

ANNALES DE L'INSTITUT FOURIER

ROBERT E. GREENE

H. WU

**Embedding of open riemannian manifolds
by harmonic functions**

Annales de l'institut Fourier, tome 25, n° 1 (1975), p. 215-235

http://www.numdam.org/item?id=AIF_1975__25_1_215_0

© Annales de l'institut Fourier, 1975, tous droits réservés.

L'accès aux archives de la revue « Annales de l'institut Fourier » (<http://annalif.ujf-grenoble.fr/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

EMBEDDING OF OPEN RIEMANNIAN MANIFOLDS BY HARMONIC FUNCTIONS

by R.E. GREENE* and H. WU*

A real-valued function f on a Riemannian manifold is called harmonic if $\Delta f \equiv 0$ where Δ is the Laplacian associated to the Riemannian metric of the manifold. The purpose of this paper is to prove the existence on any connected non-compact Riemannian manifold of dimension n of $2n + 1$ harmonic functions which taken together give a proper embedding of the manifold in $(2n + 1)$ -dimensional Euclidean space \mathbb{R}^{2n+1} . This result is conceptually related to the theorem of Behnke and Stein that every open Riemann surface is a Stein manifold since one expects the behaviour of harmonic functions and that of holomorphic functions to be closely related on a complex manifold of (complex) dimension one. On the other hand, in higher dimension the existence of proper embeddings by harmonic functions contrasts strongly with the fact that openness of a Kähler manifold is far from being a sufficient condition for it to be a Stein manifold.

The proof given here of the existence of proper embeddings by harmonic functions is divided into the following steps : § 1, the proof that at each point of a Riemannian manifold there exist harmonic functions defined in a neighborhood of the point which form a local coordinate system near the given point ; § 2, the proof, using results on approximation of locally defined harmonic functions by globally defined ones, that every open Riemannian manifold admits an embedding by harmonic functions into \mathbb{R}^{2n+1} ; and § 3, the proof that such an embedding can be found that is also a proper mapping. The arguments used in § 2 are based upon methods developed by Whitney [21].

(*) Research partially supported by National Science Foundation grants GP-27576 (first author) and GP-34785 (second author). The second author was a Sloan Fellow during the preparation of this paper.

The arguments of the present paper can be used to prove that any real analytic manifold M , compact or non-compact, with a real analytic Riemannian metric has a real analytic (proper) embedding in some Euclidean space, thus providing a new proof as well as a generalization to the non-compact case of a well-known theorem of Bochner [2]. To obtain this result in case M is non-compact, one need only observe that by the theorem of Petrovsky (see, e.g., [15, p. 225]) harmonic functions on a real analytic manifold with real analytic Riemannian metric are necessarily real analytic so that the proper embedding by harmonic functions constructed in § 3 is in this case real analytic. On the other hand, if M is a compact real analytic manifold with real analytic Riemannian metric, then the non-compact manifold $M \times \mathbb{R}$ (with the product real analytic structure and the product metric, this metric being thus real analytic) has as noted a proper real analytic embedding in some Euclidean space. The composition of this embedding of $M \times \mathbb{R}$ with the real analytic embedding $i : M \rightarrow M \times \mathbb{R}$ defined by $p \rightarrow p \times 0$ is a real analytic embedding of M in a Euclidean space. (An extension of Bochner's result to non-compact manifolds has been given by Royden [20] and Malgrange [25] using methods different from those of the present paper. The more difficult problem of finding real analytic embeddings of real analytic manifolds without assuming that the manifolds admit real analytic Riemannian metrics has been solved for compact manifolds by Morrey [13] and for both compact and non-compact manifolds by Grauert [7]).

Once it is known that a given real analytic manifold admits a proper embedding in some (finite-dimensional) Euclidean space, the Weierstrass approximation theorem may be applied to show that the manifold has a real analytic embedding in any Euclidean space in which it has a C^∞ embedding (see [7 ; pp. 470-471]). A corresponding remark applies to the question of real analytic proper embeddings. Thus the dimensionality requirements are not a point directly at issue in real analytic embedding problems. In the case of embedding by harmonic functions, the dimensionality requirements are of significance since no harmonic function analogue of the Weierstrass approximation technique indicated holds in this case.

1. Harmonic local coordinates.

PROPOSITION 1. — *If M is a Riemannian manifold of class C[∞] and dimension n and if p ∈ M is an arbitrary point of M, then there exists an open neighborhood U of p and a collection of (necessarily C[∞]) real valued functions h₁, . . . , h_n on U such that*

a) Δh_i(q) = 0 for all q ∈ U and i = 1, . . . , n.

b) if H : U → Rⁿ is defined by H(q) = (h₁(q), . . . , h_n(q)) ∈ Rⁿ for q ∈ U then H has a nonsingular Jacobian at p.

Proof. — Let (x₁, . . . , x_n) be a C[∞] local coordinate system defined in a neighborhood of p. If ℋ_p denotes the real vector space of germs of harmonic functions at p, then there is a linear mapping ℒ : ℋ_p → Rⁿ defined by

$$\mathcal{L}(\bar{h}) \rightarrow \left(\left. \frac{\partial \bar{h}}{\partial x_1} \right|_p, \dots, \left. \frac{\partial \bar{h}}{\partial x_n} \right|_p \right), \quad \bar{h} \in \mathcal{H}_p.$$

For specified functions h₁, . . . , h_n, nonsingularity at p of the Jacobian of the associated mapping H is equivalent to nonsingularity of the n × n matrix

$$\left(\left. \frac{\partial h_i}{\partial x_j} \right|_p \right) \quad \begin{matrix} i = 1, \dots, n \\ j = 1, \dots, n \end{matrix}$$

and hence is equivalent to the linear independence in Rⁿ of the vectors ℒ(̄h₁), . . . , ℒ(̄h_n), where ̄h_i = the germ of h_i at p. Thus to establish the proposition it suffices to show that the subspace ℒ(ℋ_p) of Rⁿ has dimension n or equivalently that ℒ(ℋ_p) = Rⁿ. Suppose on the contrary that ℒ(ℋ_p) ≠ Rⁿ. Then there would exist (α₁, . . . , α_n) ∈ Rⁿ

such that for every (β₁, . . . , β_n) ∈ ℒ(ℋ_p) $\sum_{i=1}^n \alpha_i \beta_i = 0$. In that case,

every ̄h ∈ ℒ(ℋ_p) would have 0 derivative at p in the direction $\sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}$. Hence the proof of the proposition will be complete if

it is established that for every vector $\vec{V} \neq 0$ in the tangent space of M at p there exists a harmonic function h defined in a neighborhood of p such that $\vec{V}h \neq 0$.

The existence of such harmonic functions is implied by a fact that was understood though not proved in detail in classical potential theory (cf. [6]). Expressed in physical terms, this fact is the observation that if C is the geodesic emanating from p with tangent \vec{V} then, provided that q is close enough to p along (the positive direction of) C , a unit charge at q exerts a force on a test charge at p which has a nonzero component in the direction of \vec{V} . Using the constructions which make up the method of the parametrix, it is possible to formulate and prove a precise version of this general observation and thus to deduce the existence of a locally defined harmonic function satisfying $\vec{V}h \neq 0$.

The following discussion will briefly outline the required construction using the notation and results of [19 ; Chapter V] (cf. [22]), to which the reader is referred for further details. Note that the case $n = 2$ requires special consideration, as a consequence of the fact $-\log r$ is the natural potential on the plane whereas r^{2-n} plays the corresponding role for $n \geq 3$. Only the case $n \geq 3$ will be discussed here ; the desired conclusion in case $n = 2$ can be deduced by similar methods (replacing r^{2-n} by $-\log r$) or can be obtained immediately from the existence of local uniformizing variables, i.e., the existence of local isothermal coordinates, on any two-dimensional Riemannian manifold ([4]).

If D is any sufficiently small neighborhood of a point $p \in M$, then there is defined on $D \times D - \{(x, x) \mid x \in D\}$ a C^∞ function $\gamma(x, y)$, the elementary kernel ("noyau elementaire"), which has the properties that it is harmonic in the first variable (when $x \neq y$) and that for each fixed y it satisfies in the distribution sense the equation $\Delta_x \gamma(x, y)|_{x=y} = \delta(y)$ where $\delta(y)$ is the Dirac distribution at y . (Thus γ is, up to the addition of a C^∞ function, the potential of a unit charge at y in the classical theory). Assume that D_1 is a neighborhood of p chosen to have compact closure in M with $\bar{D}_1 \subset D$ and to be convex in the sense that any two points in D_1 are joined by a unique minimizing geodesic, which lies in D_1 . Then the kernel γ has the further property that if x is sufficiently close to y , $x \neq y$, $x, y \in D_1$, then the derivative of γ with respect to motion of x toward y along the geodesic from x to y is nonzero. In fact, given $K > 0$, there exists an $\epsilon > 0$, such that if the distance from x to y , $x, y \in D_1$, is less than ϵ but $x \neq y$ then the x -derivative of $\gamma(x, y)$ with respect to arc length along the minimizing

geodesic from x to y is greater than K in absolute value. To see that the elementary kernel behaves this way, note that it is explicitly given as follows :

$$\gamma(x, y) = \omega(x, y) + \sum_{m=1}^{+\infty} \int_{z \in D} \omega(x, z) \wedge *_z q_m(z, y)$$

where

1) $\omega(x, y) = r^{2-n}(x, y) \times$ a C^∞ function on $D \times D$, which is nonzero for $x = y$, $r(x, y)$ denoting the Riemannian distance from x to y

and

$$2) q_1(x, y) = -\Delta_x \omega(x, y)$$

$$q_m(x, y) = \int_z q(x, z) \wedge *_z q_{m-1}(z, y) \quad m \geq 2.$$

Using from [19], Lemma 1 of § 28, Lemma 6 of § 27, and Lemma 4 of § 27, it is easily verified that the differentiation of the $\Sigma \int$ term of the expression for γ along the geodesic from x to y produces a result of order of magnitude at most that of r^{3-n} . On the other hand, x -differentiation of $\omega(x, y)$ along this geodesic produces a result of order of magnitude precisely that of $r^{1-n}(x, y)$ since a) $\omega(x, y)$ is the product of $r^{2-n}(x, y)$ and a (nonzero, for $x = y$) C^∞ function of x and y and b) the indicated derivative of $r^{2-n}(x, y)$ is exactly $(n - 2) r^{1-n}(x, y)$. Thus the x -derivative of $\gamma(x, y)$ with respect to arc length along the geodesic from x to y becomes infinite when x is sufficiently close to y . That this derivative becomes infinite as $x \rightarrow y$ uniformly in y follows from the fact that the relevant estimates involve only constants which can be chosen to vary continuously with the metric or equivalently continuously with respect to variation of y .

It follows from the observations given that if \vec{V} is any vector $\neq 0$ in M_p and C is the geodesic from p satisfying $C(0) = p, C'(0) = \vec{V}$ then for sufficiently small $t_0 > 0$ the function $q \xrightarrow{h} \gamma(q, C(t_0))$ is defined and harmonic in a sufficiently small neighborhood of p and $\vec{V}h|_p \neq 0$. \square

Remark. — Proposition 1 is actually a special case of a general theorem due to Lipman Bers, to the effect that one can obtain local

solutions of a linear elliptic equation $Pu = f$ with prescribed derivatives up to order $m - 1$ at a single point, where m is the order of P ([23 ; Lemma 5.1] and [24 ; p. 228]). However, we have included the above proof for the special case at hand because it is relatively simple and elementary.

2. Embeddings by harmonic functions.

Throughout this section and the following one, M will be a connected non-compact Riemannian manifold of class C^∞ and dimension $n \geq 2$.

Let $\mathcal{H}(M)$ be the topological space of harmonic real-valued functions on M with the coarse C^0 topology, i.e., the topology of uniform convergence on compact subsets of M . Let $C^\infty(M)$ be the space of C^∞ real-valued functions on M with the coarse C^∞ topology, i.e., the topology of uniform convergence on compact subsets of M of functional values and values of derivatives of all orders. Since an harmonic function on M is necessarily C^∞ , there is an inclusion $i : \mathcal{H}(M) \hookrightarrow C^\infty(M)$.

LEMMA 1. — *The inclusion mapping $i : \mathcal{H}(M) \rightarrow C^\infty(M)$ is a homeomorphism of $\mathcal{H}(M)$ onto a closed subspace of $C^\infty(M)$.*

Proof. — This lemma is equivalent to the following standard properties of harmonic functions, the first of which follows from the second, the second being a special case of results of Friedrichs and Gårding (see [15 ; p. 210]) :

a) If a sequence of harmonic functions converges uniformly on compact subsets of M , then the limit of the sequence is an harmonic function.

b) If a sequence of harmonic functions converges uniformly on compact sets, then the derivatives of the members of the sequence converge uniformly on compact sets to the derivatives of the limit of the sequence. \square

$C^\infty(M)$ admits a complete metric which induces its topology ([14 ; p. 22ff.,] [15 ; p. 150]). Since $\mathcal{H}(M)$ is homeomorphically

included as a closed subspace in $C^\infty(M)$, the induced metric on $\mathcal{H}(M)$ induces the topology of \mathcal{H} and is complete. In particular, the Baire category theorem holds in $\mathcal{H}(M)$ and hence also in the finite products (with product topology) $\mathcal{H}^k(M) = \mathcal{H}(M) \times \cdots \times \mathcal{H}(M)$ (k factors).

DEFINITION. — *Let P be a differential operator on M . A function $f : K \rightarrow \mathbb{R}$ defined on a compact subset K of M is said to satisfy $Pf = 0$ on K if f is the restriction to K of function f_1 defined on some open subset of M containing K and satisfying $Pf_1 = 0$ on that open subset. In particular, a function $f : K \rightarrow \mathbb{R}$ is said to be harmonic on K if f is the restriction to K of a harmonic function defined on some open subset of M containing K .*

LEMMA 2. — *If K is a compact subset of M such that $M - K$ has no compact components, then every function $f : K \rightarrow \mathbb{R}$ which is harmonic on K is the uniform limit on K of harmonic functions defined on all of M . In particular, if $p, q \in M, p \neq q$, then there is a harmonic function h on M such that $h(p) \neq h(q)$.*

Proof. — A linear elliptic operator P on M is said to have the *unique continuation property* if any two functions $u_1 : U \rightarrow \mathbb{R}, u_2 : U \rightarrow \mathbb{R}$ defined on a connected open subset U of M which satisfy (1) $Pu_1 \equiv 0$ and $Pu_2 \equiv 0$ on U and (2) $u_1 \equiv u_2$ on some open subset V of U necessarily satisfy (3) $u_1 \equiv u_2$ on all of U . Not every elliptic operator has the unique continuation property as shown by Pliś [17]. However, Aronszajn [1] and Cordes [5] have proved that all second order elliptic operators (with C^∞ coefficients) and in particular the Laplacian on M do have this property and in fact an even stronger condition of the unique continuation type than that required here. A related theorem which admits a much simpler proof and which also implies the present unique continuation property is given by Protter [18]. The first statement of the lemma is thus a consequence of the following result of Lax [11] and Malgrange [12] (see also [15 ; pp. 233-241] for a detailed discussion of the real analytic case, wherein the unique continuation property is automatic) :

If P is a C^∞ elliptic operator on a connected non-compact C^∞ manifold M and if its adjoint P^* has the unique continuation property, then, for every compact subset K of M such that

$M - K$ has no components with compact closure, any function $f : K \rightarrow \mathbb{R}$ which satisfies $Pf = 0$ on K is the limit uniformly on K of functions h defined on all of M and satisfying $Ph = 0$ everywhere on M .

The second statement of the lemma follows immediately from the first since the function which is 0 at p and 1 at q is harmonic on the set $\{p, q\}$ and $M - \{p, q\}$ is connected. \square

LEMMA 3. — *For each point $p \in M$, there exist harmonic functions $h_1, \dots, h_n : M \rightarrow \mathbb{R}$ such that the mapping*

$$q \rightarrow (h_1(q), \dots, h_n(q)) \in \mathbb{R}^n, \quad q \in M,$$

is a diffeomorphism onto its image in a neighborhood of p .

Proof. — According to Proposition 1 of § 1, there exists a neighborhood U of P together with harmonic functions h'_1, \dots, h'_n on U such that $q \rightarrow (h'_1(q), \dots, h'_n(q))$ is a diffeomorphism of U onto its image. Choose a neighborhood U_1 of p with \bar{U}_1 compact, $\bar{U}_1 \subset U$, and $M - \bar{U}_1$ connected (a geodesic ball of sufficiently small radius around p , for instance). Then Lemma 2 implies that there exist harmonic functions $h_1, \dots, h_n : M \rightarrow \mathbb{R}$ approximating h'_1, \dots, h'_n on \bar{U}_1 arbitrarily closely. From Lemma 1 it follows that the derivatives $\frac{\partial h_i}{\partial x_j}$, $i, j = 1, \dots, n$ at p relative to a fixed local coordinate system (x_1, \dots, x_n) around p can be made arbitrarily close to the corresponding derivatives $\frac{\partial h'_i}{\partial x_j}$ at p . Since the matrix $\left(\frac{\partial h'_i}{\partial x_j}\right)$ $i, j = 1, \dots, n$ is nonsingular at p by hypothesis, h_1, \dots, h_n can be chosen so that the matrix $\left(\frac{\partial h_i}{\partial x_j}\right)$ $i, j = 1, \dots, n$ is also nonsingular at p . These h_i 's satisfy the requirement of the lemma by the inverse function theorem. \square

Let K be a compact subset of M . A C^∞ mapping $F : M \rightarrow \mathbb{R}^k$ is said to be an *embedding* of K if $F|K$ is injective and if F_* is injective at every point of K .

LEMMA 4. — *Let K be a compact subset of M . There exist a finite number of harmonic functions $h_1, \dots, h_l : M \rightarrow \mathbb{R}$ such that*

the mapping $M \rightarrow R^l$ defined by $q \rightarrow (h_1(q), \dots, h_l(q))$ is an embedding of K .

Proof. — According to Lemma 3, there exists a covering of K by open sets to each of which there are associated n harmonic functions on M , which functions together define a diffeomorphism of that open set onto an open subset of R^n . Choose a finite subcover $\{U_j | j = 1, \dots, k\}$ of such an open cover of the compact set K . The union of the sets of harmonic functions associated to each open set in the subcover is a finite collection of harmonic functions, say h_1, \dots, h_{nk} . The mapping $q \rightarrow (h_1(q), \dots, h_{nk}(q))$ has an injective Jacobian at each point of K since at each point of K some subset of n of the h_1, \dots, h_{nk} defines a diffeomorphism in a neighborhood of that point. Notice that on the same grounds this mapping is injective when restricted to any one of the U_j 's, $j = 1, \dots, k$.

Now consider the compact set $K \times K - \bigcup_{i=1}^k (U_i \times U_i)$. For each (p, q) in this set there exists by Lemma 3 a harmonic function $h_{p,q}$ such that $h_{p,q}(p) \neq h_{p,q}(q)$. Then there are open neighborhoods U_p, U_q of p and q respectively such that for any $x \in U_p$ and $y \in U_q$ $h_{p,q}(x) \neq h_{p,q}(y)$. The sets $U_p \times U_q, (p, q) \in K \times K - \bigcup_{i=1}^k (U_i \times U_i)$ are an open cover of $K \times K - \bigcup_{i=1}^k (U_i \times U_i)$. Let $U_{p_j} \times U_{q_j}, j = 1, \dots, m$, be a finite subcover. Then the functions $h_{p_j, q_j}, j = 1, \dots, m$ together with h_1, \dots, h_{nk} define an injective mapping of K into R^{nk+m} : if p, q are two distinct points of K then, if some U_i contains them both, they have distinct images under one of the h_1, \dots, h_{nk} whereas if no U_i contains them both, they have distinct images under one of the h_{p_j, q_j} since then necessarily $p \in U_{p_j}$ and $q \in U_{q_j}$ for some j . Thus the finite set $h_1, \dots, h_{nk}, h_{p_1, q_1}, \dots, h_{p_m, q_m}$ of harmonic functions satisfies the condition required in the lemma. \square

The following two lemmas are a version of the projection method of Whitney [21] (see also [10 ; pp. 118-125]).

LEMMA 5. — Let $F : M \rightarrow R^k, k \geq 2n + 1$, be a C^∞ mapping with component functions F_1, \dots, F_k . Suppose that F has the property that F_* is injective at every point of a compact subset K of

M. Then the set of all $a = (a_1, \dots, a_{k-1}) \in \mathbb{R}^{k-1}$ such that the C^∞ mapping F^a of M into \mathbb{R}^{k-1} defined by

$$q \rightarrow (F_1(q) - a_1 F_k(q), \dots, F_{k-1}(q) - a_{k-1} F_k(q))$$

has $(F^a)_*$ injective at every point of K has complement of measure zero in \mathbb{R}^{k-1} .

Proof. – The linear map $P_a : \mathbb{R}^k \rightarrow \mathbb{R}^{k-1}$ defined by

$$(y_1, \dots, y_k) \rightarrow (y_1 - a_1 y_k, \dots, y_{k-1} - a_{k-1} y_k), (y_1, \dots, y_k) \in \mathbb{R}^k,$$

has kernel consisting of the set of all scalar multiples of the vector $(a_1, \dots, a_{k-1}, 1)$: in fact, this map can be interpreted geometrically as projection along this vector onto the $(k - 1)$ -dimensional subspace of \mathbb{R}^k defined by $y_k = 0$. The map $(F^a)_* : TM_p \rightarrow \mathbb{R}^{k-1}$, $p \in M$, is noninjective if and only if the kernel of P_a has a nonzero intersection with $F_*(TM_p)$ and thus if and only if $(a_1, \dots, a_{k-1}, 1) \in F_*(TM_p)$. (Here the tangent space of \mathbb{R}^k at each point is identified with \mathbb{R}^k itself in the usual fashion).

Let U be any neighborhood of K ; then the mapping of

$$\{(p, v) \mid p \in U, v \in TM_p\}$$

into \mathbb{R}^k defined by $(p, v) \rightarrow F_*v$ is C^∞ and its domain is a manifold of dimension $2n < k$. Thus its image has measure zero in \mathbb{R}^k (see, for example, [19, p. 10]). Hence the image of $\{(p, v) \mid p \in K, v \in TM_p\}$ is of measure zero in \mathbb{R}^k . Note that this image is closed under scalar multiplication. Thus the fact that it has measure zero implies that its intersection with the hyperplane defined by $y_k = 1$ has $(k - 1)$ -dimensional measure zero in this hyperplane. Thus the set of

$$a = (a_1, \dots, a_{k-1})$$

such that F^a has $(F^a)_*$ noninjective at some point of K is of measure zero in \mathbb{R}^{k-1} . \square

LEMMA 6. – Let $F : M \rightarrow \mathbb{R}^k$, $k \geq 2n + 2$, be a C^∞ mapping with component functions F_1, \dots, F_k . Suppose that F is an embedding of K. Then the set of all $a = (a_1, \dots, a_{k-1}) \in \mathbb{R}^{k-1}$ such that the C^∞ mapping F^a of M into \mathbb{R}^{k-1} defined by

$$q \rightarrow (F_1(q) - a_1 F_k(q), \dots, F_{k-1}(q) - a_{k-1} F_k(q))$$

is an embedding of K has complement of measure zero in R^{k-1} .

Proof. — In view of Lemma 5, it suffices to show that the set of $a \in R^{k-1}$ such that $F^a|K$ is not injective has measure 0 in R^{k-1} . Now $F^a(p) = F^a(q)$, $p, q \in K$, $p \neq q$, if and only if $F(p) - F(q) \neq 0$ is a scalar multiple of $(a_1, \dots, a_{k-1}, 1)$. (Here $F(p)$, $F(q)$ are considered as vectors in R^k). Let U be an open neighborhood of K . The mapping $(U \times U - \{p \times p \mid p \in U\}) \times R \rightarrow R^k$ defined by

$$(p, q, \lambda) \rightarrow \lambda(F(p) - F(q)), p, q \in U, \lambda \in R$$

is C^∞ . Since its domain is a manifold of dimension $2n + 1 < k$, its image has measure zero in R^k (again by [19, p. 10]). A fortiori, the image of $(K \times K - \{p \times p \mid p \in K\}) \times R$ also has measure 0 in R^k . This image is closed under scalar multiplication. As in the proof of Lemma 5, its having measure zero in R^k then implies its intersection with the hyperplane $y_k = 1$ has $(k - 1)$ -dimensional measure zero. Thus the set of $A = (a_1, \dots, a_{k-1})$ such that F^a is noninjective on U has measure zero in R^{k-1} . \square

For each compact subset K of M , let $\mathcal{E}(K)$ = the set of mappings $F : M \rightarrow R^{2n+1}$ in $\mathcal{H}^{2n+1}(M) = \mathcal{H}(M) \times \dots \times \mathcal{H}(M)$ ($2n + 1$ factors) which are embeddings of K , i.e., $\mathcal{E}(K)$ is the set of C^∞ mappings of M into R^{2n+1} which are embeddings of K and which have their component functions harmonic on M .

LEMMA 7. — *If K is a compact subset of M , then $\mathcal{E}(K)$ is an open dense subset of $\mathcal{H}^{2n+1}(M)$.*

Proof. — The set of C^∞ mappings $M \rightarrow R^{2n+1}$ which are embeddings of K is open in $C^\infty(M) \times \dots \times C^\infty(M)$ ($2n + 1$ factors) in the product coarse C^∞ topology ([14 ; pp. 33-34, the proof of Theorem 3.10]). The openness of $\mathcal{E}(K)$ in $\mathcal{H}^{2n+1}(M)$ follows then from the fact that the inclusion $i : \mathcal{H}^{2n+1}(M) \rightarrow C^\infty(M) \times \dots \times C^\infty(M)$ is a homeomorphism onto its image by Lemma 1.

To prove that $\mathcal{E}(K)$ is dense in $\mathcal{H}^{2n+1}(M)$, note that according to Lemma 4 there is a mapping $H_1 \in \mathcal{H}^l(M)$ of M into R^l which is an embedding of K . For any $H \in \mathcal{H}^{2n+1}(M)$,

$$H \oplus H_1 : M \rightarrow R^{2n+1} \oplus R^l = R^{2n+1+l}$$

is an embedding of K belonging to \mathcal{H}^{2n+1} . Let the component functions of H be h_1, \dots, h_{2n+1} and those of H_1 be g_1, \dots, g_l . Then repeated application of Lemma 6 implies that there exist real numbers ξ_{ij} , $1 \leq i \leq 2n+1$, $1 \leq j \leq l$ which can be chosen arbitrarily close to 0 such that the mapping in $\mathcal{H}^{2n+1}(M)$ defined by

$$q \rightarrow (h_1(q) - \sum_{j=1}^l \xi_{1j} g_j(q), h_2(q) - \sum_{j=1}^l \xi_{2j} g_j, \dots, h_{2n+1}(q) - \sum_{j=1}^l \xi_{2n+1,j} g_j)$$

is an embedding of K (cf. [8 ; p. 35]). The fact that the ξ 's may be chosen arbitrarily small immediately implies that this embedding of K may be chosen as close (in the coarse C^∞ sense) to H as desired. \square

PROPOSITION 2. — *If $\mathcal{G}(M)$ = the set of mappings H in $\mathcal{H}^{2n+1}(M)$ which have the properties*

- a) H is injective on M and
- b) H_* is injective at every point of M ,

then $\mathcal{G}(M)$ is dense in $\mathcal{H}^{2n+1}(M)$.

Proof. — Let $\{K_i \mid i = 1, 2, 3, \dots\}$ be a sequence of compact subsets of M with $K_i \subset K_{i+1}$ for all i and $\bigcup_{i=1}^{+\infty} K_i = M$. Then

$$\mathcal{G}(M) = \bigcap_{i=1}^{+\infty} \mathcal{G}(K_i) :$$

Clearly $\mathcal{G}(M) \subset \bigcap_{i=1}^{+\infty} \mathcal{G}(K_i)$. On the other hand, if $H \in \bigcap_{i=1}^{+\infty} \mathcal{G}(K_i)$ then H is injective on M because any two points $p, q \in M$ lie together in some K_i and $H|_{K_i}$ is injective for all i by assumption. Similarly, such an H must have H_* injective at every point of M since every point of M lies in some K_i .

For each i , $\mathcal{G}(K_i)$ is an open dense subset of $\mathcal{H}^{2n+1}(M)$ (Lemma 7). Since the Baire category theorem holds in $\mathcal{H}^{2n+1}(M)$,

$$\mathcal{G}(M) = \bigcap_{i=1}^{+\infty} \mathcal{G}(K_i)$$

is dense in $\mathcal{H}^{2n+1}(M)$. \square

3. Proper mappings by harmonic functions.

A C^∞ mapping $f : M \rightarrow N$ of the connected non-compact Riemannian manifold M into a Riemannian manifold N is called *proper* if, for every compact subset $K \subset N$, $f^{-1}(K)$ is a compact subset of M . If $f : M \rightarrow N$ is an injective proper mapping then f is a homeomorphism onto its image : to verify this fact, observe that the proper mapping $f : M \rightarrow f(M)$ necessarily extends to continuous bijective mapping $f^+ : M^+ \rightarrow f(M)^+$ of the one point compactification of M to the one point compactification of $f(M)$, f^+ mapping the distinguished point (at ∞) of M^+ to the distinguished point of $f(M)^+$. The continuous bijective mapping f^+ , being a map of a compact space to a Hausdorff space, is necessarily a homeomorphism, and hence f itself is a homeomorphism.

In order to use the approximation theorem for harmonic functions (Lemma 2) to construct proper mappings of M into Euclidean space by harmonic functions, a general way to construct compact subsets K of M with $M - K$ having no closure compact components is needed :

DEFINITION. — *Let K be a compact subset of M . Then the compact hull of K , denoted by \hat{K} is the union of K and all the components of $M - K$ which have compact closure.*

The following result follows from standard topological arguments ([15 ; p. 234]) :

LEMMA 8. — *If K is any compact subset of M , then \hat{K} is compact and $(\hat{K})^\wedge = \hat{K}$. If K_1 and K_2 are compact subsets of M and $K_1 \subset K_2$, then $\hat{K}_1 \subset \hat{K}_2$.*

LEMMA 9. — *Let $\{C_i \mid i = 1, 2, 3, \dots\}$ be a collection of compact connected subsets of M such that*

a) $C_{i_1} \cap C_{i_2} = \phi$ if $i_1 \neq i_2$.

b) *if K is a compact subset of M then $C_i \cap K = \phi$ for all but a finite number of $i = 1, 2, 3, \dots$.*

Then there exists a collection $\{K_k \mid k = 1, 2, 3, \dots\}$ of compact

subsets of M such that for all k $\hat{K}_k = K_k$, $K_k \subset K_{k+1}$, for each i $K_k \cap C_i = \phi$ or $C_i \subset K_k$, and $\bigcup_{k=1}^{+\infty} \overset{\circ}{K}_k = M$.

Proof. — Let $\{K_k'' \mid k = 1, 2, 3, \dots\}$ be a collection of compact subsets of M with $K_k'' \subset \overset{\circ}{K}_{k+1}''$ and $\bigcup_{k=1}^{+\infty} K_k'' = M$. Define

$$K'_k = K_k'' \cup \left(\bigcup_{i \in I_k} C_i \right),$$

where $I_k = \{i \mid C_i \cap K_k'' \neq \phi\}$. Then set $K_k = (\overset{\circ}{K}_k')^{\wedge}$. Since $K_k'' \cap C_i = \phi$ for all but a finite number of i 's, I_k is a finite set and K'_k is compact. Hence by Lemma 8 K_k is compact. Clearly $K'_k \subset K'_{k+1}$ since $K_k'' \subset K''_{k+1}$. Hence again by Lemma 8 $K_k \subset K_{k+1}$.

Now suppose $C_{i_0} \cap K_k \neq \phi$. Then since the C_i 's are disjoint, either

a) $C_{i_0} \cap K_k'' \neq \phi$, in which case $C_{i_0} \subset K_k$

or

b) $C_{i_0} \cap (K_k - K'_k) \neq \phi$ but $C_{i_0} \cap K_k'' = \phi$, i.e., C_{i_0} intersects a closure compact component of $M - K'_k$ but does not intersect K_k'' or any of the C_i , $i \in I_k$. In this case, it follows from the connectedness of C_{i_0} that C_{i_0} is contained entirely in any component of $M - K'_k$ which it intersects. Thus C_{i_0} is contained in a closure compact component of $M - K'_k$ and hence is contained in K_k .

LEMMA 10. — Let $n = \text{dimension } M$. There exists a collection of $n + 1$ (countable) families $\{C_i^j \mid i = 1, 2, 3, \dots\}$, $j = 0, 1, \dots, n$, of compact connected subsets of M , with the following properties :

a) For each $j = 0, 1, \dots, n$, $C_{i_1}^j \cap C_{i_2}^j = \phi$ if $i_1 \neq i_2$.

b) For each $j = 0, 1, \dots, n$ and every compact subset K of M , $C_i^j \cap K = \phi$ for all but a finite number of i 's.

c) For all $j = 0, 1, \dots, n$ and $i = 1, 2, 3, \dots$ if U is any connected open subset of M containing C_i^j , then $U - C_i^j$ is connected.

d) $\bigcup_{j=0}^n \left(\bigcup_{i=1}^{+\infty} C_i^j \right) = M$.

Proof. — An appropriate collection can be obtained by the following standard construction (cf. [16, p. 61]) : There exists a (countable) locally finite n -dimensional simplicial complex homeomorphic to M ([3]). Let S be such a complex and $h : S \rightarrow M$ a homeomorphism. For each $j, j = 0, 1, \dots, n$, let $\sigma_1^j, \sigma_2^j, \dots$ be the j -dimensional simplexes of S numbered arbitrarily (without repetitions) and p_1^j, p_2^j, \dots be the barycenters of $\sigma_1^j, \sigma_2^j, \dots$, respectively. Define the C_i^j to be the image under h of the closed star of p_i^j in the second barycentric subdivision of S . It is then easily verified that the C_i^j have the required properties.

LEMMA 11. — *Let $\{C_i^j | j = 0, \dots, n, i = 1, 2, 3, \dots\}$ be the collection of compact connected subsets of M obtained in Lemma 10. Then for each $j \in \{0, \dots, n\}$ and each sequence $\{\alpha_i | i = 1, 2, 3, \dots\}$ of real numbers, there exists a harmonic function $f_j : M \rightarrow \mathbb{R}$ such that $f_j(p) \geq \alpha_i$ for all $p \in C_i^j$.*

Proof. — Consider a fixed $j \in \{0, \dots, n\}$. Let $\{K_k | k = 1, 2, 3, \dots\}$ be a collection of compact subsets of M satisfying the conclusions of Lemma 9 for C_i of that lemma = the present $C_i^j, i = 1, 2, 3, \dots$. Set $D_k, k = 1, 2, 3, \dots$, equal to the union of K_k and all those C_i^j contained in $K_{k+1} - K_k$. Then, for each $k = 1, 2, 3, \dots, D_k$ is a compact set ; also, $M - D_k$ has no components with compact closure, because no C_i^j separates any open set containing it and no component of $M - K_k$ has compact closure.

Define a sequence $\{\beta_l^j | l = 1, 2, 3, \dots\}$ of real numbers by

$$\beta_l^j = \max_{i \ni C_i^j \subset K_l} \alpha_i .$$

Then the function which is $\beta_1^j + 1$ on K_1 and $\beta_2^j + 1$ on each C_i^j contained in $K_2 - K_1$ is harmonic on D_1 since K_1 and these (finitely many) C_i^j are disjoint compact subsets on M . Thus by Lemma 2 there exists a harmonic function $h_1 : M \rightarrow \mathbb{R}$ which satisfies the inequalities

$$|h_1(p) - \beta_1^j - 1| < \frac{1}{2} \quad \text{for all } p \in K_1$$

$$|h_1(p) - \beta_2^j - 1| < \frac{1}{2} \quad \text{for all } p \in D_1 - K_1 .$$

For $k > 1$, define h_k inductively as follows : The function which is h_{k-1} on K_k and $\beta_{k+1}^j + 1$ on each C_i^j contained in $K_{k+1} - K_k$ is harmonic on D_k . Let $h_k : M \rightarrow \mathbb{R}$ be any harmonic function which satisfies the inequalities

$$|h_k(p) - h_{k-1}(p)| < \frac{1}{2^k} \quad \text{for all } p \in K_k$$

$$|h_k(p) - \beta_{k+1}^j - 1| < \frac{1}{2^k} \quad \text{for all } p \in D_k - K_k .$$

The existence of such an h_k is again guaranteed by Lemma 2.

The sequence $\{h_k : M \rightarrow \mathbb{R}\}$ converges uniformly on compact subsets of M : if K is any compact subset then, since $\bigcup_{k=1}^{+\infty} \overset{\circ}{K}_k = M$ and $K_k \subset K_{k+1}$, there exists a k_0 such that $K \subset K_{k_0}$. For all $p \in K$ and $k \geq k_0$, $|h_{k+1}(p) - h_k(p)| < \frac{1}{2^{k+1}}$ from which uniform convergence on K follows. Thus the sequence $\{h_k : M \rightarrow \mathbb{R}\}$ converges to a harmonic function on M by Lemma 1. Put

$$f_j = \lim_{k \rightarrow +\infty} h_k, \quad j \in \{0, \dots, n\} .$$

If C_i^j is contained in $D_{k_0} - K_{k_0}$, $k_0 \geq 1$, then for all $p \in C_i^j$

$$|h_{k_0}(p) - \beta_{k_0+1}^j - 1| < \frac{1}{2^{k_0}}$$

and for any $k > k_0$

$$|h_k(p) - h_{k_0}(p)| < \sum_{l=k_0+1}^k \frac{1}{2^l} .$$

Thus $f_j(p) \geq \beta_{k_0+1}^j$ for any $p \in C_i^j \subset D_{k_0} - K_{k_0}$. Similarly $f_j(p) \geq \beta_1^j$ for any $p \in K_1$. It follows then from the fact that every C_i^j is either contained in K_1 or in some $D_k - K_k$ that for all $i = 1, 2, 3, \dots$, $f_j(p) \geq \alpha_i$ for any $p \in C_i^j$. \square

PROPOSITION 3. — *There exist $n + 1$ harmonic functions on M such that the mapping of M into \mathbb{R}^{n+1} obtained from these functions as components is a proper mapping.*

Proof. — If for each j one takes the sequence α_i of Lemma 11 to be $1, 2, 3, \dots$ then the functions $f_j, j = 0, \dots, n$, of that lemma have the required property. The properness of the mapping of M into R^{n+1} defined by $p \rightarrow (f_0(p), \dots, f_n(p))$ follows easily from the facts that

a) for any compact set $K \subset M, C_i^j \cap K = \phi$ for all but a finite number of pairs $(i, j) \in Z \times \{0, \dots, n\}$.

$$b) \bigcup_{j=0}^n \bigcup_{i=1}^{+\infty} C_i^j = M$$

and

c) for each $j, \lim_{i \rightarrow +\infty} \left(\inf_{p \in C_i^j} f_j(p) \right) = +\infty$ (from Lemma 11).

THEOREM. — *Any connected, non-compact Riemannian manifold M has a proper embedding by harmonic function in R^{2n+1} .*

Proof. — Let $p \rightarrow (h_1(p), \dots, h_{2n+1}(p)) \in R^{2n+1}$ be the injective immersion of M in R^{2n+1} by harmonic functions, the existence of which is guaranteed by Proposition 2. Let $f_j, j = 0, \dots, n$ be $n + 1$ harmonic functions satisfying Proposition 3. Then the $3n + 2$ harmonic functions $f_0, \dots, f_n, h_1, \dots, h_{2n+1}$ together define a proper embedding of M in $R^{3n+2} : p \rightarrow (h_1(p), \dots, h_{2n+1}(p), f_0(p), \dots, f_n(p)) \in R^{3n+2}$. By a more careful choice of the functions f_j , it can be arranged that the projection technique of Lemma 7 of § 2 can be applied to obtain a proper embedding R^{2n+1} (cf. [10 ; pp. 121-125]). Specifically, given the functions h_1, \dots, h_{2n+1} , one defines for each $j \in \{0, \dots, n\}$ a sequence $\{\alpha_i^j | i = 1, 2, 3, \dots\}$ by

$$\alpha_i^j = \max \left(2 \sup_{p \in C_i^j} \left(\max_{1 < k \leq 2n+1} |h_k(p)| \right), i \right).$$

Then for each $j \in \{0, \dots, n\}$ let f_j be the harmonic function provided by Lemma 11 applied for that j and with the sequence

$\{\alpha_i^j | i = 1, 2, 3, \dots\}$ of the Lemma being the sequence

$$\{\alpha_i^j | i = 1, 2, 3, \dots\} \text{ just defined.}$$

Now the functions f_j taken together define a proper mapping of M into R^{n+1} since $f_j \geq i$ on C_i^j . Moreover, as before the functions

$f_0, \dots, f_n, h_1, \dots, h_{2n+1}$ together define a proper injective immersion of M into \mathbb{R}^{3n+2} , which is again as before necessarily an embedding of M . The argument by repeated application of Lemma 7 which was used to prove Lemma 8 shows that real numbers ξ_{rs} , $1 \leq r \leq 2n+1, 1 \leq s \leq n+1$ can be chosen as close to 0 as desired and such that the mapping $F : M \rightarrow \mathbb{R}^{2n+1}$

$$q \xrightarrow{F} (f_0(q) - \sum_{s=1}^{n+1} \xi_{1s} h_{n+s}(q), f_1(q) - \sum_{s=1}^{n+1} \xi_{2s} h_{n+s}(q), \dots, f_n(q) - \sum_{s=1}^{n+1} \xi_{n+1,s} h_{n+s}(q), h_1(q) - \sum_{s=1}^{n+1} \xi_{n+2,s} h_{n+s}(q), \dots, h_n(q) - \sum_{s=1}^{n+1} \xi_{2n+1,s} h_{n+1}(q))$$

is an injective immersion of M . Choose such ξ_{rs} with $|\xi_{rs}| < \frac{1}{n+1}$ for all r, s . Then for $q \in C_i^j$ the fact that $f_j(q) \geq \alpha_i^j$ implies that

$$f_j(q) - \sum_{s=1}^{n+1} \xi_{j+1,s} h_{n+s}(q) \geq \alpha_i^j - (n+1) \frac{1}{n+1} \sup_{p \in C_i^j} \max_{1 \leq k \leq 2n+1} |h_k(p)| \geq \alpha_i^j - \frac{1}{2} \alpha_i^j = \frac{1}{2} \alpha_i^j.$$

That F is a proper mapping now follows from the fact that for each $j \alpha_i^j \rightarrow +\infty$ as $i \rightarrow +\infty$ together with the fact that only finitely many of the $C_i^j, j = 0, \dots, n, i = 1, 2, 3, \dots$ have a nonempty intersection with any given compact subset of M .

COROLLARY. — *On any connected non-compact Riemannian manifold M there exists a nonnegative C^∞ strictly subharmonic function $\tau : M \rightarrow \mathbb{R}$ such that $\tau^{-1}([0, a])$ is a compact subset of M for all $a \in \mathbb{R}$. (A function is by definition strictly subharmonic if its Laplacian is everywhere positive).*

Proof. — Let $H : M \rightarrow \mathbb{R}^m$ be a proper embedding of M by harmonic component functions h_1, \dots, h_m . Set $\tau = \sum_{i=1}^m h_i^2$. Then τ

clearly has the property that $\tau^{-1}([0, a])$ is compact by all $a \in \mathbb{R}$ since $\{(x_1, \dots, x_m) \in \mathbb{R}^m \mid x_1^2 + \dots + x_m^2 \leq a\}$ is compact and H is proper by hypothesis. Moreover, τ is strictly subharmonic : Let (x_1, \dots, x_n) be Riemannian normal coordinates at a point $p \in M$. Then

$$\begin{aligned} \Delta\tau|_p &= \sum_{l=1}^n \frac{\partial^2 \tau}{\partial x_l^2} \Big|_p = \sum_{l=1}^n \left[\frac{\partial^2}{\partial x_l^2} \left(\sum_{i=1}^m h_i^2 \right) \right] \Big|_p \\ &= \sum_{i=1}^m 2 \left(\sum_{l=1}^n \left(\frac{\partial h_i}{\partial x_l} \Big|_p \right)^2 + h_i \sum_{l=1}^n \frac{\partial^2 h_i}{\partial x_l^2} \Big|_p \right) \\ &= 2 \sum_{l=1}^n \sum_{i=1}^m \left(\frac{\partial h_i}{\partial x_l} \Big|_p \right)^2 \text{ since } \sum_{l=1}^n \frac{\partial^2 h_i}{\partial x_l^2} \Big|_p \\ &= \Delta h_i|_p = 0 \text{ for all } i. \end{aligned}$$

Since some set of n of the functions h_i form a local coordinate system in a neighborhood of p , not all of the derivatives $\frac{\partial h_i}{\partial x_l} \Big|_p, i = 1, \dots, m, l = 1, \dots, n$ are zero. Thus $\Delta\tau|_p > 0$. \square

The existence of a nonnegative C^∞ strictly subharmonic function with compact sublevel sets on any connected non-compact Riemannian manifold is the harmonic function analogue of the existence of a C^∞ strictly plurisubharmonic function with compact sublevel sets on a Stein manifold. The existence of such a function on a Stein manifold follows from the existence of a proper holomorphic embedding of the Stein manifold in some \mathbb{C}^m by a process whose analogue in the harmonic case is the process by which the Corollary just given is derived from the previous theorem.

The subharmonic functions which satisfy the conditions of the Corollary are subject to certain geometric restrictions. For example, if M is complete but has finite volume then such a function cannot be (uniformly) Lipschitz continuous on all of M ([9]).

BIBLIOGRAPHY

- [1] N. ARONSZAJN, A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order, *J. Math. Pures Appl.*, 36 (1957), 235-249.
- [2] S. BOCHNER, Analytic mappings of compact Riemann spaces into Euclidean space, *Duke Math. J.*, 3 (1937), 339-354.
- [3] S.S. CAIRNS, On the triangulation of regular loci, *Ann. of Math.*, 35 (1934), 579-587.
- [4] S.-S. CHERN, An elementary proof of the existence of isothermal parameters on a surface, *Proc. Amer. Math. Soc.*, 6 (1955), 771-782.
- [5] H.O. CORDES, Über die Bestimmtheit der Lösungen elliptischer Differentialgleichungen durch Anfangsvorgaben, *Nachr. Akad. Wiss. Göttingen, Math. Phys. Kl. IIa*, no 11 (1956), 239-258.
- [6] W. FELLER, Über die Lösungen der linearen partiellen Differentialgleichungen zweiter Ordnung von elliptischen Typus, *Math. Ann.*, 102 (1930), 633-649.
- [7] H. GRAUERT, On Levi's problem and the imbedding of real-analytic manifolds, *Ann. of Math.*, 68 (1958), 460-472.
- [8] R.E. GREENE, Isometric Embedding of Riemannian and Pseudo-Riemannian Manifolds, *Memoir 97, Amer. Math. Soc.*, 1970.
- [9] R.E. GREENE, and H. WU, Integrals of subharmonic functions on manifolds of nonnegative curvature (to appear).
- [10] L. HÖRMANDER, An Introduction to Complex Analysis in Several variables. D. Van Nostrand, Princeton, New Jersey, 1966.
- [11] P.D. LAX, A stability theorem for abstract differential equations and its application to the study of the local behavior of solutions of elliptic equations, *Comm. Pur Appl. Math.*, 9 (1956), 747-766.
- [12] B. MALGRANGE, Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution, *Annales de l'Institut Fourier*, 6 (1955-56), 271-355.

- [13] C.B. MORREY, The analytic imbedding of abstract real-analytic manifolds, *Ann. of Math.*, 68 (1958), 159-201.
- [14] J.R. MUNKRES, Elementary Differential Topology, Princeton University Press, Princeton, New Jersey, 1966.
- [15] R. NARASIMHAN, Analysis on Real and Complex Manifolds, North-Holland, Amsterdam, 1968.
- [16] J.F. NASH, The imbedding problem for Riemannian manifolds, *Ann. of Math.*, 63 (1956), 20-63.
- [17] A. PLIŚ, A smooth linear elliptic differential equation without a solution in a sphere, *Comm. Pure Appl. Math.*, 14 (1961), 599-617.
- [18] M.H. PROTTER, Unique continuation for elliptic equations, *Trans. Amer. Math. Soc.*, 95 (1960), 81-91.
- [19] G. de RHAM, Variétés différentiables, Hermann, Paris, 1955.
- [20] H.L. ROYDEN, The analytic approximation of differentiable mappings, *Math. Ann.*, 139 (1960), 171-179.
- [21] H. WHITNEY, Analytic coordinate systems and arcs in a manifold, *Ann. of Math.*, 38 (1937), 809-818.
- [22] H. WU, Remarks on the first main theorem in equidistribution theory. II, *J. Differential Geometry*, 2 (1968), 369-384.
- [23] L. BERS, Local behavior of solutions of general linear elliptic equations, *Comm. Pure Appl. Math.*, 8 (1955), 473-496.
- [24] L. BERS, F. JOHN, and M. SCHECHTER, Partial Differential Equations, Interscience Publishers, New York, 1964.
- [25] B. MALGRANGE, Plongement des variétés analytiques-réelles, *Bull. Soc. Math. France*, 85 (1957), 101-113.

Manuscrit reçu le 25 février 1974

Accepté par B. Malgrange.

R.E. GREENE,
Department of Mathematics
University of California
Los Angeles, Calif. 90024 (USA).

H. WU,
Department of Mathematics
University of California
Berkeley, Calif. 94720 (USA).