ANNALES DE L'INSTITUT FOURIER

ROBERT KAUFMAN

Analysis on some linear sets

Annales de l'institut Fourier, tome 21, nº 2 (1971), p. 23-29

<http://www.numdam.org/item?id=AIF_1971__21_2_23_0>

Ann. Inst. Fourier, Grenoble
21, 2 (1971), 23-29.

ANALYSIS ON SOME LINEAR SETS

by Robert KAUFMAN

0.

Let F be a compact subset of $(-\infty, \infty)$ and for each integer $N \geqslant 1$ let $\nu_N \equiv \nu(N; F)$ be the number of intervals $[kN^{-1}, (k+1)N^{-1}]$ meeting F; F is called *small* provided $\log \nu_N = o(\log N)$. The existence of small sets of « multiplicity » (M_0-sets in [6I, p. 344]) was proved in 1942 by Salem and used by Rudin [4, VIII]; a program somewhat analogous for locally compact abelian groups was completed by Varopoulos [5].

Does there exist a small set F with the property that both F and (say) $F_2 = \{x^2 : x \in F\}$ are M_0-sets? The construction of these sets doesn't seem accessible by the method of Rudin and Salem [4], nor by the Brownian motion [3]. In this note an affirmative answer is given to a more general problem.

THEOREM 1. — *Let (h_n) be a sequence of real functions of class $C^1(-\infty, \infty)$ with derivatives $h'_n > 0$. Then there is a small set F with the property that each $h_n(F)$ is an M_0-set.*

Small sets occur naturally in the construction of independent sets [3, 4, 5]; after the metrical theory of Diophantine approximation a set F is called *metrically independent* if to each integer $N \geqslant 1$ and each ε in $(0, 1)$ there is a U_0 so that the simultaneous inequalities

$$\left|\sum_{j=1}^{N} u_j x_j - v\right| < U^{-N-\varepsilon}, \quad U = \max(|u_1|, \ldots, |u_N|) > U_0$$
$$|x_i - x_j| \geqslant \varepsilon \qquad \text{for} \qquad 1 \leqslant i < j \leqslant N$$

have no solution in integers $u_1, \ldots, u_N, v$ and members $x_1, \ldots, x_N$ of F. Compare [1, VII].

Uncountable metrically independent subsets could perhaps be constructed by classical arguments, for example that of Perron [1, p. 79] or Davenport [2].

THEOREM 2. — *The set* F *determined in Theorem* 1 *can be required to have the property that each* $h_n(F)$ *be metrically independent.*

THEOREMS 1*a*, 2*a*. — *Theorems* 1 *and* 2 *remain true provided each* h_n *is monotone-continuous and* $h'_n > 0$ *almost everywhere.*

1.

In the proof of Theorem 1 we require two arrays of independent random variables $(Y_{k,m})$ and $(\xi_{k,m})$ defined on a space (Ω, P) for $1 \leqslant k < \infty$, $1 \leqslant m \leqslant k^6$. Each Y_k is uniformly distributed upon $[0, 1]$ while

$$P\{\xi_{k,m} = 1\} = \pi_k = k^{-1} = 1 - P\{\xi_{k,m} = 0\}.$$

Suppose that f is a measurable function on $(-\infty, \infty)$ and $-1 \leqslant f \leqslant 1$, and let $\mu = \pi_k E(f(Y))$; elementary calculations show that

$$E(e^{t\xi_k f(Y_k)} e^{-t\mu}) \leqslant \exp \frac{1}{2} \pi_k t^2 \exp 0(\pi_k t^3)$$

with an '0' uniform for $-1 \leqslant f \leqslant 1$, $-1 \leqslant t \leqslant 1$, $0 \leqslant \pi_k \leqslant 1$. Hence for any $z > 0$ and $1 > t > 0$

$$P\{\Big|\sum_m \xi_{k,m} - k^5\Big| > zk^5\} \leqslant 2 \exp - zk^5 t \exp \frac{1}{2} k^6 \pi_k t^2 \exp 0(\pi_k k^6 t^3).$$

Choosing $z = t = k^{-2}$ and using $\pi_k = k^{-1}$ we obtain

$$P\Big\{\Big|\sum_m \xi_{k,m} - k^5\Big| \geqslant k^3\Big\} \leqslant 0(1) \exp - \frac{1}{2} k.$$

Thus

LEMMA 1. — $\sum_{m=1}^{k^6} \xi_{k,m} = k^5 + 0(k^3)$ *almost surely in* Ω.

A sequence of random measures λ_k is now determined as follows : for any function g on $(-\infty, \infty)$

$$\int g\, d\lambda_k = k^{-2}g(0) + k^{-5} \sum_m \xi_{n,m} g(e^{-k \log^2 k} Y_{k,m}).$$

Thus in every instance $\lambda_k \geqslant 0$ and $\|\lambda_k\| \geqslant k^{-2}$; moreover $\|\lambda_k\| = 1 + 0(k^{-2})$ almost surely. Because $\Sigma e^{-k \log^2 k} < \infty$ the convolution $\lambda = \pi * \lambda_k$ converges, and F is defined to be its closed support. F is contained in at most

$$\prod_{j=1}^{k} [j^5 + 0(j^3)] = e^{0(k \log k)}$$

intervals of length $e^{-k \log^2 k}$.

Because $(k+1) \log^2 (k+1)/k \log^2 k \to 1$, this is sufficient to obtain

Lemma 2. — F *is almost surely a small set.*

Lemma 3. — *Let* $h \in C^1(-\infty, \infty)$ *and* $h' > 0$; *let* (c_m), (u_m), (v_m) *be sequences of real numbers such that*

$$|c_m| + |v_m| = 0(1) \qquad \textit{and} \qquad |u_m v_m| \to \infty.$$

Then

$$\lim_{m \to \infty} \int_0^1 \exp iu_m h(c_m + v_m t)\, dt = 0.$$

Proof. — Let g denote the C^1 function inverse to h, and let $v_m > 0$. The integral is transformed to

$$J = \int_{\alpha_m}^{\beta_m} g'(y) \exp iu_m y \,.\, v_m^{-1}\, dy,$$

where $\alpha_m = h(c_m)$, $\beta_m = h(v_m + c_m)$. A further substitution $y = y_1 + \pi u_m^{-1}$ yields

$$J = \frac{1}{2} \int_{\alpha_m}^{\beta_m} g'(y) \exp iu_m y \,.\, v_m^{-1}\, dy - \frac{1}{2} \int_{\alpha_m - \pi u_m^{-1}}^{\beta_m - \pi u_m^{-1}} g'(y + \pi u_m^{-1}) \exp iu_m y \,.\, v_m^{-1}\, dy.$$

This tends to 0 because $\beta_m - \alpha_m = 0(v_m)$ and $v_m^{-1} u_m^{-1} = o(1)$.

Proof of Theorem 1. — We show that for each function h_n $\lim_{u \to \infty} \int \exp iuh_n(s)\lambda\,(ds) = 0$, almost surely. Then $h_n(F)$ is an

M_0-set; because $h_n(F)$ is compact it is enough to prove

$$\lim_{r\to\infty} \int \exp ir^{\frac{1}{3}} h_n(s)\lambda\,(ds) = 0, \qquad r = 1, 2, 3, \ldots.$$

To each integer $r \geqslant 3$ we attach the integer $k(r)$ defined by $k(r) \leqslant \log^{\frac{1}{3}} r < k(r) + 1$ and write $\lambda_k' = \prod_{j\neq k} * \lambda_j$. Then

$$\int \exp ir^{\frac{1}{1}} h_n(s)\lambda\,(ds) = \iint \exp ir^{\frac{1}{2}} h_n(s + w)\lambda_k\,(ds)\,\lambda_k'\,(dw).$$

For each real number w in the support of λ_k' let $m(w)$ be the expected value of $\int \exp ir^{\frac{1}{2}} h_n(s + w)\lambda_k\,(ds)$. Then

$$\left|\int \exp ir^{\frac{1}{2}} h_n(s)\lambda\,(ds)\right| \leqslant \int\left|\int \exp ir^{\frac{1}{2}} h_n(s + w)\lambda_k\,(ds) - m(w)\right|\lambda_k'\,(dw) + \|\lambda_k'\| \max|m(w)|.$$

The second integral, say I, can be handled by Jensen's inequality and the estimates at the beginning of 1. Let $-1 < t < 1$ and $\Phi(x) = e^{|tx|}$. Then

$$E(\Phi(\|\lambda_k'\|^{-1}k^5 \mathrm{Re}\,I)) \leqslant 2 \exp \frac{1}{2}\, k^5t^2 0\,(\exp k^5t^3).$$

Choosing $t = k^{-\frac{1}{2}}$ we observe

$$P\left\{|\mathrm{Re}\,I| > \|\lambda_k'\| k^{-\frac{1}{2}}\right\} = P\{\Phi(\|\lambda_k'\|^{-1}k^5 \mathrm{Re}\,I) > \exp k^4\} \leqslant 2 \exp \frac{1}{2}\, k^4 \exp 0(k^{7/2}) \exp - k^4.$$

This is the general term of a convergent series, inasmuch as $k = k(r) > -1 + \log^{\frac{1}{3}} r$. Thus, almost surely in Ω, for $r > r_0$

$$\left|\mathrm{Re} \int \exp ir^{\frac{1}{2}} h_n(s)\lambda\,(ds)\right| \leqslant k^{-\frac{1}{2}} \|\lambda_k'\| + \|\lambda_k'\| \max|m(w)|$$

and of course a similar statement holds for the imaginary part of the integral. Now

$$|m(w)| \leqslant k^{-2} + \left|\int_0^1 \exp ir^{\frac{1}{2}} h_n(e^{-k \log^2 k}t + w)\,dt\right|$$

with $w = 0(1)$ and $k = k(r)$. To apply Lemma 3 we must verify $r^{\frac{1}{2}} e^{-k \log^2 k} \to \infty$ but this is plain from $k(r) < \log^{\frac{1}{3}} r$. Because $\max_k \|\lambda'_k\| < \infty$ almost surely, the proof of Theorem 1 is complete.

2.

Theorem 2 requires the construction of a random function φ in $C^\infty(-\infty, \infty)$. Let ψ be a function in $C^\infty(-\infty, \infty)$ with the properties

(i) $\psi = 0$ on $[-\infty, -2]$, $\psi = 3$ on $[2, \infty]$,
(ii) $\psi' \geqslant 0$, and $\psi' > 1$ on $(-1, 1)$.

Let (a_p) be a sequence of real numbers such that every real number belongs to infinitely many of the intervals $(a_p - p^{-1}, a_p + p^{-1})$. Finally, let (Z_p) be a sequence of independent random variables on (Ω, P), uniformly distributed upon $[0, 1]$. We define

$$\varphi(x) = \sum_{p=1}^{\infty} e^{-p}\psi(p^{-1}Z_p + p^{\frac{1}{2}}(x - a_p)) + x.$$

To each compact set F and number $\delta > 0$ there are numbers q_1 and q_2 so that

$$q_1 \geqslant 4, \qquad q_1^{\frac{1}{2}}\delta \geqslant 5, \qquad \bigcup_{p=q_1}^{q_2} (a_p - p^{-1}, a_p + p^{-1}) \supseteq F.$$

THEOREM 3. — *Let* F *be a small set and* $h \in C^1(-\infty, \infty)$, $h' > 0$; *then* $h\varphi(F)$ *is almost surely metrically independent.*

For each integer $U \geqslant 1$ we can choose a subset $S(N, U)$ of R^N so that every point in F^N has distance $< U^{-3N}$ from some point in $S(N, U)$, while card $S(N, U) \leqslant \nu^N(NU^{3N}; F)$.

Beginning with an inequality

$$\left|\sum_{j=1}^{N} u_j h\varphi(y_j) - \nu\right| < U^{-N-\varepsilon}, \qquad |h\varphi(y_j) - h\varphi(y_i)| > \varepsilon \quad (i \neq j)$$

we conclude first that $|y_i - y_j| > \eta$ for some fixed $\eta > 0$. Let $(z_1, \ldots, z_n)$ be the member of $S(N, U)$ associated to

$(y_1, \ldots, y_n)$. Then

$$(1) \quad \left|\sum_{j=1}^{N} u_j h\varphi(z_j) - \nu\right| < U^{-N-\varepsilon} + 0(U.U^{-3N}),$$
$$|z_i - z_j| > \eta - 2U^{-3N}.$$

For large U we can find $\delta < \eta - 2U^{-3N}$ and corresponding numbers q_1, q_2. Let $q_1 \leqslant p \leqslant q_2, |z_i - a_p| < p^{-1}$.

$$\left|p^{-1}Z_p + p^{\frac{1}{2}}(z_i - a_p)\right| < p^{-1} + p^{-\frac{1}{2}} < 1,$$
$$|p^{-1}Z_p + p^{\frac{1}{2}}(Z_j - a_p)| > p^{\frac{1}{2}}\delta - p^{-1} - p^{-\frac{1}{2}} > 4, \quad \text{when } j \neq i.$$

Therefore $\dfrac{\partial}{\partial Z_p} \sum_{j=1}^{N} u_j h\varphi(Z_j) = u_i \dfrac{\partial}{\partial Z_p} h\varphi(Z_i)$ exceeds $\alpha|u_i|$ in modulus, with an $\alpha > 0$ independent of $u_1, \ldots, u_n$. Hence the probability of the inequality (1) is $0(U^{-1}.U^{-N-\varepsilon})$ for each $(z_1, \ldots, z_N)$. The requirement $U = \max(|u_1|, \ldots, |u_N|)$ determines $0(U^{N-1})$ N-tuples and plainly $\nu = 0(U)$. Because F is a small set $\nu^N(NU^{3N}; F) = U^{o(1)}$ as $U \to \infty$. Theorem 3 follows from this and $\Sigma U^{-1-\varepsilon}U^{o(1)} < \infty$.

Proof of Theorem 2. — Here we use the fact that F and φ depend on independent σ-fields. F is almost surely small, whence each $h_n\varphi(F)$ is almost surely metrically independent, by Theorem 3. By Theorem 1, each $h_n\varphi(F)$ is almost surely an M_0-set and Theorem 2 is proved.

3.

Proof of Theorems 1*a and* 2*a*. — According to a theorem of Marcinkiewicz [6II, pp. 73-77], to each $\delta > 0$ there exist functions g_n in $C^1(-\infty, \infty)$ so that

$$m(h_n \neq g_n) < \delta n^{-2}, \qquad n = 1, 2, 3, \ldots.$$

At almost all points of density of the set $(h_n = g_n)$, $g_n' = h_n' > 0$. Passing to a perfect subset of the set $(g_n' > 0, g_n' = h_n', g_n = h_n)$, we can find a $\tilde{g}_n$ in $C^1(-\infty, \infty)$ such that

$$m(h_n \neq \tilde{g}_n) < 2\delta n^{-2}, \qquad n = 1, 2, 3, \ldots,$$

$\tilde{g}_n' > 0$ everywhere.

We observe next that to each $\varepsilon > 0$ there is a constant $B(\varepsilon)$ so that for all Borel sets S

$$\int_{\Omega} \lambda(S)\, dP \leqslant \varepsilon + B(\varepsilon)m(S).$$

Thus to each $\varepsilon > 0$ we can choose functions $\tilde{g}_n$ by Marcinkiewicz' theorem, so that

$$P\{\lambda(x : \tilde{g}_n\varphi(x) \neq \tilde{h}_n\varphi(x) \text{ for some } n) > \varepsilon\} < \varepsilon.$$

In proving this implication it must be observed that φ and λ are stochastically independent and $\varphi' > 1$. Writing G for the inner set in the last inequality, we know that $h_n\varphi(G' \cap F) = \tilde{g}_n\varphi(G' \cap F)$ is almost surely metrically independent and that $h_n\varphi(G' \cap F)$ is almost surely an M_0-set, if only $\lambda(G' \cap F) > 0$; and this holds for $\|\lambda\| > \varepsilon$ excepting an event of probability $< \varepsilon$. Thus Theorems 1*a* and 2*a* are derived from Theorems 1 and 2.

BIBLIOGRAPHY

[1] J. W. S. Cassels, An introduction to Diophantine approximation, Cambridge Tract 45, (1957).

[2] H. Davenport, A note on Diophantine approximation II. Mathematika 11 (1964), 50-58.

[3] J.-P. Kahane, Images browniennes des ensembles parfaits, *C.R. Acad. Sci.*, Paris 263A (1966), 613-615.

[4] J.-P. Kahane and R. Salem, Ensembles parfaits, Hermann, Paris, 1963.

[5] N. Th. Varopoulos, Sets of multiplicity in locally compact abelian groups, *Ann. Inst. Fourier* (Grenoble) 16 (1966), 123-158.

[6] A. Zygmund, Trigonometric Series, Cambridge, (1959, 1968).

Manuscrit reçu le 21 avril 1970.

Robert Kaufman
Altgeld Hall,
Department of Mathematics,
University of Illinois,
Urbana (Illinois).