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Bahman Saffari
R. C. Vaughan

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ON THE FRACTIONAL PARTS OF x/n AND RELATED SEQUENCES. I

by B. SAFFARI and R. C. VAUGHAN

1. Introduction.

1. Throughout this paper $\{x\} = x - [x]$ denotes the fractional part of the real number x. We write $\|x\| = \min_{k \in \mathbf{Z}} |x - k|$ and $e(x) = e^{2\pi i x}$.

Also, the implied constants in the O symbol of Landau and the $\gg$ and $\ll$ symbols of Vinogradov are absolute.

Finally, by a distribution function we always mean a distribution function in the sense of probability theory, defined on the real line.

2. Let (x_n) be a sequence of real numbers. The usual study of the distribution modulo 1 of (x_n) is essentially that of the distribution of the sequence $(e(x_n))$ on the circle $\mathbf{T}$. The main problems are those of investigating

(i) the existence of the asymptotic (or limit) distribution measure

$$\mu = \lim_{k \to \infty} \mu_k \tag{1.1}$$

where

$$\mu_k = \frac{1}{k} \sum_{n=1}^{k} \delta_{e(x_n)} \tag{1.2}$$

with δ_v denoting the Dirac measure at $v \in \mathbf{T}$, and

(ii) the size of the discrepancy

$$\sup_{\omega} |\mu_k(\omega) - \mu(\omega)| \tag{1.3}$$

where ω runs through those arcs of $\mathbf{T}$ whose end points have μ-measure zero.

It is classical that the existence of μ *together* with the assumption that the point $1 \in \mathbf{T}$ has μ-measure zero is equivalent to the existence of a distribution function F such that

$$(1.4) \qquad F(0+) = 0, \qquad F(1-) = 1$$

and

$$(1.5) \qquad F(\alpha) = \lim_{k\to\infty} \frac{1}{k} A([0, \alpha), k, (x_n))$$

at every α at which F is continuous, the counting function

$$(1.6) \quad A([\alpha, \beta), k, (x_n)) = \operatorname{Card}\{n : 1 \leqslant n \leqslant k, \alpha \leqslant \{x_n\} < \beta\}$$

being here defined for all real numbers α and β. The conditions (1.4) mean that F is continuous at 0 and 1, and imply that F is constant on the intervals $(-\infty, 0]$ and $[1, \infty)$. In this case F is called the asymptotic (or limit) distribution function modulo 1 of the sequence (x_n), and the discrepancy (1.3) is equal to

$$(1.7) \qquad \sup_{0\leqslant\alpha<\beta\leqslant1} \left| \frac{1}{k} A([\alpha, \beta), k, (x_n)) - (F(\beta) - F(\alpha)) \right|$$

where α and β run through the continuity points of F.

In some situations it may be more appropriate to consider the existence of the A-asymptotic distribution function modulo 1, namely the existence (outside a countable set), *and* the continuity at $\alpha = 0$ and $\alpha = 1$, of

$$(1.8) \qquad \lim_{k\to\infty} \sum_{n=1}^{k} a_{k,n} c_\alpha(x_n)$$

where

$$(1.9) \qquad c_\alpha(u) = \begin{cases} 1 & 0 \leqslant \{u\} < \alpha \\ 0 & \text{otherwise} \end{cases}$$

is the characteristic function modulo 1 of $[0, \alpha)$, and $A = (a_{k,n})$ is a positive Toeplitz matrix. Here by a positive Toeplitz matrix we mean that

$$a_{k,n} \geqslant 0, \sum_{n=1}^{\infty} a_{k,n} < \infty \qquad \text{and} \qquad \lim_{k\to\infty} \sum_{n=1}^{\infty} a_{k,n} = 1.$$

3. The sequence (x_n) is, of course, independant of k. Our object is to investigate the distribution modulo 1 of $xh(n)$ with x a large real number, $h(n)$ an arithmetical function, and the integer n belonging to $S \cap [1, k]$ where $S \subset \mathbf{N}$ and k depends on x. For our purposes it is somewhat more convenient to replace k by a real parameter y. We call $\mathscr{A} = (a_n(y) : y \in [1, \infty),\ n = 1, 2, \ldots)$ a positive Toeplitz transformation if $a_n(y) \geqslant 0$ for all n and y, $\sum_{n=1}^{\infty} a_n(y) < \infty$ for every y, and $\lim_{y\to\infty} \sum_{n=1}^{\infty} a_n(y) = 1$. We are particularly interested in the special case where the $a_n(y)$ are the simple Riesz means (R, λ_n) given by

$$\lambda_n \geqslant 0 \ (n = 1, 2, \ldots), \qquad \lambda_1 > 0 \tag{1.10}$$

and

$$a_n(y) = \begin{cases} \lambda_n \Big/ \sum_{m \leqslant y} \lambda_m & (m \leqslant y) \\ 0 & (m > y) \end{cases} \tag{1.11}$$

which we assume henceforward, although several of our proofs go through in the general case (see Appendix). Let

$$\Phi_{x,y}(\alpha, h) = \sum_{n=1}^{\infty} a_n(y) c_\alpha(xh(n)). \tag{1.12}$$

A good deal of our attention will be taken up with $h(n) = 1/n$ and we write

$$\Phi_{x,y}(\alpha) = \sum_{n=1}^{\infty} a_n(y) c_\alpha(x/n). \tag{1.13}$$

The problems arising from the study of $\Phi_{x,y}(\alpha)$ as x and $y = y(x)$ tend together to infinity are closely related to the Dirichlet divisor problem.

If there exists a distribution function Φ_h such that

$$\Phi_h(0+) = 0, \qquad \Phi_h(1-) = 1 \tag{1.14}$$

and

$$\Phi_h(\alpha) = \lim_{x\to\infty} \Phi_{x,y(x)}(\alpha, h) \tag{1.15}$$

at every α at which Φ_h is continuous, then we call Φ_h the

$\mathscr{A}$-asymptotic distribution function modulo 1. This situation is equivalent to the existence on the circle $\mathbf{T}$ of the $\mathscr{A}$-limit (or $\mathscr{A}$-asymptotic) distribution measure

$$\nu = \lim_{x\to\infty} \sum_{n=1}^{\infty} a_n(y)\, \delta_{e(xh(n))} \tag{1.16}$$

together with the fact that the point $1 \in \mathbf{T}$ has ν-measure zero. However, if there exists no distribution function Φ_h satisfying both (1.14) and (1.15), then it is more appropriate to investigate the distribution modulo 1 of $xh(n)$ via (1.16).

4. Our interest in this problem arose from investigating the asymptotic behaviour of

$$\sum_{n\leqslant y} c_\alpha(x/n).$$

During our investigation it became obvious that there were methods which could be applied in a much more general situation. In this paper we present these methods, deferring to the sequel the study of special methods.

As an example of the application of Theorem 2, consider a subset A of $\mathbf{N}^*$ such that the counting function

$$\mathrm{A}(x) = \sum_{\substack{a\leqslant x\\ a\in \mathrm{A}}} 1$$

satisfies

$$\mathrm{A}(x) = x^\sigma \mathrm{L}(x)$$

where σ is a constant with $0 < \sigma \leqslant 1$ and L is a slowly varying function, that is

$$\lim_{x\to\infty} \frac{\mathrm{L}(cx)}{\mathrm{L}(x)} = 1$$

for any positive constant c. Then

$$\lim_{x\to\infty} \frac{1}{\mathrm{A}(x)} \sum_{\substack{a\leqslant x\\ a\in \mathrm{A}}} c_\alpha(x/a) = \sum_{n=1}^{\infty} (n^{-\sigma} - (n+\alpha)^{-\sigma}). \tag{1.17}$$

Moreover, there exists a function $y_0(x)$ such that if $y > y_0(x)$

and $y = o(x)$ as $x \to \infty$, then

$$\lim_{x\to\infty} \frac{1}{A(y)} \sum_{\substack{a \leqslant y \\ a \in A}} c_\alpha(x/a) = \alpha. \tag{1.18}$$

Relation (1.18) means that the fractional parts $\{x/a\}$, where a runs over $[0, y] \cap A$, are asymptotically uniformly distributed, whereas (1.17) means that if a runs over the whole of $[0, x] \cap A$, then the $\{x/a\}$ have the asymptotic distribution function

$$\sum_{n=1}^{\infty} (n^{-\sigma} - (n + \alpha)^{-\sigma}).$$

2. Theorems and proofs.

1. We first of all state a theorem which gives a sufficient condition for the (R, λ_n)-asymptotic distribution to be uniform. This is essentially due to Erdös and Turan [1], [2] and is a finite form of Weyl's criterion. It is also possible, of course, to give a necessary condition corresponding to Weyl's criterion, and to give results when the asymptotic distribution is non-uniform but continuous, but we have no applications in mind for these.

Theorem 1 is somewhat divorced from the following theorems. However, it clearly applies to the general situation. As an application we have in mind the case

$$h(n) = \log n. \tag{2.1}$$

THEOREM 1. — *Let the discrepancy* $D_{x,y}(h)$ *be defined by*

$$D_{x,y}(h) = \sup_{0 \leqslant \alpha < \beta \leqslant 1} |\Phi_{x,y}(\beta, h) - \Phi_{x,y}(\alpha, h) - (\beta - \alpha)|. \tag{2.2}$$

Then, for any positive integer m,

$$D_{x,y}(h) < \frac{6}{m+1} + \frac{4}{\pi} \sum_{k=1}^{m} \left(\frac{1}{k} - \frac{1}{m+1}\right) \left| \sum_{n=1}^{\infty} a_n(y) e(kxh(n)) \right|. \tag{2.3}$$

Theorem 1 is a generalization of Theorem 2.2.5 of Kuipers and Niederreiter [3], and can be proved in exactly the same way.

2. The following theorem (together with the observations made in Lemmas 2, 3, 4) shows that the (R, λ_n) asymptotic distribution function modulo 1 of x/n can exist under very general conditions provided that y is not too small compared with x.

Whenever $\xi \geqslant 1$ and $\sigma \geqslant 0$ define

$$(2.4) \quad F(\alpha; \xi, \sigma) = \begin{cases} 0 & (\alpha \leqslant 0) \\ 1 & (\alpha \geqslant 1) \\ \theta(\alpha; \xi)(1 - \xi^\sigma([\xi] + \alpha)^{-\sigma}) + \xi^\sigma \sum_{k>\xi} (k^{-\sigma} - (k+\alpha)^{-\sigma}) & (0 < \alpha < 1, \sigma > 0) \\ \alpha & (0 < \alpha < 1, \sigma = 0) \end{cases}$$

where

$$(2.5) \quad \theta(\alpha; \xi) = \begin{cases} 1 & \text{if } (\xi - \alpha, \xi] \cap \mathbf{N} \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

THEOREM 2. — *Suppose that for every real number t with $0 < t < 1$ the limit*

$$(2.6) \quad \lim_{y\to\infty} \sum_{n \leqslant ty} a_n(y)$$

exists and for at least one value of t is non-zero. Then there is a non-negative real number σ such that for every real number ε with $0 < \varepsilon < \frac{1}{2}$ there is a real number $y_0(\varepsilon, \sigma) \geqslant 1$ so that whenever $y_0(\varepsilon, \sigma) \leqslant y \leqslant x$ we have

$$(2.7) \quad \Phi_{x,y}(\alpha) = F(\alpha; x/y, \sigma) + O(\varepsilon^{1+\sigma} x y^{-1}) + O(2^\sigma \varepsilon^\sigma).$$

Lemma 1 below will show that the limit (2.6) is t^σ, which defines σ. We observe that when $\sigma = 0$ Theorem 2 fails to give non-trivial information. Very likely $\Phi_{x,y}(\alpha) \to \alpha$ still holds in this case, at least when $\sum_{n \leqslant y} \lambda_n \to \infty$, but even when $\lambda_n = 1/n$ this is a deep result.

Before proceeding with the proof of Theorem 2 we state a corollary concerning the case when the integer n is allowed only to run through a shorter interval $[y, z]$.

Corollary 2.1. — *With the assumptions of Theorem* 2, *if*

$$y_0(\varepsilon, \sigma) \leqslant y < z \leqslant x/2,\ (y/z)^\sigma < 1 - \varepsilon^{2+\sigma},\ \varepsilon^z \leqslant y,$$

and $\sum_{y<n\leqslant z} \lambda_n > 0$, *then*

$$(2.8) \quad \frac{\sum_{y<n\leqslant z} \lambda_n c_\alpha(x/n)}{\sum_{y<n\leqslant z} \lambda_n} - \alpha \ll (\sigma 2^\sigma z x^{-1} + \varepsilon^{1+\sigma} x y^{-1} + 2^\sigma \varepsilon^\sigma)(1 - y^\sigma z^{-\sigma} - \varepsilon^{2+\sigma})^{-1}.$$

We remark that, in this case, the asymptotic distribution is always the uniform one, at least when $\sigma > 0$.

3. The proof of Theorem 2 requires the following lemma.

Lemma 1. — *On the hypothesis of Theorem* 2 *there is a non-negative real number* σ *such that for every real number* ε *with* $0 < \varepsilon < 1/2$ *there is a real number* $y_0(\varepsilon, \sigma) \geqslant 1$ *so that whenever* $y \geqslant y_0(\varepsilon, \sigma)$ *we have, for every* t *with* $\varepsilon \leqslant t \leqslant 1$,

$$(2.9) \quad \left| t^\sigma - \sum_{n\leqslant ty} a_n(y) \right| < \varepsilon^{2+\sigma}.$$

Proof. — The existence of (2.6) for every real number t with $0 < t < 1$ together with the assumption that for some t in this range the limit is non-zero imply that there is a non-negative real number σ such that for every t with $0 < t \leqslant 1$ we have

$$\lim_{y\to\infty} \sum_{n\leqslant ty} a_n(y) = t^\sigma.$$

Let

$$\mathrm{N} = [2e^{\varepsilon^{-2-\sigma}} \max(1, \sigma)] + 1$$

and choose $y_0(\varepsilon, \sigma) \geqslant 1$ so that if $y \geqslant y_0(\varepsilon, \sigma)$, then for every integer r with $1 \leqslant r \leqslant \mathrm{N}$ we have

$$(2.10) \quad \left| \left(\frac{r}{\mathrm{N}}\right)^\sigma - \sum_{n\leqslant ry/\mathrm{N}} a_n(y) \right| < \frac{1}{2} \varepsilon^{2+\sigma}.$$

Now choose an integer q such that

$$(2.11) \quad \frac{1}{\mathrm{N}} \leqslant \frac{q}{\mathrm{N}} < t \leqslant \frac{q+1}{\mathrm{N}} \leqslant 1,$$

which is always possible if $\varepsilon \leqslant t \leqslant 1$. Note that

$$\begin{aligned}\left(\frac{q+1}{N}\right)^{\sigma}-\left(\frac{q}{N}\right)^{\sigma}&=\int_{q/N}^{(q+1)/N}\sigma u^{\sigma-1}\,du\\&\leqslant \frac{\sigma}{N}\max\left(\left(\frac{q+1}{N}\right)^{\sigma-1},\left(\frac{q}{N}\right)^{\sigma-1}\right)\\&\leqslant \sigma\max\left(N^{-1},N^{-\sigma}\right)\leqslant \max\left(\sigma N^{-1},(e\log N)^{-1}\right)\\&<\frac{1}{2}\varepsilon^{2+\sigma}.\end{aligned}$$

Thus, by (2.10) and (2.11),

$$\begin{aligned}\sum_{n\leqslant ty}a_n(y)\leqslant\sum_{n\leqslant(q+1)y/N}a_n(y)&<\left(\frac{q+1}{N}\right)^{\sigma}+\frac{1}{2}\varepsilon^{2+\sigma}\\&<\left(\frac{q}{N}\right)^{\sigma}+\varepsilon^{2+\sigma}\leqslant t^{\sigma}+\varepsilon^{2+\sigma}\end{aligned}$$

and

$$\begin{aligned}\sum_{n\leqslant ty}a_n(y)\geqslant\sum_{n\leqslant qy/N}a_n(y)&>\left(\frac{q}{N}\right)^{\sigma}-\frac{1}{2}\varepsilon^{2+\sigma}\\&>\left(\frac{q+1}{N}\right)-\varepsilon^{2+\sigma}\geqslant t^{\sigma}-\varepsilon^{2+\sigma}.\end{aligned}$$

These last two inequalities give (2.9) as required.

4. *Proof of Theorem* 2. — Since (2.7) is trivially true when $\alpha\leqslant 0$ or $\alpha\geqslant 1$, we may assume $0<\alpha<1$. Let

$$K=\left[\frac{x}{\varepsilon y}-\alpha\right].$$

Then, by (1.13), (1.11), (1.9), Lemma 1 and (2.5),

$$\begin{aligned}\Phi_{x,y}(\alpha)&=\sum_{\frac{x}{K+\alpha}<n\leqslant y}a_n(y)c_\alpha(x/n)+O\Big(\sum_{n\leqslant 2\varepsilon y}a_n(y)\Big)\\&=\sum_{k=1}^{K}\sum_{\substack{n\leqslant y\\x/(k+\alpha)<n\leqslant x/k}}a_2(y)+O(2^{\sigma}\varepsilon^{\sigma})\\&=\theta(\alpha\,;x/y)\Big(\sum_{n\leqslant y}a_n(y)-\sum_{n\leqslant x/([x/y]+\alpha)}a_n(y)\Big)\\&\quad+\sum_{x/y\leqslant k\leqslant K}\Big(\sum_{n\leqslant x/k}a_n(y)-\sum_{n\leqslant x/(k+\alpha)}a_n(y)\Big)+O(2^{\sigma}\varepsilon^{\sigma}).\end{aligned}$$

Hence, by Lemma 1 and (2.4),

$$\Phi_{x,y}(\alpha) = F(\alpha; x/y, \sigma) + O(\varepsilon^{2+\sigma}K) + O(2^\sigma\varepsilon^\sigma) + O\left(\sum_{k>K}\left(\frac{x}{y}\right)^\sigma (k^{-\sigma} - (k+\alpha)^{-\sigma})\right).$$

The proof of (2.7) is completed by observing that $\varepsilon K \leqslant x/y$ and

$$\sum_{k>K}(k^{-\sigma} - (k+\alpha)^{-\sigma}) = \sum_{k>K}\int_k^{k+\alpha} \sigma u^{-\sigma-1}\,du \leqslant \sum_{k>K}\int_k^{k+1}\sigma u^{-\sigma-1}\,du = (K+1)^{-\sigma} < (2\varepsilon y/x)^\sigma.$$

5. *Proof of Corollary* 2.1. — We use (2.7) and Lemma 1. The condition that $(y/z)^\sigma < 1 - \varepsilon^{2+\sigma}$ means that we can assume that $\sigma > 0$. Suppose that $\xi > 1$. Then, by (2.4),

$$\begin{aligned} F(\alpha; \xi, \sigma) &\leqslant \xi^\sigma \int_{[\xi]}^{[\xi]+\alpha} u^{-\sigma}\,du + O\left(\theta(\alpha;\xi)\int_{\xi/([\xi]+\alpha)}^{1} \sigma u^{\sigma-1}\,du\right) \\ &\leqslant \alpha(\xi/[\xi])^\sigma \\ &\quad + O\left(\theta(\alpha;\xi)\sigma(1-\xi/([\xi]+\alpha))\max\left(1, \left(\frac{\xi}{[\xi]+\alpha}\right)\sigma^{-1}\right)\right) \\ &= \alpha + O(\sigma 2^\sigma \xi^{-1}). \end{aligned}$$

Similarly

$$F(\alpha; \xi, \sigma) \geqslant \xi^\sigma \int_{\xi+1}^{\xi+1+\alpha} u^{-\sigma}\,du \geqslant \alpha\left(1 + \frac{1+\alpha}{\xi}\right)^{-\sigma} \geqslant \alpha - \frac{\sigma\alpha(1+\alpha)}{\xi}.$$

Hence, if $y_0(\varepsilon, \sigma) \leqslant y \leqslant x/2$, then by (1.11), (1.13) and (2.7),

$$\sum_{n\leqslant y} \lambda_n c_\alpha(x/n) = (\alpha + O(\sigma 2^\sigma y x^{-1} + x\varepsilon^{1+\sigma}y^{-1} + 2^\sigma\varepsilon^\sigma))\sum_{n\leqslant y}\lambda_n.$$

Thus, if $y_0(\varepsilon, \sigma) \leqslant y < z \leqslant \frac{1}{2}$, then

$$\sum_{y<n\leqslant z} \lambda_n c_\alpha(x/n) = \alpha \sum_{y<n\leqslant z}\lambda_n + O\left((\sigma 2^\sigma y x^{-1} + x_\varepsilon^{1+\sigma} y^{-1} + 2^\sigma \varepsilon^\sigma)\sum_{n\leqslant y}\lambda_n\right).$$

We complete the proof of (2.8) by observing that by (1.11) and Lemma 1,

$$\Big(\sum_{n\leqslant z}\lambda_n\Big)\Big/\sum_{y<n\leqslant z}\lambda_n = \Big(1-\Big(\sum_{n\leqslant y}\lambda_n\Big)\Big/\sum_{n\leqslant z}\lambda_n\Big)^{-1} < (1-(y/z)^{\sigma}-\varepsilon^{2+\sigma})^{-1}.$$

6. In this section we make some observations concerning the nature of $F(\alpha;\xi,0)$.

Lemma 2. — *Suppose that* $0 \leqslant \alpha \leqslant 1$ *and* $\xi \geqslant 1$. *Then*

$$(2.12)\qquad F(\alpha;\xi,\sigma) = \alpha + O(\sigma 2^{\sigma}\xi^{-1}) \quad (\sigma > 0),$$

$$(2.13)\qquad \lim_{\sigma\to 0^+} F(\alpha;\xi,\sigma) = \alpha = F(\alpha;\xi,0)$$

and

$$(2.14)\qquad F(\alpha;1,\sigma) = \sum_{k=1}^{\infty}(k^{-\sigma}-(k+\alpha)^{-\sigma}) \quad (\sigma > 0).$$

By (2.14) with $\sigma = 1$, $F(\alpha;\ 1, 1) = \Gamma'(\alpha)/\Gamma(\alpha) + \gamma + 1/\alpha$ where Γ is the gamma function and γ is Euler's constant.

Proof. — The asymptotic formula (2.12) was established in the proof of (2.8), (2.13) then follows trivially, and (2.14) is immediate from (2.4).

Lemma 3. — *For each* $\xi \geqslant 1$ *and* $\sigma > 0$ *the function* $F(\alpha;\xi,0)$ *is a continuous function of* α *and is analytic on* $\mathbf{R}\setminus\{0, \{\xi\}, 1\}$ *with*

(2.15)

$$F'(\alpha) = \begin{cases} 0 & (\alpha < 0,\ \alpha > 1) \\ \sigma\xi^{\sigma}\sum_{k>\xi}(k+\alpha)^{-\sigma-1} & (0 < \alpha < \{\xi\}) \\ \sigma\xi^{\sigma}([\xi]+\alpha)^{-\sigma-1}+\sigma\xi^{\sigma}\sum_{k>\xi}(k+\alpha)^{-\sigma-1} & (\{\xi\} < \alpha < 1). \end{cases}$$

The points 0, $\{\xi\}$ *and* 1 *are angular points of* F.

Lemma 4. — *Suppose that* $0 < \alpha < 1$ *and* $\sigma > 0$. *Then considered as a function of* ξ, $F(\alpha;\xi,\sigma)$ *is continuous on* $[1,\infty)\setminus\{2, 3, 4, \ldots\}$ *and for each integer* $n \geqslant 2$,

$$(2.16)\qquad \lim_{\xi\to n^-} F(\alpha;\xi,\sigma) = n^{\sigma}\sum_{k=n+1}^{\infty}(k^{-\sigma}-(k+\alpha)^{-\sigma})$$

and

$$(2.17) \quad \lim_{\xi \to n+} F(\alpha; \xi, \sigma) = n^\sigma \sum_{k=n}^{\infty} (k^{-\sigma} - (k+\alpha)^{-\sigma}) = F(\alpha; n, \sigma).$$

7. We now establish upper and lower bounds for the mean square of $\Phi_{x,y}(\alpha) - \alpha$ which in turn imply respectively

(i) that if y is small compared with x then the only possible (R, λ_n) asymptotic distribution modulo 1 is the uniform one, and

(ii) that the discrepancy cannot be too small.

Theorem 3. — *Suppose that x_0 and x are non-negative real numbers, $y \geqslant 1$ and $0 < \alpha < 1$. Then*

$$(2.18) \quad \int_{x_0}^{x_0+x} |\Phi_{u,y}(\alpha) - \alpha|^2 \, du \leqslant \min(I_1, I_2)$$

where

$$(2.19) \quad I_1 = \frac{1}{3}(x + y^2) \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \frac{1}{m} a_{mn}(y) \right)^2$$

and

$$(2.20) \quad I_2 = \sum_{n=1}^{\infty} \left(\frac{1}{3} x + \frac{1}{2} yn \right) \left(\sum_{m=1}^{\infty} \frac{1}{m} a_{mn}(y) \right)^2.$$

This theorem can be thought of in a rather loose way as a law of the iterated logarithm. This will be discussed further in a later paper. (See [5]).

Theorem 4. — *On the hypothesis of Theorem 3,*

$$(2.21) \quad \int_{x_0}^{x_0+x} |\Phi_{u,y}(\alpha) - \alpha|^2 \, du \geqslant \max(J_1, J_2)$$

where

$$(2.22) \quad J_1 = \frac{1}{2} \pi^{-2}(x - y^2) \sum_{n=1}^{\infty} \left| \sum_{m=1}^{\infty} \frac{1}{m} a_{mn}(y)(1 - e(\alpha m)) \right|^2$$

and

$$(2.23) \quad J_2 = ((2\pi)^{-2} \sum_{n=1}^{\infty} (2x - 3yn) \left| \sum_{m=1}^{\infty} \frac{1}{m} a_{mn}(y)(1 - e(\alpha m)) \right|^2.$$

By taking the real part of the innermost sum in (2.22) and

(2.23) and then discarding all the terms with $m > 1$ one obtains in (2.21) the particularly simple lower bound $\max(L_1, L_2)$, where

$$L_1 = 2\pi^{-2}(\sin \pi\alpha)^4(x - y^2) \sum_{n=1}^{\infty} a_n^2(y)$$

and

$$L_2 = \pi^{-2}(\sin \pi\alpha)^4 \sum_{n=1}^{\infty} (2x - 3yn)a_n^2(y).$$

However, in certain circumstances this loses a factor as large as loglog y.

Corollary 4.1. — *Let the discrepancy* $D_{x,y}$ *be given by*

$$(2.24) \quad D_{x,y} = \sup_{0 \leqslant \alpha < \beta \leqslant 1} |\Phi_{x,y}(\beta) - \Phi_{x,y}(\alpha) - (\beta - \alpha)|.$$

Then

$$(2.25) \quad \int_{x_0}^{x+x_0} D_{u,y}^2 \, du \geqslant \sup_{\alpha \in [0,1]} \max(J_1, J_2).$$

By analogous methods it is possible to obtain corresponding inequalities for

$$\sum_{n=M+1}^{M+N} |\Phi_{n,y}(\alpha) - \alpha|^2$$

but the bounds obtained are more complicated and not so illuminating.

8. To prove Theorems 3 and 4 we require the following lemma which is Theorem 2 of Montgomery and Vaughan [4].

Lemma 5. — *Suppose that* $x_1, x_2, \ldots, x_R$ *are* R *distinct real numbers, and that* $v_1, v_2, \ldots, v_R$ *are* R *complex numbers. Also, let*

$$(2.26) \quad \delta = \min_{\substack{r,s \\ r \neq s}} |x_r - x_s| \quad \text{and} \quad \delta_r = \min_{\substack{s \\ s \neq r}} |x_r - x_s|.$$

Then

$$(2.27) \quad \left| \sum_{\substack{r=1 \\ r \neq s}}^{R} \sum_{s=1}^{R} \frac{v_r \bar{v}_s}{x_r - x_s} \right| \leqslant \pi \min(K_1, K_2)$$

where

$$K_1 = \delta^{-1} \sum_{r=1}^{R} |\rho_r|^2 \tag{2.28}$$

and

$$K_2 = \frac{3}{2} \sum_{r=1}^{R} |\rho_r|^2 \delta_r^{-1}. \tag{2.29}$$

9. *Proofs of Theorems* 3 *and* 4. — Let K be a positive integer. Then it is easily seen that the function $c_\alpha(u)$ given by (1.9) can be written in the form.

$$c_\alpha(u) = \alpha + \sum_{0<|k|\leq K} \frac{1 - e(-\alpha k)}{2\pi i k} e(uk) + O\left(\min\left(1, \frac{1}{K\|u\|}\right)\right) + O\left(\min\left(1, \frac{1}{K\|u - \alpha\|}\right)\right). \tag{2.30}$$

Clearly

$$\int_{x_0}^{x_0+x} \min\left(1, \frac{1}{K\left\|\frac{u}{n} - \beta\right\|}\right) du \ll (x + n)\frac{\log K}{K} \quad (0 \leq \beta \leq 1). \tag{2.31}$$

Hence, by (1.9) and (2.30),

$$\int_{x_0}^{x_0+x} \left| \sum_{n=1}^{\infty} a_n(y) c_\alpha(u/n) - \alpha \right|^2 du = I + O\left((x + y)\frac{\log K}{K}\right) \tag{2.32}$$

where

$$I = \int_{x_0}^{x_0+x} \left| \sum_{n=1}^{\infty} \sum_{\substack{0<|k|\leq K \\ (n,k)=1}} \left(\sum_{m \leq K/|k|} \frac{a_{nm}(y)(1 - e(-\alpha k m))}{2\pi i k m} \right) e\left(\frac{uk}{n}\right) \right|^2 du. \tag{2.33}$$

Clearly, if $n_j \leq y$, $0 < |k_j| \leq K$, $(n_j, k_j) = 1$ for $j = 1, 2$ and $k_1/n_1 \neq k_2/n_2$, then $|k_1/n_1 - k_2/n_2| \geq 1/(yn_1) \geq y^{-2}$.

Therefore, by (2.33) and Lemma 5,

$$I = \sum_{n=1}^{\infty} \sum_{\substack{0<|k|\leq K \\ (n,k)=1}} (x + \theta_1 y^2) \left| \sum_{m \leq K/|k|} \frac{a_{nm}(y)(1 - e(-\alpha k m))}{2\pi i k m} \right|^2 \tag{2.34}$$

and

(2.35)

$$I = \sum_{n=1}^{\infty} \sum_{\substack{0<|k|\leqslant K \\ (n,k)=1}} \left(x + \frac{3}{2}\theta_2 ny\right) \left| \sum_{m\leqslant K/|k|} \frac{a_{nm}(y)(1-e(-\alpha km))}{2\pi ikm} \right|^2$$

where $|\theta_1| \leqslant 1$, $|\theta_2| \leqslant 1$. Theorem 3 now follows from (2.32) on letting $K \to \infty$. Theorem 4 follows in the same way on discarding all the terms with $|k| \neq 1$.

Sometimes, when the simple Riesz means (R, λ_n) are specified, it may be more appropriate to use (2.34) and (2.35) rather than appeal to Theorems 3 and 4.

10. By (2.7), (2.8) and (2.13) we see that if y is small compared with x but not too small, then under very general conditions

$$\lim_{x\to\infty} \Phi_{x,y(x)}(\alpha) = \alpha. \tag{2.36}$$

We now show, as a consequence of Theorem 3, and again under very general conditions, that even if y is very small compared with x, then (2.36) still holds.

THEOREM 5. — *Suppose that* $0 < \theta < 1$, $0 < \alpha < 1$,

(2.37)

$$\lim_{y\to\infty} \left(\left(1 + y^{\frac{3\theta-1}{2\theta}}\right) \left(\sum_{n\leqslant y - y^{(3\theta-1)/2\theta}} \lambda_n \right)^{-2} \sum_{n\leqslant y} \left(\sum_{m\leqslant y/n} \frac{1}{m} \lambda_{mn} \right)^2 \right) = 0$$

and

$$\lim_{x\to\infty} \Phi_{x,x^\theta}(\alpha) \tag{2.38}$$

exists, Then

$$\lim_{x\to\infty} \Phi_{x,x^\theta}(\alpha) = \alpha. \tag{2.39}$$

We remark that (2.37) is rather a weak condition. For instance, if $\lambda_n = 1$ for every n, then it holds for every θ with $0 < \theta < 1$.

Proof. — Let y be large and define $z = y - y^{(3\theta-1)/2\theta}$. Then by Theorem 3, (1.13) and (1.11),

$$(2.40)\quad \int_{z^{1/\theta}}^{y^{1/\theta}} \left| \sum_{n\leqslant z} \lambda_n \left(c_\alpha\left(\frac{u}{n}\right) - \alpha \right) \right|^2 du \ll (y^2 + y^{1/\theta} - z^{1/\theta}) \sum_{n\leqslant y} \left(\sum_{m\leqslant y/n} \frac{1}{m} \lambda_{mn} \right)^2.$$

Furthermore, by Cauchy's inequality (inégalité de Schwarz en français !),

$$\int_{z^{1/\theta}}^{y^{1/\theta}} \left| \sum_{z<n\leqslant u^\theta} \lambda_n \left(c_\alpha\left(\frac{u}{n}\right) - \alpha \right) \right|^2 du \ll (y^{1/\theta} - z^{1/\theta})(1 + y - z) \sum_{n\leqslant y} \lambda_n^2.$$

Hence, by (2.40),

$$(2.41)\quad \int_{z^{1/\theta}}^{y^{1/\theta}} \left| \sum_{n\leqslant u^\theta} \lambda_n \left(c_\alpha\left(\frac{u}{n}\right) - \alpha \right) \right|^2 du \ll \left(y^2 + (y^{1/\theta} - z^{1/\theta}) \left(1 + y^{\frac{3\theta-1}{2\theta}}\right)\right) \sum_{n\leqslant y} \left(\sum_{m\leqslant y/n} \frac{1}{m} \lambda_{mn} \right)^2.$$

It is easily verified that

$$y^2 \ll (y^{1/\theta} - z^{1/\theta}) y^{(3\theta-1)/2\theta}.$$

Thus, by (2.41) and (2.37),

$$\inf_{z^{1/\theta}\leqslant u\leqslant y^{1/\theta}} |\Phi_{u,u^\theta}(\alpha) - \alpha| \to 0 \quad \text{as} \quad y \to \infty.$$

This gives the desired result.

3. Appendix.

1. Theorem 1 does not require that the $a_n(y)$ be the simple Riesz means (R, λ_n). It is valid provided that

$$\sum_{n=1}^{\infty} a_n(y) = 1.$$

2. Theorem 2 can be generalized in the following way. We say that the positive Toeplitz transformation $\mathcal{A} = (a_n(y))$ has asymptotic (or limit) distribution function φ with respect to the ordinary Cesaro method (C, 1) if there exists a distribution function φ such that

$$(3.1)\quad \lim_{y\to\infty} \sum_{n\leqslant ty} a_n(y) = \varphi(t)$$

at every t at which φ is continuous. For example, if the $a_n(y)$ are the simple Riesz means (R, λ_n) and if φ exists, then by Lemma 1 it is either a continuous function given by

$$\text{(3.2)} \qquad \varphi(t) = \begin{cases} 0 & (t \leqslant 0) \\ t^\sigma & (0 < t < 1) \quad (\text{with } \sigma > 0), \\ 1 & (t \geqslant 1) \end{cases}$$

or is one of the « Heaviside » functions Y_0 and Y_1, where $\mathrm{Y}_a(t) = 0$ if $t < a$, $\mathrm{Y}_a(t) = 1$ if $t \geqslant a$. (In the general case, necessarily $\varphi(t) = 0$ for $t < 0$). On examining the proof of Theorem 2, one sees that provided φ exists, is continuous and satisfies $\varphi(0) = 0$, $\varphi(1) = 1$, then it is possible to replace Theorem 2 by a similar but more general statement. In particular $\mathrm{F}(\alpha; \xi, \sigma)$ is to be replaced by

$$\text{(3.3)} \quad \mathrm{G}(\alpha; \xi, \varphi) = \begin{cases} 0 & (\alpha \leqslant 0) \\ 1 & (\alpha \geqslant 1) \\ \theta(\alpha, \xi)\left(1 - \varphi\left(\dfrac{\xi}{[\xi]+\alpha}\right)\right) + \displaystyle\sum_{k>\xi}\left(\varphi\left(\frac{\xi}{k}\right) - \varphi\left(\frac{\xi}{k+\alpha}\right)\right) \\ \quad (\text{when } 0 < \alpha < 1), \end{cases}$$

but some care is needed with the error terms. Besides the above example where φ is given by (3.2), there are other interesting instances in which φ exists.

3. Theorems 3 and 4 do not require the $a_n(y)$ to be the simple Riesz means (R, λ_n). They remain valid without modification provided that $a_n(y) = 0$ for $n > y$. Otherwise, there are extra error-terms involving $\sum_{n>y} a_n(y)$. Thus one can still obtain meaningful information in case $\lim_{y\to\infty} \sum_{n>y} a_n(y) = 0$.

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Proposé par J.-P. Kahane.

B. SAFFARI,
Université de Paris-Sud
Orsay.

R. C. VAUGHAN,
University of Michigan
Ann Arbor.
and Imperial College
London.