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EVERY COMPACT SET IN C^n IS A GOOD COMPACT SET

by Jan-Erik BJORK

Introduction.

The aim of this note is to establish a conjecture of A. Martineau in [1]. Before we state the Main Theorem which settles this conjecture we introduce some notations.

We denote the n-dimensional complex vector space by C^n and we assume that n coordinates $z_1 \ldots z_n$ are given. By a polynomial we always mean a polynomial in the coordinate functions $z_1 \ldots z_n$. Next we let W be an open set in C^n. We denote by O(W) the algebra of all holomorphic functions on W. If K is a compact subset of W we shall put $O_K(W) = \{f \in O(W)$: there exists a sequence (P_n) of polynomials and some open neighborhood U_f of K such that $\lim |P_n - f|_{U_f} = 0$, i.e. f can be uniformly approximated by polynomials in some neighborhood of K$\}$.

It is obvious that $O_K(W)$ is a subalgebra of O(W). But it is not clear that $O_K(W)$ is a closed subalgebra of the Frechet algebra O(W), where O(W) is equipped with the topology of uniform convergence on compact subsets of W. The Main Theorem shows that $O_K(W)$ is a closed subalgebra of O(W), which means that K is a good compact set in the sense of ([1], p. 18, Definition 1.10).

MAIN THEOREM. — *Let* K *be a compact set in* C^n *and let* W *be an open set containing* K. *Then there exists an open neighborhood* U_0 *of* K *such that if* $f \in O_K(W)$, *then* $\lim |P_n - f|_{U_0} = 0$ *for some sequence of polynomials.*

Before we begin the proof we remark that we shall use some basic ideas which are developped in ([2], Chapter 1, Section G), and except for a direct application of the Oka-Weil Theorem concerning polynomial approximation in C^n no deep methods of several complex variables are used. The discussion which follows contains the preliminary steps towards the proof of the Main Theorem, and it is entirely based on the content from [2].

Preliminaries.

Let W be an open subset of C^n and suppose that A is a closed subalgebra of the Frechet algebra O(W). We also assume that A contains the polynomials and that A is closed under derivation, i.e. if $f \in A$ then $\delta f/\delta z_i \in A$ for each $i = 1 \ldots n$. We denote by Spec(A) the collection of all non zero continuous complex-valued homomorphisms on the Frechet algebra A. The continuity means that to each point x in Spec(A) there exists a compact set S in W such that $|x(f)| \leqslant |f|_S$ for all f in A. Each element f of A may be considered as a complex-valued function $\hat{f}$ on Spec(A) if we define $\hat{f}(x) = x(f)$. Because A contains the polynomials we can determine a map π from Spec(A) into C^n as follows. If $x \in \mathrm{Spec}(A)$ then the map which sends each polynomial P into $\hat{P}(x)$ determines a unique point $\pi(x)$ in C^n for which $\hat{P}(x) = P(\pi(x))$ holds for all polynomials.

Next we use the fact that A is closed under derivations to obtain some properties of the map π. Take a point $x_0 \in \mathrm{Spec}(A)$ and suppose that $|\hat{f}(x_0)| \leqslant |f|_K$ holds for all f in A and for some compact set K in W. We choose $\varepsilon > 0$ so small that if $K_\varepsilon = \{z \in C^n : d(z, K) \leqslant \varepsilon\}$, then $K_\varepsilon \subset W$. Here we are using the metric

$$d(z, w) = \sup\{|z_i - w_i| : i = 1 \ldots n\}$$

in C^n. If we now take a point z in C^n for which $d(z, 0) < \varepsilon$ and if we define the map $L_z : f \longrightarrow \Sigma\, T_k(f) z^k : k = (k_1 \ldots k_n)$ and where $T_k(f) = (D^k f)\hat{}\,(x_0)/k!$, then L_z is a homomorphism on A for which $|L_z(f)| \leqslant |f|_{K_\varepsilon}$ holds. (See p. 47 in [2]). It follows that L_z determines a point $x(z)$ in Spec(A), and here $\pi(x(z)) = \pi(x_0) + z$ holds.

We take the sets (defined for large values of n)

$$W_n(x_0) = \{x(z) : d(z, 0) < 1/n\}$$

as a basis for neighborhoods around the point x_0 in Spec(A). Since x_0 is an arbitrary point in Spec(A) this enables us to define a topology in Spec(A). In this way Spec(A) becomes a locally compact space and the map π defines a local homeomorphism from Spec(A) into C^n. It can also be proved that π defines an analytic structure in Spec(A) in which the functions from A become analytic. Using this fact a classical argument by Weierstrass shows that Spec(A) is a metric space too.

The open set W can be identified with an open subset of Spec(A), for each point evaluation in W determines a homomorphism on A. We denote this open subset of Spec(A) with $j(W)$ and we notice that the restriction of π to the set $j(W)$ maps $j(W)$ homeomorphically onto W. If $z \in W$ we denote by $j(z)$ the unique point in $j(W)$ for which $\pi(j(z)) = z$ holds. With these notations we see that $f(z) = \hat{f}(j(z))$ holds for all f in A.

We remark here that the topology introduced on Spec(A) above actually coincides with the weak A-topology on Spec(A). This implies for example that Spec(A) is a Stein manifold (See p. 55, Theorem 18 in [2]). The result is deep and is originally due to K. Oka when $A = O(W)$ and proved for a general A by E. Bishop. However we shall not need this result in the proof of the Main Theorem.

Proof of the Main Theorem.

Firstly we show that $O_K(W)$ is a subalgebra of $O(W)$ which is closed under derivation. For if $f \in O_K(W)$ and if $\lim |P_n - f|_{U_f} = 0$ for some open neighborhood U_f of K, then it follows that

$$\lim |\delta P_n/\delta z_i - \delta f/\delta z_i|_V = 0$$

if V is a subset of U_f such that its closure $cl(V)$ is contained in U_f. So if we let V be an open neighborhood of K such that $cl(V) \subset U_f$, then it follows that $\delta f/\delta z_i \in O_K(W)$.

We denote by A the closure of $O_K(W)$ in the Frechet algebra $O(W)$. Hence A is a subalgebra of $O(W)$ satisfying the conditions of the preceding discussion. So now we can introduce the set Spec(A) and the associated map π from Spec(A) into C^n. The idea of the proof which follows is to derive some properties of the map π. Firstly we shall need the following property of A.

Let $f \in A$. Since K is a compact subset of W it follows that $\lim |g_n - f|_K = 0$ for some sequence (g_n) in $O_K(W)$. To each g_n we can determine a polynomial P_n such that $|P_n - g_n|_K < 1/n$. It follows that $\lim |P_n - f|_K = 0$, i.e. we have proved that each function in A can be uniformly approximated by polynomials on K.

Now we introduce the set

$$\mathrm{Hull}(K) = \{x \in \mathrm{Spec}(A) : |\hat{f}(x)| \leqslant |f|_K \text{ for all } f \text{ in } A\}.$$

We claim that π determines a homeomorphism from Hull(K) onto P(K), the polynomially convex hull of K in C^n.

Firstly we take a point z in P(K). If $f \in A$ then $\lim |P_n - f|_K = 0$ for some sequence of polynomials (P_n). Since $z \in P(K)$ it follows that $\lim |P_n(z) - P_m(z) \leqslant \lim |P_n - P_m|_K = 0$, so that $\lim P_n(z)$ exists. So if we define $L(f) = \lim P_n(z)$, where (P_n) are polynomials such that $\lim |f - P_n|_K = 0$, then we see that L is a continuous complex-valued homomorphism on A. Hence $L(f) = \hat{f}(x)$ for some point x in Hull(K) and here $\pi(x) = z$ holds. So we have proved that π maps Hull(K) onto P(K). Suppose next that $x_1, x_2 \in \mathrm{Hull}(K)$ are such that $\pi(x_1) = \pi(x_2) = z$ for some z in C^n. Since $|P(z)| = |\hat{P}(x_i)| \leqslant |P|_K$ for all polynomials it follows that $z \in P(K)$. Suppose now that $f \in A$ and let $\lim |f - P_n|_K = 0$ for some polynomials P_n. Since $x_i \in \mathrm{Hull}(K)$ it follows that $\lim |\hat{f}(x_i) - \hat{P}_n(x_i)| \leqslant \lim |f - P_n|_K = 0$. Hence

$$\hat{f}(x_i) = \lim \hat{P}_n(x_i) = \lim P_n(z)$$

holds, so it follows that $\hat{f}(x_1) = \hat{f}(x_2)$. Since this holds for all f in A we conclude that $x_1 = x_2$ in Spec(A). Notice that we also have proved that $\pi(x_1) \in P(K)$ here. Together with the previous result it follows that $\pi(\mathrm{Hull}(K)) = P(K)$ and if $z \in P(K)$ then the set $\pi^{-1}(z) \cap \mathrm{Hull}(K)$ consists of one point.

Next a topological consideration will show that there exists an open neighborhood U of Hull(K) in Spec(A) such that the restriction of π to the set U maps U homeomorphically onto the open set $\pi(U)$ in C^n. For if no such U exists then we can find a sequence (x_n, y_n) of pairs of points in Spec(A) for which $\pi(x_n) = \pi(y_n)$, while x_n and y_n both converge to the set Hull(K). Since we already know that π maps Hull(K) homeomorphically onto P(K) it follows that Hull(K) is a compact subset of Spec(A). So we can pass to a subsequence and assume that $\lim x_n = x_0$ and that $\lim y_n = y_0$ both exists in Spec(A). Clearly x_0 and y_0 belong to Hull(K). We also have $\pi(x_0) = \lim \pi(x_n) = \lim \pi(y_n) = \pi(y_0)$. Hence $x_0 = y_0$ follows, and now we have derived a contradiction because π is a local homeomorphism in a neighborhood of x_0.

So we have proved that there exists an open set U in Spec(A) such that Hull(K) is contained in U while π maps U homeomorphically onto the open set $\pi(U)$ in C^n. Suppose now that $f \in A$. Then we define a function f_0 on $\pi(U)$ as follows. If $z \in \pi(U)$ we put $f_0(z) = \hat{f}(x(z))$, where $x(z)$ is the unique point in the set $\pi^{-1}(z) \cap U$. The properties of π show that f_0 is analytic on $\pi(U)$. Clearly P(K) is contained in $\pi(U)$ so we can choose a compact polynomially convex set S, where P(K) is contained in the interior of S while S is contained in $\pi(U)$. Then the Oka-Weil Theorem shows that if $f \in A$ then there exists a sequence (P_n) of polynomials for which $\lim |P_n - f_0|_S = 0$.

Now we can use the set S above to finish the proof. Let $f \in A$ so that f_0 is determined on $\pi(U)$. If $z \in K$ we see that

$$f_0(z) = \hat{f}(j(z)) = f(z) \ ,$$

where $f(z)$ denotes the original value of the function f considered as an element of O(W). We claim that there exists an open set U_1 in C^n, where $K \subset U_1 \subset (\pi(U) \cap W)$ and for which $f_0(z) = f(z)$ holds for all f in A and for all $z \in U_1$.

For let z_0 be a point in K. Then $j(z_0) \in U$ so we can choose an open neighborhood Δ of $j(z_0)$ in Spec(A) such that Δ is contained in the set $j(W) \cap U$. It follows that if $z \in \pi(\Delta)$ and if $x(z)$ is the point in U for which $\pi(x(z)) = z$, then $x(z) = j(z)$ must hold. Hence $f_0(z) = \hat{f}(x(z)) = \hat{f}(j(z)) = f(z)$. Since $\pi(\Delta)$ is an open set in C^n here we can cover K by a finite union of such open sets and obtain the set U_1.

(1) The proof gives a sharper version of the Main Theorem as follows. Let f O(W) be such that f together with all its derivatives can be uniformly approximated by polynomials on K. Then f can be uniformly approximated on U_0 by polynomials.

Finally we put $U_0 = U_1 \cap \mathrm{int}(S)$, so that U_0 is an open neighborhood of K in C^n. If now $f \in A$ then we can determine a sequence (P_n) of polynomials for which $\lim |P_n - f_0|_S = 0$. It follows that $\lim |P_n - f|_{U_0} = 0$, so that U_0 gives the required neighborhood in the Main Theorem.

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