Université Grenoble Alpes

ANNALES DE L'INSTITUT FOURIER

Irene Schwarz

On quotients of $\overline{\mathcal{M}}_{g,n}$ by certain subgroups of S_n

Tome 72, nº 4 (2022), p. 1417-1435.

https://doi.org/10.5802/aif.3496

Les *Annales de l'Institut Fourier* sont membres du Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org e-ISSN : 1777-5310

Ann. Inst. Fourier, Grenoble
72, 4 (2022) 1417-1435

ON QUOTIENTS OF $\overline{\mathcal{M}}_{g,n}$ BY CERTAIN SUBGROUPS OF S_n

by Irene SCHWARZ

Abstract. — We investigate when certain quotients of the compactified moduli space of n-pointed genus g curves $\overline{\mathcal{M}}^G := \overline{\mathcal{M}}_{g,n}/G$ are of general type or, on the contrary, uniruled, for a fairly broad class of subgroups G of the symmetric group S_n which act by permuting the n marked points. We show that the property of being of general type only depends on the transpositions contained in G. Furthermore, in the case that G is the full symmetric group S_n or a product $S_{n_1} \times \cdots \times S_{n_m}$, we find a narrow transitional band in which $\overline{\mathcal{M}}^G$ changes its behaviour from being of general type to its opposite, i.e. being uniruled, as n increases. As an application we consider the universal difference variety $\overline{\mathcal{M}}_{g,2n}/S_n \times S_n$.

Résumé. — Nous analysons quand certains quotients de l'espace compactifié des modules $\overline{\mathcal{M}}^G := \overline{\mathcal{M}}_{g,n}/G$ de courbes de genre g marquées en n points sont de type général, ou au contraire, *uniruled*, pour une classe assez large de sous-groupes G du groupe symétrique S_n agissant par permutation des points marqués. On montre que la propriété d'être de type général ne dépend que des transpositions contenues dans G. Dans le cas où G est le groupe symétrique S_n ou un produit $S_{n_1} \times \cdots \times S_{n_m}$ on trouve une bande étroite de transition où $\overline{\mathcal{M}}^G$ passe du type général au cas *uniruled* quand n augmente. Comme application, on considère la variété de différences universelles $\overline{\mathcal{M}}_{g,2n}/S_n \times S_n$.

1. Introduction

In this paper we shall consider a class of quotients of $\overline{\mathcal{M}}_{g,n}$, the compactified moduli space of n-pointed genus g complex curves, by certain subgroups of the symmetric group S_n which act by permuting the marked points. We recall from [16] that (for $2g-2+n \geqslant 3$) S_n is the group of automorphisms both for the coarse moduli space $\overline{\mathcal{M}}_{g,n}$ and its associated Deligne Mumford stack. This generalizes recent work of Bruno and Mella

Keywords: Moduli spaces, algebraic curves, Kodaira dimension.
2020 *Mathematics Subject Classification:* 14H10, 14H51.

and older work of Earl, Kra and Korkmaz, see the references given in [16]. Our aim is to analyse under which conditions such quotients are of general type or, in a complementary case, uniruled or even unirational. As usual, we do this by using the Kodaira dimension.

We were led to investigating the questions addressed in the present paper by analysing an analogous problem for the compactified moduli space $\overline{\mathcal{N}}_{g,n}$ of n-nodal genus g curves in [17]. Here $\overline{\mathcal{N}}_{g,n} = \overline{\mathcal{M}}_{g,2n}/G$ where the special group G is also a subgroup of S_{2n}, namely the semidirect product $G := (\mathbb{Z}_2)^n \rtimes S_n$. In view of the great importance of n-nodal curves, e.g. in the deformation type arguments used in the proof of the Brill–Noether theorem, this problem was directly motivated by geometry.

Our proof in [17], however, let us realize that there are some related results for *general* quotients of $\overline{\mathcal{M}}_{g,n}$ which in some aspects are *different* from the special case of $\overline{\mathcal{N}}_{g,n}$. In particular, it is important that $G := (\mathbb{Z}_2)^n \rtimes S_n$ is a semidirect product and *not* a product of subgroups. The main point of this paper is to prove first results in this direction for a class of general quotients.

For $G \subset S_n$, we denote the quotient by this action as $\overline{\mathcal{M}}_{g,n}^G := \overline{\mathcal{M}}_{g,n}/G$ and we suppress the subscript (g,n) if we feel it unnecessary within a given context. Then the natural quotients induce the chain of surjective morphisms of schemes

$$\overline{\mathcal{M}}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}^G \longrightarrow \overline{\mathcal{M}}_{g,n}^{S_n} \tag{1.1}$$

which gives the following ordering for the Kodaira dimension

$$\kappa(\overline{\mathcal{M}}_{g,n}) \geqslant \kappa(\overline{\mathcal{M}}_{g,n}^G) \geqslant \kappa(\overline{\mathcal{M}}_{g,n}^{S_n}). \tag{1.2}$$

In fact, this is a trivial version of the subadditivity of the Kodaira dimension. We will show in Theorem 2.1 that the singularities of these moduli spaces do not impose adjunction conditions and therefore their Kodaira dimension is the Kodaira–Iitaka dimension of their canonical class. However, for any surjective morphism of schemes $f : X \to Y$ and any line bundle L on Y we get an inclusion $H^0(Y,L) \to H^0(X, f^*L)$ implying $\kappa(X, f^*L) \geqslant \kappa(Y,L)$. Choosing $L = K_Y$ and observing that $K_X \geqslant f^*K_Y$ we get equation (1.2).

Since all algebraic varieties in (1.1) have the same dimension, one gets

$$\begin{aligned} \overline{\mathcal{M}}_{g,n}^{S_n} \text{ of general type } &\Longrightarrow \overline{\mathcal{M}}_{g,n}^G \text{ of general type} \\ &\Longrightarrow \overline{\mathcal{M}}_{g,n} \text{ of general type.} \end{aligned} \tag{1.3}$$

By the same argument, one gets

Remark 1.1. — For any subgroup H of G one has

$$\overline{\mathcal{M}}^G \text{ of general type } \Longrightarrow \overline{\mathcal{M}}^H \text{ of general type.} \tag{1.4}$$

We also remark that it follows from equation (1.2) that, if the Kodaira dimension of $\overline{\mathcal{M}}_{g,n}$ is already minimal, it stays minimal when quotienting by a subgroup G of S_n. In particular, this holds for the special cases analysed in [5] where $\overline{\mathcal{M}}_{g,n}$ is actually rational. In this paper we focus on the complementary case.

For explicit calculation we shall need the canonical class $K_{\overline{\mathcal{M}}^G}$ of $\overline{\mathcal{M}}^G$ which we calculate by pullback to the well known moduli space $\overline{\mathcal{M}}_{g,n}$. For this purpose we need the ramification divisor of the quotient map.

Proposition 1.2. — *Denoting by $(i\ j)$ the transposition in S_n interchanging i and j the ramification divisor R of the quotient map $\pi : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}^G$ is given by*

$$R = \sum_{(i\ j)\in G} \delta_{0,\{i,j\}}. \tag{1.5}$$

Here the (standard) definition of the boundary divisor $\delta_{0,\{i,j\}}$ is given in Section 2 below, which in particular introduces all divisors needed in this paper. The proposition is shown in equation (2.2).

Now the well known explicit formula for the canonical divisor $K_{\overline{\mathcal{M}}_{g,n}}$ gives

Corollary 1.3. — *The pullback $K_G := \pi^*(K_{\overline{\mathcal{M}}^G})$ to $\overline{\mathcal{M}}_{g,n}$ is given by*

$$K_G = K_{\overline{\mathcal{M}}_{g,n}} - R = 13\lambda + \psi - 2\delta - \sum_{(i\ j)\in G} \delta_{0,\{i,j\}} \tag{1.6}$$

As we can see the canonical class of $\overline{\mathcal{M}}^G$ depends only on the transpositions in G. This allows us to conclude

Theorem 1.4. — *Let G and H be two subgroups of S_n containing the same transpositions, i.e. $(i\ j) \in G \Leftrightarrow (i\ j) \in H$ for any transposition $(i\ j)$. Then $\overline{\mathcal{M}}^G$ is of general type if and only if $\overline{\mathcal{M}}^H$ is of general type.*

Proof. — We will show in Section 2, Proposition 2.2, that a quotient $\overline{\mathcal{M}}^G$ is of general type if and only if we can write K_G, the pullback of the canonical class along the quotient map, as the sum of an ample and an effective G-invariant divisor. Assume that $K_G = A + E$ is such a decomposition and that the subgroup H has the same transpositions as G. Then, by Corollary 1.3, $K_G = K_H$ is both G- and H-invariant and therefore

$$K_H = \frac{1}{|H|}\sum_{h\in H} h(K_G) = \frac{1}{|H|}\sum_{h\in H} h(A) + \frac{1}{|H|}\sum_{h\in H} h(E)$$

is a decomposition of K_H into H-invariant divisors. □

This theorem is especially useful, when the group G contains no transpositions at all, allowing us to relate the quotient $\overline{\mathcal{M}}^G$ to the well known moduli space $\overline{\mathcal{M}}_{g,n}$. For the sake of the reader we recall:

Proposition 1.5. — *The moduli space $\overline{\mathcal{M}}_{g,n}$ is of general type for $g \geqslant 22$ or for $n \geqslant n_{\min}(g)$ given in the following table:*

Table 1.1.

g	4	5	6	7	8	9	10	11	12
$n_{\min}$	16	15	16	15	14	13	11	12	11
g	13	14	15	16	17	18	19	20	21
$n_{\min}$	11	10	10	9	9	9	7	6	4

This collects results from [7, 8, 12, 15] and covers all cases known up to now. As a corollary to Theorem 1.4 we now obtain:

Corollary 1.6. — *If G has no transpositions then the quotient $\overline{\mathcal{M}}^G$ is of general type if and only if $\overline{\mathcal{M}}_{g,n}$ is of general type.*

In particular, this covers the case where G is cyclic (and different from $\mathbb{Z}_2$) or the cardinality $|G|$ is odd. It also covers the largest non-trivial subgroup of S_n, the alternating group A_n (in fact G contains no transpositions, if and only if it is a subgroup of A_n). This has a geometric interpretation: The set of n-tuples of n fixed different points on a genus g curve carries a notion of *orientation*: Two n-tuples have the same orientation, if they are mapped one to another by an even permutation. Taking the quotient by S_n corresponds to passing from n-pointed curves to n-marked curves, while taking the quotient by $\mathcal{A}_n$ means passing to curves marked in n points with orientation. Under the first action the property of being of general type might change, but it is invariant under the second.

If the subgroup G does contain transpositions, the situation is more complicated.

We can see this by studying the quotient of $\overline{\mathcal{M}}_{g,n}$ by the entire symmetric group S_n. This quotient is of special interest, both because it is comparatively easy to understand and because it has two important modular interpretations. It can be interpreted as the moduli space $\widetilde{\mathcal{M}}_{g,n}$ of

n-marked curves of genus g, i.e. stable curves with an unordered set of n marked points. Furthermore, it is birational to the moduli space $\mathcal{C}_{g,n}$ parametrizing effective divisors of degree n on stable curves of genus g. This space is usually called the universal symmetric product of degree n.

These modular interpretations allow us to study the Kodaira dimension of this moduli space. First let us consider the forgetful map $\overline{\mathcal{M}}^{S_n} = \widetilde{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_g$ forgetting the marked points. The fibre over a curve C is birational to $C_n := \mathrm{Sym}^n(C) := C^n/S_n$ which is well known to be of general type for $n < g$ (see e.g. [14]). Now, whenever the base $\overline{\mathcal{M}}_g$ is of general type (which it is at least for $g \geqslant 22$), then it follows from the subadditivity of the Kodaira dimension that the moduli space $\overline{\mathcal{M}}^{S_n} = \widetilde{\mathcal{M}}_{g,n}$ (and thus, by Remark 1.1, the quotient $\overline{\mathcal{M}}^G$ for any subgroup G of S_n) is of general type for all $n < g$.

On the other hand the fibres C_n of the forgetful map $\mathcal{C}_{g,n} \to \overline{\mathcal{M}}_g$ are effective divisors of degree n on the curve C. Since the Riemann–Roch theorem implies that any effective divisor of degree $d > g$ lies in some $\mathfrak{g}^1_d$, the moduli space $\mathcal{C}_{g,n}$ is trivially uniruled for all $n > g$. Since being uniruled is a birationally invariant property the same must be true for the quotient $\overline{\mathcal{M}}^{S_n}$.

There are also some known results for $n = g$ or for $n < g$ when g is too small for $\overline{\mathcal{M}}_g$ to be of general type. Let us summarize these results.

Proposition 1.7. — *The space $\overline{\mathcal{M}}^{S_n}$ is of general type if*

(i) $g \geqslant 22, \quad n < g,$ *or*
(ii) $13 \leqslant g \leqslant 21$, *and* $n_{\min}(g) \leqslant n < g$, *where* $n_{\min}(g)$ *is given in the following table*

Table 1.2.

g	12	13	14	15	16	17	18	19	20	21
$n_{\min}$	10	11	10	10	9	9	9	7	6	4

Furthermore, $\overline{\mathcal{M}}^{S_n}$ is

(1) *uniruled, if $n > g$ (for any g) or $g \in \{10, 11\}$ with $n \neq g$*
(2) *unirational, if $g < 10, n \leqslant g$.*
(3) *For $g \geqslant 12$ the Kodaira dimension $\kappa(\overline{\mathcal{M}}_{g,g}) = 3g-3$ is intermediary.*

Here the result of (i) and the first assertion in (1) follow as explained above. The table in (ii) is proven in [11] by explicit computation. The second assertion of (1) is proved in [9] which also contains (2) and (3).

Here we see an interesting phenomenon: While the Kodaira dimension of $\overline{\mathcal{M}}_{g,n}$ (the moduli space of n-pointed curves) increases monotonically with n, this is not true for $\widetilde{\mathcal{M}}_{g,n}$ (the moduli space of n-marked curves). Here in fact the Kodaira dimension first increases and then (for $n > g$) sharply drops to $-\infty$.

For a subgroup $G \subset S_n$ the Kodaira dimension of the quotient $\overline{\mathcal{M}}^G$ must always lie between these two extremes. In particular, $\overline{\mathcal{M}}^G$ is always of general type for g large enough and $n < g$. For $n \geqslant g$ the Kodaira dimension of $\overline{\mathcal{M}}^G$ depends very much on the group G.

In the following theorems we shall analyse, for a large class of groups G, when $\overline{\mathcal{M}}^G$ is of general type or, on the contrary, uniruled. We remark that the transition point for switching from general type to uniruled will depend on the group G. This allows for n being arbitrarily large for fixed g. First, considerations similar to those preceding Proposition 1.7 give us the following result.

Theorem 1.8. — *Fix a partition $n = n_1 + \cdots + n_m$ and set $G = S_{n_1} \times \cdots \times S_{n_m}$. Then $\overline{\mathcal{M}}^G_{g,n}$ is*

1. *of general type for $g \geqslant 22$, $\max\{n_1, \dots, n_m\} < g$ and*
2. *uniruled for $\max\{n_1, \dots, n_m\} > g$.*

When g is so small that $\overline{\mathcal{M}}_g$ is not of general type and $\max\{n_1, \dots, n_m\} \leqslant g$ we need explicit computations with the canonical class. From a technical point of view, this is the most ambitious result of this paper. We remark that by Remark 1.1 any result on being of general type also holds for any subgroup $H \subset G$.

Theorem 1.9. — *Fix a partition $n = n_1 + \cdots + n_m$, set $G = S_{n_1} \times \cdots \times S_{n_m}$ and let $g \leqslant 21$. Then $\overline{\mathcal{M}}^G_{g,n}$ is of general type if*

1. *$\max\{n_1, \dots, n_m\} \leqslant g - 2$ and $f_m(g; n_1, \dots, n_m) \leqslant 13$, where f_m is the function defined in equation (4.7) of Section 4 below.*
2. *$\max\{n_1, \dots, n_m\} \leqslant g - 1$ and $f_m(g; n_1, \dots, n_m, L_1, \dots, L_m) \leqslant 13$, where f_m is the function defined in equation (4.12) of Section 4 below (depending on a choice of divisor classes $L_1, \dots, L_m$ as described at the end of Section 4.*

Furthermore, $\overline{\mathcal{M}}^G$ still has non-negative Kodaira dimension if $\max\{n_1, \dots, n_m\} \leqslant g$ and $f_m(g; n_1, \dots, n_m) \leqslant 13$.

A geometric interpretation of this result is similar to the interpretation for $\mathcal{A}_n$: the partition $P : n = n_1 + \cdots + n_m$ induces the group $G = G_P = S_{n_1} \times \cdots \times S_{n_m}$, and the action of G_P maps an n-pointed genus g curve to a curve with markings in the m sets of $n_1, \dots, n_m$ unordered points (considered as an ordered m-tuple of sets) which we may call a P-marked curve. Thus Theorem 1.8 and Theorem 1.9 state when the moduli space of P-marked genus g curves is of general type or on the contrary is uniruled.

Since there is no upper bound on the number of summands m in the partition of n, an inspection of the defining equation for f_m shows that the values of n may tend to infinity, provided the subgroup G is chosen appropriately. As in the above case for $m = 2$, Riemann–Roch establishes a (small) transitional band beyond which $\overline{\mathcal{M}}_{g,n}^G$ becomes uniruled. We emphasize that the existence of this transitional band (for any *fixed* subgroup G) is different from the result for $\overline{\mathcal{N}}_{g,n}$ proved in [17]: $\overline{\mathcal{N}}_{g,n} = \overline{\mathcal{M}}_{g,2n}/(\mathbb{Z}_2)^n \rtimes S_n$ is of general type for all values of n, if $g \geqslant 22$. This is perfectly compatible with Theorem 1.8, since $G := (\mathbb{Z}_2)^n \rtimes S_n$ with its action on the $2n$ marked points is *not* given by a direct product subgroup of S_{2n}, as required in Theorem 1.8. To understand this from a more conceptual point of view, however, is wide open at present. We emphasize that it is not merely the size of the group which is relevant: The alternating group $G = \mathcal{A}_n$ might be taken arbitrarily large and still $\overline{\mathcal{M}}^G$ will preserve general type, while taking the quotient by much smaller groups, e.g. $G = S_{g+1}$ will turn $\overline{\mathcal{M}}^G$ into being uniruled.

As an application we consider $m = 2$ and the special case $G = S_n \times S_n$. This quotient has a geometric interpretation as the universal difference variety, i.e. the fibre of the map $\overline{\mathcal{M}}^G \to \overline{\mathcal{M}}_g$ over a smooth curve C is birational to the image of the difference map $C_n \times C_n \to J^0(C), (D, E) \mapsto D - E$, see e.g. [4].

Then Theorem 1.8 combined with Theorem 1.9 (for a proper choice of divisors) gives the following.

PROPOSITION 1.10. — *The universal difference variety $\overline{\mathcal{M}}_{g,2n}/S_n \times S_n$ is of general type for $g \geqslant 22$, $\quad n < g$, or, in the low-genus case, if $10 \leqslant g \leqslant 21$ and $n_{\min}(g) \leqslant n < g$ where $n_{\min}(g)$ is specified in the following table*

Table 1.3.

g	10	11	12	13	14	15	16	17	18	19	20	21
$n_{\min}$	7	8	8	7	7	7	6	6	7	5	4	3

Furthermore the universal difference variety is uniruled for $n > g$ and has non-negative Kodaira dimension for $n_{\min}(g) \leqslant n \leqslant g$.

This proposition amplifies the results of [10], which considers the universal difference variety in the special case $n = \lceil \frac{g}{2} \rceil$. We emphasize that the result in Table 1.3 for $g = 13$ and $n = 7$ is taken from [10]; in view of the sharp coupling between g and n the authors are able to use in this special case an additional divisor, which is not applicable in the other cases and which is not contained in our Section 3. All other cases in Table 1.3 follow from our Theorem 1.9.

The outline of the paper is as follows. In Section 2 we introduce notation and some preliminary results, in Section 3 we introduce the class of divisors used in our proof. Here we basically recall, for the sake of the reader, some material from [17]. In Section 4 we prove Theorem 1.8, Theorem 1.9 and Proposition 1.10. The use of a small program in computer algebra is appropriate to check our calculations.

2. Preliminaries and Notation

The aim of this section is to develop a criterion for $\overline{\mathcal{M}}^G$ being of general type. This requires a basic understanding of the Picard group $\mathrm{Pic}(\overline{\mathcal{M}}^G)$ and an explicit description of the boundary divisors and tautological classes on $\overline{\mathcal{M}}^G$ which we shall always consider as G-invariant divisors on $\overline{\mathcal{M}}_{g,n}$ (any such divisor descends to a divisor on $\overline{\mathcal{M}}^G$). For results on $\overline{\mathcal{M}}_{g,n}$ we refer to the book [3] (containing in particular the relevant results from the papers [1] and [2]). All Picard groups are taken with rational coefficients and, in particular, we identify the Picard group on the moduli stack with that of the corresponding coarse moduli space.

In particular, we recall the notion of the Hodge class λ on $\overline{\mathcal{M}}_{g,n}$, which automatically is S_n- (and therefore G-)invariant and thus gives the Hodge class λ on $\overline{\mathcal{M}}^G$ (where, by the usual abuse of notation, we denote both classes by the same symbol; this abuse of notation is continued throughout the paper).

In order to describe the relevant boundary divisors on $\overline{\mathcal{M}}_{g,n}$, we recall that Δ_0 (sometimes also called Δ_{irr}) on $\overline{\mathcal{M}}_g$ is the boundary component consisting of all (classes of) stable curves of arithmetical genus g, having at least one nodal point of order 0, i.e. with the property that ungluing the curve at this node preserves connectedness. Furthermore, Δ_i, for $1 \leqslant i \leqslant \lfloor \frac{g}{2} \rfloor$, denotes the boundary component of curves possessing a node of order i (i.e. ungluing at this point decomposes the curve in two connected components of arithmetical genus i and $g - i$ respectively). Similarly, on $\overline{\mathcal{M}}_{g,n}$ and for any subset $S \subset \{1, \ldots, n\}$, we denote by $\Delta_{i,S}, 0 \leqslant i \leqslant \lfloor \frac{g}{2} \rfloor$, the boundary component consisting of curves possessing a node of order

i such that after ungluing the connected component of genus i contains precisely the marked points labeled by S. Note that, if S contains at most 1 point, one has $\Delta_{0,S} = \emptyset$ (the existence of infinitely many automorphisms on the projective line technically violates stability). Thus, in that case, we shall henceforth consider $\Delta_{0,S}$ as the zero divisor.

We shall denote by $\delta_i, \delta_{i,S}$ the rational divisor classes of $\Delta_i, \Delta_{i,S}$ in $\mathrm{Pic}(\overline{\mathcal{M}}_g)$ and $\mathrm{Pic}(\overline{\mathcal{M}}_{g,n})$, respectively. Note that δ_0 is also called δ_{irr} in the literature, but we shall reserve the notation δ_{irr} for the pull-back of δ_0 under the forgetful map $\pi : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_g$.

We write δ for the sum of all boundary divisors and set $\delta_{i,s} = \sum_{|S|=s} \delta_{i,S}$. We remark that a single $\delta_{i,S}$ is not G-invariant (for a subgroup G of S_n), but the divisor $\sum_{g\in G} \delta_{i,g(S)}$, averaged by the action of G, obviously is. In particular δ and $\delta_{i,s}$ are always S_n-invariant. We shall use such an averaging in the proof of Theorem 1.9.

Next we recall the notion of the point bundles $\psi_i, 1 \leqslant i \leqslant n$, on $\overline{\mathcal{M}}_{g,n}$. Informally, the line bundle ψ_i (sometimes called the cotangent class corresponding to the label i) is given by choosing as fibre of ψ_i over a point $[C; x_1, \dots, x_n]$ of $\overline{\mathcal{M}}_{g,n}$ the cotangent line $T^v_{x_i}(C)$. For later use we also set

$$\omega_i := \psi_i - \sum_{S\subset\{1,\dots,n\}, S\ni i} \delta_{0,S}, \tag{2.1}$$

and introduce $\psi = \sum_{i=1}^n \psi_i$. Clearly, the class ψ is S_n-invariant.

Now observe that the ramification divisor (class) of the quotient map $\pi : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}^G$ is precisely

$$R = \sum_{(i,j)\in G} \delta_{0,\{i,j\}}, \tag{2.2}$$

which is the statement of 1.2. In fact, ramification requires existence of a non-trivial automorphism belonging to G, and by standard results this only occurs in the presence of the projective line with 2 marked points that are swapped by this automorphism. The non-trivial automorphism is then the transposition (of the labels) of these two marked points.

Furthermore, the Hurwitz formula for the quotient map π gives

$$K := \pi^*(K_{\overline{\mathcal{M}}^G}) = K_{\overline{\mathcal{M}}_{g,n}} - R = 13\lambda + \psi - 2\delta - \sum_{(i,j)\in G} \delta_{0,\{i,j\}}. \tag{2.3}$$

As a first step in the direction of our criterion we need the following result on the geometry of the moduli space $\overline{\mathcal{M}}^G$.

Theorem 2.1. — *For any subgroup G of S_n, the singularities of $\overline{\mathcal{M}}^G$ do not impose* adjunction conditions, *i.e. if $\rho : \widetilde{\mathcal{M}}^G \to \overline{\mathcal{M}}^G$ is a resolution* *of singularities, then for any $\ell \in \mathbb{N}$ there is an isomorphism*

$$\rho^* : H^0\Big((\overline{\mathcal{M}}^G)_{\mathrm{reg}}, K^{\otimes \ell}_{(\overline{\mathcal{M}}^G)_{\mathrm{reg}}}\Big) \longrightarrow H^0(\widetilde{\mathcal{M}}^G, K^{\otimes \ell}_{\tilde{\mathcal{M}}^G}). \tag{2.4}$$

Here $(\overline{\mathcal{M}}^G)_{\mathrm{reg}}$ denotes the set of regular points of $\overline{\mathcal{M}}^G$, considered as a projective variety and $K_{\tilde{\mathcal{M}}^G}, K_{(\overline{\mathcal{M}}^G)_{\mathrm{reg}}}$ denote the canonical classes on $\widetilde{\mathcal{M}}^G$ and $(\overline{\mathcal{M}}^G)_{\mathrm{reg}}$.

The proof follows the lines of the proof of Theorem 1.1 in [9]. We shall briefly review the argument. A crucial input is Theorem 2 of the seminal paper [13] which proves that the moduli space $\overline{\mathcal{M}}_g$ has only canonical singularities. The proof relies on the Reid–Tai criterion: Pluricanonical forms (i.e. sections of $K^{\otimes \ell}$) extend to the resolution of singularities, if for any automorphism σ of an object of the moduli space the so-called *age* satisfies $age(\sigma) \geqslant 1$. The proof in [9] then proceeds to verify the Reid–Tai criterion for the quotient of $\overline{\mathcal{M}}_{g,n}$ by the full symmetric group S_n. Here one specifically has to consider those automorphisms of a given curve which act as a permutation of the marked points. For all those automorphisms the proof in [9] verifies the Reid–Tai criterion. Thus, in particular, the criterion is verified for all automorphisms which act on the marked points as an element of some subgroup of S_n. Thus, the proof in [9] actually establishes the existence of only canonical singularities for any quotient $\overline{\mathcal{M}}_{g,n}/G$ where G is a subgroup of S_n. Clearly, this is our theorem.

With this result we can show the following criterion for the quotient $\overline{\mathcal{M}}^G$ to be of general type.

Proposition 2.2. — *Let $\pi : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}^G$ be the quotient map and $K_{\overline{\mathcal{M}}^G}$ be the canonical class on $\overline{\mathcal{M}}^G$. Then $\overline{\mathcal{M}}^G$ is of general type if and only if*

$$K_G := \pi^* K_{\overline{\mathcal{M}}^G} = A + E, \tag{2.5}$$

where A and E are rational G-invariant divisor classes on $\overline{\mathcal{M}}_{g,n}$, A is ample and E is effective.

Proof. — Theorem 2.1 implies that the Kodaira dimension of $\overline{\mathcal{M}}^G$ equals the Iitaka dimension of the canonical class $K_{\overline{\mathcal{M}}^G}$. In particular, $\overline{\mathcal{M}}^G$ is of general type if and only if $K_{\overline{\mathcal{M}}^G}$ is big, i.e. a positive linear combination of an ample and an effective rational class on $\overline{\mathcal{M}}^G$. In order to pull this criterion back to $\overline{\mathcal{M}}_{g,n}$ we recall the following exercise from Hartshorne's book: If $f : X \to Y$ is a finite surjective morphism of proper schemes over a Noetherian ring, then a divisor A on Y is ample if and only if f^*A is ample on X. We use this for $f = \pi$ and recall that a divisor on $\overline{\mathcal{M}}_{g,n}$ descends to $\overline{\mathcal{M}}^G$ if and only if it is G-invariant. A G-invariant effective divisor on $\overline{\mathcal{M}}_{g,n}$ corresponds to an effective divisor on $\overline{\mathcal{M}}^G$. Thus equation (2.5) decomposes the canonical class equivalently on either $\overline{\mathcal{M}}_{g,n}$ or on $\overline{\mathcal{M}}^G$. Identifying the Kodaira dimension with the Iitaka dimension of the canonical class finishes the proof. □

We remark that it suffices to replace the ample divisor A by any big divisor since this is itself the sum of an ample and an effective divisor. Thus we will use Proposition 2.2 mostly for the big divisor $A = \epsilon\psi$ for some $\epsilon > 0$.

Proposition 2.3. — *The class ψ on $\overline{\mathcal{M}}_{g,n}$ is the pull-back of a divisor class on $\overline{\mathcal{M}}^G$ which is big and nef.*

Proof. — Farkas and Verra have proven in Proposition 1.2 of [11] that the S_n-invariant class ψ descends to a big and nef divisor class $N_{g,n}$ on the quotient space $\overline{\mathcal{M}}_{g,n}/S_n$. Consider the sequence of natural projections $\overline{\mathcal{M}}_{g,n} \xrightarrow{\pi} \overline{\mathcal{M}}^G \xrightarrow{\nu} \overline{\mathcal{M}}^{S_n}$. Then $\nu^*(N_{g,n})$ is a big and nef divisor on $\overline{\mathcal{M}}^G = \overline{\mathcal{M}}_{g,n}/G$ and $\pi^*(\nu^*(N_{g,n}) = \psi$. □

We thus obtain a sufficient condition: If K is a positive multiple of ψ + some effective G-invariant divisor class on $\overline{\mathcal{M}}_{g,n}$, then $\overline{\mathcal{M}}^G$ is of general type.

3. Divisors

In this section we introduce the relevant S_n-invariant effective divisors on $\overline{\mathcal{M}}_{g,n}$. First we recall the following standard result.

Proposition 3.1. — *Let $f : X \to Y$ be a morphism of projective schemes, $D \subset Y$ be an effective divisor and assume that $f(X)$ is not contained in D. Then $f^*(D)$ is an effective divisor on X.*

In our case the assumption of this proposition is fulfilled automatically since we only consider surjective maps.

We shall need invariant divisors on $\overline{\mathcal{M}}_{g,n}$. Rather than exhibiting them directly by explicit definitions, we shall simply recall from the literature the existence of special divisors with small *slope*: If $g+1$ is not prime, then there is an effective S_n-invariant divisor class D on $\overline{\mathcal{M}}_g$ (of Brill–Noether type) of slope

$$s(D) = 6 + \frac{12}{g+1}, \tag{3.1}$$

while for $g+1$ odd (which trivially includes the case $g+1$ being prime) there is an effective S_n-invariant divisor class D on $\overline{\mathcal{M}}_g$ (of Giesecker–Petri type) of slope

$$s(D) = 6 + \frac{14g+4}{g^2+2g}, \tag{3.2}$$

see [6]. For a few cases ($g = 10, 12, 16, 21$) it has been shown in [12] (for $g = 12$) and [7] (for the other 3 cases) that there exist special effective invariant divisors $D = D_g$ with even smaller slope, i.e.

$$s(D_g) = \begin{cases} 7 & g = 10 \\ 6 + \frac{563}{642} & g = 12 \\ 6 + \frac{41}{61} & g = 16 \\ 6 + \frac{197}{377} & g = 21. \end{cases} \tag{3.3}$$

We shall need them in the proof of Theorem 1.9.

Finally, we need divisors of Weierstrass-type, and these we have to introduce explicitly. We recall from [15, Section 5], the divisors $W(g; a_1, \ldots, a_m)$ on $\overline{\mathcal{M}}_{g,m}$, where $a_i \geqslant 1$ and $\sum a_i = g$. They are given by the locus of curves C with marked points $p_1, \ldots, p_m$ such that there exists a $\mathfrak{g}^1_g$ on C containing $\sum_{1 \leqslant i \leqslant m} a_i p_i$. We want to minimize the distance between the weights a_i. Thus we decompose $g = km + r$, with $r < m$, and set

$$\begin{aligned} \widetilde{W}_{g,m} &= W(g; a_1, \ldots, a_m), \\ & \qquad a_j = k+1 \ \ (1 \leqslant j \leqslant r), \quad a_j = k \ \ (r+1 \leqslant j \leqslant m). \end{aligned} \tag{3.4}$$

This gives, in view of [15, Theorem 5.4],

$$\begin{aligned} \widetilde{W}_{g,m} = -\lambda &+ \sum_{i=1}^{r} \frac{(k+1)(k+2)}{2}\omega_i + \sum_{i=r+1}^{m} \frac{k(k+1)}{2}\omega_i - 0 \cdot \delta_{\mathrm{irr}} \\ &- \sum_{i,j \leqslant r} (k+1)^2 \delta_{0,\{i,j\}} - \sum_{i \leqslant r, j > r} k(k+1)\delta_{0,\{i,j\}} - \sum_{i,j > r} k^2 \delta_{0,\{i,j\}} \\ &- \text{higher order boundary terms,} \end{aligned} \tag{3.5}$$

where *higher order* means a positive linear combination of $\delta_{i,S}$ where either $i > 0$ or $|S| > 2$.

From $\widetilde{W}_{g,m}$ we want to generate a G-invariant divisor class $\widetilde{W}_{g,n,m}$ on $\overline{\mathcal{M}}_{g,n}$, by summing over appropriate pullbacks. Thus we let S, T be disjoint subsets of $\{1, \ldots, n\}$ with $|S| = r$ and $|T| = m - r$ (recall that r is fixed by the decomposition $g = mk + r$) and let

$$\pi_{S,T} : \overline{\mathcal{M}}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,m} \tag{3.6}$$

be a projection (i.e. a surjective morphism of projective varieties) mapping the class $[C;q_1,\dots,q_n]$ to $[C;p_1,\dots,p_m]$, where the points q_i labeled by S are sent to the points $p_1,\dots,p_r$ (all with weights $a_i=k+1$) and the points labeled by T are sent to the points $p_{r+1},\dots,p_m$ (all with weights equal to k). Clearly, for fixed g, there are precisely $\binom{n}{r}\binom{n-r}{m-r}$ such projections. With this notation, we introduce

$$\widetilde{W}_{g,n,m} := \sum_{S,T}\pi^*_{S,T}\widetilde{W}_{g,m} = -w_\lambda\lambda + w_\psi\psi + 0\cdot\delta_{\mathrm{irr}} - \sum_{s\geqslant 2} w_s\delta_{0,s} - \text{higher order boundary terms,} \tag{3.7}$$

where *higher order* denotes a positive linear combination of boundary divisors $\delta_{i,S}$ with $i\geqslant 1$ and

$$w_s \geqslant s w_\psi \geqslant 3w_\psi \qquad \text{for } s\geqslant 3, \tag{3.8}$$

$$w_\lambda = \binom{n}{r}\binom{n-r}{m-r}, \tag{3.9}$$

$$w_\psi = \binom{n-1}{r-1}\binom{n-r}{m-r}\frac{(k+1)(k+2)}{2} + \binom{n-1}{r}\binom{n-r-1}{m-r-1}\frac{k(k+1)}{2}, \tag{3.10}$$

$$w_2 = 2w_\psi + \binom{n-2}{r-2}\binom{n-r}{m-r}(k+1)^2 + 2\binom{n-2}{r-1}\binom{n-r-1}{m-r-1}k(k+1) + \binom{n-2}{r}\binom{n-r-2}{m-r-2}k^2. \tag{3.11}$$

Equation (3.10) is proved by applying pullback to (3.5), using $\omega := \sum_{i=1}^n \omega_i$,

$$\sum_{S,T}\sum_{i=1}^{r}\pi^*_{S,T}\omega_i = \binom{n}{r}\binom{n-r}{m-r}\frac{r}{n}\omega = \binom{n-1}{r-1}\binom{n-r}{m-r}\omega \tag{3.12}$$

and

$$\sum_{S,T}\sum_{i=r+1}^{m}\pi^*_{S,T}\omega_i = \binom{n}{r}\binom{n-r}{m-r}\frac{m-r}{n}\omega = \binom{n-1}{r}\binom{n-r-1}{m-r-1}\omega, \tag{3.13}$$

noting that equation (2.1) implies

$$\omega = \psi - \sum_S |S|\delta_{0,S}. \tag{3.14}$$

The sums over the pullbacks of the boundary divisors are computed by similar combinatorial considerations which we leave to the reader. Note that both the summand $2w_\psi$ on the right hand side of (3.11) and the bound in (3.8) are generated by the change of basis given in (2.1).

Next, for the proof of Theorem 1.9 it will be convenient to renormalize the divisor $\widetilde{W}_{g,n,m}$ in such a way that the coefficient of ψ is equal to 1. We thus introduce $W_{g,n} = \frac{1}{w_\psi}\widetilde{W}_{g,m}$ and find, setting $m = \min\{g,n\}$,

(3.15)
$$W_{g,n} = a(g,n)\lambda + \psi + 0 \cdot \delta_{\mathrm{irr}} - \sum_{s\geqslant 2} b_s\delta_{0,s} - \text{higher order boundary terms},$$

where *higher order* denotes a positive linear combination of boundary divisors $\delta_{i,S}$ with $i \geqslant 1$,

$$a(g,n) = \begin{cases} \frac{2n}{(k+1)(g+r)} & g = kn + r, r < n \\ \frac{n}{g} & n > g \end{cases} \tag{3.16}$$

$b_2 = b(g,n)$ with

$$b(g,n) = \begin{cases} 2 + \frac{2}{n-1}\frac{r(r-1)(k+1)^2+2r(n-r)k(k-r)+(n-r)(n-r-1)k^2}{r(k+1)(k+2)+(n-r)k(k+1)} & \\ & g = kn + r, r < n \\ 2 + \frac{g-1}{n-1} & n > g \end{cases} \tag{3.17}$$

and $b_s > b_2$ for all $s > 2$.

In addition, we shall use the anti ramification divisor classes from [11, Section 2], to obtain (by straightforward though somewhat lenghty algebraic computation) the existence of effective divisor classes T_g on $\overline{\mathcal{M}}_{g,g-1}$ satisfying

$$T_g = -\frac{g-7}{g-2}\lambda + \psi - \frac{1}{2g-4}\delta_{\mathrm{irr}} - \left(3 + \frac{1}{2g-4}\right)\delta_{0,2} + h.t. \tag{3.18}$$

where the higher order terms $h.t.$ denote a linear combination of all other boundary divisors with coefficients $\leqslant -2$.

Furthermore, normalizing the divisor classes in [11, Theorem 3.1], one obtains for $g \geqslant 1$ and any $1 \leqslant m \leqslant g/2$ effective divisor classes $F_{g,m}$ on $\overline{\mathcal{M}}_{g,n}$ (with $n = g - 2m$) satisfying

$$F_{g,m} = a\lambda + \psi - b_{\mathrm{irr}}\delta_{\mathrm{irr}} - b_{0,2}\delta_{0,2} + h.t., \tag{3.19}$$

where, as above, the higher order terms *h.t.* denote a linear combination of all other boundary divisors with coefficients $\leqslant -2$ and

$$a = \frac{n}{n-1}\left(\frac{10m}{g-2} + \frac{1-g}{g-m}\right), \qquad b_{0,2} = 3 + \frac{(g-n)(n+1)}{(g+n)(n-1)}, \quad b_{\mathrm{irr}} = \frac{nm}{(g-2)(n-1)} \tag{3.20}$$

Finally, to cover the case where g and n have different parity, we set $n = g-2m+1$ and pull back $F_{g,m}$ given in eqution 3.19 in all possible ways to $\overline{\mathcal{M}}_{g,n}$ (via a forgetful map forgetting one of the marked points). Summing all these divisor classes and then normalizing gives an effective divisor class $\widetilde{F}_{g,m}$ on $\overline{\mathcal{M}}_{g,n}$ satisfying

$$\widetilde{F}_{g,m} = a\lambda + \psi - b_{\mathrm{irr}}\delta_{\mathrm{irr}} - b_{0,2}\delta_{0,2} + h.t., \tag{3.21}$$

where, as above, the higher order terms *h.t.* denote a linear combination of all other boundary divisors with coefficients $\leqslant -2$ and

$$a = \frac{n}{n-2}\left(\frac{10m}{g-2} + \frac{1-g}{g-m}\right), \qquad b_{0,2} = 3 + \frac{g-n-1}{g+n-1}, \quad b_{\mathrm{irr}} = \frac{nm}{(g-2)(n-2)}. \tag{3.22}$$

4. Proofs

Proof of Theorem 1.8. — This proof proceeds along similar lines as the results for $\overline{\mathcal{M}}^{S_n}$ explained in the introduction.

Let us set $G = S_{n_1}\times\cdots\times S_{n_m}$ and consider the forgetful map $\overline{\mathcal{M}}^G \to \overline{\mathcal{M}}_g$ forgetting all marked points. We recall that $\overline{\mathcal{M}}^G$ will be uniruled when the general fibre of this morphism is. Likewise, when both the base $\overline{\mathcal{M}}_g$ and the general fibre are of general type, then by subadditivity of the Kodaira dimension the same must hold for $\overline{\mathcal{M}}^G$.

Now note that the general fibre of this morphism is birational to

$$C^n/G \simeq (C^{n_1}/S_{n_1}) \times \cdots \times (C^{n_m}/S_{n_m}).$$

For $\max\{n_1,\ldots,n_m\} > g$ at least one factor C^{n_i}/S_{n_i} is uniruled and thus the product C^n/G must be uniruled as well. Likewise $\max\{n_1,\ldots,n_m\} < g$ implies that each factor C^{n_i}/S_{n_i} is of general type, and thus is the product. It remains to recall that the base $\overline{\mathcal{M}}_g$ is of general type for $g \geqslant 22$. □

Proof Theorem 1.9. — This is more technical and more challenging. Since g is so small that the base $\overline{\mathcal{M}}_g$ is no longer of general type we require explicit computations. We will decompose the canonical class into the sum of an ample and an effective divisor class and use Proposition 2.2 to conclude that $\overline{\mathcal{M}}^G$ is of general type. As the ample divisor A from equation (2.5) we will use a positive multiple of the S_n-invariant divisor ψ. So the difficulty will lie in finding a suitable G-invariant effective divisor class E.

For assertion (1) we take Weierstrass divisors $W_k = W_{g,n_k}$ on each $\overline{\mathcal{M}}_{g,n_k}$ and $W = W_{g,n}$ on $\overline{\mathcal{M}}_{g,n}$, with coefficients $a(g,n_k), b(g,n_k)$ and $a(g,n), b(g,n)$ respectively, see Section 3 equation (3.15). For each summand n_k in the partition $n = \sum_{1\leqslant k\leqslant m} n_k$ we denote by $A_k \subset \{1,\dots,n\}$ the set of n_k labels on which the corresponding factor S_{n_k} from the direct product $G = S_{n_1}\times\cdots\times S_{n_m}$ acts; these sets are fixed by the embedding of G into S_n. We denote by $\pi_k : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n_k}$ the forgetful map forgetting all points except those labelled by A_k. In order to calculate $\pi_k^* W_k$ we introduce some notation.

For any sets $S \subset \{1,\dots,n\}$ we define

$$\delta_{i,s}^{S,\ell} := \sum_{|T\cap S|=\ell,|T|=s} \delta_{i,T} \tag{4.1}$$

and denote by $\pi_S : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,|S|}$ the natural forgetful map, forgetting all points except for those labelled by S. With this notation, by the usual abuse of notation explained in Section 2, one easily finds

Proposition 4.1. — *The pull-back divisors are* $\pi_S^*(\lambda) = \lambda$, $\quad \pi_S^*(\delta_{\mathrm{irr}}) = \delta_{\mathrm{irr}}$ *and*

$$\pi_S^*(\psi) = \sum_{i\in S}\psi_i - \sum_{s=2}^{n-n_k+1} \delta_{0,s}^{S,1}, \qquad \pi_S^*(\delta_{i,s}) = \sum_{\ell\geqslant 0} \delta_{i,s+\ell}^{S,s}. \tag{4.2}$$

Now let us observe that in the notation above the labels i, j belong to different sets A_k, A_ℓ if and only if the transposition $(i\ j)$ is not in G. Therefore the divisor $L := \sum_{1\leqslant k\leqslant m} \pi_k^* W_k$ has the decomposition

$$\begin{aligned} L = -\sum_{1\leqslant k\leqslant m} a(g,n_k)\lambda + \psi - 2\sum_{(i\ j)\notin G} \delta_{0,\{i,j\}} + 0\delta_{\mathrm{irr}} \\ - \sum_{1\leqslant k\leqslant m} b(g,n_k) \sum_{i,j\in S_k} \delta_{0,\{i,j\}} + h.t, \end{aligned} \tag{4.3}$$

where $h.t.$ denotes a (higher order) sum of boundary divisors, each multiplied with coefficients < -2. In addition we consider

$$W = -a(g,n)\lambda + \psi - b(g,n)\sum_{i,j}\delta_{0,\{i,j\}} + h.t, \quad D = s\lambda - \delta_{\mathrm{irr}} + h.t., \tag{4.4}$$

where $D = D_g$ is chosen with minimal slope $s = s(g)$ (see the list of divisors with small slope in (3.1)–(3.3)) and set

$$\epsilon := \min\{b(g,n_k) - 3 \,|\, k \in \{1,\ldots,m\} \text{ with } n_k \geqslant 2\}. \tag{4.5}$$

Clearly, $\epsilon > 0$ if and only if $\max\{n_1,\ldots,n_m\} \leqslant g-2$. Combining equations (4.3), (4.4), (4.5) (see also 3.15) one obtains the decompositions

$$K_G \geqslant 2D + \frac{1}{1+\epsilon}L + \frac{2\epsilon}{b(g,n)(1+\epsilon)}W + \eta\psi, \qquad \eta := \frac{\epsilon}{1+\epsilon}(1 - \frac{2}{b(g,n)}) > 0, \tag{4.6}$$

provided one has the inequality

$$f_m(g,n_1,\ldots,n_m) := 2s(g) - \frac{1}{1+\epsilon}\sum_{1\leqslant k\leqslant m} a(g,n_k) - \frac{2\epsilon}{b(g,n)(1+\epsilon)}a(g,n) \leqslant 13. \tag{4.7}$$

Since the divisor class ψ is ample and all divisors in equation (4.6) are effective and S_n-invariant, the proof boils down to checking the inequality (4.7).

Note that for $\max\{n_1,\ldots,n_m\} \in \{g-1,g\}$ we get $\epsilon = \eta = 0$, which proves that K_G is at least effective and thus gives non-negative Kodaira dimension.

To treat the additional case $\max\{n_1,\ldots,n_m\} = g-1$ in case (2) we need more general divisors $L_1,\ldots,L_m$. The function f_m in equation (4.7) will then depend on these divisors, destroying the explicit form of f_m given in equation (4.7).

Instead of the family of (generalized) Weierstrass divisors W_k on $\overline{\mathcal{M}}_{g,n_k}$, for $1 \leqslant k \leqslant m$, we use divisors L_k on $\overline{\mathcal{M}}_{g,n_k}$, for $1 \leqslant k \leqslant m$, having a decomposition

$$L_k = a_k\lambda + \psi - b_{k,\mathrm{irr}}\delta_{\mathrm{irr}} - b_k\delta_{0,2} + h.t. \tag{4.8}$$

where $b_k > 3$ and the higher order terms $h.t.$ denote a linear combination of all other boundary divisors with coefficients $\leqslant -2$. Setting (analog to the above) $L := \sum_{1\leqslant k\leqslant m} \pi_k^* L_k$ we obtain

$$L = -\sum_{1\leqslant k\leqslant m} a_k\lambda + \psi - 2\sum_{(i\ j)\notin G} \delta_{0,\{i,j\}} - \sum_{1\leqslant k\leqslant m} b_{k,\mathrm{irr}}\delta_{\mathrm{irr}} - \sum_{1\leqslant k\leqslant m} b_k \sum_{i,j\in S_k} \delta_{0,\{i,j\}} + h.t, \tag{4.9}$$

with $h.t.$ as above. This is analog to (4.3).

In this notation, we already have for shortness's sake suppressed dependence on g, n. Using the same convention in equation (4.4) (thus simply writing a, b in the decomposition of W) and introducing

$$\epsilon := \min\{b_k - 3 \mid k \in \{1,\dots,m\} \text{ with } n_k \geqslant 2\}, \tag{4.10}$$

which is (4.5) with $b(g, n_k)$ replaced by b_k and writing $\alpha_+ := \max\{\alpha, 0\}$, we obtain the decomposition

$$K_G \geqslant \left(2 - \frac{1}{1+\epsilon}\sum_k b_{k,\mathrm{irr}}\right)_+ D + \frac{1}{1+\epsilon}L + \frac{2\epsilon}{b(1+\epsilon)}W + \eta\psi, \quad \text{where} \quad \eta := \frac{\epsilon}{1+\epsilon}\left(1 - \frac{2}{b}\right) > 0, \tag{4.11}$$

provided one has the inequality

$$\begin{aligned} &f_m(g, n_1, \dots, n_m, L_1, \dots, L_m) \\ &:= \left(2 - \frac{1}{1+\epsilon}\sum_k b_{k,\mathrm{irr}}\right)_+ s + \frac{1}{1+\epsilon}\sum_{1\leqslant k\leqslant m} a_k - \frac{2\epsilon}{b(1-\epsilon)}a \leqslant 13. \end{aligned} \tag{4.12}$$

This finishes the proof of Theorem 1.9. □

Proof of Proposition 1.10. — The assertion on uniruledness and on being of general type for $g \geqslant 22$ are a direct application of Theorem 1.8.

When $n \leqslant g - 2$ and $(g, n) \neq (20, 4)$ we can apply Theorem 1.9 (1).

To cover the remaining cases, we shall use Theorem 1.9 (2) with the following choice of divisors L_k for $k \in \{1, 2\}$ (clearly $m = 2$ for the difference variety) in equation (4.12).

In case $n = g - 1$, we choose $L_1 = L_2 = T_g$, defined in equation (3.18). It is straightforward to check that $f_m \leqslant 13$ in this case.

Finally, in case $g = 20$ and $n = 4$, we choose $L_1 = L_2 = F_{20,8}$. Again, since now all divisor classes are explicit, it is straightforward to check $f_m \leqslant 13$. □

BIBLIOGRAPHY

[1] E. Arbarello & M. Cornalba, "The Picard groups of the moduli spaces of curves", *Topology* **26** (1987), no. 2, p. 153-171.

[2] ———, "Calculating cohomology groups of moduli spaces of curves via algebraic geometry", *Publ. Math., Inst. Hautes Étud. Sci.* **88** (1998), no. 1, p. 97-127.

[3] E. Arbarello, M. Cornalba & P. Griffiths, *Geometry of algebraic curves: Volume II. With a contribution by Joseph Daniel Harris*, Grundlehren der Mathematischen Wissenschaften, vol. 268, Springer, 2011.

[4] E. Arbarello, M. Cornalba, P. Griffiths & J. Harris, *Geometry of algebraic curves: Volume I*, Grundlehren der Mathematischen Wissenschaften, vol. 267, Springer, 1985.

[5] G. Casnati & C. Fontanari, "On the rationality of moduli spaces of pointed curves", *J. Lond. Math. Soc.* **75** (2007), no. 3, p. 582-596.

[6] D. Eisenbud & J. Harris, "The Kodaira dimension of the moduli space of curves of genus$\geqslant$23", *Invent. Math.* **90** (1987), no. 2, p. 359-387.

[7] G. Farkas, "Koszul divisors on moduli spaces of curves", *Am. J. Math.* **131** (2009), no. 3, p. 819-867.

[8] G. Farkas, D. Jensen & S. Payne, "The Kodaira dimensions of M22 and M23", `https://arxiv.org/abs/2005.00622`, 2020.

[9] G. Farkas & A. Verra, "The classification of universal Jacobians over the moduli space of curves", *Comment. Math. Helv.* **88** (2013), no. 3, p. 587-611.

[10] ———, "The universal difference variety over $\overline{\mathcal{M}}_g$", *Rend. Circ. Mat. Palermo* **62** (2013), no. 1, p. 97-110.

[11] ———, "The universal theta divisor over the moduli space of curves", *J. Math. Pures Appl.* **100** (2013), no. 4, p. 591-605.

[12] ———, "The geometry of the moduli space of odd spin curves", *Ann. Math.* **180** (2014), no. 3, p. 927-970.

[13] J. Harris & D. Mumford, "On the Kodaira dimension of the moduli space of curves", *Invent. Math.* **67** (1982), no. 1, p. 23-86.

[14] A. Kouvidakis, "On some results of Morita and their application to questions of ampleness", *Math. Z.* **241** (2002), no. 1, p. 17-33.

[15] A. Logan, "The Kodaira dimension of moduli spaces of curves with marked points", *Am. J. Math.* **125** (2003), no. 1, p. 105-138.

[16] A. Massarenti, "The automorphism group of $\overline{\mathcal{M}}_{g,n}$", *J. Lond. Math. Soc.* **89** (2014), no. 1, p. 131-150.

[17] I. Schwarz, "On the Kodaira dimension of the moduli space of nodal curves", `https://arxiv.org/abs/1811.01193`, 2018.

Manuscrit reçu le 14 juillet 2019,
révisé le 2 février 2021,
accepté le 5 mars 2021.

Irene SCHWARZ
schwarzi@math.hu-berlin.de
Humboldt Universität Berlin
Institut für Mathematik
Rudower Chausee 25
12489 Berlin (Germany)