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Jason M. STARR, Zhiyu TIAN & Runhong ZONG

Weak Approximation for Fano Complete Intersections in Positive Characteristic

WEAK APPROXIMATION FOR FANO COMPLETE INTERSECTIONS IN POSITIVE CHARACTERISTIC

by Jason M. STARR, Zhiyu TIAN & Runhong ZONG (*)

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ABSTRACT. — For a smooth curve B over a field $k = \bar{k}$ with $\mathrm{char}(k) = p$, for every complete intersection X_B in $B \times_{\mathrm{Spec}\,k} \mathbb{P}^n_k$ of type $(d_1, \ldots, d_c)$, we prove weak approximation of adelic points of X_B by $k(B)$-points at all places of (strong) potentially good reduction, if the Fano index is $\geqslant 2$ and if $p > \max(d_1, \ldots, d_c)$. This also applies to specializations of complex Fano manifolds with Picard rank 1 and Fano index 1 away from "bad primes".

RÉSUMÉ. — Pour une courbe lisse B sur un corps $k = \bar{k}$ de caractéristique positive p, pour chaque intersection complète X_B dans $B \times_{\mathrm{Spec}\,k} \mathbb{P}^n_k$ de type $(d_1, \ldots, d_c)$, nour prouvons l'approximation faible des points adeliques de X_B par des $k(B)$-points sur toutes les places de forte réduction potentiellement bonne, si l'indice de Fano est au moins deux et si $p > \max(d_1, \ldots, d_c)$. Cela s'applique également aux spécialisations des variétés de Fano complexes de nombre de Picard de rang 1 et d'indice de Fano 1 en dehors de l'ensemble des mauvais nombres premiers.

1. Statement of the Theorem

Let k be an algebraically closed field with $\mathrm{char}(k) = p \geqslant 0$. Let B be a connected, smooth k-curve. By Tsen's Theorem, [21], for positive integers n and $(d_1, \ldots, d_c)$, every complete intersection in $\mathbb{P}^n$ of type $(d_1, \ldots, d_c)$ over $k(B)$ has at least one $k(B)$-point if $i := n + 1 - (d_1 + \cdots + d_c)$ is positive. Then the complete intersection is *Fano*, and i is the *Fano index*. This degree bound in Tsen's theorem is sharp, as one can easily check that the Calabi–Yau hypersurface defined by $\sum_{i=0}^{n-1} t^i X_i^n = 0$ in $\mathbb{P}^{n-1}_{k(t)}$ has no rational points. There are singular Fano complete intersections with a unique $k(B)$-point (e.g. take the projective cone over the Calabi–Yau hypersurface as above).

QUESTION 1.1. — *In the smooth case, is there more than one rational point; are there infinitely many rational points; are they Zariski dense; are they adelically dense?*

For a flat, projective B-scheme X_B, the topology on the set of adelic points,

$$X_B(\mathbb{A}_{B/k}) := \prod_{b \in B(k)} X_B(\mathrm{Spec}\, \widehat{\mathcal{O}}_{B,b}),$$

is the coarsest such that for every $b \in B(k)$ and for every integer $e \geqslant 0$, the map

$$\rho_{b,e} : X_B(\mathbb{A}_{B/k}) \longrightarrow X_B(\mathrm{Spec}\, \widehat{\mathcal{O}}_{B,b}/\mathfrak{m}_{B,b}^{e+1}\widehat{\mathcal{O}}_{B,b}),$$

is continuous for the discrete topology on the codomain. The B-scheme X_B satisfies *weak approximation* if $X_B(k(B))$ is dense in this topological space.

The B-scheme X_B is a *good reduction* complete intersection if there exists a faithful, étale B-scheme, $\mathcal{B} \to B$, and a $\mathcal{B}$-smooth complete intersection in $\mathbb{P}_{\mathcal{B}}^n$ whose geometric generic fibers equal the geometric generic fiber of X_B.

The weaker notion of a *(strong) potentially good reduction* complete intersection over B allows $\mathcal{B}$ to be a smooth, tame, Deligne–Mumford k-stack together with an k-morphism, $\mathcal{B} \to B$, whose coarse moduli space is a faithful, étale B-scheme. The stack is *tame* if p is prime to the order of each inertia group.

THEOREM 1.2. — *Weak approximation holds for (strong) potentially good reduction complete intersections if $p > \max(d_1, \ldots, d_c)$ and $i > 1$, resp. if $p > \max(d_1, \ldots, d_c)$ and p prime to the Gromov–Witten invariant of 1-pointed conics if $i = 1$.*

Remark 1.3. — See [17, Theorem 8] for an improvement.

In characteristic 0, the *Weak Approximation Conjecture* of Hassett and Tschinkel predicts that weak approximation holds at all places for every flat, projective B-scheme whose geometric generic fiber is smooth and *separably rationally connected*. The Weak Approximation Conjecture in characteristic 0 was proved by Hassett and Tschinkel at places of good reduction, [10], and it was proved by the second and third authors at places of (strong) potentially good reduction, [19]. In particular, this proves Theorem 1.2 in characteristic 0.

1.1. Structure of the proof

Theorem 1.2 follows from several results about specialization to positive characteristic of properties of Fano manifolds in characteristic 0. The strongest of these properties is *separable rational connectedness*. For a (strong) potentially good reduction B-scheme, weak approximation holds if the geometric fibers of the smooth family over $\mathcal{B}$ are separably rationally connected, [19].

To prove specialization of separable rational connectedness, we use work of the second author, [18], deducing separable rational connectedness from *stability* of the tangent bundle and *separable uniruledness*. The main new ingredient of this article is a theorem about specialization of separable uniruledness, Theorem 3.10. We also prove a theorem about specialization of stability of the tangent bundle for Fano manifolds with cyclic Picard group and Fano index $i = 1$, Theorem 3.21, using several results about specialization for Picard groups, specialization of algebraic de Rham cohomology groups, and specialization of the torsion order, which are due to Gounelas–Javanpeykar, [6], due to Totaro, [20], and due to Voisin, [22]. These three theorems revolve around *decomposition of the diagonal* and *torsion order*, which we quickly review.

Remark 1.4. — After this article is written, some stronger results about specialization of separable rational connectedness is obtained in [16], leading to an improvement of some results of this article.

Acknowledgments

2. Decomposition of the diagonal and torsion order

Let K be a field. Let X be a smooth, geometrically connected, m-dimensional, projective K-scheme. In the Chow group $\mathrm{CH}_m(X \times_{\mathrm{Spec}\,K} X)$,

denote by CH'_m the subgroup generated by all prime m-cycles $Z \subset X \times_{\mathrm{Spec}\,K} X$ such that $\mathrm{pr}_1(Z)$ is a proper subvariety of X, and denote by CH''_m the image of the flat pullback homomorphism,

$$\mathrm{pr}_2^* : \mathrm{CH}_0(X) \longrightarrow \mathrm{CH}_m(X \times_{\mathrm{Spec}\,K} X).$$

DEFINITION 2.1. — *For each integer $\tau > 0$, a τ-decomposition of the diagonal is an ordered pair $(Z', Z'') \in \mathrm{CH}'_m \times \mathrm{CH}''_m$ such that $\tau \cdot [\Delta_X]$ equals $Z' + Z''$.*

DEFINITION 2.2 ([4]). — *The torsion order, $\mathrm{Tor}(K, X)$, is the greatest common divisor of all $\tau > 0$ such that there exists a τ-decomposition of the diagonal, or 0 if there is no such τ.*

There are cycle class maps from Chow groups into étale ℓ-adic cohomology for every prime ℓ different from the characteristic. More generally, there are also cycle class maps in fppf topology, cf. [6, Remark A.3]. Thus, there is an action of correspondences on the ℓ-adic cohomology of X. The action of the correspondence $[\Delta_X]$ is the identity, and the action of $\tau \cdot [\Delta_X]$ is the "multiplication by τ" map. On the other hand, the Gysin pushforward maps for proper subvarieties shift the cohomological degree by $\geqslant 2$. Thus, a τ-decomposition of the diagonal has strong consequences for ℓ-adic cohomology. In particular, consider the cycle class map,

$$\mathrm{cycle}_\ell^1 : \mathrm{Pic}(X_{\overline{K}}) \otimes \mathbb{Z}_\ell \longrightarrow H^2_{\mathrm{fppf}}(X_{\overline{K}}, \mathbb{Z}_\ell(1)).$$

THEOREM 2.3 ([6, Theorem A.1, Lemma 3.1]). — *If the torsion order of $X_{\overline{K}}$ is a positive integer τ, then the Abelian variety $\mathrm{Pic}^0(X_{\overline{K}})$ is zero, the cycle class map cycle_ℓ^1 is an isomorphism for every prime ℓ, and the ℓ-torsion in $\mathrm{Pic}(X_{\overline{K}})$ is zero for every ℓ prime to τ.*

Proof. — The only assertion not explict in [6] is the vanishing of $\mathrm{Pic}^0(X_{\overline{K}})$ and of the ℓ-torsion, $H^1_{\mathrm{fppf}}(X_{\overline{K}}, \mu_\ell)$, for all ℓ not dividing τ. As proved there, the multiplication by τ map on ℓ-adic cohomology (of degree $\geqslant 1$) equals a sum of Gysin pushforward maps for proper subvarieties. Each Gysin map has image in the subgroup of cohomological degree $\geqslant 2$. Since the ℓ-adic cohomology group is ℓ-torsion, and since ℓ does not divide τ, the multiplication by τ map is an isomorphism. Thus, $H^1_{\mathrm{fppf}}(X_{\overline{K}}, \mu_\ell)$ vanishes.

In particular, the ℓ-torsion in $\mathrm{Pic}(X_{\overline{K}})$ vanishes for every prime ℓ not dividing τ, i.e., for all but finitely many primes. Since every Abelian variety over $\overline{K}$ has nonvanishing ℓ-torsion for all primes ℓ different from the characteristic, the Abelian variety $\mathrm{Pic}^0(X_{\overline{K}})$ is zero. $\qquad\square$

As a corollary, the maximal *solvable* prime-to-τ quotient of $\pi_1(X_{\overline{K}}, x_0)^{p'}$ is trivial.

Remark 2.4. — We mention a simple example here. Let X be a smooth projective complex surface admitting an N-decomposition of the diagonal. Then we claim that N can be chosen to be the smallest positive integer that annihilates the torsion of $H_1(X, \mathbb{Z})$ (clearly any N has to be a multiple of this). To see this, simply note that there is an algebraic cycle Z supported in $X \times D$ for some divisor D of X such that $N(\Delta_X - x \times X - Z)$ is homologically trivial. By the result of Bloch–Srinivas ([2]), for a codimesion 2 cycle on a variety admitting a rational decomposition of the diagonal, homological equivalence is the same as algebraic equivalence and algebraic equivalence modulo rational equivalence is parameterized by the intermediate Jacobian. We can apply this to $X \times X$. Since X admits a decomposition of the diagonal, X has no non-trivial odd degree rational cohomology. Thus the intermediate Jacobian of $X \times X$ is trivial. It follows that $N(\Delta_X - x \times X - Z)$ is rationally equivalent to 0. This example means that it is almost impossible to find a stronger restriction on π_1 in general.

THEOREM 2.5 ([20, Lemma 2.2]). — *If the torsion order of $X_{\overline{K}}$ is positive and prime to the characteristic of K, then $H^0(X, \Omega^r_{X/K})$ is zero for every $r > 0$.*

Proof. — The hypothesis is stated differently in [20]: Totaro requires universal triviality of $\mathrm{CH}_0 \otimes \mathbb{Q}$ in characteristic 0, respectively universal triviality of CH_0 /p in characteristic p. Totaro's hypothesis follows from the hypothesis above.

Write Z'' as $\mathrm{pr}_2^* W$ for $W \in \mathrm{CH}_0(X_{\overline{K}})$. Then for every field extension $E/\overline{K}$, the action of the correspondence

$$\mathrm{pr}_{2,*}(Z' \cap \mathrm{pr}_1^*(\,\cdot\,)) : \mathrm{CH}_0(X_E) \longrightarrow \mathrm{CH}_0(X_E),$$

is the zero homomorphism, since every zero-cycle is rationally equivalent to one that is the complement of the supporting divisor D of Z'. Thus, the "multipliction by τ" map on $\mathrm{CH}_0(X_E)$ has image in the cyclic subgroup generated by the pullback of W. It follows that $\mathrm{CH}_0 \otimes \mathbb{Q}$ is universally trivial. For every prime p that does not divide τ, also CH_0 /p is universally trivial. $\qquad\square$

When the characteristic $p > 0$ does not divide τ, Theorem 2.5 implies vanishing of the p-torsion in $\mathrm{Pic}(X_{\overline{K}})$ avoiding the use of fppf cohomology in [6].

LEMMA 2.6. — *If the characteristic p is positive and if $H^0(X_{\overline{K}}, \Omega_{X_{\overline{K}}/\overline{K}})$ vanishes, e.g., if p is prime to the torsion order, then $\mathrm{Pic}(X_{\overline{K}})$ has vanishing p-torsion.*

Proof. — This uses a fragment of the theory of Cartier isomorphisms. Let $\mathcal{L}$ be an invertible $\mathcal{O}_{X_{\overline{K}}}$-module, and let s be an isomorphism of $\mathcal{O}_{X_{\overline{K}}}$-modules,

$$s : \mathcal{L}^{\otimes p} \longrightarrow \mathcal{O}_{X_{\overline{K}}}.$$

Every point of $X_{\overline{K}}$ is contained in a Zariski open affine U on which there exists a trivializing isomorphism of $\mathcal{O}_U$-modules,

$$t_U : \mathcal{O}_U \longrightarrow \mathcal{L}_U.$$

The composite $s \circ t_U^{\otimes p}$ is multiplication by a section $f_U \in \mathbb{G}_m(U)$. Multiplying t_U by $g_U \in \mathbb{G}_m(U)$ modifies f_U to $f_U g_U^p$. In particular, the logarithmic derivative $f_U^{-1} \mathrm{d} f_U \in \Omega_{X_k/k}(U)$ is independent of the choice of trivializations. Thus, there exists a global section α of $\Omega_{X_{\overline{K}}/\overline{K}}$ such that for every (U, t_U), the logarithmic derivative $f_U^{-1} \mathrm{d} f_U$ equals α.

If α equals 0, then every $\mathrm{d} f_U$ equals 0. Thus, étale locally, f_U equals g_U^p for some unique g_U. There exists an étale cover $X_{\overline{K}}' \to X_{\overline{K}}$ and a unique trivialization t of $\mathcal{L}$ on $X_{\overline{K}}'$ such that $s \circ t^{\otimes p}$ is the identity. The uniqueness of the trivialization, combined with étale descent, implies that t descends to a unique trivialization of $\mathcal{L}$ on $X_{\overline{K}}$ such that $s \circ t^{\otimes p}$ is the identity. In particular, if $H^0(X_{\overline{K}}, \Omega_{X_{\overline{K}}/\overline{K}})$ vanishes, then there is no p-torsion in $\mathrm{Pic}(X_{\overline{K}})$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

For X smooth, the Chow group, $\mathrm{CH}^1(X_{\overline{K}})$, equals the Picard group, $\mathrm{Pic}(X_{\overline{K}})$. Consider the dual Abelian group to the Picard group,

$$\mathrm{Pic}(X_{\overline{K}})^\vee := \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}(X_{\overline{K}}), \mathbb{Z}).$$

For every 1-cycle α in $X_{\overline{K}}$, there is an element in this group associating to every invertible sheaf the corresponding degree on the curve,

$$\mathrm{cycle}_1(\alpha) : \mathrm{Pic}(X_{\overline{K}}) \longrightarrow \mathbb{Z}, \quad [\mathcal{L}] \longmapsto \deg_\alpha(\mathcal{L}).$$

This defines a cycle class homomorphism,

$$\mathrm{cycle}_1 : \mathrm{CH}_1(X_{\overline{K}}) \longrightarrow \mathrm{Pic}(X_{\overline{K}})^\vee, \quad \alpha \longmapsto \mathrm{cycle}_1(\alpha).$$

The kernel equals the group of numerically trivial 1-cycles, and it contains the group of 1-cycles that are algebraically equivalent to 0.

DEFINITION 2.7. — *For a 1-cycle α, the* index *equals the index in $\mathbb{Z}$ of the image subgroup of* $\mathrm{cycle}_1(\alpha)$.

THEOREM 2.8 ([22, Theorem 3.4 (ii)]). — *The image of* cycle_1 *contains* $\tau \cdot \mathrm{Pic}(X_{\overline{K}})^\vee$, *for τ equal to the torsion order.*

Proof. — The statement is vacuous if τ equals 0, thus assume that τ is positive. By Theorem 2.3, the Abelian variety $\mathrm{Pic}^0(X_{\overline{K}})$ is zero. Thus, $\mathrm{Pic}(X_{\overline{K}})^{\vee}$ is a finite free Abelian group.

Via flat pullback of cycles, intersection product of cycles, and proper pushforward of cycles, there is an action of correspondences on the Chow group. In particular, this restricts to an action of correspondences on the Picard group, $\mathrm{Pic}(X_{\overline{K}}) = \mathrm{CH}^1(X_{\overline{K}})$. This action factors through rational equivalence of correspondences. This, in turn, induces a transpose action on the dual group $\mathrm{Pic}(X_{\overline{K}})^{\vee}$. This makes $\mathrm{Pic}(X_{\overline{K}})^{\vee}$ into a module for the ring of correspondences. Similarly the Chow group $\mathrm{CH}_1(X_{\overline{K}})$ is a module for this ring. By the projection formula, cycle_1 is a module homomorphism.

The image of $\mathrm{CH}_0(X_{\overline{K}})$ in $\mathrm{CH}_n(X_{\overline{K}} \times_{\mathrm{Spec}\,\overline{K}} X_{\overline{K}})$ acts trivially on the Picard group since the image of the cycle under the first projection is a zero-cycle. Similarly, for the normalization of a prime divisor in $X_{\overline{K}}$,

$$e : D \longrightarrow X_{\overline{K}}$$

for every prime n-cycle,

$$Z \subset D \times_{\mathrm{Spec}\,\overline{K}} X_{\overline{K}}),$$

the action of $(e \times \mathrm{Id})_*[Z]$ on $\mathrm{Pic}(X_{\overline{K}})$ is zero if the fiber of Z over the generic point of D is empty.

Thus, assume that Z dominates D. Then the generic fiber is a curve in $X_{\overline{K}} \times_{\mathrm{Spec}\,\overline{K}} \mathrm{Spec}\,\overline{K}(D)$. Denote the curve class by α. The action of $(e \times \mathrm{Id})_*[Z]$ on $\mathrm{Pic}(X_{\overline{K}})$ is

$$\mathrm{Pic}(X_{\overline{K}}) \longrightarrow \mathrm{Pic}(X_{\overline{K}}), \quad \mathcal{L} \longmapsto \langle \mathcal{L}, \mathrm{cycle}_1(\alpha)\rangle \mathcal{O}_X(D).$$

Therefore, the transpose action has image contained in the span of $\mathrm{cycle}_1(\alpha)$.

Since Z' is a sum of such correspondences, the image of the corresponding action is contained in the image of cycle_1. Therefore the multiplication by τ map has image contained in the image of cycle_1, i.e., $\tau \cdot \mathrm{Pic}(X_{\overline{K}})^{\vee}$ is contained in the image of cycle_1. $\qquad\square$

3. Separable Uniruledness

There are several variants of uniruledness and rational (chain) connectedness. Let K be a field. Denote by $\overline{K}$ the separable closure of K. Let X be an integral, n-dimensional, projective K-scheme X.

DEFINITION 3.1. — *For each integer $r > 0$, an r-uniruling of X over K is a triple (M, j, σ) of an integral, quasi-projective K-scheme M, a morphism of K-schemes,*

$$j : Y \longrightarrow X \times_{\mathrm{Spec}\,K} M, \quad h := \mathrm{pr}_X \circ j, \quad \pi := \mathrm{pr}_M \circ j,$$

and a section of the r-fold fiber product over M of Y,

$$\sigma : M \longrightarrow Y \times_M \cdots \times_M Y, \quad \sigma_i := \mathrm{pr}_i \circ \sigma,$$

such that π is proper and flat with geometric generic fibers that are connected, at-worst-nodal curves of genus 0, and such that the following morphism to the r-fold fiber product of X is dominant and generically finite,

$$h^{(r)} : M \xrightarrow{\sigma} Y \times_M \cdots \times_M Y \xrightarrow{h \times \cdots \times h} X \times_{\mathrm{Spec}\,K} \cdots \times_{\mathrm{Spec}\,K} X,$$

$$\mathrm{pr}_i \circ h^{(r)} = h \circ \sigma_i.$$

The r-uniruling is smooth *(sometimes* unsplit*) if π is smooth. A smooth r-uniruling is* separable *if $h^{(r)}$ is generically smooth.*

DEFINITION 3.2. — *The K-variety is* uniruled, resp. separably uniruled, *if there exists a 1-uniruling, resp. a separable 1-uniruling. The K-variety is* rationally chain connected, resp. rationally connected, separably rationally connected, *if there exists a 2-uniruling, resp. a smooth 2-uniruling, a separable 2-uniruling.*

DEFINITION 3.3 (Uniruling index). — *For every r-uniruling, the* curve class *in $\mathrm{Pic}(X_{\overline{K}})^{\vee}$ equals the cycle class of the h-pushforward of all geometric fibers of π.*

Following [12, Definition IV.1.7.3], the r-uniruling index, $u_r(K, X)$, is the greatest common divisor of $\deg(h^{(r)})$ for all r-unirulings, and 0 if there exists no r-uniruling. The r-smooth index, $u_r^{\mathrm{sm}}(K, X)$, resp. r-separable index, $u_r^{\mathrm{sep}}(K, X)$, is the greatest common divisor of $\deg(h^{(r)})$ for all r-unirulings that are smooth, resp. that are separable.

DEFINITION 3.4. — *Let $f : \mathbb{P}^1 \to X$ be a non-constant morphism to a variety X. We say f is a* free *(resp.* very free*) curve if $f(\mathbb{P}^1) \subset X^{\mathrm{sm}}$ and*

$$f^* T_X \cong \bigoplus_{i=1}^{\dim X} \mathcal{O}_{\mathbb{P}^1}(a_i), \quad a_i \geqslant 0 \quad (\text{resp. } a_i > 0)$$

The following theorem is well-known.

THEOREM 3.5 ([12, Chapter IV, Theorem 1.9, Theorem 3.7]). — *Let X be a smooth projective variety defined over an algebraically closed field. Then X is separably uniruled (resp. separably rationally connected) if and only if there exists a free (resp. very free) curve.*

3.1. Separable Rational Connectedness and Weak Approximation

Separable rational connectedness implies weak approximation at places of (strong) potentially good reduction by two theorems of the second and third authors.

THEOREM 3.6 ([19, Theorem 1.3]). — *For every algebraically closed field k, for every finite cyclic group Γ with $p \nmid \#\Gamma$, every smooth, projective k-scheme with a Γ-action is Γ-separably rationally connected if and only if it is separably rationally connected.*

A smooth, projective, connected k-scheme X with an action of Γ by k-isomorphisms is Γ-*separably rationally connected* if the fixed locus X^Γ is nonempty and for every k-point $(x_0, x_\infty) \in X^\Gamma \times_{\operatorname{Spec} k} X^\Gamma$, there exists a Γ-equivariant, very free k-morphism,

$$u : (\mathbb{P}^1_k, 0, \infty) \longrightarrow (X, x_0, x_\infty),$$

where Γ acts faithfully on $\mathbb{P}^1_k$ by scaling by roots of unity.

THEOREM 3.7 ([19, Theorem 1.5]). — *For every smooth, connected curve B over a field k, for every flat, projective B-scheme, weak approximation holds at all places of (strong) potentially good reduction such that the (base change) smooth fiber is separably rationally connected.*

Remark 3.8. — In [19, Theorem 1.5] there is a hypothesis that k has characteristic 0. However, this hypothesis is only used to *deduce* that the closed fiber is separably rationally connected. The explicit tree of rational curves and infinitesimal deformation theory from [19, Section 4] is valid in arbitrary characteristic.

QUESTION 3.9. — *For a smooth, proper scheme over a DVR, if the generic fiber is separably rationally connected, is the closed fiber separably rationally connected?*

One of the main theorems of the article answers the question positively in a special case, the proof of which is in Section 6.

Theorem 3.10. — *Let R be a DVR with residue characteristic $p > 0$. Let X_R be a smooth, projective R-scheme whose generic fiber is a complete intersection of type $(d_1, \ldots, d_c)$ and Fano index $i > 0$.*

If $i > 1$ and $p > \max(d_1, \ldots, d_c)$, then the closed fiber is separably uniruled by free lines and rationally connected by free curves. It is separably rationally connected if it is a complete intersection.

Remark 3.11. — By [16, Corollary 9], the smooth closed fiber is always separably rationally connected under these assumptions.

Remark 3.12. — There is a related result proved by the "rational curves working group" of the AIM Workshop *Rational Subvarieties in Positive Characteristic*, 2016. Among other results, this working group proved that for every smooth, degree d hypersurface $X_k \subset \mathbb{P}_k^n$ with $i \geqslant 2$, if $p > (d!)((d!) - 1)^{n-d-1}$, then every line in X_k containing a sufficiently general point is a free line. This is relevant to positive characteristic extensions of the Debarre–de Jong conjecture (where existence of free lines is not sufficient). The method of proof of this result is quite different from the method of proof of Theorem 3.10, following [9, Section 4.2] rather than Corollary 5.3.

The inequality in the theorem is nearly sharp.

Definition 3.13. — *For an integer $d > 1$, the p-adic valuation of d is the unique integer $v \geqslant 0$ such that $d = p^v e$ for a p-prime integer e. The integer d is p-special, resp. p-nonspecial, if $1 \leqslant e < p$, resp. if $e > p$. An ordered tuple of positive integers $(d_1, \ldots, d_c)$ is p-special if for every $d_i \geqslant p$ that is p-divisible, resp. p-prime, the integer d_i is p-special, resp. the integer $d_i + 1$ is p-special. Otherwise, it is p-nonspecial.*

In Section 7, we will prove the following.

Proposition 3.14 (Non-existence of Free Lines). — *Let k be an algebraically closed field k of characteristic p. For every p-nonspecial $(d_1, \ldots, d_c)$ such that $\max(d_1, \ldots, d_c) \geqslant p$, for every n such that Fano index is $\geqslant 2$, there exists a smooth complete intersection in $\mathbb{P}_k^n$ of type $(d_1, \ldots, d_c)$, yet with no free lines.*

3.2. Specialization of Separable Uniruledness

Separable uniruledness in characteristic 0 implies separable uniruledness for smooth specializations over a field whose characteristic is prime to an

explicit integer D, and there is a similar result for stability of the tangent bundle and for separable rational connectedness.

Let B be a connected, smooth, quasi-projective scheme over $\operatorname{Spec}\mathbb{Z}$. For every integer $D > 1$, denote by $B_D \subset B$ the open subscheme obtained by inverting D, i.e., the inverse image in B of $\operatorname{Spec}\mathbb{Z}[1/D]$. Let $X_B \to B$ be a smooth, projective morphism of relative dimension n with geometrically connected fibers. Denote by $\operatorname{Spec} K \to B$ the generic point, and denote the generic fiber of X_B by $X = \operatorname{Spec} K \times_B X_B$.

In Section 5.2, we will prove the following specialization result of separable uniruledness.

THEOREM 3.15 (Separable Uniruledness in Mixed Characteristic). — *If X is geometrically uniruled, then for $D = u_1(K, X)$, every fiber over B_D is separably uniruled.*

3.3. Stability of the Tangent Bundle

By the following theorem of the second author, separable uniruledness implies free rational connectedness for manifolds with cyclic Picard group, resp. it implies separable rational connectedness for manifolds with cyclic Picard group and *stable* tangent bundle.

For a smooth k-scheme X_k, the torsnion-free, coherent $\mathcal{O}_X$-modules that are *reflexive* are precisely the pushforwards to X of locally free sheaves from those open subschemes $U \subset X_k$ whose complement has codimension $\geqslant 2$.

For a smooth, projective k-scheme X_k whose Picard group is generated by an ample invertible sheaf $\mathcal{O}_X(1)$, for every reflexive $\mathcal{O}_X$-module $\mathcal{E}$ of rank $r > 0$, the exterior power $\bigwedge^r_{\mathcal{O}_X} \mathcal{E}$ equals $\mathcal{O}_X(d)$ on an open subset whose complement has codimension $\geqslant 2$. The integer d is the *degree* of $\mathcal{E}$. The *slope* is the fraction $\mu(\mathcal{E}) = d/r$. Finally, $\mathcal{E}$ is *stable*, resp. *semistable*, if $\mu(\mathcal{F}) < \mu(\mathcal{E})$, resp. if $\mu(\mathcal{F}) \leqslant \mu(\mathcal{E})$, for every nonzero, reflexive $\mathcal{O}_X$-submodule $\mathcal{F}$ of $\mathcal{E}$ with rank $< r$.

THEOREM 3.16 ([18, Theorem 5, Corollary 9]). — *Over every field $k = \overline{k}$, every smooth, projective k-variety X_k that is separably uniruled with cyclic Picard group is freely rationally connected. It is separably rationally connected if T_{X_k} is stable. In particular, a smooth complete intersection in $\mathbb{P}^n_k$ is separably rationally connected if and only if it is separably uniruled.*

Here we say a variety is *freely rationally connected* if there is a family of free curves $\pi : Y \to M, h : Y \to X$ such that $h^{(2)} : Y \times_M Y \to X \times X$ is dominant.

Remark 3.17. — One can compare this result with [16, Theorem 3], where a sufficient and necessary condition for X to be separably rationally connected is found.

QUESTION 3.18. — *In characteristic 0, does every Fano manifold with cyclic Picard group have a stable tangent bundle? Does this hold in positive characteristic?*

For an algebraically closed field $\overline{K}$ of characteristic 0, there is a general result of Miles Reid for Fano manifolds $X_{\overline{K}}$ with cyclic Picard group whose Fano index equals 1.

THEOREM 3.19 ([14, Theorem 3]). — *For $\overline{K}$ of characteristic 0 and a Fano manifold $X_{\overline{K}}$ with cyclic Picard group whose Fano index equals 1, the tangent bundle is stable.*

We prove two cases of this for specializations to positive characteristic. The first case uses Theorem 2.3 and Theorem 2.5 of Gounelas–Javanpeykar, [6], resp. of Totaro, [20]. This version applies in the ramified case and bounds the "bad primes" as divisors of the torsion order, $\mathrm{Tor}(\overline{K}, X_{\overline{K}})$.

The second case uses the method of Kato, Fontaine–Messing, and Deligne–Illusie on degenerations of the Hodge-de Rham spectral sequence (Fröhlicher spectral sequence) and crystalline cohomology, [5]. This version applies only in the unramified case, it identifies the "bad primes" as all primes $p \leqslant \dim(X_{\overline{K}})$ and all prime orders of torsion elements of $H^*(X_{\overline{K}})$, and it includes a hypothesis that there is no "extra" torsion in crystalline cohomology (it is unknown whether such specializations of Fano manifolds have "extra" torsion).

Let R be a complete DVR whose fraction field K has characteristic 0 and whose residue field k is perfect of characteristic p (e.g., the Witt vectors $W(k)$ of the residue field). Let X_R be a smooth, projective R-scheme of relative dimension n with connected geometric fibers. There is an associated specialization map of Picard groups of geometric fibers,

$$\mathrm{spec}_{X_R/R} : \mathrm{Pic}(X_{\overline{K}}) \longrightarrow \mathrm{Pic}(X_{\bar{k}}).$$

This map is an isomorphism if the connected components of $\mathrm{Pic}_{X_R/R}$ are finite, étale R-schemes.

The proof of the next two theorems are in Section 4.2.

THEOREM 3.20 (Surjectivity of the Picard specialization map). — *Assume that the torsion order $\mathrm{Tor}(\overline{K}, X_{\overline{K}})$ is positive.*

    (1) *If $\mathrm{Tor}(\overline{K}, X_{\overline{K}})$ is prime to p, then the specialization map $\mathrm{spec}_{X_R/R}$ is surjective and $H^0(X_k, \Omega^r_{X_k/k})$ vanishes for $r > 0$.*

(2) If $p > \dim(X_{\overline{K}})$, if R equals $W(k)$, and if the crystalline cohomology $W(k)$-module $H^*(X/W(k), W(k))$ is p-torsion-free, then the connected components of $\mathrm{Pic}_{X_R/R}$ are finite, étale R-schemes, and for every $r > 0$ both $H^0(X_k, \Omega^r_{X_k/k})$ and $H^r(X_k, \mathcal{O}_{X_k})$ vanish.

THEOREM 3.21 (Stability of Tangent Bundles in Mixed Characteristic). — Assume that the geometric generic fiber $X_{\overline{K}}$ has effective first Chern class generating $\mathrm{Pic}(X_{\overline{K}})$ (i.e. X is Fano, of Picard rank 1, Fano index 1). The geometric closed fiber has stable tangent bundle if one of the following conditions hold.

(1) The torsion order $\mathrm{Tor}(\overline{K}, X_{\overline{K}})$ is prime to p.
(2) We have that $p > \dim(X_{\overline{K}})$, that R equals $W(k)$, and that the crystalline cohomology $W(k)$-module $H^*(X/W(k), W(k))$ is p-torsion-free. In this case, all deformations of X_k are unobstructed.

Once we know the separable rational connectedness, the following weak approximation result is immediate (see Section 5.2).

COROLLARY 3.22 (Weak Approximation for Picard Rank 1 and Fano Index 1). — Assume the hypotheses above.

(1) In case (1), define $D := \mathrm{Tor}(\overline{K}, X_{\overline{K}}) \cdot u_1(\overline{K}, X_{\overline{K}})$. If D is prime to p (e.g., if $u_2(\overline{K}, X_{\overline{K}})$ is prime to p), then X_k is separably rationally connected.
(2) In case (2), define $D := (n!) \cdot u_1(K, X)$. If D is prime to p, then X_k is separably rationally connected.

In both cases, if p is prime to D, weak approximation holds at all points of B of (strong) potentially good reduction where the (base change) smooth fiber is isomorphic to $X_{\overline{k}}$ as above.

Remark 3.23. — In the second of the two cases, since deformations of X_k are unobstructed, existence of at least one B-section follows by deforming to characteristic 0 and applying [7]. However, weak approximation certainly does not follow from [7]. Moreover, in the first case, even existence of one B-section requires the full proof.

This corollary applies to index-1 Fano complete intersections in $\mathbb{P}^n$, in weighted projective spaces, in Grassmannians, and in other projective homogeneous varieties with cyclic Picard group. However, bounding the integer $u_1(k, X)$ seems difficult. In fact, it seems difficult to explicitly compute the genus-0, Gromov–Witten invariant $\langle \tau_{e-2}(\eta_X) \rangle_{0,e}$ associated to free curves of minimal $c_1(T_X)$-degree $e \geqslant 2$; this Gromov–Witten invariant is an integer, and $u_1(k, X)$ is a divisor of this integer.

For Fano hypersurfaces of Fano index 1, the minimal degree e equals 2, and the Gromov–Witten invariant $\langle \eta_X \rangle_{0,2}$ has been computed, [3]. Recall that the *Catalan number* is defined by

$$C_d = \frac{1}{d+1} \binom{2d}{d}.$$

THEOREM 3.24 (Weak Approximation for Index 1 Fano Hypersurfaces). — *For every integer $d \geqslant 4$, for the unique positive integer $n = d$ such that the Fano index equals 1, smooth, degree-d hypersurfaces in $\mathbb{P}_k^n$ are separably uniruled by conics and separably rationally connected if $p > d$ and p does not divide $(d+1)C_d - 2^d$. Also, Theorem 1.2 holds in this case.*

The proof can be found in Section 6.2.

4. Proofs of Theorems 3.20 and 3.21

4.1. Vanishing results

The key step is a vanishing theorem. As in the statement of Theorems 3.20 and 3.21, let X_R be a smooth, projective scheme of relative dimension n over R, a complete mixed characteristic DVR. Denote by K, resp. by k, the fraction field, resp. the residue field. Assume that k is perfect of characteristic p. First we prove the vanishing result from Theorem 3.20.

PROPOSITION 4.1 (Hodge Coniveau in Mixed Characteristic). — *Assume that the geometric generic fiber $X_{\overline{K}}$ has positive torsion order $\mathrm{Tor}(\overline{K}, X_{\overline{K}})$.*

  (1) *if the torsion order is prime to p, then the geometric closed fiber has vanishing cohomology groups $H^0(X_k, \Omega^r_{X_k/k})$ for every $r > 0$.*
  (2) *If $p > \dim(X_{\overline{K}})$, if R equals the Witt vectors $W(k)$, and if the crystalline cohomology $W(k)$-module $H^*(X/W(k), W(k))$ is p-torsion-free, then for every $r > 0$, both $H^0(X_k, \Omega^r_{X_k/k})$ and $H^r(X_k, \mathcal{O}_{X_k})$ vanish.*

Proof. — Existence of a τ-decomposition of the diagonal for $\tau > 0$ implies the Hodge cohomology of the geometric generic fiber has coniveau $\geqslant 1$, i.e., for every $r > 0$ both $H^0(X_{\overline{K}}, \Omega^r_{X_{\overline{K}}/\overline{K}})$ and $H^r(X_{\overline{K}}, \mathcal{O}_{X_{\overline{K}}})$ vanish for the geometric generic fiber $X_{\overline{K}}$.

Next assume that p is prime to the torsion order. By Voisin's theorem on specialization of decompositions of the diagonal, also p is prime to the torsion over of $X_{\bar{k}}$. Then vanishing of $H^0(X_k, \Omega^r_{X_k/k})$ follows from Totaro's theorem, Theorem 2.5, [20].

Finally, assume that $p > n$, assume that R equals the Witt vectors $W(k)$, and assume that the crystalline cohomology $W(k)$-module $H^*(X/W(k), W(k))$ is p-torsion-free, i.e., it is a finite, free $W(k)$-module. Since $n < p$, and since the crystalline cohomology is p-torsion-free, the cohomology groups $H^q(X_R, \Omega^r_{X_R/R})$ are finite free R-modules, compatibly with arbitrary base change, cf. [5]. Thus, vanishing of $H^0(X_{\overline{K}}, \Omega^r_{X_{\overline{K}}/\overline{K}})$ and $H^r(X_{\overline{K}}, \mathcal{O}_X)$ for the geometric generic fiber $X_{\overline{K}}$ implies the same for the closed fiber. $\qquad\square$

LEMMA 4.2. — *For a smooth, projective scheme X_R with connected geometric fibers over R, a complete DVR with fraction field K and residue field k, the relative Picard R-scheme is a countable disjoint union of finite, étale R-schemes provided $H^1(X_k, \mathcal{O}_{X_k})$ and $H^2(X_k, \mathcal{O}_{X_k})$ vanish. If $\mathrm{char}(k)$ equals p and if $H^0(X_k, \Omega_{X_k/k})$ vanishes, then $\mathrm{Pic}(X_{\bar{k}})$ is p-torsion-free.*

Proof. — If $H^r(X_k, \mathcal{O}_X)$ vanishes for $r = 1, 2$, then every invertible sheaf on the geometric closed fiber deforms uniquely to an invertible sheaf over the geometric generic fiber by infinitesimal deformation theory. Since X_R is proper and smooth over R, also the relative Picard scheme satisfies the valuative criterion of properness. It is a countable increasing union of open and closed subschemes that are finite and étale which are indexed by Hilbert polynomials. The final part is a restatement of Lemma 2.6. $\qquad\square$

THEOREM 4.3. — *Let K be a field with separable closure $\overline{K}$. Let X be a smooth, projective, geometrically connected K-scheme.*

(1) *For every integer r, if there exists an $r + 1$-uniruling of X, then there exists an r-uniruling, and $u_r(\overline{K}, X_{\overline{K}})$ divides $u_{r+1}(\overline{K}, X_{\overline{K}})$. If $\mathrm{char}(K)$ equals 0, then $u_r(\overline{K}, X_{\overline{K}})$ divides $u_2(\overline{K}, X_{\overline{K}})^r$. In this case, $u_2(\overline{K}, X_{\overline{K}})$ and $u_r(\overline{K}, X_{\overline{K}})$ have the same prime divisors for every $r \geqslant 2$.*

(2) *(Bloch–Srinivas) If $u_2(\overline{K}, X_{\overline{K}})$ is nonzero, then $\mathrm{Tor}(\overline{K}, X_{\overline{K}})$ is a (nonzero) divisor of $u_2(\overline{K}, X_{\overline{K}})$.*

Proof.

(1). — Perform a base change from K to $\overline{K}$. Let $((h, \pi), \sigma)$ be an $(r+1)$-uniruling of degree d. For a sufficiently general $x \in X(\overline{K})$, denote by M' the inverse image of $\{x\} \times X^r$ under $h^{(r+1)}$. Denote $M' \times_M Y$ by Y', with its projection $\pi' : Y' \to M'$. Denote the composite $M' \times_M Y \to Y \xrightarrow{h} X$ by h'. Denote by σ' the M'-morphism to the r-fold fiber product

$$M' \longrightarrow Y' \times_{M'} \cdots \times_{M'} Y', \quad \mathrm{pr}_Y \circ \mathrm{pr}_i \circ \sigma' = \sigma_i|_{M'}.$$

Then $((h', \pi'), \sigma')$ is an r-uniruling of degree d.

In characteristic 0, let $((h, \pi), \sigma)$ be a 2-uniruling such that of degree d. Let $r \geqslant 1$ be an integer. Fix an r-tuple of points in $\mathbb{P}^1_K$,

$$(t_1, \ldots, t_r) \in \mathbb{P}^1_K \times_{\mathrm{Spec}\,K} \cdots \times_{\mathrm{Spec}\,K} \mathbb{P}^1_K.$$

For the generic point $(x, (y_1, \ldots, y_r))$ of $X \times_{\mathrm{Spec}\,K} X^r$, geometrically there are d curves in the 2-uniruling that connect x to y_i for $i = 1, \ldots, r$.

Consider those combs of genus 0 curves in X whose handle, $\mathbb{P}^1$, is contracted to x, and with r teeth, whose i^{th} tooth is a curve of the 2-uniruling connecting x to y_i and intersecting the handle at t_i. For $i = 1, \ldots, r$, the number of possibilities for the ith tooth is d. Thus, the number of such combs is d^r. Of course every Galois orbit has order divisible by d. Also, the sum of the orders equals d^r. Thus, the greatest common divisor of these orders is divisible by d and divides d^r.

Since the characteristic equals 0, and since $(x, (y_1, \ldots, y_r))$ is the generic point, each of the teeth is a very free rational curve. Thus, this comb gives a smooth point of the moduli space of r-pointed, genus-0 stable maps to X that evaluate to $(y_1, \ldots, y_r)$. It follows that the smooth locus of the generic fiber of the evaluation morphism for r-pointed, genus-0 stable maps has a closed point whose residue field has degree dividing d^r, as an extension of the residue field of $(x, (y_1, \ldots, y_r))$. Up to taking residuals of this closed point in complete intersections zero-cycles in the smooth locus of this fiber, every dense open subset of the smooth locus of the generic fiber has index dividing d^r.

(2). — This is the usual Bloch–Srinivas argument, [2]. □

4.2. Proofs of Theorems 3.20 and 3.21

Proof of Theorem 3.20. — The vanishing statement is Proposition 4.1. Lemma 4.2 immediately implies the second case. Also, Lemma 4.2 implies vanishing of the p-torsion subgroup of the Picard group in the first case.

Theorem 2.3 implies vanishing of the prime-to-p torsion in both cases, and it implies that the cokernel of the specialization map is a finite p-group. Thus the Picard groups equal the Néron–Severi groups. The specialization map is surjective on torsion in the Néron–Severi group. Finally, the cokernel of the induced map of torsion-free quotients is a finite p-group. Thus, it only remains to prove surjectivity of the induced map of Néron–Severi groups.

If the map of torsion-free quotients is not surjective, then there exists a free quotient $\widetilde{\chi} : \mathrm{NS}_{X_{\bar{k}}/\bar{k}} \to \mathbb{Z}$ that restricts on $\mathrm{NS}_{X_{\overline{K}}/\overline{K}}$ as $p\chi$ for some $\chi \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{NS}_{X_{\overline{K}}/\overline{K}}, \mathbb{Z})$. By Voisin's theorem, Theorem 2.8, [22], there exists a one-cycle in $X_{\overline{K}}$ whose cycle class equals χ. Up to making R bigger, this one-cycle specializes to a one-cycle on $X_{\bar{k}}$. Then $\widetilde{\chi}$ equals p times the cycle class of this specialized one-cycle, contradicting that $\widetilde{\chi}$ is surjective. This contradiction proves that the specialization map is surjective on torsion-free quotients of the Néron–Severi group. $\square$

Proof of Theorem 3.21. — The proof follows closely the proof of [14, Theorem 3]. By the above, we can assume that $H^0(X_{\bar{k}}, \Omega^r_{X_{\bar{k}}/\bar{k}})$ vanishes for every $r > 0$, and we can assume that the specialization map on Picard groups is an isomorphism. In the remainder of the proof, replace k by $\bar{k}$, i.e., assume that k is algebraically closed.

Let $\mathcal{F}$ be a nonzero, reflexive $\mathcal{O}_{X_k}$-submodule of $\Omega^1_{X/k}$ of rank $q < n$. Then $\bigwedge^q_{\mathcal{O}_X} \mathcal{F}$ is an $\mathcal{O}_{X_k}$-submodule of $\Omega^q_{X/k}$ of rank 1. Since X_k is k-smooth, also $\Omega^1_{X/k}$ is locally free, hence $\Omega^q_{X/k}$ is locally free. Thus, this rank 1 submodule factors through its determinant.

The geometric Picard group is generated by the class $H_k = c_1(T_{X/k})$. Thus, the determinant equals $\mathcal{O}_{X_k}(eH_k)$ for some integer e. If $e \geqslant 0$, then the determinant is effective, so that also $H^0(X_k, \Omega^1_{X/k})$ is nonzero, contrary to the previous paragraph. Thus, e is negative. So the slope $\mu(\mathcal{F})$ is $\leqslant -1/q$, and this is strictly less than $-1/n = \mu(\Omega^1_{X/k})$. Thus, $\Omega^1_{X/k}$ is stable. Taking duals, also T_X is stable.

Finally, in the second case we prove unobstructedness of deformations. In this case, since the relative Picard scheme is cyclic, and since there is an ample invertible sheaf by hypothesis, in fact the dual of the relative dualizing sheaf is ample, i.e., the relative dualizing sheaf is "antiample". By Raynaud's theorem, [5, Corollarie 2.8], also $H^{n-2}(X_k, \Omega^1_{X/k} \otimes \mathcal{L})$ vanishes for every antiample invertible sheaf $\mathcal{L}$. For $\mathcal{L}$ equal to $\omega_{X/k}$, Serre duality then gives vanishing of the obstruction group,

$$H^2(X_k, T_{X/k})^{\vee} \cong H^{n-2}(X_k, \Omega^1_{X/k} \otimes \omega_{X/k}). \qquad \square$$

5. Proof of Theorem 3.15

5.1. Specialization of separable uniruledness

Let R be a DVR with residue field k. Let $h_R : Y_R \to X_R$ be a surjective, projective, generically finite morphism of flat, finitely presented R-schemes.

LEMMA 5.1. — *If X_k is integral, then there exists a dense open subset W of X_R containing the generic point η of X_k such that W is regular, such that $h_R^{-1}(W)$ is Cohen–Macaulay, and such that $h_R^{-1}(W) \to W$ is finite and flat.*

Proof. — Since R is a DVR, and since X_R is a finite type R-scheme, the regular locus X_R^{reg} is an open subscheme of X_R, [8, Corollaire 6.12.6]. Since X_R is R-flat, also the intersection of X_R^{reg} with X_k equals the regular locus X_k^{reg} of X_k, [8, Proposition 6.5.1]. Since X_k is integral, the regular locus X_k^{reg} contains the generic point $\eta = \operatorname{Spec} k(X_k)$. Thus, without loss of generality, replace X_k by X_k^{reg}, and assume that X_k is regular. Up to replacing X_R by the unique connected component containing η, also assume that X_R is integral. Then X_R is R-flat of constant fiber dimension d.

By Chevalley's theorem, there is a maximal, open subscheme U of X_R over which h_R is finite. Since Y_R is R-flat, every associated point of Y_R maps to the generic point of $\operatorname{Spec} R$. Since h_R is generically finite, this associated point also maps to U. If the image does not equal the generic point of X_R, then the closure of the image is an R-flat, proper closed subset of X_R, hence it does not contain the generic point η of X_k. Up to replacing X_R by the open complement of such proper images, assume that every associated point of Y_R maps to the generic point of X_R, e.g., every irreducible component $Y_{R,i}$ surjects to X_R. Since h_R is generically finite, also $Y_{R,i}$ is R-flat of constant fiber dimension d. By Krull's Hauptidealsatz, both Y_k and X_k have pure dimension d. Thus, for every irreducible component $Y_{k,i}$ of Y_k on which h_k has positive fiber dimension $\geqslant 1$, the closed image $h_k(Y_{k,i})$ has dimension $\leqslant d-1$. Since X_k is integral of dimension d, this closed image is a proper closed subset. Thus, up to shrinking X_R once more, assume that h_k is also generically finite. In other words, U contains η. Thus, up to replacing X_R by U, assume that h_R is finite.

By Auslander's theorem, [8, Proposition 6.11.2 (i), Corollaire 6.11.3], the Cohen–Macaulay locus Y_R^{CM} of Y_R is an open subscheme of Y_R. Since Y_R is R-flat, and since R is Cohen–Macaulay, the intersection of Y_R^{CM} with Y_k equals the Cohen–Macaulay locus Y_k^{CM}, [8, Corollaire 6.3.5 (ii)]. By the Generic Flatness Theorem, there exists a dense open subset V of X_k over

which h_k is flat; denote the closed complement of V by C. Since Y_k has pure dimension d, and since h_k is finite, also $h_k^{-1}(V)$ is a dense open in Y_k. Since this dense open is flat over a regular scheme, it is Cohen–Macaulay, [8, Proposition 6.1.5]. Thus, Y_k^{CM} is dense in Y_k. Up to replacing X_R by the open complement of C, assume that Y_R^{CM} contains Y_k. Then the closed complement D of Y_R^{CM} in Y_R maps finitely to a closed subset of X_R whose intersection with X_k is empty. Thus, up to replacing X_R by the open complement of the closed image of D, assume that Y_R^{CM} equals Y_R, i.e., Y_R is Cohen–Macaulay.

Finally, since Y_R is Cohen–Macaulay, since X_R is regular, and since h_R is finite and dominant, also h_R is finite and flat, [8, Proposition 6.1.5]. $\qquad\square$

PROPOSITION 5.2. — *With hypotheses as in the previous lemma, if the generic degree of h_R is prime to $\mathrm{char}(k)$, then there exists an irreducible component $Y_{k,i}$ of Y_k, with its reduced structure, such that $Y_{k,i} \to X_k$ is surjective, generically étale and tame, i.e., the finite degree is prime to $\mathrm{char}(k)$.*

Proof. — By the previous lemma, up to shrinking X_R, assume that X_R is regular and that h_R is finite and flat. Denote by $\deg(h_R)$ the generic degree of h_R.

The fiber Y_η of h_R over η is a finite $k(X_k)$-scheme whose length as an $k(X_k)$-module equals $\deg(h_R)$. This length equals the sum over all points $y \in Y_\eta$ of the product of the length at y of Y_η by the degree $[\kappa(y) : k(X_k)]$. Since this sum is prime to p, there exists at least one point y such that $[\kappa(y) : k(X_k)]$ is prime to p. In particular, since the degree of this finite field extension is prime to p, this finite field extension is separable. Thus the closure $Y_{k,i}$ of this point in Y_k is an irreducible component such that the induced morphism to X_k is dominant and generically étale. $\qquad\square$

Now we are ready to prove closedness of separable uniruledness under suitable assumptions. Let M_R be an integral, flat, projective R-scheme, X_R a smooth, projective R-scheme and let

$$\pi_R : Y_R \longrightarrow M_R, \quad h_R : Y_R \longrightarrow X_R,$$

be projective R-morphisms such that π is flat with geometric generic fiber isomorphic to $\mathbb{P}^1$, and with h_R surjective and generically finite of degree d. We also fix a relatively ample line bundle $\mathcal{O}_{X_R}(1)$ so that lines means a rational curve of $\mathcal{O}_{X_R}(1)$-degree 1.

COROLLARY 5.3. — *If d is prime to $\mathrm{char}(k)$, then for at least one generic point η of the closed fiber M_k, at least one component of the fiber Y_η is a*

free curve in X_k. In particular, if the geometric generic fiber of π_R gives a free line in X_K, then also the closed fiber X_k contains free lines.

Proof. — By the previous proposition, there exists an integral closed subscheme $Y_{k,i}$ of Y_k such that the restriction,

$$h_{k,i} : Y_{k,i} \longrightarrow X_k,$$

is surjective, generically étale, and tame. Since h_R is a generically finite morphism between integral schemes, this closed subscheme contains a nonempty open U subscheme of Y_k. Up to shrinking, assume that $h_{k,i}$ is étale on U. Then U is k-smooth. Since π_k is flat, the image V of U in M_k is an integral, smooth k-scheme that is an open subscheme of M_k. Thus, the closure $M_{k,i}$ of V is an irreducible component of M_k.

For the generic point η of $M_{k,i}$, the closed subscheme $Y_{k,i}$ is the closure of an irreducible component $Y_{k,i,\eta}$ of the π-fiber over η. Finally, since $h_{k,i}$ is étale, the curve $Y_{k,i,\eta}$ is a free rational curve in X_k.

As an irreducible component of the full fiber, the degree of $Y_{k,i}$ is at most the degree of the generic fiber of π. Thus, if the generic fiber is a line, also $Y_{k,i}$ is a line. $\qquad\square$

Remark 5.4. — One could compare Corollary 5.3 with the theorem of Matsusaka, which says that given a projective scheme over a DVR, if the geometric generic fiber is ruled, then at least one component of the central fiber is ruled ([13, Section 4.5, p. 111–114]). Despite the similarity in the statement, the proofs are quite different.

5.2. Proof of Theorem 3.15

Proof of Theorem 3.15. — Every point b of B_D maps to a point of $\operatorname{Spec} \mathbb{Z}[1/D]$, either $\operatorname{Spec} \mathbb{Q}$ or $\operatorname{Spec}(\mathbb{Z}/p\mathbb{Z})$ for a prime integer p not dividing D.

Consider first the case of $\operatorname{Spec}(\mathbb{Z}/p\mathbb{Z})$. The closure C of $\{b\}$ in $B \times \operatorname{Spec}(\mathbb{Z}/p\mathbb{Z})$ is an integral, quasi-projective scheme over $\mathbb{Z}/p\mathbb{Z}$. Since $\mathbb{Z}/p\mathbb{Z}$ is perfect, this scheme is generically smooth. Thus, after replacing B by a dense open subscheme that contains b, assume that b is the generic point of an integral, closed subscheme C of $B \times \operatorname{Spec}(\mathbb{Z}/p\mathbb{Z})$ that is smooth over $\operatorname{Spec}(\mathbb{Z}/p\mathbb{Z})$.

Since C and $B \times \operatorname{Spec}(\mathbb{Z}/p\mathbb{Z})$ are smooth over $\operatorname{Spec}(\mathbb{Z}/p\mathbb{Z})$, the closed immersion of C in B is a regular embedding of some codimension, say b. Also $\operatorname{Spec} \mathbb{Z}/p\mathbb{Z}$ is itself a regular embedding into $\operatorname{Spec} \mathbb{Z}$ of codimension 1.

Thus, C is a regular embedding into B_D of codimension $b + 1$. Denote the blowing up along C by

$$\nu : \widetilde{B}_D \longrightarrow B_D.$$

This is a regular scheme that is flat over $\operatorname{Spec}\mathbb{Z}[1/D]$. Moreover, the exceptional divisor $E = \nu^{-1}(C)$ is a (Zariski) locally trivial projective space bundle of relative dimension b over C.

Consider the pullback X_E of X_M. Assume that there exists a pair of $\operatorname{Spec}(\mathbb{Z}/p\mathbb{Z})$-morphisms

$$h_E : Y_E \longrightarrow X_E, \quad \pi_E : Y_E \longrightarrow M_E,$$

such that h_E is dominant, generically étale, and tame over X_E, and such that π_E is projective and flat with geometric generic fiber isomorphic to $\mathbb{P}^1$. For the maximal open subscheme M_E^o over which π_E is smooth, the inverse image Y_E^o is a dense open subscheme.

By Lemma 5.1, there is a dense open subscheme of X_E over which Y_E is flat. Similarly, since the smooth locus is open, there is a dense open subscheme such that the inverse image in Y_E is contained in the dense open Y_E^o, and such that h_E is étale on the inverse image open. Since X_E is flat over E, the image of this dense open subscheme contains a dense open subscheme of E.

Since E is a projective space bundle over C, there exists a rational section $C \dashrightarrow E$ whose image intersects this dense open subscheme. Shrink B_D further so that this rational section is regular on C. The pullback of $Y_E \to E$ by this section gives a diagram,

$$h_C : Y_C \longrightarrow C, \quad \pi_C : Y_C \longrightarrow M_C$$

such that h_C is dominant, generically étale, and tame over C. Thus, to prove the existence of Y_C, it suffices to find Y_E, i.e., it suffices to prove the result after replacing B_D by $\widetilde{B}_D$ and after replacing C by E.

Denote by R the local ring of $\widetilde{B}_D$ at the generic point η of E, and denote by X_R the pullback of X_B. Since C is smooth over $\operatorname{Spec}(\mathbb{Z}/p\mathbb{Z})$ and since B_D is smooth over $\operatorname{Spec}\mathbb{Z}$, also pR equals the maximal ideal of R, and the residue field equals the function field of E. The fraction field of R equals K, the function field of B_D, and the generic fiber of X_R equals the generic fiber X.

By hypothesis, p is prime to $u_1(K, X)$. By the definition of $u_1(k, X)$, there exists a dominant, generically finite morphism of degree d prime to p,

$$h_K^o : Y_K^o \longrightarrow X_K,$$

and there exists a proper, smooth k-morphism of relative dimension 1,

$$\pi_K^o : Y_K^o \longrightarrow M_K^o,$$

whose geometric generic fiber is isomorphic to $\mathbb{P}^1$. Up to shrinking M_K^o, assume that π_K^o is smooth. Then the relative dualizing sheaf ω_π has dual $\omega_\pi^\vee$ that is π_K^o-very ample with vanishing higher direct images, and with pushforward compatible with arbitrary base change. Up to shrinking M_K^o further, assume that the pushforward is free of rank 3. Then Y_K^o is isomorphic to a M_K^o-flat Cartier divisor in $M_K^o \times_{\operatorname{Spec} K} \mathbb{P}_K^2$ of relative degree 2. The graph of h_K^o gives a closed immersion of Y_K^o in $M_K^o \times_{\operatorname{Spec} K} \mathbb{P}_K^2 \times_{\operatorname{Spec} K} X$.

By Nagata compactification, there exists a flat, projective scheme M_R' over R whose generic fiber contains M_K^o as a dense open. Denote by Y_R' the closure in $M_R' \times_{\operatorname{Spec} R} \mathbb{P}_R^2 \times_{\operatorname{Spec} R} X_R$ of Y_K^o. This is flat over a dense open subscheme of M_R' that contains M_K^o. There is a blowing up $M_R \to M_R'$ such that the strict transform Y_R^{pre} of Y_R' is an M_R-flat closed subscheme of $M_R \times_{\operatorname{Spec} R} \mathbb{P}_R^2 \times_{\operatorname{Spec} R} X_R$ (e.g., take M_R to be the closure of the graph of the induced R-rational transformation from M_R to the R-relative Hilbert scheme of $\mathbb{P}_R^2 \times_{\operatorname{Spec} R} X_R$). By Corollary 5.3, there exists a free curve in the closed fiber X_k. By standard limit arguments, for a dense open subscheme of E, there is a family of free curves in the fibers of $X_E \to E$ over this dense open subscheme.

The proof in case b equals $\operatorname{Spec} \mathbb{Q}$ is similar and easier. $\qquad\square$

Proof of Corollary 3.22. — The hypotheses of Theorem 3.21 all apply. Thus, the tangent bundle of X_k is stable. By the previous proof, also X_k contains free rational curves. By Theorem 3.16, the closed fiber X_k is separably rationally connected. Thus, for a fibration over a smooth, affine, connected $\bar{k}$-curve B such that at least one $\bar{k}$-fiber is $\bar{k}$-isomorphic to the base change of X_k as above, there exists a B-section by [11]. Finally, by [19], weak approximation holds at every $\bar{k}$-point of B of (strongly) potentially good reduction such that the smooth (potential) fiber is $\bar{k}$-isomorphic to the base change of such X_k. $\qquad\square$

6. Free Lines on Complete Intersections

6.1. Specialization of (free) lines for complete intersections

Let R be a commutative ring with 1. Let X_R be a projective R-scheme; for simplicity, assume that X_R is R-flat. Let $\mathcal{O}(1)$ be an R-ample invertible sheaf on X_R. Let g, n, and e be nonnegative integers. For every R-scheme

T, a *genus-g, n-pointed stable map to X_R of $\mathcal{O}(1)$-degree e* over T is a datum

$$\zeta = (\pi : C \longrightarrow T, (\sigma_i : T \longrightarrow C)_{i=1,\ldots,n}, u : C \longrightarrow X_R)$$

of a proper, flat morphism π of relative dimension 1 whose geometric fibers are connected, reduced, at-worst-nodal curves, of an ordered n-tuple (σ_i) of pairwise disjoint sections of π with image in the smooth locus of π, and of an R-morphism u such that the log relative dualizing sheaf,

$$\omega_{\pi,\sigma} := \omega_\pi \left(\sum_{i=1}^n \sigma_i(T) \right)$$

is u-ample (*stability*) and such that $u^*\mathcal{O}(1)$ has relative degree e over T. For a genus-g, n-pointed stable map to X_R of $\mathcal{O}(1)$-degree e,

$$\widetilde{\zeta} = \left(\widetilde{\pi} : \widetilde{C} \longrightarrow T, (\widetilde{\sigma}_i : T \longrightarrow \widetilde{C})_{i=1,\ldots,n}, \widetilde{u} : \widehat{C} \longrightarrow X_R \right),$$

a *2-morphism* from ζ to $\widetilde{\zeta}$ is a T-isomorphism $\phi : C \to \widetilde{C}$ such that $\widetilde{u} \circ \phi$ equals u and such that $\phi \circ \sigma_i$ equals $\widetilde{\sigma}_i$ for every $i = 1, \ldots, n$. For every R-morphism $f : T' \to T$ the *f-pullback* $f^*\zeta$ of ζ equals

$$\left(\mathrm{pr}_{T'} : T' \times_T C \longrightarrow T', ((\mathrm{Id}_{T'}, \sigma_i) : T' \longrightarrow T' \times_T C)_{i=1,\ldots,n}, \right.$$
$$\left. u \circ \mathrm{pr}_C : T' \times_T C \longrightarrow X_R \right).$$

Altogether, these operations define a stack in groupoids $\overline{\mathcal{M}}_{g,n}((X_R/R, \mathcal{O}(1)), e)$ over the category of R-schemes.

THEOREM 6.1 ([1, Theorem 2.8]). — *The stack in groupoids $\overline{\mathcal{M}}_{g,n}((X_R/R, \mathcal{O}(1)), e)$ is an algebraic (Artin) stack with finite diagonal that is R-proper. There is a coarse moduli space,*

$$f_{g,n,e} : \overline{\mathcal{M}}_{g,n}((X_R/R, \mathcal{O}(1)), e) \longrightarrow \overline{\mathrm{M}}_{g,n}((X_R/R, \mathcal{O}(1)), e),$$

and $\overline{\mathrm{M}}_{g,n}((X_R/R, \mathcal{O}(1)), e)$ is a projective R-scheme. The maximal open subscheme on which f is an isomorphism parameterizes stable maps that have only the identity automorphism.

COROLLARY 6.2. — *For $g = 0$, for $e = 1$, and for every integer $n \geqslant 0$, the morphism $f_{0,n,1}$ is an isomorphism, i.e., $\overline{\mathcal{M}}_{0,n}((X_R/R, \mathcal{O}(1)), e)$ is a projective R-scheme.*

Remark 6.3. — If n is 0, and if the line bundle $\mathcal{O}(1)$ is very ample, the moduli space $\overline{\mathcal{M}}_{0,n}((X_R/R, \mathcal{O}(1)), e)$ can be realized as a closed subscheme of the Grassmaniann $G(2, V)$ parameterizing all lines in the projective space $\mathbb{P}(V)$.

Proof. — This is well-known, but we include a proof for completeness. It suffices to check for every algebraically closed field, $R \to k$, every stable map defined over k has trivial automorphism group. Consider first the case that n equals 0. For a genus-0 stable map $u : C \to X_k$ of degree 1, there is at most one irreducible component on which $u^*\mathcal{O}(1)$ has positive degree. Since every other component must be contracted by u, the stability hypothesis implies that C is irreducible. Since C is a connected, at-worst-nodal curve of arithmetic genus 0, in fact C is smooth. For every sufficiently positive integer d, the invertible sheaf $\mathcal{O}(d)$ is very ample on X, so that the stable map gives a morphism $C \to \mathbb{P}^{N_d}$ of degree d. The automorphism of every such morphism is cyclic of order dividing d. Choosing d among a set of sufficiently positive, relatively prime integers, the automorphism group of the stable map is trivial.

For a stable map with $n > 0$,

$$(C, (q_1, \ldots, q_n), u : C \longrightarrow X_k),$$

there is a unique irreducible component C_i of u such that $u^*\mathcal{O}(1)$ has degree 1, and every automorphism acts as the identity on C_i by the previous paragraph. Let $q_{n+1}, q_{n+2}, q_{n+3} \in C_i$ be distinct k-points that are different from the finitely many nodes and marked points that are contained in C_i. Then every automorphism of the stable map is, in particular, an automorphism of the marked curve $(C, (q_1, \ldots, q_n, q_{n+1}, q_{n+2}, q_{n+3}))$. The stability condition for stable maps implies that this $n+3$-pointed curve is also stable. Since the stack $\overline{\mathcal{M}}_{0,n+3}$ is a projective scheme, the claim follows. $\qquad\square$

For each finite sequence of positive integers, $(d_1, \ldots, d_c)$, let D be the product of $d_i!$ for $i = 1, \ldots, c$. Let n be the unique integer such that the Fano index equals 2, i.e., $n = 1 + (d_1 + \cdots + d_c)$. Let B' be the affine space over $\operatorname{Spec} \mathbb{Z}$ that parameterizes ordered c-tuples $(g_1, \ldots, g_c)$ of homogeneous polynomials g_i in variables $(t_0, \ldots, t_n)$ such that $\deg(g_i) = d_i$. Let $B \subset B'$ be the dense open subscheme that parameterizes such c-tuples whose Jacobian ideal has empty zero scheme, i.e., such that $\operatorname{Zero}(g_1, \ldots, g_c)$ is a smooth complete intersection of type $(d_1, \ldots, d_c)$ in $\mathbb{P}^n$. Let $X_B \subset \mathbb{P}^n_B$ be the universal smooth, complete intersection of type $(d_1, \ldots, d_c)$ and Fano index 2.

PROPOSITION 6.4. — *For the $\mathbb{Z}[1/D]$-scheme B_D, every point parameterizes a smooth complete intersection X in $\mathbb{P}^n$ of type $(d_1, \ldots, d_c)$ and Fano index 2 such that there exists an integral closed subscheme Y of the space $\overline{\mathcal{M}}_{0,1}((X, \mathcal{O}(1)), 1)$ of pointed lines in X with $Y \to X$ surjective, generically étale and tame.*

Proof. — By the proof of Theorem 3.15, it suffices to prove that the evaluation morphism for 1-pointed lines is dominant and generically finite of degree D for the generic complete intersection in characteristic 0.

For the function field $K = \mathbb{Q}(M_{\mathbb{Q}})$ and the generic complete intersection $X_K \subset \mathbb{P}^n_K$ of type $(d_1, \ldots, d_c)$, the tangent direction of each line at the marked point defines a closed immersion

$$\tau : \overline{\mathcal{M}}_{0,1}((X_K/K, \mathcal{O}(1)), 1) \hookrightarrow \mathbb{P}_{X_K}(T_{X_K/K}),$$

whose image is, relative to the projection to X_K, generically a complete intersection of type $(2, 3, \ldots, d_1, 2, 3, \ldots, d_2, \ldots, 2, 3, \ldots, d_c)$, cf. [12, Exercise V.4.10.5, p. 272], [15, Lemma 4.3]. In particular, the codimension of this complete intersection equals $(d_1 - 1) + \cdots + (d_c - 1) = (d_1 + \cdots + d_c) - c$. Of course the projective bundle of the tangent bundle has relative dimension $n - c - 1$. Thus, since n equals $1 + (d_1 \cdots + d_c)$, this complete intersection Y_K has relative dimension 0 over X_K, i.e., it is generically finite over X_K. Finally, by Bézout's theorem, the relative degree of this generically finite map equals $(d_1!) \cdots (d_c!) = D$. □

6.2. Proof of main theorems

Proof of Theorem 3.10. — By Corollary 5.3 to prove that X_k is separably uniruled by lines, it suffices to construct an irreducible closed subscheme $Y_{\mathrm{Frac}(R)}$ of the $\mathrm{Frac}(R)$-fiber of $\overline{\mathcal{M}}_{0,1}((X_R/R, \mathcal{O}(1)), 1)$ such that the induced morphism $Y_{\mathrm{Frac}(R)} \to X_{\mathrm{Frac}(R)}$ is surjective and generically étale of degree prime to p. This is proved by induction on n for every integer at least $n_0 := 1 + (d_1 + \cdots + d_c)$.

The base case is when $n = n_0$, in which case $Y_{\mathrm{Frac}(R)}$ exists by Proposition 6.4. By way of induction, assume that $n \geqslant n_0 + 1$ and assume that the result holds for $n - 1$. Denote by K the function field of $\mathbb{P}^1$ over $\mathrm{Frac}(R)$. Since $\mathrm{Frac}(R)$ is an infinite field, by Bertini's theorem, there exists a pencil of hyperplane sections, $\widetilde{X}_{\mathbb{P}^1} \subset X_{\mathrm{Frac}(R)} \times \mathbb{P}^1$, whose generic fiber $\widetilde{X}_K$ is K-smooth. Thus $\widetilde{X}_K$ is a smooth complete intersection of type. By the induction hypothesis, there exists an integral closed subscheme $\widetilde{Y}_K$ in $\overline{\mathcal{M}}_{0,1}((\widetilde{X}_K/K, \mathcal{O}(1)), 1)$ such that $\widetilde{Y}_K \to \widetilde{X}_K$ is surjective and generically étale of degree prime to p. Define $\widetilde{Y}_{\mathbb{P}^1}$ to be the closure of $\widetilde{Y}_K$ in $\overline{\mathcal{M}}_{0,1}((\widetilde{X}_{\mathbb{P}^1}/\mathbb{P}^1, \mathcal{O}(1)), 1)$. By covariance of the stack of stable maps, there is an induced proper $\mathrm{Frac}(R)$-morphism,

$$\overline{\mathcal{M}}_{0,1}((\widetilde{X}_{\mathbb{P}^1}/\mathbb{P}^1, \mathcal{O}(1)), 1) \longrightarrow \overline{\mathcal{M}}_{0,1}(X_{\mathrm{Frac}(R)}/\mathrm{Frac}(R)), 1),$$

i.e., lines in the hyperplane sections of $X_{\mathrm{Frac}(R)}$ give lines in $X_{\mathrm{Frac}(R)}$ (that happen to be contained in the specified hyperplane section). Define $Y_{\mathrm{Frac}(R)}$ to be the image of $\widetilde{Y}_{\mathbb{P}^1}$ in $\overline{\mathcal{M}}_{0,1}((X_{\mathrm{Frac}(R)}/\mathrm{Frac}(R), \mathcal{O}(1)), 1)$. Since the generic fiber of $Y_{\mathrm{Frac}(R)} \to X_{\mathrm{Frac}(R)}$ equals the generic fiber of $\widetilde{Y}_{\mathbb{P}^1} \to \widetilde{X}_{\mathbb{P}^1}$, also $Y_{\mathrm{Frac}(R)}$ is surjective, generically étale, and of degree prime to p over $X_{\mathrm{Frac}(R)}$. Thus, by induction on n, there always exists an integral closed subscheme $Y_{\mathrm{Frac}(R)}$ of $\overline{\mathcal{M}}_{0,1}((X_{\mathrm{Frac}(R)}/\mathrm{Frac}(R), \mathcal{O}(1)), 1)$ that is generically étale of degree prime to p over $X_{\mathrm{Frac}(R)}$.

By [6], the closed fiber of X_R has cyclic Picard group. Since the closed fiber is separably uniruled, by [18, Theorem 5, Corollary 9], the closed fiber is also freely rationally connected, and the closed fiber is even separably rationally connected when it is a complete intersection. □

Proof of Theorem 1.2. — By hypothesis, for a tame, Galois base change by $\widehat{\mathcal{O}}_{B,b} \to R$, for a Γ-equivariant modification, the closed fiber over b is a smooth complete intersection in $\mathbb{P}^n$ of type $(d_1, \ldots, d_c)$. By Theorem 3.10, the closed fiber is separably rationally connected. By Theorem 3.7, the original family satisfies weak approximation at b. □

Proof of Theorem 3.24. — We apply Corollary 3.22. By [4] and [15], the torsion order $\mathrm{Tor}(\overline{K}, X_{\overline{K}})$ divides $d!$, and existence of lines on such hypersurfaces implies that $i_1(\overline{K}, X_{\overline{K}})$ equals 1. Thus, it suffices to prove that for the generic Fano hypersurface X in $\mathbb{P}_K^d$ of degree d and Fano index 1 in characteristic 0, the radical of the uniruling index $u_1(K, X)$ divides $(d!)((d+1)C_d - 2^d)$. This is similar to the proof of Theorem 3.10, except using pointed conics in place of pointed lines, i.e., using

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,1}((X_K/K, \mathcal{O}(1)), 2) \longrightarrow X_K.$$

Note that this requires working in characteristic prime to 2, but this is already implied by the hypothesis on the characteristic. In the case of a Fano hypersurface of index 1 in characteristic 0, the generic degree e of this map is computed in [3],

$$e = \frac{(d!)^2}{2^{d+1}} \left((d+1)C_d - 2^d \right).$$

As in the proof of Theorem 1.2, separable rational connectedness implies weak approximation at places of (strong) potentially good reduction via Theorem 3.7. □

7. Complete Intersections with No Free Lines

For all integers $n, d \geqslant$, denote by $g_{n,d}$ the degree-d Fermat homogeneous polynomial
$$g_{n,d}(t_0, \dots, t_n) = t_0^d + \dots + t_n^d.$$

LEMMA 7.1. — *For every algebraically closed field k of characteristic p prime to d, the zero scheme X_k of $g_{n,d}$ in $\mathbb{P}_k^n$ is a k-smooth hypersurface. If $d+1$ is p-nonspecial (Definition 3.13) and if $n \geqslant d$, then every irreducible component of every fiber of the evaluation morphism,*
$$\mathrm{ev} : \overline{\mathcal{M}}_{0,1}((X_k/k, \mathcal{O}(1)), 1) \longrightarrow X_k,$$
has dimension strictly larger than the expected fiber dimension $n - d - 1$.

Proof. — Consider $\overline{\mathcal{M}}_{0,2}((X_k/k, \mathcal{O}(1)), 1)$ with its natural evaluation morphism to $\mathbb{P}_k^n \times \mathbb{P}_k^n = \mathrm{Proj}\, k[s_0, \dots, s_n] \times \mathrm{Proj}\, k[t_0, \dots, t_n]$. The image is the common zero locus of the collection of bihomogeneous polynomials
$$g_{n,d,\ell}(s_0, \dots, s_n, t_0, \dots, t_n) = \sum_{j=0}^n \binom{d}{\ell} s_j^{d-\ell} t_j^\ell, \quad 0 \leqslant \ell \leqslant d.$$

If any binomial coefficient $\binom{d}{\ell}$ is zero, then at least one $g_{n,d,\ell}$ is identically zero. Then, by Krull's Hauptidealsatz, every irreducible component has dimension strictly larger than the expected dimension. Since the difference of dimensions of domain and image is a lower bound on the dimension of every irreducible component of every fiber of a morphism, also every irreducible component of every fiber of ev has dimension strictly larger than the expected fiber dimension.

By hypothesis, $d+1$ is nonspecial, i.e., $d+1 = p^{v-1}e$ for a p-prime integer $e > p$. By the Division Algorithm, d equals $p^v a + r$ for an integer $a \geqslant 0$ and an integer r with $0 \leqslant r < p^v$. Since $d+1 = p^{v-1}e > p^v$, it follows that $a \geqslant 1$. Since d is p-prime, also $r \geqslant 1$. Since p^v does not divide $d+1$, also $r < p^v - 1$. Thus,
$$(s+t)^d = (s+t)^{p^v a}(s+t)^r = (s^{p^v} + t^{p^v})^a (s+t)^r$$
$$= \left(\sum_{b=0}^a \binom{a}{b} s^{p^v(a-b)} t^{p^v b} \right) \left(\sum_{q=0}^r \binom{r}{q} s^{r-q} t^q \right)$$
$$= \sum_{q=0}^r \sum_{b=0}^a \binom{r}{q}\binom{a}{b} s^{p^v(a-b)+r-q} t^{p^v b + q}.$$

In particular, the coefficient of the monomial $s^{d-\ell}t^\ell$ is nonzero only if ℓ is congruent modulo p^v to $0, \dots, r$, i.e., only for $r+1$ of the total p^v congruence

classes. Since $r + 1 < p^v$, there are some binomial coefficients that are identically zero. $\square$

PROPOSITION 7.2. — *Let k be an algebraically closed field of characteristic p. Let $(d_1, \ldots, d_c)$ be positive integers such that there exists at least one i with d_i a p-prime, p-nonspecial (Definition 3.13) integer $\geqslant p$. Let $n \geqslant 1 + (d_1 + \cdots + d_c)$. Then there exists a k-smooth complete intersection in $\mathbb{P}_k^n$ of type $(d_1, \ldots, d_c)$ of Fano index $\geqslant 2$ that has no free lines.*

Proof. — Choose g_i to be g_{n,d_i}. By the previous lemma and the hypothesis that $d_i \not\equiv 0 \pmod{p}$, the zero scheme is k-smooth. By Bertini's Theorem, for general choice of homogeneous equation g_j for $j \neq i$, the zero scheme $X_k = \mathrm{Zero}(g_1, \ldots, g_c)$ is a k-smooth complete intersection of type $(d_1, \ldots, d_c)$ in $\mathbb{P}_k^n$. Yet, by the previous lemma, every irreducible component of every fiber of

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,1}((X_k/k, \mathcal{O}(1)), 1) \longrightarrow X_k,$$

as a subscheme of

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,1}((Y_k/k, \mathcal{O}(1)), 1) \longrightarrow Y_k, Y_k = (g_i = 0) \subset \mathbb{P}_k^n$$

defined by $(d_1 + \ldots + d_{i-1} + d_{i+1} + \ldots + d_c)$-equations, has dimension $\geqslant [n - 1 - (d_1 + \cdots + d_c)] + 1$. For a pointed free line, the evaluation morphism is smooth with relative dimension equal to the expected dimension, $[n - 1 - (d_1 + \cdots + d_c)]$. Thus, there exists no free line in X_k. $\square$

For every integer $q \geqslant 1$, for $d = qp$, for every integer n, denote by $g_{n;d}$ the following degree-d homogeneous polynomial,

$$g_{n;d} = t_0 t_n^{qp-1} + \sum_{j=0}^{n-1} t_j^{qp-1} t_{j+1}.$$

If p divides $n + 1$, define $\widetilde{g}_{n;d} = g_{n;d} + t_0^d$.

LEMMA 7.3. — *For every algebraically closed field k of characteristic p, for every p-divisible integer $d > 0$, for every integer n such that $n + 1$ is p-prime, resp. such that $n + 1$ is p-divisible, the zero scheme X_k of $g_{n;d}$ is k-smooth, resp. the zero scheme X_k of $\widetilde{g}_{n;d}$ is k-smooth. If the integer d is not p-special (Definition 3.13), and if $n \geqslant d$, then every irreducible component of every fiber of the evaluation morphism,*

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,1}((X_k/k, \mathcal{O}(1)), 1) \longrightarrow X_k,$$

has dimension strictly larger than the expected fiber dimension $n - d - 1$.

Proof. — Consider the indices of variables as elements in $\mathbb{Z}/(n+1)\mathbb{Z}$. Then the partial derivative with respect to t_i of $g_{n;d}$ and $\widetilde{g}_{n;d}$ equals

$$\partial_{t_i} g_{n;d} = \partial_{t_i} \widetilde{g}_{n;d} = t_{i-1}^{qp-1} - t_i^{qp-2} t_{i+1}.$$

In particular, multiplying by t_i gives

$$t_i \partial_{t_i} g_{n;d} = t_{i-1}^{qp-1} t_i - t_i^{qp-1} t_{i+1}.$$

Thus, modulo the ideal of partial derivatives,

$$I = \langle \partial g_{n;d} \rangle = \langle \partial \widetilde{g}_{n;d} \rangle,$$

every monomial with nonzero coefficient in $g_{n;d}$ is congruent to a common element, say $s = t_0^{qp-1} t_1$,

$$t_i^{qp-1} t_{i+1} \cong s \pmod{I}, \quad i = 0, \ldots, n,$$

Summing over all i,

$$g_{n;d} \cong (n+1)s \pmod{I}, \quad g_{n;d} \cong t_0^{pq} + (n+1)s \pmod{I}.$$

If $n+1$ is not divisible by p, then for every i the element $t_i^{qp-1} t_{i+1}$ is in the Jacobian ideal of $g_{n;d}$. Thus the radical of the Jacobian ideal contains $t_i^{qp-2} t_{i+1}$. Using the partial derivative above, the radical of the Jacobian ideal also contains t_{i-1}^{qp-1}, and thus t_{i-1}. Since the radical of the Jacobian ideal contains t_{i-1} for every i, the Jacobian ideal contains $\langle t_0^e, \ldots, t_n^e \rangle$ for some integer $e \geqslant 1$. Thus, when $n+1$ is not divisible by p, the zero locus of $g_{n;d}$ is smooth.

When $n+1$ is divisible by p, then $\widetilde{g}_{n;d}$ is congruent to t_0^{pr} modulo I. Thus, the Jacobian ideal contains t_0^{pr}. Using the partial derivative above, the radical of the Jacobian ideal contains t_0. Using the partial derivatives for t_n and t_{n-1}, also the radical contains t_n and t_{n-1}. Then using the partial derivatives for t_{n-2} and t_{n-3}, also the radical contains t_{n-2} and t_{n-3}, etc. Finally, the radical of the Jacobian ideal contains every t_i, $i = 0, \ldots, n$. So the Jacobian ideal contains $\langle t_0^e, \ldots, t_n^e \rangle$ for some integer $e \geqslant 1$. Thus, when $n+1$ is divisible by p, the zero locus of $\widetilde{g}_{n;d}$ is smooth.

As in the proof of Lemma 7.1, in the k-algebra $k[s_0, \ldots, s_n, t_0, \ldots, t_n]$ write,

$$g_{n;d}(s_0 + t_0, \ldots, s_n + t_n) = \sum_{\ell=0}^{d} g_{n;d,\ell}(s_0, \ldots, s_n, t_0, \ldots, t_n),$$

respectively, if p divides $n+1$,

$$\widetilde{g}_{n;d}(s_0 + t_0, \ldots, s_n + t_n) = \sum_{\ell=0}^{d} \widetilde{g}_{n;d,\ell}(s_0, \ldots, s_n, t_0, \ldots, t_n),$$

where $g_{n;d,\ell}$, resp. $\widetilde{g}_{n;d,\ell}$, is bihomogeneous in $(s_0, \ldots, s_n)$ and $(t_0, \ldots, t_n)$ of bidegree $(\ell, d - \ell)$. If any $g_{n;d,\ell}$ is identically zero, then by Krull's Hauptidealsatz every irreducible component of every fiber of ev has dimension strictly larger than the expected dimension.

Now assume that the p-multiple, $d = pq$, is p-nonspecial. Then d equals $p^{v-1}e$ for a p-prime integer $e > p$. By the Division Algorithm, d equals $p^v a + r$ for an integer $a \geqslant 0$ and an integer r with $0 \leqslant r < p^v$. Since $d = p^{v-1}e > p^v$, it follows that $a \geqslant 1$. Since p^v does not divide d, also $r \geqslant 1$. Since p^{v-1} does divide d, r is $\leqslant p^v - p^{v-1}$. Thus,

$$(s+t)^{d-1} = (s+t)^{p^v a}(s+t)^{r-1} = (s^{p^v} + t^{p^v})^a (s+t)^{r-1}$$

$$= \left(\sum_{b=0}^{a} \binom{a}{b} s^{p^v(a-b)} t^{p^v b} \right) \left(\sum_{m=0}^{r-1} \binom{r-1}{m} s^{r-1-m} t^m \right)$$

$$= \sum_{m=0}^{r-1} \sum_{b=0}^{a} \binom{r-1}{m} \binom{a}{b} s^{p^v(a-b)+r-1-m} t^{p^v b+m}.$$

In particular, the coefficient of the monomial $s^{d-1-\ell} t^\ell$ is nonzero only if ℓ is congruent modulo p^v to $0, \ldots, r-1$. Similarly, the summand

$$(s_j + t_j)^{d-1}(s_{j+1} + t_{j+1})$$

$$= (s_{j+1} + t_{j+1}) \sum_{m=0}^{r-1} \sum_{b=0}^{a} \binom{r-1}{m} \binom{a}{b} s_j^{p^v(a-b)+r-1-m} t_j^{p^v b+m},$$

has nonzero bihomogeneous component of bidegree $(d - \ell, \ell)$ only if ℓ is congruent modulo p^v to $0, \ldots, r$. Summing over all integers j, the bihomogeneous component $g_{n;d,\ell}$ of bidegree $(d - \ell, \ell)$ is nonzero only if ℓ is congruent modulo p^v to $0, \ldots, r$. Since $r \leqslant p^v - p^{v-1}$, this is at most $p^v + 1 - p^{v-1}$ of the total p^v congruence classes. So, when d is p-nonspecial and $n + 1$ is p-prime, every irreducible component of every fiber of ev has dimension strictly larger than the expected dimension.

When $n + 1$ is divisible by p, then $\widetilde{g}_{n;d}$ has one additional monomial $t_0^d = t_0^{p^{v-1}e}$. By the Binomial Theorem,

$$(s_0 + t_0)^{p^{v-1}e} = \sum_{b=0}^{e} \binom{e}{b} s_0^{d-p^{v-1}b} t_0^{p^{v-1}b}.$$

Thus, the bihomogeneous component of bidegree $(d - \ell, \ell)$ is nonzero only if ℓ is a multiple of p^{v-1}. In particular, the excluded congruence classes $p^v - p^{v-1} < \ell < p^v$ for nonzero $g_{n;d,\ell}$ is still excluded for $\widetilde{g}_{n;d,\ell}$. Every prime p is $\geqslant 2$. Also, by hypothesis, d is divisible by p, so that p^{v-1} is at least $p \geqslant 2$. Thus, the number $p^{v-1} - 1$ of excluded congruence classes is

$\geqslant 1$. Thus, as above, every irreducible component of every fiber of ev has dimension strictly larger than the expected dimension. $\qquad\square$

PROPOSITION 7.4. — Let k be an algebraically closed field of characteristic p. Let $(d_1, \ldots, d_c)$ be positive integers such that there exists at least one i with d_i a p-divisible, p-nonspecial (Definition 3.13) integer $\geqslant p$. Let $n \geqslant 1 + (d_1 + \cdots + d_c)$. Then there exists a k-smooth complete intersection in $\mathbb{P}_k^n$ of type $(d_1, \ldots, d_c)$ of Fano index $\geqslant 2$ that has no free lines.

Proof. — The proof is the same as the proof of Proposition 7.2, but using Lemma 7.3 in place of Lemma 7.1. $\qquad\square$

Together, Propositions 7.2 and 7.4 establish Proposition 3.14.

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Jason M. STARR
Department of Mathematics
Stony Brook University
Stony Brook, NY 11794 (USA)
jstarr@math.sunysb.edu

Zhiyu TIAN
Beijing International Center for Mathematical
Research
Peking University
No.5 Yiheyuan Road
Beijing, 100871 (China)
zhiyutian@bicmr.pku.edu.cn

Runhong ZONG
Department of Mathematics
Nanjing University
204 Meng Minwei Building
No. 22 Hankou Road
Nanjing, Jiangsu, 210093 (China)
rzong@nju.edu.cn