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Vanishing cohomology and Betti bounds for complex projective hypersurfaces

VANISHING COHOMOLOGY AND BETTI BOUNDS FOR COMPLEX PROJECTIVE HYPERSURFACES

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ABSTRACT. — We employ the formalism of vanishing cycles and perverse sheaves to introduce and study the vanishing cohomology of complex projective hypersurfaces. As a consequence, we give upper bounds for the Betti numbers of projective hypersurfaces, generalizing those obtained by different methods by Dimca in the isolated singularities case, and by Siersma–Tibăr in the case of hypersurfaces with a 1-dimensional singular locus. We also prove a supplement to the Lefschetz hyperplane theorem for hypersurfaces, which takes the dimension of the singular locus into account, and we use it to give a new proof of a result of Kato.

RÉSUMÉ. — Nous utilisons le formalisme des cycles évanescents et des faisceaux pervers pour introduire et étudier la cohomologie évanescente des hypersurfaces projectives. Nous déduisons des majorants pour les nombres de Betti des hypersurfaces projectives, en généralisant ceux obtenus avec des méthodes différentes par Dimca dans le cas des singularités isolées, et par Siersma–Tibăr dans le cas des hypersurfaces avec lieu singulier de dimension 1. Nous prouvons aussi un complément au théorème de la section hyperplane de Lefschetz pour les hypersurfaces qui tient compte de la dimension du lieu singulier, et nous l'utilisons pour donner une nouvelle preuve du résultat de Kato.

1. Introduction. Results

Let $V = \{f = 0\} \subset \mathbb{C}P^{n+1}$ be a reduced complex projective hypersurface of degree d, with $n \geqslant 1$. By the classical Lefschetz Theorem, the inclusion map $j : V \hookrightarrow \mathbb{C}P^{n+1}$ induces cohomology isomorphisms

$$(1.1) \qquad j^k : H^k\left(\mathbb{C}P^{n+1}; \mathbb{Z}\right) \xrightarrow{\cong} H^k(V; \mathbb{Z}) \quad \text{for all} \quad k < n,$$

and a primitive monomorphism for $k = n$ (e.g., see [2, Theorem 5.2.6]). Moreover, if $s = \dim V_{\text{sing}} < n$ is the complex dimension of the singular locus of V (with $\dim \emptyset = -1$), then Kato [4] showed that (see also [2, Theorem 5.2.11])

$$(1.2) \qquad H^k(V; \mathbb{Z}) \cong H^k\left(\mathbb{C}P^{n+1}; \mathbb{Z}\right) \quad \text{for all} \quad n + s + 2 \leqslant k \leqslant 2n,$$

and the homomorphism j^k induced by inclusion is given in this range (and for k even) by multiplication by $d = \deg(V)$. It therefore remains to study the cohomology groups $H^k(V; \mathbb{Z})$ for $n \leqslant k \leqslant n + s + 1$.

In the case when $V \subset \mathbb{C}P^{n+1}$ is a *smooth* degree d hypersurface, the above discussion yields that $H^k(V; \mathbb{Z}) \cong H^k(\mathbb{C}P^n; \mathbb{Z})$ for all $k \neq n$. This is in fact the only information we take as an input in this note (it also suffices to work with (1.1), its homology counterpart, and Poincaré duality). The Universal Coefficient Theorem also yields in this case that $H^n(V; \mathbb{Z})$ is free abelian, and its rank $b_n(V)$ can be easily deduced from the formula for the Euler characteristic of V (e.g., see [10, Proposition 10.4.1]):

$$(1.3) \qquad \chi(V) = (n + 2) - \frac{1}{d}\left[1 + (-1)^{n+1}(d-1)^{n+2}\right].$$

Specifically, if $V \subset \mathbb{C}P^{n+1}$ is a smooth degree d projective hypersurface, one has:

$$(1.4) \qquad b_n(V) = \frac{(d-1)^{n+2} + (-1)^{n+1}}{d} + \frac{3(-1)^n + 1}{2}.$$

The case when V has only isolated singularities was studied by Dimca [1, 2], (see also [13] and [17]) while projective hypersurfaces with a one-dimensional singular locus have been more recently considered by Siersma–Tibăr [17].

In the singular case, let us fix a Whitney stratification $\mathcal{V}$ of V and consider a one-parameter smoothing of degree d, namely

$$V_t := \{f_t = f - tg = 0\} \subset \mathbb{C}P^{n+1} \quad (t \in \mathbb{C}),$$

for g a general polynomial of degree d. Here, the meaning of "general" is that the hypersurface $W := \{g = 0\}$ is smooth and transverse to all strata in the stratification $\mathcal{V}$ of V. Then, for $t \neq 0$ small enough, all the V_t are smooth and transverse to the stratification $\mathcal{V}$. Let

$$B = \{f = g = 0\}$$

be the base locus (axis) of the pencil. Consider the incidence variety

$$V_D := \left\{(x, t) \in \mathbb{C}P^{n+1} \times D \,\middle|\, x \in V_t\right\}$$

with D a small disc centered at $0 \in \mathbb{C}$ so that V_t is smooth for all $t \in D^* := D \setminus \{0\}$. Denote by $\pi : V_D \to D$ the proper projection map, and note that $V = V_0 = \pi^{-1}(0)$ and $V_t = \pi^{-1}(t)$ for all $t \in D^*$. In what follows we write V for V_0 and use V_t for a smoothing of V (i.e., with $t \in D^*$). In this setup, one can define the Deligne vanishing cycle complex of the family π, see [10, Section 10.3] for a quick introduction. More precisely, one has a bounded constructible complex

$$\varphi_\pi \mathbb{Z}_{V_D} \in D^b_c(V)$$

on the hypersurface V, whose hypercohomology groups fit into a long exact sequence (called the *specialization sequence*):

(1.5)
$$\cdots \to H^k(V;\mathbb{Z}) \xrightarrow{sp^k} H^k(V_t;\mathbb{Z}) \xrightarrow{\alpha^k} \mathbb{H}^k\left(V;\varphi_\pi \mathbb{Z}_{V_D}\right)$$
$$\to H^{k+1}(V;\mathbb{Z}) \xrightarrow{sp^{k+1}} \cdots$$

The maps sp^k are called the *specialization* morphisms, while the α^k's are usually referred to as the *canonical* maps. For any integer k, we define

$$H^k_\varphi(V) := \mathbb{H}^k\left(V;\varphi_\pi \mathbb{Z}_{V_D}\right)$$

and call it the kth *vanishing cohomology group* of V. This is an invariant of V, i.e., it does not depend on the choice of a particular smoothing of degree d (since all smooth hypersurfaces of a fixed degree are diffeomorphic). By its very definition, the vanishing cohomology measures the difference between the topology of a given projective hypersurface V and that of a smooth hypersurface of the same degree.

Remark 1.1. — Since the incidence variety $V_D = \pi^{-1}(D)$ deformation retracts to $V = \pi^{-1}(0)$, and the *specialization map* $sp^k : H^k(V;\mathbb{Z}) \to H^k(V_t;\mathbb{Z})$ of (1.5) factorizes as

$$H^k(V;\mathbb{Z}) \xrightarrow{\cong} H^k(V_D;\mathbb{Z}) \longrightarrow H^k(V_t;\mathbb{Z})$$

with $H^k(V_D;\mathbb{Z}) \to H^k(V_t;\mathbb{Z})$ induced by the inclusion map, it follows readily that the vanishing cohomology of V can be identified with the relative cohomology of the pair (V_D, V), i.e.,

(1.6)
$$H^k_\varphi(V) \cong H^{k+1}(V_D, V_t;\mathbb{Z}).$$

In particular, the groups $H^k_\varphi(V)$ are the cohomological version of the *vanishing homology groups*

$$H^\vee_k(V) := H_k(V_D, V_t;\mathbb{Z})$$

introduced and studied in [17] in special situations. For the purpose of computing Betti numbers of projective hypersurfaces, the two "vanishing"

theories yield the same answer, but additional care is needed to handle torsion when computing the actual integral cohomology groups.

Our first result gives the concentration degrees of the vanishing cohomology of a projective hypersurface in terms of the dimension of the singular locus.

THEOREM 1.2. — *Let $V \subset \mathbb{C}P^{n+1}$ be a reduced complex projective hypersurface with $s = \dim V_{\mathrm{sing}}$ the complex dimension of its singular locus. Then*

$$(1.7) \qquad H_{\varphi}^{k}(V) \cong 0 \quad \text{for all integers} \quad k \notin [n, n+s].$$

Moreover, $H_{\varphi}^{n}(V)$ is a free abelian group.

In view of Remark 1.1, one gets by Theorem 1.2 and the Universal Coefficient Theorem the concentration degrees of the vanishing homology groups $H_{k}^{\curlyvee}(V)$ of a projective hypersurface in terms of the dimension of its singular locus:

COROLLARY 1.3. — *With the above notations and assumptions, we have that*

$$(1.8) \qquad H_{k}^{\curlyvee}(V) \cong 0 \quad \text{for all integers} \quad k \notin [n+1, n+s+1].$$

Moreover, $H_{n+s+1}^{\curlyvee}(V)$ is free.

Remark 1.4. — In the case when the projective hypersurface $V \subset \mathbb{C}P^{n+1}$ has a 1-dimensional singular locus, it was shown in [17, Theorem 4.1] that $H_{k}^{\curlyvee}(V) \cong 0$ for all $k \neq n+1, n+2$. Moreover, [17, Theorem 6.1] shows that in this case one also has that $H_{n+2}^{\curlyvee}(V)$ is free. So, Corollary 1.3 provides a generalization of the results of [17] to projective hypersurfaces with arbitrary singularities. Nevertheless, the methods used in its proof are fundamentally different from those in [17].

As a consequence of Theorem 1.2, the specialization sequence (1.5) together with the fact that the integral cohomology of a smooth projective hypersurface is free, yield the following result on the integral cohomology of a complex projective hypersurface (where the estimate on the nth Betti number uses formula (1.4)):

COROLLARY 1.5. — *Let $V \subset \mathbb{C}P^{n+1}$ be a degree d reduced projective hypersurface with a singular locus V_{sing} of complex dimension s. Then:*
  (i) $H^{k}(V; \mathbb{Z}) \cong H^{k}(V_{t}; \mathbb{Z}) \cong H^{k}(\mathbb{C}P^{n}; \mathbb{Z})$ *for all integers $k \notin [n, n+s+1]$.*
  (ii) $H^{n}(V; \mathbb{Z}) \cong \ker(\alpha^{n})$ *is free.*

(iii) $H^{n+s+1}(V;\mathbb{Z}) \cong H^{n+s+1}(\mathbb{C}P^n;\mathbb{Z}) \oplus \operatorname{coker}(\alpha^{n+s})$.

(iv) $H^k(V;\mathbb{Z}) \cong \ker(\alpha^k) \oplus \operatorname{coker}(\alpha^{k-1})$ *for all integers* $k \in [n+1, n+s]$, $s \geqslant 1$.

In particular,

$$b_n(V) \leqslant b_n(V_t) = \frac{(d-1)^{n+2} + (-1)^{n+1}}{d} + \frac{3(-1)^n + 1}{2},$$

and

$$b_k(V) \leqslant \operatorname{rank} H_\varphi^{k-1}(V) + b_k(\mathbb{C}P^n) \quad \text{for all} \quad k \in [n+1, n+s+1], \ s \geqslant 0.$$

The homological version of the specialisation sequence (1.5) identifies to the long exact sequence of the pair (V_D, V_t), namely:

$$\cdots \to H_{k+1}(V_t;\mathbb{Z}) \to H_{k+1}(V_D;\mathbb{Z}) \to H_{k+1}^{\curlyvee}(V;\mathbb{Z}) \xrightarrow{\alpha_k} H_k(V_t;\mathbb{Z}) \to \cdots$$

The inclusions $V_t \hookrightarrow V_D \hookrightarrow \mathbb{C}P^{n+1} \times D$ induce in homology a commutative triangle, where $H_k(V_t;\mathbb{Z}) \to H_k(\mathbb{C}P^{n+1} \times D;\mathbb{Z})$ is injective for $k \neq n$ (by the Lefschetz Theorem for $k < n$, and it is multiplication by d for $k > n$, see, e.g., Remark 5.5 for the homological version of the proof of Theorem 5.3). This shows that the morphisms $H_k(V_t;\mathbb{Z}) \to H_k(V_D;\mathbb{Z})$ is also injective for all $k \neq n$, and therefore $\alpha_k = 0$ for $k \neq n$. Consequently, the above long exact sequence splits into a 5-term exact sequence, and short exact sequences:

$$0 \to H_{n+1}(V_t;\mathbb{Z}) \to H_{n+1}(V;\mathbb{Z}) \to H_{n+1}^{\curlyvee}(V;\mathbb{Z})$$

(1.9) $\quad \xrightarrow{\alpha_n} H_n(V_t;\mathbb{Z}) \to H_n(V;\mathbb{Z}) \to 0.$

$$0 \to H_k(V_t;\mathbb{Z}) \to H_k(V_D;\mathbb{Z}) \to H_k^{\curlyvee}(V;\mathbb{Z}) \to 0 \qquad \text{for} \quad k \geqslant n+1.$$

We then get the following homological version of Corollary 1.5((i)-(iv)), with the same upper bounds for Betti numbers, but with an interesting improvement for (iii) and (iv) showing more explicitly the dependence of the homology of V on the vanishing homology groups:

COROLLARY 1.6. — *Let* $V \subset \mathbb{C}P^{n+1}$ *be a degree d reduced projective hypersurface with a singular locus V_{sing} of complex dimension s. Then:*

(i') $H_k(V;\mathbb{Z}) \cong H_k(V_t;\mathbb{Z}) \cong H_k(\mathbb{C}P^n;\mathbb{Z})$ *for all $k \leqslant n-1$ and all* $k \geqslant n+s+2$.

(ii') $H_n(V;\mathbb{Z}) \cong \operatorname{coker}(\alpha_n)$.

(iii') $H_{n+1}(V;\mathbb{Z}) \cong \ker(\alpha_n) \oplus H_{n+1}(\mathbb{C}P^n;\mathbb{Z})$.

(iv') $H_k(V;\mathbb{Z}) \cong H_k^{\curlyvee}(V;\mathbb{Z}) \oplus H_k(\mathbb{C}P^n;\mathbb{Z})$, *for all $n+2 \leqslant k \leqslant n+s+1$, whenever $s \geqslant 1$, and $H_{n+s+1}(V;\mathbb{Z})$ is free.*

The ranks of the (possibly non-trivial) vanishing (co)homology groups can be estimated in terms of the local topology of singular strata and of their generic transversal types by making use of the hypercohomology spectral sequence. Such estimates can be made precise for hypersurfaces with low-dimensional singular loci. Concretely, as special cases of Corollaries 1.5 and 1.6, in Section 3 we recast Siersma–Tibăr's [17] result for $s \leqslant 1$, and in particular Dimca's [1, 2] computation for $s = 0$. Concerning the estimation of the rank of the highest interesting (co)homology group, we prove the following general result:

THEOREM 1.7. — *Let $V \subset \mathbb{C}P^{n+1}$ be a degree d reduced projective hypersurface with a singular locus V_{sing} of complex dimension s. For each connected stratum $S_i \subseteq V_{\mathrm{sing}}$ of top dimension s in a Whitney stratification of V, let $F_i^{\pitchfork}$ denote its transversal Milnor fiber with corresponding Milnor number $\mu_i^{\pitchfork}$. Then:*

$$(1.10) \qquad b_{n+s+1}(V) \leqslant 1 + \sum_i \mu_i^{\pitchfork},$$

and the inequality is strict for $n + s$ even.

In fact, the inequality in (1.10) is deduced from

$$(1.11) \qquad b_{n+s+1}(V) \leqslant 1 + \mathrm{rank}\ H_{\varphi}^{n+s}(V),$$

together with

$$(1.12) \qquad \mathrm{rank}\ H_{\varphi}^{n+s}(V) \leqslant \sum_i \mu_i^{\pitchfork},$$

and the inequality (1.11) is strict for $n + s$ even. For further refinements of Theorem 1.7, see Remark 3.1. Note also that if $s = 0$, i.e., V has only isolated singularities, then $\mu_i^{\pitchfork}$ is just the usual Milnor number of such a singularity of V.

Let us remark that if the projective hypersurface $V \subset \mathbb{C}P^{n+1}$ has singularities in codimension 1, i.e., $s = n - 1$, then $b_{n+s+1}(V) = b_{2n}(V) = r$, where r denotes the number of irreducible components of V. Indeed, in this case, one has (e.g., see [2, (5.2.9)]):

$$(1.13) \qquad H^{2n}(V; \mathbb{Z}) \cong \mathbb{Z}^r.$$

In particular, Theorem 1.7 yields the following generalization of [17, Corollary 7.6]:

Corollary 1.8. — *If the reduced projective hypersurface $V \subset \mathbb{C}P^{n+1}$ has singularities in codimension 1, then the number r of irreducible components of V satisfies the inequality:*

$$(1.14) \qquad r \leqslant 1 + \sum_i \mu_i^{\pitchfork}.$$

Remark 1.9. — Note that if the projective hypersurface $V \subset \mathbb{C}P^{n+1}$ is a rational homology manifold, then the Lefschetz isomorphism (1.1) and Poincaré duality over the rationals yield that $b_i(V) = b_i(\mathbb{C}P^n)$ for all $i \neq n$. Moreover, $b_n(V)$ can be deduced by computing the Euler characteristic of V, e.g., as in [10, Section 10.4].

The computation of Betti numbers of a projective hypersurface which is a rational homology manifold can be deduced without appealing to Poincaré duality by using the vanishing cohomology instead, as the next result shows:

Proposition 1.10. — *If the projective hypersurface $V \subset \mathbb{C}P^{n+1}$ is a $\mathbb{Q}$-homology manifold, then $H_\varphi^k(V) \otimes \mathbb{Q} \cong 0$ for all $k \neq n$. In particular, in this case one gets: $b_i(V) = b_i(V_t) = b_i(\mathbb{C}P^n)$ for all $i \neq n$, and $b_n(V) = b_n(V_t) + \operatorname{rank} H_\varphi^n(V)$.*

At this end, we note that Corollary 1.5(i) reproves Kato's isomorphism (1.2) about the integral cohomology of V, by using only the integral cohomology of a smooth hypersurface (for this it suffices to rely only on the Lefschetz isomorphism (1.1), its homological version, and Poincaré duality). In Section 5, we give a new proof of Kato's result (see Theorem 5.3), which relies on the following supplement to the Lefschetz hyperplane section theorem for hypersurfaces, which may be of independent interest:

Theorem 1.11. — *Let $V \subset \mathbb{C}P^{n+1}$ be a reduced complex projective hypersurface with $s = \dim V_{\text{sing}}$ the complex dimension of its singular locus. (By convention, we set $s = -1$ if V is nonsingular.) Let $H \subset \mathbb{C}P^{n+1}$ be a generic hyperplane. Then*

$$(1.15) \qquad H^k(V, V \cap H; \mathbb{Z}) = 0 \quad \text{for} \quad k < n \quad \text{and} \quad n + s + 1 < k < 2n.$$

Moreover, $H^{2n}(V, V \cap H; \mathbb{Z}) \cong \mathbb{Z}^r$, where r is the number of irreducible components of V, and $H^n(V, V \cap H; \mathbb{Z})$ is (torsion-)free.

Note that the vanishing (1.15) for $k < n$ is equivalent to the classical Lefschetz hyperplane section theorem. The proof of (1.15) for $n + s + 1 < k < 2n$ reduces to understanding the homotopy type of the complement of a smooth affine hypersurface transversal to the hyperplane at infinity; see [7, Corollary 1.2] for such a description. Homological counterparts of

Theorem 1.11 and of Kato's result are also explained in Section 5, see Corollary 5.2 and Remark 5.5.

Finally, let us note that similar techniques apply to the study of Milnor fiber cohomology of complex hypersurface singularity germs. This is addressed by the authors in the follow-up paper [11] (see also [16] for the case of 1-dimensional singularities).

Acknowledgements

2. Concentration degrees of vanishing cohomology

The proof of Theorem 1.2 makes use of the formalism of perverse sheaves and their relation to vanishing cycles, see [3, 10] for a brief introduction.

2.1. Proof of Theorem 1.2

By definition, the incidence variety V_D is a complete intersection of pure complex dimension $n + 1$. It is non-singular if $V = V_0$ has only isolated singularities, but otherwise it has singularities where the base locus $B = V \cap W$ of the pencil $\{f_t\}_{t \in D}$ intersects the singular locus $\Sigma := V_{\text{sing}}$ of V.

If $\mathbb{Z}_{V_D}$ denotes the constant sheaf with stalk $\mathbb{Z}$ on the complete intersection V_D, a result of Lê [6] implies that the complex $\mathbb{Z}_{V_D}[n + 1]$ is a perverse sheaf on V_D. It then follows that $\varphi_\pi \mathbb{Z}_{V_D}[n]$ is a $\mathbb{Z}$-perverse sheaf on $\pi^{-1}(0) = V$ (see, e.g., [10, Theorem 10.3.13] and the references therein).

Recall that the stalk of the cohomology sheaves of $\varphi_\pi \mathbb{Z}_{V_D}$ at a point $x \in V$ are computed by (e.g., see [10, (10.20)]):

$$(2.1) \qquad \mathcal{H}^j \left(\varphi_\pi \underline{\mathbb{Z}}_{V_D} \right)_x \cong H^{j+1} \left(B_x, B_x \cap V_t; \mathbb{Z} \right),$$

where B_x denotes the intersection of V_D with a sufficiently small ball in some chosen affine chart $\mathbb{C}^{n+1} \times D$ of the ambient space $\mathbb{C}P^{n+1} \times D$ (hence B_x is contractible). Here $B_x \cap V_t = F_{\pi,x}$ is the Milnor fiber of π at x. Let us now consider the function

$$h = f/g : \mathbb{C}P^{n+1} \setminus W \to \mathbb{C}$$

where $W := \{g = 0\}$, and note that $h^{-1}(0) = V \setminus B$ with $B = V \cap W$ the base locus of the pencil. If $x \in V \setminus B$, then in a neighborhood of x one can describe V_t $(t \in D^*)$ as

$$\{x | f_t(x) = 0\} = \{x | h(x) = t\},$$

i.e., as the Milnor fiber of h at x. Note also that h defines V in a neighborhood of $x \notin B$. Since the Milnor fiber of a complex hypersurface singularity germ does not depend on the choice of a local equation (e.g., see [2, Remark 3.1.8]), we can therefore use h or a local representative of f when considering Milnor fibers (of π) at points in $V \setminus B$. From here on we will use the notation F_x for the Milnor fiber of the hypersurface singularity germ (V, x), and we note for future reference that the above discussion also yields that F_x is a manifold, which moreover is contractible if $x \in V \setminus B$ is a smooth point.

It was shown in [14, Proposition 5.1] (see also [12, Proposition 4.1] or [17, Lemma 4.2]) that there are no vanishing cycles along the base locus B, i.e.,

$$(2.2) \qquad \varphi_\pi \mathbb{Z}_{V_D} |_B \simeq 0.$$

Therefore, if $u : V \setminus B \hookrightarrow V$ is the open inclusion, we get that

$$(2.3) \qquad \varphi_\pi \mathbb{Z}_{V_D} \simeq u_! u^* \varphi_\pi \mathbb{Z}_{V_D}.$$

Since pullback to open subvarieties preserves perverse sheaves, we note that $u^* \varphi_\pi \mathbb{Z}_{V_D}[n]$ is a perverse sheaf on the *affine* variety $V \setminus B$. Artin's vanishing theorem for perverse sheaves (e.g., [15, Corollary 6.0.4]) then implies that:

$$
\begin{aligned}
H^k_\varphi(V) &:= \mathbb{H}^k\left(V; \varphi_\pi \mathbb{Z}_{V_D}\right) \\
&\cong \mathbb{H}^{k-n}\left(V; \varphi_\pi \mathbb{Z}_{V_D}[n]\right) \\
(2.4) \qquad &\cong \mathbb{H}^{k-n}\left(V; u_! u^* \varphi_\pi \mathbb{Z}_{V_D}[n]\right) \\
&\cong \mathbb{H}^{k-n}_c\left(V \setminus B; u^* \varphi_\pi \mathbb{Z}_{V_D}[n]\right) \\
&\cong 0
\end{aligned}
$$

for all $k - n < 0$, or equivalently, for all $k < n$.

Contractibility of Milnor fibers at smooth points of $V \setminus B$ implies that the support of $\varphi_\pi \mathbb{Z}_{V_D}$ is in fact contained in $\Sigma \setminus B$, with Σ denoting as before the singular locus of V. In particular, if $v : \Sigma \setminus B \hookrightarrow V \setminus B$ is the closed inclusion, then

$$(2.5) \qquad u^* \varphi_\pi \mathbb{Z}_{V_D} \simeq v_! v^* u^* \varphi_\pi \mathbb{Z}_{V_D}.$$

Next, consider the composition of inclusion maps

$$\Sigma \setminus B \xhookrightarrow{q} \Sigma \xhookrightarrow{p} V$$

with $p \circ q = u \circ v$. By using (2.3) and (2.5), we get:

$$
\begin{aligned}
\varphi_\pi \underline{\mathbb{Z}}_{V_D} &\simeq u_! v_! v^* u^* \varphi_\pi \underline{\mathbb{Z}}_{V_D} \\
&\simeq (u \circ v)_! (u \circ v)^* \varphi_\pi \underline{\mathbb{Z}}_{V_D} \\
&\simeq (p \circ q)_! (p \circ q)^* \varphi_\pi \underline{\mathbb{Z}}_{V_D} \\
&\simeq p_! q_! q^* p^* \varphi_\pi \underline{\mathbb{Z}}_{V_D} \\
&\simeq p_* p^* \varphi_\pi \underline{\mathbb{Z}}_{V_D},
\end{aligned}
$$

(2.6)

where the last isomorphism uses the fact that $p^* \varphi_\pi \underline{\mathbb{Z}}_{V_D}$ is supported on $\Sigma \setminus B$, hence $p^* \varphi_\pi \underline{\mathbb{Z}}_{V_D} \simeq q_! q^* p^* \varphi_\pi \underline{\mathbb{Z}}_{V_D}$. Since the support of the perverse sheaf $\varphi_\pi \underline{\mathbb{Z}}_{V_D}[n]$ on V is contained in the closed subset Σ, we get that $p^* \varphi_\pi \underline{\mathbb{Z}}_{V_D}[n]$ is a perverse sheaf on Σ (e.g., see [10, Corollary 8.2.10]). Since the complex dimension of Σ is s, the support condition for perverse sheaves together with the hypercohomology spectral sequence yield that

$$
\mathbb{H}^\ell \left(\Sigma; p^* \varphi_\pi \underline{\mathbb{Z}}_{V_D}[n] \right) \cong 0
$$

for all $\ell \notin [-s, s]$. This implies by (2.6) that

(2.7) $H^k_\varphi(V) = \mathbb{H}^{k-n} \left(V; \varphi_\pi \underline{\mathbb{Z}}_{V_D}[n] \right) \cong \mathbb{H}^{k-n} \left(\Sigma; p^* \varphi_\pi \underline{\mathbb{Z}}_{V_D}[n] \right) \cong 0$

for all $k \notin [n-s, n+s]$.

The desired concentration degrees for the vanishing cohomology is now obtained by combining (2.4) and (2.7).

Let us finally show that $H^n_\varphi(V)$ is free. Fix a Whitney stratification $\mathcal{V}$ of V, so that $V \setminus \Sigma$ is the top stratum. (Note that together with $\pi^{-1}(D^*)$, this also yields a Whitney stratification of V_D.) Since W intersects V transversally (i.e., W intersects each stratum S in $\mathcal{V}$ transversally in $\mathbb{C}P^{n+1}$), we can assume without any loss of generality that the base locus $B = V \cap W$ is a closed union of strata of $\mathcal{V}$. Next, we have by (2.4) that

$$
H^n_\varphi(V) \cong \mathbb{H}^0_c \left(V \setminus B; u^* \varphi_\pi \underline{\mathbb{Z}}_{V_D}[n] \right),
$$

with

$$
\mathcal{P} := u^* \varphi_\pi \underline{\mathbb{Z}}_{V_D}[n]
$$

a $\mathbb{Z}$-perverse sheaf on the affine variety $V \setminus B$ and $u : V \setminus B \hookrightarrow V$ the open inclusion. In particular, this implies that if $S \in \mathcal{V}$ is any stratum in $V \setminus B$ with inclusion $i_S : S \hookrightarrow V \setminus B$ then $\mathcal{H}^k(i_S^* \mathcal{P}) \simeq 0$ for all integers $k < -\dim_\mathbb{C} S$. By the Artin–Grothendieck type result of [15, Corollary 6.0.4], in order to show that $\mathbb{H}^0_c(V \setminus B; \mathcal{P})$ is free it suffices to check that the perverse sheaf $\mathcal{P}$ satisfies the following costalk condition (see[1] [15, Example 6.0.2 (3)]):

(2.8) $\mathcal{H}^{-\dim_\mathbb{C} S} \left(i_S^! \mathcal{P} \right)_x$ is free

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[1] We thank Jörg Schürmann for indicating the relevant references to us.

for any point x in any stratum S in $V \setminus B$ with inclusion $i_S : S \hookrightarrow V \setminus B$. Let us now fix a stratum $S \in \mathcal{V}$ contained in $V \setminus B$ and let $x \in S$ be a point with inclusion map $k_x : \{x\} \hookrightarrow S$. Consider the composition $i_x := i_S \circ k_x : \{x\} \hookrightarrow V \setminus B$. Using the fact that

$$k_x^* i_S^! \simeq k_x^! i_S^! \left[2 \dim_{\mathbb{C}} S \right] \simeq i_x^! \left[2 \dim_{\mathbb{C}} S \right]$$

(e.g., see [15, Remark 6.0.2(1)]), the condition (2.8) for $x \in S$ is equivalent to the following:

$$(2.9) \qquad \mathcal{H}^{\dim_{\mathbb{C}} S} \left(i_x^! \mathcal{P} \right) \quad \text{is free.}$$

In fact, the above discussion applies to any algebraically constructible complex $\mathcal{F}^{\cdot} \in {}^p D^{\geqslant 0}$, with $({}^p D^{\leqslant 0}, {}^p D^{\geqslant 0})$ denoting the perverse t-structure on $D_c^b(V \setminus B)$. Furthermore, in our setup (i.e., working with PID coefficients and having finitely generated stalk cohomology) $\mathcal{F}^{\cdot} \in {}^p D^{\geqslant 0}$ satisfies the additional costalk condition (2.8) (or, equivalently, (2.9)) if and only if the Verdier dual $\mathcal{D}\mathcal{F}^{\cdot}$ satisfies $\mathcal{D}\mathcal{F}^{\cdot} \in {}^p D^{\leqslant 0}$.

Let $i : V = V_0 \hookrightarrow V_D$ denote the closed inclusion, and consider the following *variation triangle* for the projection map $\pi : V_D \to D$:

$$(2.10) \qquad i^! [1] \longrightarrow \varphi_\pi \xrightarrow{var} \psi_\pi \xrightarrow{[1]}$$

with ψ_π denoting the corresponding nearby cycle functor for π (e.g., see [15, (5.90)]). Apply the functor $u^! = u^*$ to the triangle (2.10), and the apply the resulting triangle of functors to the complex $\underline{\mathbb{Z}}_{V_D}[n]$ to get the following triangle of constructible complexes on $V \setminus B$:

$$(2.11) \qquad \begin{aligned} \mathcal{Z} := u^! i^! \underline{\mathbb{Z}}_{V_D}[n+1] &\longrightarrow \mathcal{P} := u^* \varphi_\pi \underline{\mathbb{Z}}_{V_D}[n] \\ &\longrightarrow \mathcal{R} := u^* \psi_\pi \underline{\mathbb{Z}}_{V_D}[n] \xrightarrow{[1]} \end{aligned}$$

Let $x \in S$ be a point in a stratum of $V \setminus B$ with inclusion map $i_x : \{x\} \hookrightarrow V \setminus B$ as before, and apply the functor $i_x^!$ to the triangle (2.11) to get the triangle:

$$(2.12) \qquad i_x^! \mathcal{Z} \longrightarrow i_x^! \mathcal{P} \longrightarrow i_x^! \mathcal{R} \xrightarrow{[1]}$$

The cohomology long exact sequence associated to (2.12) contains the terms

$$\cdots \longrightarrow \mathcal{H}^{\dim_{\mathbb{C}} S} \left(i_x^! \mathcal{Z} \right) \longrightarrow \mathcal{H}^{\dim_{\mathbb{C}} S} \left(i_x^! \mathcal{P} \right) \longrightarrow \mathcal{H}^{\dim_{\mathbb{C}} S} \left(i_x^! \mathcal{R} \right) \longrightarrow \cdots$$

Since the category of (torsion-)free abelian groups is closed under extensions, in order to prove (2.9) it suffices to check that $\mathcal{H}^{\dim_{\mathbb{C}} S}(i_x^! \mathcal{Z})$ and $\mathcal{H}^{\dim_{\mathbb{C}} S}(i_x^! \mathcal{R})$ are (torsion-)free. (Note that, in fact, all costalks in question are finitely generated.)

Let us first show that $\mathcal{H}^{\dim_{\mathbb{C}} S}(i_x^! \mathcal{Z})$ is free. Regard the stratum S containing x as a stratum in V_D, and let $r_x : \{x\} \to V_D$ be the point inclusion, i.e., $r_x = i \circ u \circ i_x$. So $i_x^! \mathcal{Z} = r_x^! \underline{\mathbb{Z}}_{V_D}[n+1]$. Recall that $\underline{\mathbb{Z}}_{V_D}[n+1]$ is a $\mathbb{Z}$-perverse sheaf on V_D, i.e., $\underline{\mathbb{Z}}_{V_D}[n+1] \in {}^p D^{\leqslant 0}(V_D) \cap {}^p D^{\geqslant 0}(V_D)$. As already indicated above, in order to show that $\mathcal{H}^{\dim_{\mathbb{C}} S}(r_x^! \underline{\mathbb{Z}}_{V_D}[n+1])$ is free it suffices to verify that $\mathcal{D}(\underline{\mathbb{Z}}_{V_D}[n+1]) \in {}^p D^{\leqslant 0}(V_D)$, or equivalently, $\mathcal{D}\underline{\mathbb{Z}}_{V_D} \in {}^p D^{\leqslant -n-1}(V_D)$. This fact is a consequence of [15, Definition 6.0.4, Example 6.0.11], where it is shown that the complete interesection V_D has a *rectified homological depth* equal to its complex dimension $n+1$.

Next note that, due to the local product structure, the Milnor fiber F_x of the hypersurface singularity germ (V, x) with $x \in S$ has the homotopy type of a finite CW complex of real dimension $n - \dim_{\mathbb{C}} S$. In particular, $H_{n-\dim_{\mathbb{C}} S}(F_x; \mathbb{Z})$ is free. Since by the costalk calculation (cf. [15, (5.92)]) and Poincaré duality we have for $x \in S$ that

$$(2.13) \qquad \mathcal{H}^{\dim_{\mathbb{C}} S}\left(i_x^! \mathcal{R}\right) \cong H_c^{n+\dim_{\mathbb{C}} S}(F_x; \mathbb{Z}) \cong H_{n-\dim_{\mathbb{C}} S}(F_x; \mathbb{Z}),$$

it follows that $\mathcal{H}^{\dim_{\mathbb{C}} S}(i_x^! \mathcal{R})$ is free. This completes the proof of Theorem 1.2.

2.2. Proof of Proposition 1.10

Since V is a $\mathbb{Q}$-homology manifold, it follows by standard arguments involving the Hamm fibration (e.g., see [2, Theorem 3.2.12]) that V_D is also a $\mathbb{Q}$-homology manifold (with boundary). Thus $\underline{\mathbb{Q}}_{V_D}[n+1]$ is a self-dual $\mathbb{Q}$-perverse sheaf on V_D. Moreover, since $\varphi_\pi[-1]$ commutes with the Verdier dualizing functor (see [8, Theorem 3.1] and the references therein), we get that $\mathcal{Q} := \varphi_\pi \underline{\mathbb{Q}}_{V_D}[n]$ is a Verdier self-dual perverse sheaf on V. Using the Universal Coefficients Theorem, we obtain:

$$H_\varphi^k(V) \otimes \mathbb{Q} = \mathbb{H}^{k-n}(V; \mathcal{Q}) \cong \mathbb{H}^{k-n}(V; \mathcal{D}\mathcal{Q}) \cong \mathbb{H}^{n-k}(V; \mathcal{Q})^\vee$$
$$= \left(H_\varphi^{2n-k}(V) \otimes \mathbb{Q}\right)^\vee.$$

The desired vanishing follows now from Theorem 1.2.

3. Bounds on Betti numbers of projective hypersurfaces

In this section, we prove Theorem 1.7 and specialize it, along with Corollary 1.5, in the case when the complex dimension s of the singular locus is $\leqslant 1$.

3.1. Proof of Theorem 1.7

Proof of Theorem 1.7. — Let $\Sigma := V_{\mathrm{sing}}$ be the singular locus of V, of complex dimension s, and fix a Whitney stratification $\mathcal{V}$ of V so that $V \setminus \Sigma$ is the top open stratum. We have by Corollary 1.5 (or by the specialization sequence (1.5)) that

$$b_{n+s+1}(V) \leqslant 1 + \mathrm{rank}\ H_\varphi^{n+s}(V).$$

So it suffices to show that

(3.1) $$\mathrm{rank}\ H_\varphi^{n+s}(V) \leqslant \sum_i \mu_i^{\pitchfork},$$

where the summation on the right-hand side runs over the top s-dimensional connected strata S_i of Σ, and $\mu_i^{\pitchfork}$ denotes the corresponding transversal Milnor number for such a stratum S_i.

If $s = 0$, an easy computation shows that (3.1) is in fact an equality, see (3.10) below. Let us next investigate the case when $s \geqslant 1$.

For any $\ell \leqslant s$, denote by Σ_ℓ the union of strata in Σ of complex dimension $\leqslant \ell$. In particular, we can filter Σ by closed (possibly empty) subsets

$$\Sigma = \Sigma_s \supset \Sigma_{s-1} \supset \cdots \supset \Sigma_0 \supset \Sigma_{-1} = \emptyset.$$

Let

$$U_\ell := \Sigma_\ell \setminus \Sigma_{\ell-1}$$

be the union of ℓ-dimensional strata, so $\Sigma_\ell = \sqcup_{k \leqslant \ell} U_k$. (Here, $\sqcup$ denotes disjoint union.) Recall that the smooth hypersurface $W = \{g = 0\}$ was chosen so that it intersects each stratum in Σ transversally.

In the notations of the proof of Theorem 1.2, it follows from equations (2.4) and (2.5) that:

$$H_\varphi^{n+s}(V) \cong \mathbb{H}_c^{n+s}\left(V \setminus B; u^* \varphi_\pi \underline{\mathbb{Z}}_{V_D}\right) \cong \mathbb{H}_c^{n+s}\left(\Sigma \setminus B; v^* u^* \varphi_\pi \underline{\mathbb{Z}}_{V_D}\right),$$

with $B = V \cap W$ the axis of the pencil, and with $v : \Sigma \setminus B \hookrightarrow V \setminus B$ and $u : V \setminus B \hookrightarrow V$ the inclusion maps. We also noted that either h or a local representative of f can be used when considering Milnor fibers of π at points in $V \setminus B$. For simplicity, let us use the notation

$$\mathcal{R} := v^* u^* \varphi_\pi \underline{\mathbb{Z}}_{V_D} \in D_c^b(\Sigma \setminus B),$$

and consider the part of the long exact sequence for the compactly supported hypercohomology of $\mathcal{R}$ associated to the disjoint union

$$\Sigma \setminus B = (U_s \setminus B) \sqcup (\Sigma_{s-1} \setminus B)$$

involving $H_\varphi^{n+s}(V)$, namely:

$$\cdots \to \mathbb{H}_c^{n+s}\left(U_s \setminus B; \mathcal{R}\right) \to H_\varphi^{n+s}(V) \to \mathbb{H}_c^{n+s}\left(\Sigma_{s-1} \setminus B; \mathcal{R}\right) \to \cdots$$

We claim that

$$(3.2) \qquad\qquad \mathbb{H}_c^{n+s}\left(\Sigma_{s-1} \setminus B; \mathcal{R}\right) \cong 0,$$

so, in particular, there is an epimorphism:

$$(3.3) \qquad\qquad \mathbb{H}_c^{n+s}\left(U_s \setminus B; \mathcal{R}\right) \twoheadrightarrow H_\varphi^{n+s}(V).$$

In order to prove (3.2), consider the part of the long exact sequence for the compactly supported hypercohomology of $\mathcal{R}$ associated to the disjoint union

$$\Sigma_{s-1} \setminus B = (U_{s-1} \setminus B) \sqcup (\Sigma_{s-2} \setminus B)$$

involving $\mathbb{H}_c^{n+s}(\Sigma_{s-1} \setminus B; \mathcal{R})$, namely:

$$\cdots \to \mathbb{H}_c^{n+s}\left(U_{s-1} \setminus B; \mathcal{R}\right) \to \mathbb{H}_c^{n+s}\left(\Sigma_{s-1} \setminus B; \mathcal{R}\right)$$
$$\to \mathbb{H}_c^{n+s}\left(\Sigma_{s-2} \setminus B; \mathcal{R}\right) \to \cdots$$

We first show that

$$(3.4) \qquad\qquad \mathbb{H}_c^{n+s}\left(U_{s-1} \setminus B; \mathcal{R}\right) \cong 0.$$

Indeed, the (p,q)-entry in the E_2-term of the hypercohomology spectral sequence computing $\mathbb{H}_c^{n+s}(U_{s-1} \setminus B; \mathcal{R})$ is given by

$$E_2^{p,q} = H_c^p\left(U_{s-1} \setminus B; \mathcal{H}^q(\mathcal{R})\right),$$

and we are interested in those pairs of integers (p,q) with $p+q = n+s$. Since a point in a $(s-1)$-dimensional stratum of V has a Milnor fiber which has the homotopy type of a finite CW complex of real dimension $n-s+1$, it follows that

$$\mathcal{H}^q(\mathcal{R})|_{U_{s-1} \setminus B} \simeq 0 \quad \text{for any} \quad q > n-s+1.$$

Also, by reasons of dimension, we have that $E_2^{p,q} = 0$ if $p > 2s-2$. In particular, the only possibly non-trivial entries on the E_2-page of the above spectral sequence are those corresponding to pairs (p,q) with $p \leqslant 2s-2$ and $q \leqslant n-s+1$, none of which add up to $n+s$. This proves (3.4). If $s=1$, this completes the proof of (3.2) since $\Sigma_{-1} = \emptyset$. If $s > 1$, the long exact sequences for the compactly supported hypercohomology of $\mathcal{R}$ associated to the disjoint union

$$\Sigma_\ell \setminus B = (U_\ell \setminus B) \sqcup (\Sigma_{\ell-1} \setminus B),$$

$0 \leqslant \ell \leqslant s-1$, can be employed to reduce the proof of (3.2) to showing that

$$(3.5) \qquad\qquad \mathbb{H}_c^{n+s}\left(U_\ell \setminus B; \mathcal{R}\right) \cong 0$$

for all $0 \leqslant \ell \leqslant s - 1$. To prove (3.5), we make use of the hypercohomology spectral sequence whose E_2-term is computed by

$$E_2^{p,\,q} = H_c^p \left(U_\ell \setminus B; \mathcal{H}^q(\mathcal{R}) \right),$$

and we are interested again in those pairs of integers (p, q) with $p + q = n + s$. Since a point in an ℓ-dimensional stratum of V has a Milnor fiber which has the homotopy type of a finite CW complex of real dimension $n - \ell$, it follows that

$$\mathcal{H}^q(\mathcal{R})|_{U_\ell \setminus B} \simeq 0 \quad \text{for any} \quad q > n - \ell.$$

Moreover, by reasons of dimension, $E_2^{p,q} = 0$ if $p > 2\ell$. So the only possibly non-trivial entries on the E_2-page are those corresponding to pairs (p, q) with $p \leqslant 2\ell$ and $q \leqslant n - \ell$, none of which add up to $n + s$. This proves (3.5), and completes the proof of (3.2) in the general case.

In order to prove (3.1), we make use of the epimorphism (3.3) as follows. Recall that, in our notations, $U_s \setminus B$ is a disjoint union of connected strata $S_i \setminus B$ of complex dimension s. Each $S_i \setminus B$ has a generic transversal Milnor fiber $F_i^{\pitchfork}$, which has the homotopy type of a bouquet of $\mu_i^{\pitchfork}$ $(n - s)$-dimensional spheres. So the integral cohomology of $F_i^{\pitchfork}$ in concentrated in degree $n - s$. Moreover, for each i, there is a local system $\mathcal{L}_i^{\pitchfork}$ on $S_i \setminus B$ with stalk $\widetilde{H}^{n-s}(F_i^{\pitchfork}; \mathbb{Z})$, whose monodromy is usually refered to as the *vertical monodromy*. This is exactly the restriction of the constructible sheaf $\mathcal{H}^{n-s}(\mathcal{R})$ to $S_i \setminus B$. It then follows from the hypercohomology spectral sequence computing $\mathbb{H}_c^{n+s}(U_s \setminus B; \mathcal{R})$ and by Poincaré duality that

$$(3.6) \quad \mathbb{H}_c^{n+s} \left(U_s \setminus B; \mathcal{R} \right) \cong \bigoplus_i H_c^{2s} \left(S_i \setminus B \, ; \mathcal{L}_i^{\pitchfork} \right) \cong \bigoplus_i H_0 \left(S_i \setminus B \, ; \mathcal{L}_i^{\pitchfork} \right)$$

which readily gives (3.1). □

Remark 3.1. — Note that the upper bound on $b_{n+s+1}(V)$ can be formulated entirely in terms of coinvariants of vertical monodromies along the top dimensional singular strata of V. Indeed, if in the notations of the above proof we further let h_i^v denote the vertical monodromy along $S_i \setminus B$, then each term on the right-hand side of (3.6) is computed by the coinvariants of h_i^v, i.e., $H_0(S_i \setminus B; \mathcal{L}_i^{\pitchfork}) \cong \widetilde{H}^{n-s}(F_i^{\pitchfork}; \mathbb{Z})_{h_i^v}$. Note that the latter statement, when combined with (3.3), yields an epimorphism

$$(3.7) \qquad \bigoplus_i \widetilde{H}^{n-s} \left(F_i^{\pitchfork}; \mathbb{Z} \right)_{h_i^v} \twoheadrightarrow H_\varphi^{n+s}(V),$$

the summation on the left hand side being over the top dimensional singular strata of V. One can, moreover, proceed like in [11] and give a more precise dependence of all (possibly non-trivial) vanishing cohomology groups

$H_\varphi^k(V)$, $n \leqslant k \leqslant n + s$, in terms of the singular strata of V. We leave the details to the interested reader.

3.2. Isolated singularities

Assume that the projective hypersurface $V \subset \mathbb{C}P^{n+1}$ has only isolated singularities (i.e., $s = 0$). Then the incidence variety V_D is smooth since the pencil has an empty base locus, and the projection $\pi : V_D \to D$ has isolated singularities exactly at the singular points of V. The only non-trivial vanishing homology group, $H_{n+1}^\curlyvee(V)$, is free, and is computed as:

$$(3.8) \qquad H_{n+1}^\curlyvee(V) \cong \bigoplus_{x \in V_{\mathrm{sing}}} \widetilde{H}_n(F_x; \mathbb{Z}) \cong \bigoplus_{x \in V_{\mathrm{sing}}} \mathbb{Z}^{\mu_x},$$

where F_x denotes the Milnor fiber of the isolated hypersurface singularity germ (V, x), with corresponding Milnor number μ_x. The second isomorphism follows from the fact that F_x has the homotopy type of a bouquet of μ_x n-spheres.

The 5-term exact sequence (1.9) then reads as:

$$0 \to H_{n+1}(V_t; \mathbb{Z}) \to H_{n+1}(V; \mathbb{Z}) \to \bigoplus_{x \in V_{\mathrm{sing}}} \widetilde{H}_n(F_x; \mathbb{Z})$$

(3.9)

$$\overset{\alpha_n}{\to} H_n(V_t; \mathbb{Z}) \to H_n(V; \mathbb{Z}) \to 0.$$

Therefore Corollary 1.5 (i)–(iii), together with the following bound via Theorem 1.7:

$$b_{n+1}(V) \leqslant 1 + \sum_{x \in V_{\mathrm{sing}}} \mu_x.$$

recover [17, Proposition 2.2], which in turn is a homology counterpart of Dimca's result [2, Theorem 5.4.3]. In fact, Dimca's result was formulated in cohomology, and it is a direct consequence of the specialization sequence (1.5) via Theorem 1.2, together with the observation that the only non-trivial vanishing cohomology group, $H_\varphi^n(V)$, is computed as:

$$(3.10) \qquad H_\varphi^n(V) \cong \bigoplus_{x \in V_{\mathrm{sing}}} \widetilde{H}^n(F_x; \mathbb{Z}).$$

Remark 3.2. — Let us recall here that if $V \subset \mathbb{C}P^{n+1}$ is a degree d reduced projective hypersurface with only isolated singularities, then its Euler characteristic is computed by the formula (e.g., see [2, Exercise 5.3.7 (i) and Corollary 5.4.4] or [10, Proposition 10.4.2]):

$$(3.11) \quad \chi(V) = (n + 2) - \frac{1}{d} \left[1 + (-1)^{n+1}(d-1)^{n+2} \right] + (-1)^{n+1} \sum_{x \in V_{\mathrm{sing}}} \mu_x,$$

with μ_x denoting as before the Milnor number of the isolated hypersurface singularity germ (V, x). In particular, if V is a projective *curve* (i.e., $n = 1$), then $H_0(V; \mathbb{Z}) \cong \mathbb{Z}$, $H_2(V; \mathbb{Z}) \cong \mathbb{Z}^r$, with r denoting the number of irreducible components of V, and $H_1(V; \mathbb{Z})$ is a free group whose rank is computed from (3.11) by the formula:

$$(3.12) \qquad b_1(V) = r + 1 + d^2 - 3d - \sum_{x \in V_{\mathrm{sing}}} \mu_x.$$

3.3. 1-dimensional singular locus

This particular case was treated in homology in [17, Proposition 7.7]. Let us recall the preliminaries, in order to point out once more that in this paper we have transposed them to a fully general setting.

One starts with $V \subset \mathbb{C}P^{n+1}$, a degree d projective hypersurface with a singular locus $\Sigma := V_{\mathrm{sing}}$ of complex dimension 1. The singular locus Σ consists of a union of irreducible projective curves Σ_i and a finite set I of isolated singular points. Each curve Σ_i has a generic transversal type of transversal Milnor fiber

$$F_i^{\pitchfork} \simeq \bigvee_{\mu_i^{\pitchfork}} S^{n-1}$$

with corresponding transversal Milnor number $\mu_i^{\pitchfork}$. Each Σ_i also contains a finite set S_i of special points of non-generic transversal type. One endows V with the Whitney stratification whose strata are:

- the isolated singular points in I,
- the special points in $S = \bigcup_i S_i$,
- the (top) one-dimensional components of $\Sigma \setminus S$,
- the open stratum $V \setminus \Sigma$.

The genericity of the pencil $\{V_t\}_{t \in D}$ implies that the base locus B intersects each Σ_i in a finite set B_i of general points, which are not contained in $I \cup S_i$. The total space V_D of the pencil has in this case only isolated singularities (corresponding to the points where B intersects Σ), and the projection $\pi : V_D \to D$ has a 1-dimensional singular locus $\Sigma \times \{0\}$.

With the above specified landscape, the Siersma–Tibăr result [17, Proposition 7.7] reads now as the specialisation for $s = 1$ of Corollary 1.6, together with the bound provided by Theorem 1.7.

4. Examples

In this section we work out a few specific examples. In particular, in § 4.1 we show that the upper bound given by Theorem 1.7 is sharp, § 4.2 deals with a hypersurface which is a rational homology manifold, while § 4.3 discusses the case of a projective cone on a singular curve. However, as pointed out in [1] already in the case of isolated singularities, it is difficult in general to compute the integral cohomology of a hypersurface by means of Corollary 1.5. It is therefore important to also develop alternative methods for exact calculations of cohomology and/or Betti numbers, e.g., see [2] for special situations.

4.1. Singular quadrics

Let n and q be integers satisfying $4 \leqslant q \leqslant n + 1$, and let

$$f_q(x_0, \ldots x_{n+1}) = \sum_{0 \leqslant i, j \leqslant n+1} q_{ij} x_i x_j$$

be a quadric of rank $q := \mathrm{rank}(Q)$ with $Q = (q_{ij})$. The singular locus Σ of the quadric hypersurface $V_q = \{f_q = 0\} \subset \mathbb{C}P^{n+1}$ is a linear space of complex dimension $s = n + 1 - q$ satisfying $0 \leqslant s \leqslant n - 3$. The generic transversal type for $\Sigma = \mathbb{C}P^s$ is an A_1-singularity, so $\mu^{\pitchfork} = 1$. Theorem 1.7 yields that

(4.1) $b_{n+s+1}(V_q) \leqslant 2.$

In what follows, we show that if the rank q is even (i.e., $n + s + 1$ is even), the upper bound on $b_{n+s+1}(V_q)$ given in (4.1) is sharp. Indeed, in our notation, the quadric V_q is a projective cone with vertex Σ over a smooth quadric $W_q \subset \mathbb{C}P^{n-s}$. Moreover, since $n - s \geqslant 3$, the homotopy version of the Lefschetz hyperplane theorem yields that W_q is simply-connected (see, e.g., [2, Theorem 1.6.5]). Let $U = V_q \setminus \Sigma$ and consider the long exact sequence

$$\cdots \to H_c^k(U; \mathbb{Z}) \to H^k(V_q; \mathbb{Z}) \to H^k(\Sigma; \mathbb{Z}) \to H_c^{k+1}(U; \mathbb{Z}) \to \cdots$$

Note that projecting from Σ gives U the structure of a vector bundle of rank $s + 1$ over W_q. Let $p : U \to W_q$ denote the bundle map. Then

$$H_c^k(U; \mathbb{Z}) \cong H^k(W_q; Rp_! \mathbb{Z}_U)$$

can be computed by the corresponding hypercohomology spectral sequence (i.e., the compactly supported Leray–Serre spectral sequence of the map p),

with $E_2^{a,b} = H^a(W_q; R^b p_! \mathbb{Z}_U)$. Since $\pi_1(W_q) = 0$, the local system $R^b p_! \mathbb{Z}_U$ is constant on W_q with stalk $H_c^b(\mathbb{C}^{s+1}; \mathbb{Z})$. Since the latter is $\mathbb{Z}$ if $b = 2s + 2$ and 0 otherwise, the above spectral sequence yields isomorphisms $H_c^k(U; \mathbb{Z}) \cong H^{k-2-2s}(W_q; \mathbb{Z})$ if $k \geqslant 2s + 2$ and $H_c^k(U; \mathbb{Z}) \cong 0$ if $k < 2s + 2$. On the other hand, $H^k(\Sigma; \mathbb{Z}) = 0$ if $k > 2s$, so the above long exact sequence yields:

$$(4.2) \qquad H^k(V_q; \mathbb{Z}) \cong \begin{cases} H^k(\Sigma; \mathbb{Z}) & 0 \leqslant k \leqslant 2s \\ 0 & k = 2s + 1 \\ H^{k-2-2s}(W_q; \mathbb{Z}) & 2s + 2 \leqslant k \leqslant 2n. \end{cases}$$

Since W_q is a smooth quadric, its integral cohomology is known from (1.1), (1.2) and (1.4). Altogether, this gives:

$$(4.3) \qquad H^k(V_q; \mathbb{Z}) \cong \begin{cases} 0 & k \text{ odd} \\ \mathbb{Z} & k \text{ even}, \ k \neq n + s + 1 \\ \mathbb{Z}^2 & k = n + s + 1 \text{ even}. \end{cases}$$

4.2. One-dimensional singular locus with a two-step filtration

Let $V = \{f = 0\} \subset \mathbb{C}P^4$ be the 3-fold in homogeneous coordinates $[x : y : z : t : v]$, defined by

$$f = y^2 z + x^3 + tx^2 + v^3.$$

The singular locus of V is the projective line $\Sigma = \{[0 : 0 : z : t : 0] \mid z, t \in \mathbb{C}\}$. By (1.1), we get: $b_0(V) = 1$, $b_1(V) = 0$, $b_2(V) = 1$. Since V is irreducible, (1.13) yields: $b_6(V) = 1$. We are therefore interested to understand the Betti numbers $b_3(V)$, $b_4(V)$ and $b_5(V)$.

It was shown in [9, Example 6.1] that V has a Whitney stratification with strata:

$$S_3 := V \setminus \Sigma, \quad S_1 := \Sigma \setminus [0:0:0:1:0], \quad S_0 := [0:0:0:1:0],$$

giving V a two-step filtration $V \supset \Sigma \supset [0:0:0:1:0]$.

The transversal singularity for the top singular stratum S_1 is the Brieskorn type singularity $y^2 + x^3 + v^3 = 0$ at the origin of $\mathbb{C}^3$ (in a normal slice to S_1), with corresponding transversal Milnor number $\mu_1^{\pitchfork} = 4$. So Theorem 1.7 yields that $b_5(V) \leqslant 5$, while Corollary 1.5 gives $b_3(V) \leqslant 10$. As we will indicate below, the actual values of $b_3(V)$ and $b_5(V)$ are zero.

It was shown in [9, Example 6.1] that the hypersurface V is in fact a $\mathbb{Q}$-homology manifold, so it satisfies Poincaré duality over the rationals. In

particular, $b_5(V) = b_1(V) = 0$ and $b_4(V) = b_2(V) = 1$. To determine $b_3(V)$, it suffices to compute the Euler characteristic of V, since $\chi(V) = 4 - b_3(V)$. Let us denote by $Y \subset \mathbb{C}P^4$ a smooth 3-fold which intersects the Whitney stratification of V transversally. Then (1.3) yields that $\chi(Y) = -6$ and we have by [10, (10.40)] that

$$(4.4) \qquad \chi(V) = \chi(Y) - \chi(S_1 \setminus Y) \cdot \mu_1^{\pitchfork} - \chi(S_0) \cdot (\chi(F_0) - 1),$$

where F_0 denotes the Milnor fiber of V at the singular point S_0. As shown in [9, Example 6.1], $F_0 \simeq S^3 \vee S^3$. So, using the fact that the general 3-fold Y intersects S_1 at 3 points, we get from (4.4) that $\chi(V) = 4$. Therefore, $b_3(V) = 0$, as claimed. Moreover, since $H^3(V; \mathbb{Z})$ is free, this also shows that in fact $H^3(V; \mathbb{Z}) \cong 0$.

Remark 4.1. — Note that the hypersurface of the previous example has the same Betti numbers as $\mathbb{C}P^3$. This fact can also be checked directly, by noting that the monodromy operator acting on the reduced homology of the Milnor fiber of f at the origin in $\mathbb{C}^5$ has no eigenvalue equal to 1 (see [2, Corollary 5.2.22]).

More generally, consider a degree d homogeneous polynomial $g(x_0, \ldots, x_n)$ with associated Milnor F_g such that the monodromy operator h_* acting on $\widetilde{H}_*(F_g; \mathbb{Q})$ is the identity. Then the hypersurface $V = \{g(x_0, \ldots, x_n) + x_{n+1}^d = 0\} \subset \mathbb{C}P^{n+1}$ has the same $\mathbb{Q}$-(co)homology as $\mathbb{C}P^n$. For example, the hypersurface $V_n = \{x_0 x_1 \ldots x_n + x_{n+1}^{n+1} = 0\}$ has singularities in codimension 2, but the same $\mathbb{Q}$-(co)homology as $\mathbb{C}P^n$. However, V_n does not have in general the $\mathbb{Z}$-(co)homology of $\mathbb{C}P^n$; indeed, $H^3(V_2; \mathbb{Z})$ contains 3-torsion (cf. [2, Proposition 5.4.8]).

4.3. Projective cone on a curve

The projective curve $C = \{xyz = 0\} \subset \mathbb{C}P^2$ has three irreducible components and three singularities of type A_1 (each having a corresponding Milnor number equal to 1). Therefore, by Remark 3.2 and formula (3.12), the integral cohomology of C is given by:

$$H^0(C; \mathbb{Z}) \cong \mathbb{Z}, \quad H^1(C; \mathbb{Z}) \cong \mathbb{Z}, \quad H^2(C; \mathbb{Z}) \cong \mathbb{Z}^3.$$

The projective cone on C is the surface $V = \{xyz = 0\} \subset \mathbb{C}P^3$. The singular locus of V consists of three projective lines intersecting at the point $[0 : 0 : 0 : 1]$, each having a (generic) transversal singularity of type

A_1, i.e., with corresponding transversal Milnor number equal to 1. By [2, (5.4.18)], we have that

$$H^k(V;\mathbb{Z}) \cong H^{k-2}(C;\mathbb{Z}), \quad \text{for all} \quad k \geqslant 2.$$

Together with (1.1), this yields:

(4.5)
$$H^0(V;\mathbb{Z}) \cong \mathbb{Z}, \ H^1(V;\mathbb{Z}) \cong 0, \ H^2(V;\mathbb{Z}) \cong \mathbb{Z},$$
$$H^3(V;\mathbb{Z}) \cong \mathbb{Z}, \ H^4(V;\mathbb{Z}) \cong \mathbb{Z}^3.$$

By Theorem 1.2, the only non-trivial vanishing cohomology groups of V are $H^2_\varphi(V)$, which is free, and $H^3_\varphi(V)$. These can be explicitly computed by using (1.4), (1.9) and (4.5), to get:

$$H^2_\varphi(V) \cong \mathbb{Z}^7, \ H^3_\varphi(V) \cong \mathbb{Z}^2$$

(compare with [17, Example 7.5]).

5. Supplement to the Lefschetz hyperplane theorem and applications

In this section, we give a new proof of Kato's result mentioned in the Introduction. Our proof is different from that of [2, Theorem 5.2.11], and it relies on a supplement to the Lefschetz hyperplane section theorem (Theorem 1.11), which is proved in Theorem 5.1 below.

5.1. A supplement to the Lefschetz hyperplane theorem

In this section, we prove the following result of Lefschetz type:

THEOREM 5.1. — *Let* $V \subset \mathbb{C}P^{n+1}$ *be a reduced complex projective hypersurface with* $s = \dim V_{\text{sing}}$ *the complex dimension of its singular locus. (By convention, we set* $s = -1$ *if* V *is nonsingular.) Let* $H \subset \mathbb{C}P^{n+1}$ *be a generic hyperplane (i.e., transversal to a Whitney stratification of* V*), and denote by* $V_H := V \cap H$ *the corresponding hyperplane section of* V*. Then*

(5.1) $\quad H^k(V, V_H;\mathbb{Z}) = 0 \quad \text{for} \quad k < n \quad \text{and} \quad n+s+1 < k < 2n.$

Moreover, $H^{2n}(V, V_H;\mathbb{Z}) \cong \mathbb{Z}^r$*, where* r *is the number of irreducible components of* V*, and* $H^n(V, V_H;\mathbb{Z})$ *is (torsion-)free.*

Proof. — Let us first note that the long exact sequence for the cohomology of the pair (V, V_H) together with (1.13) yield that:

$$H^{2n}(V, V_H; \mathbb{Z}) \cong H^{2n}(V; \mathbb{Z}) \cong \mathbb{Z}^r.$$

Moreover, we have isomorphisms:

$$H^k(V, V_H; \mathbb{Z}) \cong H_c^k(V^a; \mathbb{Z}),$$

where $V^a := V \setminus V_H$. Therefore, the vanishing in (5.1) for $k < n$ is a consequence of the Artin vanishing theorem (e.g., see [15, Corollary 6.0.4]) for the perverse sheaf $\underline{\mathbb{Z}}_{V^a}[n]$ (cf. [6]) on the affine hypersurface V^a obtained from V by removing the hyperplane section V_H. Indeed,

$$H_c^k(V^a; \mathbb{Z}) = \mathbb{H}_c^{k-n}(V^a; \underline{\mathbb{Z}}_{V^a}[n]) \cong 0$$

for all $k - n < 0$. (Note that vanishing in this range is equivalent to the classical Lefschetz hyperplane section theorem.)

Since V is reduced, we have that $s < n$. If $n = s + 1$ then $n + s + 1 = 2n$ and there is nothing else to prove in (5.1). So let us now assume that $n > s + 1$. For $n + s + 1 < k < 2n$, we have the following sequence of isomorphisms:

$$
\begin{aligned}
H^k(V, V_H; \mathbb{Z}) &\cong H^k(V \cup H, H; \mathbb{Z}) \\
(5.2) \qquad\qquad &\cong H_{2n+2-k}\left(\mathbb{C}P^{n+1} \setminus H, \mathbb{C}P^{n+1} \setminus (V \cup H); \mathbb{Z}\right) \\
&\cong H_{2n+1-k}\left(\mathbb{C}P^{n+1} \setminus (V \cup H); \mathbb{Z}\right),
\end{aligned}
$$

where the first isomorphism follows by excision, the second is an application of the Poincaré–Alexander–Lefschetz duality, and the third follows from the cohomology long exact sequence of a pair. Set

$$U = \mathbb{C}P^{n+1} \setminus (V \cup H),$$

and let $L = \mathbb{C}P^{n-s}$ be a generic linear subspace (i.e., transversal to both V and H). Then, by transversality, $L \cap V$ is a nonsingular hypersurface in L, transversal to the hyperplane at infinity $L \cap H$ in L. Therefore, $U \cap L = L \setminus (V \cup H) \cap L$ has the homotopy type of a wedge

$$U \cap L \simeq S^1 \vee S^{n-s} \vee \ldots \vee S^{n-s},$$

e.g., see [7, Corollary 1.2]. Thus, by the Lefschetz hyperplane section theorem (applied $s + 1$ times), we obtain:

$$H_i(U; \mathbb{Z}) \cong H_i(U \cap L; \mathbb{Z}) \cong 0$$

for all integers i in the range $1 < i < n - s$. Substituting $i = 2n + 1 - k$ in (5.2), we get that $H^k(V, V_H; \mathbb{Z}) \cong 0$ for all integers k in the range $n + s + 1 < k < 2n$.

It remains to show that $H^n(V, V_H; \mathbb{Z}) \cong H^n_c(V^a; \mathbb{Z}) \cong \mathbb{H}^0_c(V^a; \mathbb{Z}_{V^a}[n])$ is (torsion-)free. This follows as in the proof of Theorem 1.2 since the affine hypersurface V^a has rectified homological depth equal to its complex dimension n. This completes the proof of the theorem. $\qquad\square$

Theorem 5.1 and the Universal Coefficient Theorem now yield the following consequence:

COROLLARY 5.2. — *In the notations of Theorem 5.1 we have that:*

$$(5.3) \qquad H_k(V, V_H; \mathbb{Z}) = 0 \quad for \quad k < n \quad and \quad n + s + 1 < k < 2n.$$

Moreover, $H_{2n}(V, V_H; \mathbb{Z}) \cong \mathbb{Z}^r$, where r is the number of irreducible components of V.

5.2. Kato's theorem for hypersurfaces

The isomorphism (1.2) from the introduction was originally proved by Kato [4], and it holds more generally for complete intersections. We derive it here as a consequence of Theorem 5.1.

THEOREM 5.3 (Kato). — *Let $V \subset \mathbb{C}P^{n+1}$ be a reduced degree d complex projective hypersurface with $s = \dim V_{\mathrm{sing}}$ the complex dimension of its singular locus. (By convention, we set $s = -1$ if V is nonsingular.) Then*

$$(5.4) \qquad H^k(V; \mathbb{Z}) \cong H^k\left(\mathbb{C}P^{n+1}; \mathbb{Z}\right) \quad for \; all \quad n + s + 2 \leqslant k \leqslant 2n.$$

Moreover, if $j : V \hookrightarrow \mathbb{C}P^{n+1}$ denotes the inclusion, the induced cohomology homomorphisms

$$(5.5) \qquad j^k : H^k\left(\mathbb{C}P^{n+1}; \mathbb{Z}\right) \longrightarrow H^k(V; \mathbb{Z}), \quad n + s + 2 \leqslant k \leqslant 2n,$$

are given by multiplication by d if k is even.

Proof. — The statement of the theorem is valid only if $n \geqslant s + 2$, so in particular we can assume that V is irreducible and hence $H^{2n}(V; \mathbb{Z}) \cong \mathbb{Z}$. Moreover, the fact that j^{2n} is multiplication by $d = \deg(V)$ is true regardless of the dimension of singular locus, see [2, (5.2.10)]. If $n = s + 2$ there is nothing else to prove, so we may assume (without any loss of generality) that $n \geqslant s + 3$.

We next proceed by induction on s.

If V is nonsingular (i.e., $s = -1$), the assertions are well-known for any $n \geqslant 1$. We include here a proof for completeness. The isomorphism (5.4) can be obtained in this case from the Lefschetz isomorphism (1.1), its homology analogue, and Poincaré duality. The statement about j^k can also

be deduced from (1.1) and Poincaré duality, but we include here a different argument inspired by [2]. Consider the isolated singularity at the origin for the affine cone $CV \subset \mathbb{C}^{n+2}$ on V, and the corresponding link $L_V := S^{2n+3} \cap CV$, for S^{2n+3} a small enough sphere at the origin in $\mathbb{C}^{n+2}$. Then L_V is a $(n-1)$-connected closed oriented manifold of real dimension $2n+1$, so its only possibly nontrivial integral (co)homology appears in degrees 0, n, $n+1$ and $2n+1$. The Hopf fibration $S^1 \hookrightarrow S^{2n+3} \longrightarrow \mathbb{C}P^{n+1}$ induces by restriction to CV a corresponding Hopf fibration for V, namely $S^1 \hookrightarrow L_V \longrightarrow V$. Then for any $n+1 \leqslant k \leqslant 2n-2$, the cohomology Gysin sequences for the diagram of fibrations

$$
\begin{array}{ccc}
S^{2n+3} & \longrightarrow & \mathbb{C}P^{n+1} \\
\uparrow & & \uparrow \\
L_V & \longrightarrow & V
\end{array}
$$

yield commutative diagrams (with $\mathbb{Z}$-coefficients):
(5.6)

$$
\begin{array}{ccccccc}
H^{k+1}\left(S^{2n+3}\right) & \longrightarrow & H^k\left(\mathbb{C}P^{n+1}\right) & \xrightarrow[\cong]{\psi} & H^{k+2}\left(\mathbb{C}P^{n+1}\right) & \longrightarrow & H^{k+2}\left(S^{2n+3}\right) \\
\downarrow & & j^k \downarrow & & j^{k+2} \downarrow & & \downarrow \\
H^{k+1}\left(L_V\right) & \longrightarrow & H^k(V) & \xrightarrow[\cong]{\psi_V} & H^{k+2}(V) & \longrightarrow & H^{k+2}\left(L_V\right)
\end{array}
$$

Here, if $k = 2\ell$ is even, the isomorphism ψ is the cup product with the cohomology generator $a \in H^2(\mathbb{C}P^{n+1}; \mathbb{Z})$, and similarly, ψ_V is the cup product with $j^2(a)$. The assertion about j^k follows now from (5.6) by decreasing induction on ℓ, using the fact mentioned at the beginning of the proof that j^{2n} is given by multiplication by d.

Let us next choose a generic hyperplane $H \subset \mathbb{C}P^{n+1}$ (i.e., H is transversal to a Whitney stratification of V), and set as before $V_H = V \cap H$. It then follows from Theorem 5.1 and the cohomology long exact sequence of the pair (V, V_H) that $H^{2n-1}(V; \mathbb{Z}) \cong 0$. It therefore remains to prove (5.4) and the corresponding assertion about j^k for k in the range for $n+s+2 \leqslant k \leqslant 2n-2$. Let us consider the commuting square

$$
\begin{array}{ccc}
V_H & \xrightarrow{\delta} & H = \mathbb{C}P^n \\
\gamma \downarrow & & \downarrow \\
V & \xrightarrow[j]{} & \mathbb{C}P^{n+1}
\end{array}
$$

and the induced commutative diagram in cohomology:

(5.7)
$$
\begin{array}{ccc}
H^k\left(\mathbb{C}P^{n+1};\mathbb{Z}\right) & \xrightarrow{\ j^k\ } & H^k(V;\mathbb{Z}) \\
{\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\gamma^k} \\
H^k\left(\mathbb{C}P^{n};\mathbb{Z}\right) & \xrightarrow[\ \delta^k\ ]{} & H^k\left(V_H;\mathbb{Z}\right)
\end{array}
$$

By Theorem 5.1 and the cohomology long exact sequence of the pair (V, V_H) we get that γ^k is an isomorphism for all integers k in the range $n + s + 2 \leqslant k \leqslant 2n - 2$. Moreover, since $V_H \subset \mathbb{C}P^n$ is a degree d reduced projective hypersurface with a $(s-1)$-dimensional singular locus (by transversality), the induction hypothesis yields that $H^k(V_H;\mathbb{Z}) \cong H^k(\mathbb{C}P^n;\mathbb{Z})$ for $n + s \leqslant k \leqslant 2n - 2$ and that, in the same range and for k even, the homomorphism δ^k is given by multiplication by d. The commutativity of the above diagram (5.7) then yields (5.4) for all integers k satisfying $n + s + 2 \leqslant k \leqslant 2n - 2$, and the corresponding assertion about the induced homomorphism j^k for k even in the same range. This completes the proof of the Theorem 5.3. □

Remark 5.4. — Let us remark here that the proof of Kato's theorem in [2, Theorem 5.2.11] relies on the Kato–Matsumoto result [5] on the connectivity of the Milnor fiber of the singularity at the origin of the affine cone $CV \subset \mathbb{C}^{n+2}$.

Remark 5.5. — One can prove the homological version of Theorem 5.3 in the similar manner, namely by using Corollary 5.2 instead of Theorem 5.1. This yields the isomorphisms:

(5.8) $\qquad H_k(V;\mathbb{Z}) \cong H_k\left(\mathbb{C}P^{n+1};\mathbb{Z}\right) \quad \text{for all} \quad n + s + 2 \leqslant k \leqslant 2n,$

and the homomorphisms induced by the inclusion $j : V \hookrightarrow \mathbb{C}P^{n+1}$ in homology are given in this range (and for k even) by multiplication by $d = \deg(V)$.

Remark 5.6. — We already noted that Theorem 1.2 yields the isomorphism (1.2) of Kato's theorem (see Corollary 1.5 (i)). On the other hand, Kato's Theorem 5.3 may be used to obtain a weaker version of Theorem 1.2 by more elementary means. Indeed, in the notations from the Introduction consider the diagram:

$$
\begin{array}{ccccccc}
H^k\left(\mathbb{C}P^{n+1};\mathbb{Z}\right) & \xrightarrow{\ \cong\ } & H^k\left(\mathbb{C}P^{n+1}\times D;\mathbb{Z}\right) & \xrightarrow{\ b^k\ } & H^k\left(V_D;\mathbb{Z}\right) & \xrightarrow{\ c^k\ } & H^k\left(V_t;\mathbb{Z}\right) \\
& & & & \downarrow{\scriptstyle\cong} & & \\
& & & & H^k(V;\mathbb{Z}) & &
\end{array}
$$

and let $a^k := c^k \circ b^k$. By Theorem 5.3, we have that:

(i) a^k is the multiplication by d if $k > n$ even and an isomorphism for $k < n$;

(ii) b^k is the multiplication by d if $n + s + 2 \leqslant k \leqslant 2n$ (k even) and an isomorphism for $k < n$.

Therefore, c^k is an isomorphism if $n + s + 2 \leqslant k \leqslant 2n$ or $k < n$. The cohomology long exact sequence of the pair (V_D, V_t) then yields that $H_\varphi^k(V) \cong H^{k+1}(V_D, V_t; \mathbb{Z}) \cong 0$ for all integers $k \notin [n-1, n+s+1]$.

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