to the group $\langle\{a \otimes a \;[\mathrm{mod}\;2] \mid a \in \widetilde{\Gamma}\}\rangle$ if and only if $\widetilde{\Gamma} = 2\widetilde{\Gamma}$. Then ι is surjective if and only if $\widetilde{\Gamma} = 2\widetilde{\Gamma}$.

PROPOSITION 6.49. — *The group homomorphism ι is injective.*

Proof. — For every $f \in \mathrm{IET}(\Gamma)$ we denote by $[f]$ its image in $\mathrm{IET}(\Gamma)_{\mathrm{ab}}$. Thanks to Theorem 5.23 we know that $[f] \in 2\,\mathrm{IET}(\Gamma)_{\mathrm{ab}}$ if and only if $\varepsilon_\Gamma(f) \in 2{}^{\ominus}\bigwedge^2_{\mathbb{Z}} \Gamma$. Hence to prove the statement it is enough to prove that for every $f \in \mathrm{IET}(\Gamma)$ such that $\varepsilon_\Gamma^{\bowtie}(f) = 0$ and $\mathrm{proj}_{\mathrm{Ker}(\varepsilon_\Gamma^{\bowtie})}(f) = 0$ we have $\varepsilon_\Gamma(f) \in 2{}^{\ominus}\bigwedge^2_{\mathbb{Z}} \Gamma$. We use notations of inversions defined in Definition 6.16.

By Corollary 2.14 there exists $n \in \mathbb{N}$ and a $\mathbb{Z}$-linearly independent family $\{l_1, l_2, \ldots, l_n\}$ of Γ_+ and $n_{i,j} \in \mathbb{Z}$ such that $\mathcal{E}_{f,1} = \sum_{i,j} n_{i,j} l_i \otimes l_j$. The equality $\mathrm{proj}_{\mathrm{Ker}(\varepsilon_\Gamma^\bowtie)}(f) = 0$ gives us that $n_{i,i} = 0$. We have $\varepsilon_\Gamma^\bowtie(f) = \sum_{i \neq j}(n_{i,j} + n_{j,i})l_i \otimes l_j \ [\mathrm{mod}\ 2] = 0$. We deduce that 2 divides $(n_{i,j} + n_{j,i})$ for every $1 \leqslant i \neq j \leqslant n$. We obtain that:

$$
\begin{aligned}
\varepsilon_\Gamma(f) &= \sum_{i \neq j} n_{i,j} l_i \wedge l_j \\
&= \sum_{i < j} (n_{i,j} - n_{j,i}) l_i \wedge l_j \\
&= \sum_{i < j} (n_{i,j} + n_{j,i}) l_i \wedge l_j - 2 \sum_{i < j} n_{j,i} l_i \wedge l_j
\end{aligned}
$$

We deduce that 2 divides $\varepsilon_\Gamma(f)$ and this gives the result. $\qquad\square$

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