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## A CHARACTERIZATION OF WEAKLY SEQUENTIALLY COMPLETE BANACH LATTICES

by A. W. Wickstead

Meyer-Nieberg ([5], Korollar I.8) has given a number of properties of a Banach lattice,  $E$ , that are equivalent to weak sequential completeness of the underlying Banach space. Among these is that  $E$  is a band in  $E^{**}$ ; and from [4b], Theorem 39.1 this is equivalent to  $E = (E^*)^\times$ , the space of order bounded order continuous linear functionals on  $E^*$ , the (ordered) Banach dual of  $E$  (we follow [5] for terminology). We give a further equivalence that was first proved for  $L^1(\mu)$  ( $\mu$  a  $\sigma$ -finite measure) by J. P. R. Christensen ([2], Theorem 4). Our tools include a representation theorem for a class of vector lattice due to Fremlin ([3], Theorem 6) and the following theorem of Christensen ([2], Theorem 2).

**THEOREM 1.** — *Let  $N$  be the natural numbers and  $K = \{0, 1\}^N$  with the product topology. If  $\varphi$  is a real valued finitely additive set function on the subsets of  $N$  it may be regarded in an obvious way as a function on  $K$ . If  $\varphi$  is measurable as such a function then  $\varphi$  is countably additive as a set function.*

**THEOREM 2.** — *A Banach lattice  $E$  is weakly sequentially complete if and only if every  $\sigma(E^*, E)$ -Borel measurable linear functional on  $E^*$  is  $\sigma(E^*, E)$ -continuous.*

As the sequential  $\sigma(E^{**}, E^*)$ -closure of  $E$  in  $E^{**}$  consists of  $\sigma(E^*, E)$ -Borel measurable linear functionals «if» is obvious.

Conversely suppose  $E$  is weakly sequentially complete, and  $L$  is a  $\sigma(E^*, E)$ -Borel measurable linear functional

on  $E^*$ . We must show that  $L$  is induced by an element of  $E$ . As  $E$  is weakly sequentially complete it follows from [5], Korollar I.8 and the remark above that this is equivalent to showing that  $L$  is order bounded and order continuous, i.e. if the net  $(f_\gamma)$  in  $E^*$  is directed downward to 0 then  $L(f_\gamma) \rightarrow 0$ .

$L$  is certainly norm measurable and hence ([1], Theorem 2) norm bounded. As  $E^*$  is a Banach lattice,  $L$  is certainly order bounded, so we must show  $L$  is order continuous.

Without loss of generality we may suppose  $f_0 \geq f_\gamma \geq 0$  for all  $\gamma$  and restrict our attention to the band,  $B$ , in  $E^*$  generated by  $f_0$  which is  $\sigma(E^*, E)$ -closed, as  $E$  has an order continuous norm (this is equivalent to  $E$  being an ideal in  $E^{**}$  using [4b], Theorem 39.1, and this is certainly true as  $E$  is a band in  $E^{**}$ ) by [4a], Theorem 36.2. The topology  $\sigma(E^*, E)$  on  $B$  is the same as  $\sigma(B, E/B^0)$  where  $B^0$  is the annihilator of  $B$  in  $E$ , so we may limit our attention to the Banach lattice  $E/B^0$  and its Banach dual  $B$ ; i.e. we limit our attention to the case that  $E^*$  has a weak order unit.

Using [3], Theorem 6, we may find a locally compact Hausdorff space  $\Sigma$  and a Radon measure  $\mu$  on  $\Sigma$  such that  $E^*$  is vector lattice isomorphic to a lattice ideal in  $M(\mu)$ , the space of all equivalence classes of  $\mu$ -measurable extended real valued functions on  $\Sigma$ . We identify  $E^*$  with this ideal. Also by [3], Theorem 7,  $E = E^{**}$  may be identified with the ideal  $\{x \in M(\mu) : \int_\Sigma f x d\mu < \infty \text{ for all } f \in E^*\}$ . Further as  $E^*$  has a weak order unit we may suppose  $1_\Sigma \in E^*$ , and hence  $\chi_A \in E^*$  for all Borel sets  $A \subset \Sigma$ .

Fix  $\alpha_i \in \mathbf{R}_+$  and  $A_i$  Borel subsets of  $\Sigma$  ( $i = 1, 2, \dots$ ), such that  $\sum_{i=1}^{\infty} \alpha_i \chi_{A_i} \in E^*$ . We claim

$$L(\sum \alpha_i \chi_{A_i}) = \sum L(\alpha_i \chi_{A_i}).$$

Define  $\varphi$  on subsets  $M$  of  $N$  by  $\varphi(M) = L(\sum_{i \in M} \alpha_i \chi_{A_i})$ , which is defined, as  $E^*$  is an ideal in  $M(\mu)$ . Clearly  $\varphi$  is finitely additive as  $L$  is linear. The map

$$\theta : M \longmapsto \sum_{i \in M} \alpha_i \chi_{A_i}$$

is continuous for the  $\sigma(E^*, E)$  topology on  $E^*$  and the product topology on  $K$ . This is because if  $x \in E$  then

$$\theta(N)(|x|) = \int_{\Sigma} \left( \sum_N \alpha_i \chi_{A_i} \right) |x| d\mu$$

is finite, so given  $\varepsilon > 0$  we can find a finite set  $F \subset N$  with  $\left| \int_{\Sigma} \left( \sum_{N \setminus F} \alpha_i \chi_{A_i} \right) x d\mu \right| < \varepsilon$ . If  $M_{\gamma}, M \subset N$  and  $M_{\gamma} \rightarrow M$  for the product topology we can find  $\gamma_0$  such that  $\gamma \geq \gamma_0$  implies  $M_{\gamma} \cap F = M \cap F$ . Thus  $\gamma \geq \gamma_0$  implies

$$|\theta(M_{\gamma})(x) - \theta(M)(x)| < \varepsilon;$$

i.e.  $\theta(M_{\gamma}) \rightarrow \theta(M)$  for  $\sigma(E^*, E)$ . Hence  $\varphi = L \circ \theta$  is measurable as a real valued function on  $K$ , so is countably additive as a set function on  $N$ , by Theorem 1, which proves the claim.

Define  $\nu$  on the Borel sets in  $\Sigma$  by  $\nu(A) = L(\chi_A)$ , which is meaningful as  $\chi_A \in E^*$ . If  $A_i$  are disjoint Borel sets then  $\chi_{\cup A_i} = \sum \chi_{A_i}$ , and the above claim (with  $\alpha_i = 1$ ) shows that  $\nu$  is countably additive. If  $\mu(A) = 0$  then  $\chi_A = 0$  (as an element of  $E^*$ ) so  $\nu(A) = L(\chi_A) = 0$ . We may thus apply the Radon-Nikodym theorem to find  $y \in L^1(\mu)$  with  $\nu(A) = \int_A y d\mu$  for all Borel subsets  $A$  of  $\Sigma$  ( $y$  is integrable as  $f_1 = \chi_{\{\sigma \in \Sigma: \varphi(\sigma) > 0\}} \in E^*$  and

$$L(f_1) = \int_{\Sigma} y^+ d\mu < \infty).$$

We must next show that  $L(f) = \int_{\Sigma} fy d\mu$  for all  $f \in E^*$ . This will show that  $y \in E^{**}$ , and hence that  $L$  is order continuous. If  $f \in E_+^*$  (it is no loss of generality to assume this) and  $\varepsilon > 0$  we may find Borel sets  $A_i$  and  $\alpha_i \geq 0$  with  $\sum \alpha_i \chi_{A_i} \leq f \leq \sum \alpha_i \chi_{A_i} + \varepsilon 1_{\Sigma}$ , and hence (as  $E^*$  is a Banach lattice)  $\|\sum \alpha_i \chi_{A_i} - f\| \leq \varepsilon \|1_{\Sigma}\|$ . We have

$$\begin{aligned} L(\sum \alpha_i \chi_{A_i}) &= \sum \alpha_i L(\chi_{A_i}) = \sum \alpha_i \int_{\Sigma} \chi_{A_i} d\nu \\ &= \sum \alpha_i \int_{\Sigma} y \chi_{A_i} d\mu = \int_{\Sigma} (\sum \alpha_i \chi_{A_i}) y d\mu \end{aligned}$$

(this last equality follows from Lebesgues' dominated conver-

gence theorem). As we have seen,  $L$  is bounded, so

$$\left| \int_{\Sigma} fy \, d\mu - L(f) \right| \leq \left| \int_{\Sigma} fy \, d\mu - \int_{\Sigma} (\Sigma \alpha_i \chi_{A_i}) y \, d\mu \right| \\ + |L(\Sigma \alpha_i \chi_{A_i}) - L(f)| \leq \varepsilon \|y\|_1 + \varepsilon \|L\| \|1_{\Sigma}\|.$$

Thus  $L(f) = \int_{\Sigma} fy \, d\mu$  for all  $f \in E_+^*$ , completing the proof.

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