NICOLAS TH. VAROPOULOS

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SOME REMARKS ON Q-ALGEBRAS

par Nicolas Th. VAROPOULOS

0. Introduction.

We shall follow J. Wermer [1] and say that R a Banach algebra is a Q-algebra if there exists A a uniform algebra (i.e. a closed subalgebra of $\mathbf{C}(X)$ for some compact space X; in our definition we do not necessarily assume that a uniform algebra contains an identity) and $I \triangleleft A$ a closed ideal of A such that $R \cong A/I$.

In this note we shall prove a number of (fairly superficial) results on Q-algebras. The first thing we do is to state the following:

Criterion.

Let R be a commutative Banach algebra, then R is a Q-algebra if and only if there exists $C > 0$ some constant with the following property:

— For any $p \geqslant 1$ positive integer, any choice

$$x_1, x_2, \ldots, x_p \in R$$

of p elements in the unit ball of the algebra and any

$$P(z_1, z_2, \ldots, z_p)$$

homogeneous polynomial of p variables (of positive degree) we have:

$$\|P(x_1, x_2, \ldots, x_p)\|_R \leqslant C^{\deg P}\|P\|_\infty$$

where

$$\|P\|_\infty = \sup_{\substack{|z_j|\leqslant 1 \\ 1\leqslant j\leqslant p}} |P(z_1, z_2, \ldots, z_p)|.$$

The above criterion is very close to the criterion already developed by A.M. Davie [2], the proof is also analogous.

We introduce now the following:

DÉFINITION 0.1. — *Let* R *be a Banach algebra, we shall say that* R *is an injective algebra if the linear mapping induced by the algebra multiplication*

$$m : \mathrm{R} \otimes \mathrm{R} \to \mathrm{R} \quad (m(x \otimes y) = xy;\, x, y \in \mathrm{R})$$

is continuous for the injective norm of the tensor product $\mathrm{R} \otimes_\varepsilon \mathrm{R}$ [3].

We can state then the following:

THEOREM 1. — *Every commutative injective algebra is a* Q-*algebra.*

To state our next theorem we must start with a reminder on intermediate spaces (cf. e.g. [4]). For us an *interpolation pair* of Banach algebras will be a pair of complex Banach algebras R^0 and R^1 continuously and algebraically embedded in a complex topological algebra V in such a way that the subspace $\mathrm{R}^0 + \mathrm{R}^1 \subset \mathrm{V}$ of V is a *subalgebra* of V. We shall denote by $\| \ \|_0$ and $\| \ \|_1$ the norms of R^0 and R^1 respectively, the space $\mathrm{R}^0 + \mathrm{R}^1$ ($\subset$ V) can then be assigned with a naturel Banach norm (cf. [4] § 1). We shall assume that with that norms $\mathrm{R}^0 + \mathrm{R}^1$ becomes a Banach algebra.

We can consider then as in [4] § 2 $\mathfrak{Z} = \mathfrak{Z}(\mathrm{R}^0, \mathrm{R}^1)$ the space of $\mathrm{R}^0 + \mathrm{R}^1$ valued functions $f(z)$ defined in the strip $\{z \in \mathbf{C};\ 0 \leqslant \mathrm{Re} z \leqslant 1\}$ continuous and bounded with respect to the norm of $\mathrm{R}^0 + \mathrm{R}^1$ analytic in $\{z \in \mathbf{C};\ 0 < \mathrm{Re} z < 1\}$ and such that $f(it) \in \mathrm{R}^0$ is R^0-continuous and tends to zero as $|t| \to \infty$, $f(1 + it) \in \mathrm{R}^1$ is R^1-continuous and tends to zero as $|t| \to \infty$.

The space $\mathfrak{Z}$ is then an algebra under pointwise multiplication and if we norm it by:

$$\|f\|_{\mathfrak{Z}} = \max \left[\sup_t \|f(it)\|_0,\ \sup_t \|f(1+it)\|_1\right] \tag{1}$$

it becomes a Banach algebra.

For every real number s $(0 \leqslant s \leqslant 1)$ we denote then by $\mathfrak{N}_s = \{f \in \mathfrak{Z}; f(s) = 0\}$ (cf. [4], § 3) and we define $R_s = \mathfrak{Z}/\mathfrak{N}_s$ which is also then a Banach algebra, we call R_s the s-intermediate algebra between R^0 and R^1.

As an example we may consider the algebras

$$l^p \ (1 \leqslant p \leqslant +\infty)$$

of p-summable (or bounded for $p = +\infty$) sequences under pointwise multiplication. It is well known then (cf. [4] § 13) that for two values of $p = \alpha, \beta \, (\alpha < \beta)$ the two algebras l^α and l^β form an interpolation pair $(V = l^\infty)$ the intermediate algebras obtained then are all the algebras l^γ, $\alpha \leqslant \gamma \leqslant \beta$.

We can now state our next

THEOREM 2. — *Let* R^0 *and* R^1 *be two Q-algebras that form an interpolation pair then, for all* $0 \leqslant s \leqslant 1$ *the intermediate algebra* R_s *is also a Q-algebra.*

We can finally state the following:

THEOREM 3. — *Let* R *be a Q-algebra and let* $C > 0$ *and* $\alpha \geqslant 0$ *be two constants and* $f \in R$ *some element of the algebra such that*

$$\|f^n\|_R \leqslant Cn^\alpha \quad (n \geqslant 1).$$

For any polynomial of one variable then $P(z) = \sum_{n=1}^{m} a_n z^n$ *and any positive* $\varepsilon > 0$ *we have*

$$\|P(f)\|_R \leqslant C_1 \sup_{|z| \leqslant 1} \left| \sum_{n=1}^{m} a_n n^{2\alpha+\varepsilon} z^n \right|$$

where C_1 *is a constant that depends only on* C, α *and* ε.

This theorem is of course a partial generalization of the theorem of J. Wermer [1].

Let us denote by A_α $(\alpha \geqslant 0)$ the algebra of functions on $\mathbf{T} = \mathbf{R}(\text{mod } 2\pi)$ of the form

$$f(\theta) = \sum_{\nu=-\infty}^{+\infty} a_\nu e^{i\nu\theta}; \qquad \sum_{\nu=-\infty}^{+\infty} |a_\nu|(1 + |\nu|^\alpha) < +\infty$$

we have then.

PROPOSITION 0.1.

(i) A_α *is an injective algebra for all* $\alpha > \frac{1}{2}$, *and it is* not *injective for all* $0 \leqslant \alpha \leqslant \frac{1}{2}$.

(ii) l^1 *is an injective algebra. (This has been pointed out to me by S. Kaijser).*

We have finally the following

COROLLARY.

(i) A_α *is a Q-algebra for all* $\alpha > \frac{1}{2}$

(ii) A_α *is not a Q-algebra for all* $\alpha \leqslant \frac{1}{2}$. *(The special case* $\alpha = 0$ *is due to* J. Wermer [1]).

(iii) l^p *is a Q-algebra for all* $1 \leqslant p \leqslant \infty$ *(The special case* $1 \leqslant p \leqslant 2$ *is already due to* A. M. Davie [2]).

1. Proof of the criterion.

The criterion is a direct consequence of the following lemma which is due to I. G. Craw.

Craw's lemma.

Let R be a commutative Banach algebra and let us suppose that there exist two constants $M > 0$ and $\delta > 0$ such that for any choice of p elements in the δ-ball of R, $x_1, x_2, \ldots, x_p$ ($\|x_j\| \leqslant \delta$, $1 \leqslant j \leqslant p$) and any $P(z_1, z_2, \ldots, z_p)$ polynomial with $P(0) = 0$ we have

$$\|P(x_1, x_2, \ldots, x_p)\|_R \leqslant M \sup \{|P(z_1, z_2, \ldots, z_p)|\,;\ |z_j| \leqslant 1,\ 1 \leqslant j \leqslant p\}.$$

Then R is a Q-algebra.

The proof of the lemma is easy and can at any rate be found in [2]. It is also easy to show that if an algebra satisfies the conditions of our criterion it also satisfies the conditions of Craw's lemma. It is finally trivial to verify that any Q-algebra satisfies the conditions of the criterion with $C = 1$ already. For a more elaborate treatment of a very analogous situation we refer the reader to [2]. For an even more precise theorem cf. [7].

2. Injective algebras.

PROPOSITION 2.1.

(i) *Let* R *be a Banach algebra and let us suppose that there exists some constant* C *such that for any* $f \in \mathrm{R}$ *there exist* $\mathrm{S}, \mathrm{P}_n, \mathrm{Q}_n \in \mathrm{R}'$ $(n \geqslant 1)$ *such that:*

$$|\langle \mathrm{S}, f\rangle| \geqslant \|f\|; \qquad \sum_{n=1}^{\infty} \|\mathrm{P}_n\|\|\mathrm{Q}_n\| \leqslant \mathrm{C};$$
$$\langle \mathrm{S}, xy\rangle = \sum_{n=1}^{\infty} \langle \mathrm{P}_n, x\rangle\langle \mathrm{Q}_n, y\rangle, \qquad \forall x, y \in \mathrm{R}$$

then R *is an injective algebra.*

(ii) *Let* R *be a commutative injective algebra then there exists a constant* K *such that for any* $n \geqslant 1$ *and any* $\mathrm{S} \in \mathrm{R}'_1$ (= *the unit ball of the dual of* R) *there exists* $\mu \in \mathrm{M}(\mathrm{R}'_1)$ *some Radon measure such that*

$$\langle \mathrm{S}, x_1 x_2 \ldots x_n\rangle = \int_{\mathrm{R}'_1} \langle \mathrm{T}, x_1\rangle\langle \mathrm{T}, x_2\rangle \ldots \langle \mathrm{T}, x_n\rangle \, d\mu(\mathrm{T}), \qquad \forall x_1, x_2, \ldots, x_n \in \mathrm{R}; \tag{2}$$

and $\|\mu\| \leqslant \mathrm{K}^n$.

Proof. — Both (i) and (ii) are immediate consequences of the definition of the $\otimes_\varepsilon$-norm.

For (ii) observe that we may suppose *w.l.o.g.* that

$$x_1 = x_2 = \cdots = x_n;$$

(2) is then obtained by an obvious symmetrization process, based on the fact that there exists a numerical constant $c > 0$ such that for all $n \geqslant 1$ we can express the monomial $z_1 z_2 \ldots z_n$ as a sum of powers $\sum_s \lambda_s \left(\sum_{j=1}^{n} \alpha_j^s z_j\right)^n$ where $\sum_{j=1}^{n} |\alpha_j^s| \leqslant 1$ and $\Sigma |\lambda_s| \leqslant c^n$; e.g. $z_1 z_2 = \left(\frac{z_1 + z_2}{2}\right)^2 - \left(\frac{z_1 - z_2}{2}\right)^2$.

Proof of Theorem 1. — Let P be a homogeneous polynomial of p variables and let $x_1, x_2, \ldots, x_p \in \mathrm{R}$ be p elements of the unit ball of the injective algebra R we have then for any $S \in R_1'$ using (2):

$$|\langle S, P(x_1, x_2, \ldots, x_p)\rangle| = \left|\int_{R_1'} P(\langle T, x_1\rangle, \langle T, x_2\rangle, \ldots, \langle T, x_p\rangle)\, d\mu(T)\right| \leqslant \|\mu\|\, \|P\|_\infty \leqslant K_1^{\deg P}\|P\|_\infty.$$

This together with our criterion proves the theorem.

Obvious examples of injective algebras are of course supplied by the algebras of bounded differentiable (or lipschitz) functions on any manifold, this is seen by our proposition 2.1 (i) above. Less obvious ones can be given among the A_α algebras, indeed we have

Proof of proposition 0.1 (i). — Let us denote by $\| \ \|_\pi = \| \ \|_\pi^{(N)}$ the projective norm on the tensor product $C(I_N) \hat{\otimes} C(I_N)$ where $I_N (N \geqslant 1)$ denotes the finite space of $2N + 1$ points $I_N = \{-N, -N+1, \ldots, N\}$ (for a systematic study of $\| \ \|_\pi$ cf. [5]) and let us denote by $\tilde{V}$ the Banach space of double sequences $a = \{a_{n,m} \in \mathbf{C};\ n, m \in Z\}$ such that $\|\{a_{n,m} \in \mathbf{C};\ |n|, |m| \leqslant N\}\|_\pi \underset{N \to \infty}{=} 0(1)$ with its natural norm (cf. [8]). The thing to observe about V is that for any double sequence a we have

$$\text{(L)} \qquad \|a\|_{\tilde{V}} \leqslant C \sup_n \left(\sum_{m=-\infty}^{+\infty} |a_{n,m}|^2\right)^{\frac{1}{2}}$$

where C is a numerical constant, this is a result of J. E. Littlewood (for a proof cf. [5] ch. 6). Our proposition is now a consequence of the following

Lemma. — *Let* $\alpha \geqslant 0$ *and let us denote*

$$T_x^{(\alpha)} = \left\{ t_{\nu,\mu}^{(x)} = x_{\nu+\mu} \left[\frac{1 + |\nu + \mu|}{(1 + |\nu|)(1 + |\mu|)}\right]^\alpha \right\};\ \nu, \mu \in \mathbf{Z},$$
$$\forall x = \{x_\nu;\ \nu \in \mathbf{Z}\} \in l^\infty;$$

$$f_\alpha(z) = \sum_{n=0}^{\infty} \frac{a_n}{(1 + |n|)^\alpha} z^n;\ \forall f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ analytic in } \{|z| < 1\}.$$

We have then:

(i) A_α *is an injective algebra if and only if* $T_x^{(\alpha)} \in \tilde{V}$ $(\forall x \in l^\infty)$.

(ii) *If* $\alpha > \frac{1}{2}$ *then* $T_x^{(\alpha)} \in \tilde{V}$ $(\forall x \in l^\infty)$.

(iii) *If* $T_x^{(\alpha)} \in \check{V}$ *for all* $x \in l^\infty$ (*and some fixed* $\alpha \geqslant 0$) *then for all* $f, \varphi \in H^2$ (*the Hardy class*) *there exists some* $\psi \in H^2$ *such that*:

$$f_\alpha \varphi_\alpha = \psi_\alpha, \qquad \|\psi\|_2 \leqslant C\|f\|_2\|\varphi\|_2$$

where C *is independent of* f, φ *and* ψ.

(iv) *If the conclusion of* (iii) *holds then* $\alpha > \frac{1}{2}$.

Proof of Lemma. — (i) is an immediate consequence of proposition 2.1 applied to $x = e^{in\theta}$, $y = e^{im\theta}$. (ii) is an obvious double use of (L).

To see (iii) we first observe that the hypothesis that there exists a constant C (depending only on α) such that for every $y = \{y_{\mu,\nu};\ \nu,\ \mu \geqslant 0\} \in l^2 \hat{\otimes} l^2$ (= the completion of $l^2 \otimes l^2$ for the projective $\otimes_\pi$-norm) we have:

$$\|\{y_{\nu,\mu} t_{\nu,\mu}^{(x)};\ \nu,\ \mu \geqslant 0\}\|_{l^2 \hat{\otimes} l^2} \leqslant C\|x\|_\infty \|y\|_{l^2 \hat{\otimes} l^2}.$$

By passage to Fourier transforms this implies that for all $f, \varphi \in H^2$ if we denote by $\theta = f_\alpha \varphi_\alpha = \sum_{n=0}^{\infty} b_n z^n$ we have

$$\left\| \sum_{n=0}^{\infty} b_n x_n (1+n)^\alpha z^n \right\|_{H^1} \leqslant C\|x\|_\infty \|f\|_2 \|\varphi\|_2$$

and $x \in l^\infty$ being arbitrary if we put $x_n = \pm 1$ and take the expectation we obtain that

$$\sum_{n=0}^{\infty} |b_n|^2 (1+n)^{2\alpha} \leqslant 100\ C^2 \|f\|_2^2 \|\varphi\|_2^2$$

which proves (iii).

To see (iv) let us observe that the conclusion of (iii) simply says that H^2 can be assigned with a commutative unitary Banach algebra structure for wich the mappings

$$M_z:\ f \to f_\alpha(z) \quad (|z| < 1)$$

are multiplicative linear functionals. But this implies that $\|M_z\| \leqslant 1$ and that therefore we have

$$|f_\alpha(z)| \leqslant \|f\|_2 \quad \forall f \in H^2, \qquad |z| < 1$$

and this is clearly only possible if $\alpha > \frac{1}{2}$ (it suffices to try it on $f(z) = \sum_{n=1}^{\infty} n^{\frac{1}{2}} (\log n)^{\frac{1}{2}+\varepsilon} z^n$).

Proof of proposition 0.1 (ii). — Let us use the standard notation and denote by $l^1 \widehat{\widehat{\otimes}} l^1$ the completion of $l^1 \otimes_\varepsilon l^1$; $l^1 \widehat{\widehat{\otimes}} l^1$ can then be identified with a space of double sequences $\underline{\alpha} = \{\alpha_{\nu,\mu};\ \nu, \mu \geqslant 0\}$ and to prove our proposition it suffices then to show that for all $\underline{\alpha}$ in the unit ball of $l^1 \widehat{\widehat{\otimes}} l^1$ we have $\sum_{\nu=0}^{\infty} |\alpha_{\nu\nu}| \leqslant 1$. But by the very definition of the $\otimes_\varepsilon$-norm we see that this fact is an immediate consequence of the following

LEMMA. — *Let us denote by* $G = \mathbf{Z}(m)$ *the group of integers* (mod m) *where* m *is an arbitrary integers and let* $f \in \mathbf{C}(G)$ *be a function on* G (i.e. *a choice of* m *values*). *There exists then* $F \in V = \mathbf{C}(G) \widehat{\otimes} \mathbf{C}(G)$ *such that*

$$F(g_1, g_2) = 0, \quad \forall g_1, g_2 \in G, \quad g_1 \neq g_2;$$
$$F(g, g) = f(g); \quad \|F\|_V \leqslant \|f\|_\infty.$$

Proof of the lemma. — Let us denote by $\hat{G}$ the character group of G, it suffices to set then

$$F(g, h) = f(g) \left(\frac{1}{m} \sum_{\chi \in \hat{G}} \chi(g)\overline{\chi(h)} \right); \qquad g, h \in G.$$

(For a general treatment of the algebra $V = \mathbf{C}(G) \widehat{\otimes} \mathbf{C}(G)$ cf. [5].)

3. Proofs of theorems 2 and 3.

Indeed this is an immediate consequence of the definition $R_s = \mathcal{Z}/\mathcal{N}_s$ (cf. § 0) provided that we prove that $\mathcal{Z}$ is a Q-algebra.

But this fact is a consequence of our criterion applied to the norm of $\mathcal{Z}$ (cf. [1]) since both R^0 and R^1 are by hypothesis Q-algebras.

We shall now prove the following:

PROPOSITION 3.1. — *Let* H *be some Hilbert space and let* $T \in \mathcal{L}(H)$ *be such that*

$$\|T^n\| \leqslant cn^\alpha \quad (n \geqslant 1)$$

where $c > 0$ *and* $\alpha \geqslant 0$. *For every* $\varepsilon > 0$ *then and every* $P(z) = \sum_{n=1}^{N} a_n z^n$ *we have*

$$(4) \qquad \|P(T)\| \leqslant C \sup_{|z|\leqslant 1} \left| \sum_{n=1}^{N} a_n n^{2\alpha+\varepsilon} z^n \right| = C \|P_{2\alpha+\varepsilon}\|_\infty$$

where C *is a constant that depends only on* c, α *and* ε.

Proof. — In the proof we may suppose that $0 \leqslant \alpha \leqslant \frac{1}{2}$ and $2\alpha + \varepsilon < 1$ for otherwise the proposition is an easy consequence of classical results on Riesz potentials and Bernstein's theorem [6].

Let us as usual denote by T^* the adjoined operator of T and by $\langle , \rangle$ the scalar product in H, let also $h \in H$ be some fixed element of H and let us define:

$$\mathbf{F}(t) = \sum_{n=0}^{\infty} A_n^{-\alpha-\frac{1}{2}(1+\varepsilon)} (T^n h) e^{int} \in L^2(\mathbf{T};\ H)$$
$$\mathbf{F}_*(t) = \sum_{n=0}^{\infty} A_n^{-\alpha-\frac{1}{2}(1+\varepsilon)} (T^{*n} h) e^{int} \in L^2(\mathbf{T};\ H)$$

(cf. (7) bellow) where we define the A_n^β by:

$$(1-x)^{-\beta-1} = \sum_{n=0}^{\infty} A_n^\beta x^n \quad \beta \neq -1, -2, -3, \ldots, (|x| < 1)$$

and where $L^2(\mathbf{T}; H)$ denotes of course the space of H-valued L^2 functions on $\mathbf{T}$. Taking into account then the obvious identity:

$$A_n^{\beta+\gamma+1} = \sum_{p+q=n} A_p^\beta A_q^\gamma$$

valid for all admissible β and γ we see that:

$$(5) \qquad \Phi(t) = \langle \mathbf{F}_*(t), \mathbf{F}(t) \rangle = \sum_{n=0}^{\infty} A^{-2\alpha-\varepsilon} \langle T^n h, h \rangle e^{int}$$

and that

(6) $$\Phi(t) \in L^1(T); \quad \|\Phi(t)\|_{L^1} \leqslant C.$$

where C depends only on c, α and ε. But (5) and (6) imply then that

$$|\langle P(T)h, h\rangle| = \frac{1}{2\pi}\left|\int_0^{2\pi} P^*(e^{i\theta})\overline{\Phi(e^{i\theta})}\, d\theta\right| \leqslant C\|P^*\|_\infty$$

where $P^*(e^{i\theta}) = \sum_{n=1}^{N} (A_n^{-2\alpha-\varepsilon})^{-1} a_n e^{in\theta}$; but if we take then into account that for every admissible β we have:

(7) $$A_n^\beta = \frac{n^\beta}{\Gamma(\beta+1)}\left[1 + 0\left(\frac{1}{n}\right)\right].$$

We see that $\|P^*\|_\infty \leqslant K\|P_{2\alpha+\varepsilon}\|$ (where K depends only on α and ε) and our result follows.

Theorem 3 follows now at once from proposition 3.1 if we use the technique in [1] of representing a Q-algebra as an algebra of operators in some appropriate Hilbert space. A *direct* proof of Theorem 3 in the same lines as above can also be given, but we felt that proposition 3.1 presented some independent interest.

Proof of corollary 2 (ii). — If we suppose that $0 \leqslant \alpha < \frac{1}{2}$ and apply theorem 3 to the element $f(x) = e^{ix} \in A_\alpha(T)$; it is then clear that $\|f^n\|_{A_\alpha} = n^\alpha$ $(n \geqslant 1)$, we conclude therefore that if A_α were a Q-algebra we would have

$$\|P(e^{ix})\|_{A_\alpha} \leqslant C\|P_{2\alpha+\varepsilon}(e^{ix})\|_\infty$$

for every polynomial P which is manifestly false because it implies the false assertion $A_{2\alpha+\varepsilon} \subsetneq A_\alpha$ (for all $\varepsilon > 0$) [6]. The case $\alpha = \frac{1}{2}$ is more delicate, cf. [7].

It is worth remarking perhaps that a more elaborate technique in the proof of proposition 3.1 can improve the growth $n^{2\alpha+\varepsilon}$ in the second member of (4) (and therefore also in Theorem 3) to $n^{2\alpha}(\log n)^{1+\varepsilon}$ or even to $n^{2\alpha}\log n(\log\log n)^{1+\varepsilon}$ e.c.t. The exponent 2α of $n^{2\alpha}$ in Theorem 3 is however best possible i.e. it cannot be replaced by anything smaller; indeed if it could, the argument we used to prove corollary 2 (ii) and the fact that A_α *is* a Q-algebra for all $\alpha < \frac{1}{2}$ would supply a contradiction.

To conclude we remark that the other parts of the corollary are immediate, (i) is a consequence of Theorem 1 and proposition 0.1 (i); and (iii) is obtained by interpolating between l^1 and l^∞ which are Q-algebras.

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Nicolas Th. VAROPOULOS,
Université de Paris XI,
Mathématiques, Bât. 425,
91-Orsay.