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A CHARACTERIZATION OF WEAKLY SEQUENTIALLY COMPLETE BANACH LATTICES

by A. W. Wickstead

Meyer-Nieberg ([5], Korollar I.8) has given a number of properties of a Banach lattice, E , that are equivalent to weak sequential completeness of the underlying Banach space. Among these is that E is a band in E^{**} ; and from [4b], Theorem 39.1 this is equivalent to $E = (E^*)^\times$, the space of order bounded order continuous linear functionals on E^* , the (ordered) Banach dual of E (we follow [5] for terminology). We give a further equivalence that was first proved for $L^1(\mu)$ (μ a σ -finite measure) by J. P. R. Christensen ([2], Theorem 4). Our tools include a representation theorem for a class of vector lattice due to Fremlin ([3], Theorem 6) and the following theorem of Christensen ([2], Theorem 2).

THEOREM 1. — *Let N be the natural numbers and $K = \{0, 1\}^N$ with the product topology. If φ is a real valued finitely additive set function on the subsets of N it may be regarded in an obvious way as a function on K . If φ is measurable as such a function then φ is countably additive as a set function.*

THEOREM 2. — *A Banach lattice E is weakly sequentially complete if and only if every $\sigma(E^*, E)$ -Borel measurable linear functional on E^* is $\sigma(E^*, E)$ -continuous.*

As the sequential $\sigma(E^{**}, E^*)$ -closure of E in E^{**} consists of $\sigma(E^*, E)$ -Borel measurable linear functionals «if» is obvious.

Conversely suppose E is weakly sequentially complete, and L is a $\sigma(E^*, E)$ -Borel measurable linear functional

on E^* . We must show that L is induced by an element of E . As E is weakly sequentially complete it follows from [5], Korollar I.8 and the remark above that this is equivalent to showing that L is order bounded and order continuous, i.e. if the net (f_γ) in E^* is directed downward to 0 then $L(f_\gamma) \rightarrow 0$.

L is certainly norm measurable and hence ([1], Theorem 2) norm bounded. As E^* is a Banach lattice, L is certainly order bounded, so we must show L is order continuous.

Without loss of generality we may suppose $f_0 \geq f_\gamma \geq 0$ for all γ and restrict our attention to the band, B , in E^* generated by f_0 which is $\sigma(E^*, E)$ -closed, as E has an order continuous norm (this is equivalent to E being an ideal in E^{**} using [4b], Theorem 39.1, and this is certainly true as E is a band in E^{**}) by [4a], Theorem 36.2. The topology $\sigma(E^*, E)$ on B is the same as $\sigma(B, E/B^0)$ where B^0 is the annihilator of B in E , so we may limit our attention to the Banach lattice E/B^0 and its Banach dual B ; i.e. we limit our attention to the case that E^* has a weak order unit.

Using [3], Theorem 6, we may find a locally compact Hausdorff space Σ and a Radon measure μ on Σ such that E^* is vector lattice isomorphic to a lattice ideal in $M(\mu)$, the space of all equivalence classes of μ -measurable extended real valued functions on Σ . We identify E^* with this ideal. Also by [3], Theorem 7, $E = E^{**}$ may be identified with the ideal $\{x \in M(\mu) : \int_\Sigma f x d\mu < \infty \text{ for all } f \in E^*\}$. Further as E^* has a weak order unit we may suppose $1_\Sigma \in E^*$, and hence $\chi_A \in E^*$ for all Borel sets $A \subset \Sigma$.

Fix $\alpha_i \in \mathbf{R}_+$ and A_i Borel subsets of Σ ($i = 1, 2, \dots$), such that $\sum_{i=1}^{\infty} \alpha_i \chi_{A_i} \in E^*$. We claim

$$L(\sum \alpha_i \chi_{A_i}) = \sum L(\alpha_i \chi_{A_i}).$$

Define φ on subsets M of N by $\varphi(M) = L(\sum_{i \in M} \alpha_i \chi_{A_i})$, which is defined, as E^* is an ideal in $M(\mu)$. Clearly φ is finitely additive as L is linear. The map

$$\theta : M \longmapsto \sum_{i \in M} \alpha_i \chi_{A_i}$$

is continuous for the $\sigma(E^*, E)$ topology on E^* and the product topology on K . This is because if $x \in E$ then

$$\theta(N)(|x|) = \int_{\Sigma} \left(\sum_N \alpha_i \chi_{A_i} \right) |x| d\mu$$

is finite, so given $\varepsilon > 0$ we can find a finite set $F \subset N$ with $\left| \int_{\Sigma} \left(\sum_{N \setminus F} \alpha_i \chi_{A_i} \right) x d\mu \right| < \varepsilon$. If $M_{\gamma}, M \subset N$ and $M_{\gamma} \rightarrow M$ for the product topology we can find γ_0 such that $\gamma \geq \gamma_0$ implies $M_{\gamma} \cap F = M \cap F$. Thus $\gamma \geq \gamma_0$ implies

$$|\theta(M_{\gamma})(x) - \theta(M)(x)| < \varepsilon;$$

i.e. $\theta(M_{\gamma}) \rightarrow \theta(M)$ for $\sigma(E^*, E)$. Hence $\varphi = L \circ \theta$ is measurable as a real valued function on K , so is countably additive as a set function on N , by Theorem 1, which proves the claim.

Define ν on the Borel sets in Σ by $\nu(A) = L(\chi_A)$, which is meaningful as $\chi_A \in E^*$. If A_i are disjoint Borel sets then $\chi_{\cup A_i} = \sum \chi_{A_i}$, and the above claim (with $\alpha_i = 1$) shows that ν is countably additive. If $\mu(A) = 0$ then $\chi_A = 0$ (as an element of E^*) so $\nu(A) = L(\chi_A) = 0$. We may thus apply the Radon-Nikodym theorem to find $y \in L^1(\mu)$ with $\nu(A) = \int_A y d\mu$ for all Borel subsets A of Σ (y is integrable as $f_1 = \chi_{\{\sigma \in \Sigma: \varphi(\sigma) > 0\}} \in E^*$ and

$$L(f_1) = \int_{\Sigma} y^+ d\mu < \infty).$$

We must next show that $L(f) = \int_{\Sigma} fy d\mu$ for all $f \in E^*$. This will show that $y \in E^{**}$, and hence that L is order continuous. If $f \in E^*_+$ (it is no loss of generality to assume this) and $\varepsilon > 0$ we may find Borel sets A_i and $\alpha_i \geq 0$ with $\sum \alpha_i \chi_{A_i} \leq f \leq \sum \alpha_i \chi_{A_i} + \varepsilon 1_{\Sigma}$, and hence (as E^* is a Banach lattice) $\|\sum \alpha_i \chi_{A_i} - f\| \leq \varepsilon \|1_{\Sigma}\|$. We have

$$\begin{aligned} L(\sum \alpha_i \chi_{A_i}) &= \sum \alpha_i L(\chi_{A_i}) = \sum \alpha_i \int_{\Sigma} \chi_{A_i} d\nu \\ &= \sum \alpha_i \int_{\Sigma} y \chi_{A_i} d\mu = \int_{\Sigma} (\sum \alpha_i \chi_{A_i}) y d\mu \end{aligned}$$

(this last equality follows from Lebesgues' dominated conver-

gence theorem). As we have seen, L is bounded, so

$$\left| \int_{\Sigma} fy \, d\mu - L(f) \right| \leq \left| \int_{\Sigma} fy \, d\mu - \int_{\Sigma} (\Sigma \alpha_i \chi_{A_i}) y \, d\mu \right| \\ + |L(\Sigma \alpha_i \chi_{A_i}) - L(f)| \leq \varepsilon \|y\|_1 + \varepsilon \|L\| \|1_{\Sigma}\|.$$

Thus $L(f) = \int_{\Sigma} fy \, d\mu$ for all $f \in E_+^*$, completing the proof.

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