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# TRANSITIVE RIEMANNIAN ISOMETRY GROUPS WITH NILPOTENT RADICALS

by Carolyn GORDON [1]

## 1. Introduction.

This paper addresses the problem of describing the full isometry group $I(M)$ of a homogeneous Riemannian manifold $M$ in terms of a given connected transitive subgroup $G$. This problem has been investigated by several authors in case $G$ is compact – see in particular Oniščik [6] and Ozeki [7] – and by the present author [3] for $G$ semisimple or at least reductive with compact radical. Less is known for solvable $G$, although Wilson [8] has recently established the normality of $G$ in $I(M)$ when $G$ is nilpotent. In this contribution, we utilize these results on compact, semisimple, and nilpotent groups to study the case in which $G$ is any connected Lie group with nilpotent radical. We will restrict our attention to $I_0(M)$, the identity component of $I(M)$.

We reformulate the problem in a slightly more general context. For $G$ and $M$ as above, $I_0(M)$ is the product $I_0(M) = GL$ of $G$ with the isotropy subgroup $L$ at a point of $M$. $L$ is compact and contains no normal subgroups of $I_0(M)$. We will describe all connected Lie groups of the form $A = GL$, $G$ connected with nilpotent radical and $L$ compact, omitting the latter condition on $L$.

The main results appear in Sections 2 and 3. In Section 2 we describe the Levi factors of $A$, establishing that the noncompact parts of suitable Levi factors of $G$ and $A$ coincide. A weaker relationship is obtained between the compact parts. We then examine in Section 3 the structure of the Lie algebra of $A$, paying particular attention to its radical.

Section 4 extends these results in case $G \cap L$ is trivial. In terms of our original problem, this is the case of a simply transitive isometry action of $G$

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on a manifold $M$. Finally as a consequence of the results of Sections 2 and 3, we note in Section 5 a sufficient condition on the structure of $G$ to insure normality of $G$ in $A$.

## 2. Description of the Levi factors.

*Notation* (2.1). – Given connected Lie groups $A$ and $G$ with $G \subset A$, choose Levi factors $G_{ss}$ and $A_{ss}$ of $G$ and $A$ with $G_{ss} \subset A_{ss}$ (see Jacobson [5], pp. 91-93). Denote by $\mathfrak{a}$, $\mathfrak{g}$, $\mathfrak{a}_{ss}$, and $\mathfrak{g}_{ss}$ the Lie algebras of $A$, $G$, $A_{ss}$, and $G_{ss}$, respectively. Write

$$\mathfrak{a}_{ss} = \mathfrak{a}_{nc} \oplus \mathfrak{a}_c \qquad \text{and} \qquad \mathfrak{g}_{ss} = \mathfrak{g}_{nc} \oplus \mathfrak{g}_c$$

where $\mathfrak{a}_{nc}$ and $\mathfrak{g}_{nc}$ are semisimple of the noncompact type, i.e., all simple ideals of $\mathfrak{a}_{nc}$ and $\mathfrak{g}_{nc}$ are noncompact, and $\mathfrak{a}_c$ and $\mathfrak{g}_c$ are compact. Let $A_{nc}$, $A_c$, $G_{nc}$ and $G_c$ be the connected subgroups of A with Lie algebras $\mathfrak{a}_{nc}$, $\mathfrak{a}_c$, $\mathfrak{g}_{nc}$, and $\mathfrak{g}_c$. We have Levi decompositions

$$A = (A_{ss})(\mathrm{rad}(A)) \qquad \text{and} \qquad G = (G_{ss})(\mathrm{rad}(G))$$

with $A_{ss} = A_{nc}A_c$ and $G_{ss} = G_{nc}G_c$.

THEOREM (2.2). – *Let the connected Lie group* $A$ *be a product* $A = GL$ *of a connected subgroup* $G$ *with nilpotent radical and a compact subgroup* $L$. *Then in the notation* (2.1), $A_{nc} = G_{nc}$.

*Proof.* – We need only show that $\mathfrak{a}_{nc} = \mathfrak{g}_{nc}$. Let

$$\pi_{nc} : \mathfrak{a} \to \mathfrak{a}_{nc} \qquad \text{and} \qquad \pi_c : \mathfrak{a} \to \mathfrak{a}_c$$

be the homomorphic projections relative to the decomposition

$$\mathfrak{a} = \mathfrak{a}_{nc} + \mathfrak{a}_c + \mathrm{rad}(\mathfrak{a}).$$

$\pi_c(\mathfrak{g}_{nc}) = \{0\}$ since $\mathfrak{a}_c$ contains no noncompact semisimple subalgebras, so $\mathfrak{g}_{nc} \subset \mathfrak{a}_{nc}$.

Let $A' = A/(A_c\,\mathrm{rad}(A))$ and let $\pi : A \to A'$ be the natural projection For any subgroup $H$ of $A$, we will denote $\pi(H)$ by $H'$. The Lie algebra of $A'$ may be identified with $\mathfrak{a}_{nc}$ and the differential $(d\pi)_e$ with $\pi_{nc}$. $G'_{nc}$ then has Lie algebra $\mathfrak{g}_{nc}$. Letting $N = \mathrm{rad}(G)$,

$$(1) \qquad G' = G'_{nc}G'_cN'$$

with $N'$ nilpotent, and $A' = G'L'$.

Modding out a discrete normal subgroup if necessary, we may assume $A'$ has finite center. Let $U'$ be a maximal compact subgroup of $A'$ containing $G'_c$. A conjugate of $L'$ lies in $U'$, so

$$A' = G'U' = (G'_{nc}N')U'$$

by (1). Under a left-invariant Reimannian metric, $A'/U'$ is a symmetric space of non-positive sectional curvature with no Euclidean factor (see Helgason [4], pp. 241-253) on which $G'_{nc}N'$ acts transitively and effectively by isometries. We now use the characterization by Azencott and Wilson of isometry groups transitive on manifolds of non-positive sectional curvature. By [1], Proposition (2.5), given any Iwasawa subgroup $S'_1$ of $G'_{nc}$, there exists a closed subgroup $S'_2$ of $N'$, normal in $G'_{nc}N'$, such that $S'_1S'_2$ is a closed simply-connected solvable subgroup of $A'$ acting simply transitively on $A'/U'$. The Lie algebra $\mathfrak{g}_{nc} + \mathfrak{s}'_2$ of $G'_{nc}S'_2$ is a « basic isometry algebra » (see [2], pp. 27-29), so Theorem (4.6) and Proposition (5.3), part (i), of [2] together contradict the nilpotency of $\mathfrak{s}'_2$, unless $\mathfrak{s}'_2 = \{0\}$. Hence $S'_1$ and consequently $G'_{nc}$ act transitively on $A'/U'$, and $A' = G'_{nc}U'$. Since both $A'$ and $G'_{nc}$ are semisimple of the noncompact type, $A' = G'_{nc}$ ([3], Proposition (3.3)) and $\mathfrak{a}_{nc} = \mathfrak{g}_{nc}$. □

We now describe $\mathfrak{a}_c$. For $L_{ss}$ the (unique) Levi factor of $L$, $hL_{ss}h^{-1} \subset A_{ss}$ for some $h \in A$. Note that $A = G(hLh^{-1})$, so there is no loss of generality in assuming that $L_{ss} \subset A_{ss}$.

*Notation* (2.3). — If $\mathfrak{u}$ is a compact Lie algebra, the unique Levi factor $[\mathfrak{u},\mathfrak{u}]$ of $\mathfrak{u}$ will be denoted $\mathfrak{u}_{ss}$.

PROPOSITION (2.4). — *Let the connected Lie group* $A$ *be a product* $A = GL$ *of a connected subgroup* $G$ *with nilpotent radical and a compact subgroup* $L$ *with Lie algebra denoted by* $\mathfrak{l}$. *Using notation* (2.1) *and* (2.3),

$$\mathfrak{a}_c = \mathfrak{g}_c + \pi_c(\mathfrak{l}_{ss}) \tag{2}$$

*where* $\pi_c : \mathfrak{a} \to \mathfrak{a}_c$ *is the projection along* $\mathfrak{a}_{nc} + \mathrm{rad}(\mathfrak{a})$.

*Replacing* $L$ *by a conjugate so that* $\mathfrak{l}_{ss} \subset \mathfrak{a}_{ss}$,

$$\mathfrak{a}_{ss} = \mathfrak{g}_{ss} + \mathfrak{l}_{ss}. \tag{3}$$

*Proof.* — Since $\mathfrak{a}_c = \pi_c(\mathfrak{g}) + \pi_c(\mathfrak{l})$ and $\mathfrak{a}_c$ is compact and semisimple, we have

$$\mathfrak{a}_c \doteq (\pi_c(\mathfrak{g}))_{ss} + (\pi_c(\mathfrak{l}))_{ss} \tag{4}$$

(see Oniščik [6], Theorem (1.1)).

$$[\mathfrak{g}_c, \mathfrak{a}_{nc}] = \{0\} \text{ by Theorem (2.2), so}$$
$$\mathfrak{g}_c \subset \mathfrak{a}_c \quad \text{and} \quad \pi_c(\mathfrak{g}) = \mathfrak{g}_c + \pi_c(\mathrm{rad}(\mathfrak{g})).$$

$\pi_c(\mathrm{rad}(\mathfrak{g}))$ is a solvable ideal in the compact algebra $\pi_c(\mathfrak{g})$, hence is central. Thus $(\pi_c(\mathfrak{g}))_{ss} = \mathfrak{g}_c$ and (4) now implies (2). (3) follows from (2) and Theorem (2.2). □

We note that the work of Oniščik [6] on decompositions of compact Lie algebras may be applied to (2) to further analyze $\mathfrak{a}_c$.

## 3. Description of the radical.

THEOREM (3.1). – *Let the connected Lie group* A *be a product* A = GL *of a connected subgroup* G *and a compact subgroup* L, *and suppose the radical of* G *is nilpotent. We use notation* (2.1) *and denote the radicals of* $\mathfrak{a}$ *and* $\mathfrak{g}$ *by* $\mathfrak{s}$ *and* $\mathfrak{n}$, *respectively. Then :*

(*a*) $\mathfrak{n}$ *is the sum of ideals* $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$ *where* $\mathfrak{n}_1 := \mathfrak{n} \cap \mathfrak{a}_{ss}$ *is central in* $\mathfrak{g}$ *and* $[\mathfrak{g},\mathfrak{n}] \subset \mathfrak{n}_2$.

(*b*) $\mathfrak{s}$ *is a vector space direct sum* $\mathfrak{s} = \mathfrak{u} + \mathfrak{n}'_2$ *of an abelian subalgebra* $\mathfrak{u}$, *compactly imbedded in* $\mathfrak{a}$, *and an ideal* $\mathfrak{n}'_2$ *containing* $[\mathfrak{g},\mathfrak{n}]$.

(*c*) $[\mathfrak{a},\mathfrak{s}] \subset \mathfrak{n}'_2$ *and* $[\mathfrak{g}_{ss},\mathfrak{s}] = [\mathfrak{g}_{ss},\mathfrak{n}]$.

(*d*) *There exists an isomorphism*

$$\psi : \mathfrak{g}_{ss} + \mathfrak{n}_1 + \mathfrak{n}'_2 \to \mathfrak{g}$$

*which maps* $\mathfrak{n}'_2$ *onto* $\mathfrak{n}_2$ *and restricts to the identity map on* $[\mathfrak{g},\mathfrak{g}] + \mathfrak{n}_1$.

*Remarks* (3.2). – (1) $\mathfrak{n}_1$ is in general non-trivial. For example, the unitary group $G = U(n)$ acts transitively on the sphere $SO(2n)/SO(2n-1)$. $U(n)$ has non-trivial radical whereas $A = SO(2n)$ is semisimple. Hence $\mathfrak{n}_1 = \mathfrak{n} \neq \{0\}$.

Theorems (2.2) and (3.1) imply $\mathfrak{g}_{nc} \oplus \mathfrak{n}'_2$ is an $\mathfrak{a}$-ideal isomorphic to $\mathfrak{g}_{nc} \oplus \mathfrak{n}_2$. Thus one might also ask whether $\mathfrak{n}_1$ can be non-zero when $\mathfrak{g}_c = \{0\}$. The answer is again yes. Let H be a connected semisimple Lie group of the noncompact type containing a connected compact semisimple subgroup K. Set

$$A = H \times K$$
$$G = H \times N$$

where N is a non-trivial connected abelian subgroup of K, and

$$L = \{(h,h) \in A : h \in K\}.$$

Then G is transitive on A/L and again $\mathfrak{n}_1 = \mathfrak{n} \neq \{0\}$.

(2) By part (*b*), $\mathfrak{n}'_2 = \mathfrak{n}_2$ in case $[\mathfrak{g},\mathfrak{n}] = \mathfrak{n}$. However, in the proof of Proposition (5.2), we will construct a class of examples in which $\mathfrak{n}'_2 \neq \mathfrak{n}_2$.

*Proof of Theorem* (3.1). – The center of a Lie algebra $\mathfrak{h}$ will be denoted $z(\mathfrak{h})$. We will make frequent use of the fact that if $\mathfrak{u}$ is a compactly imbedded subalgebra of $\mathfrak{a}$, then the operators $\mathrm{ad}_{\mathfrak{a}}X$, $X \in \mathfrak{u}$, are all skew-symmetric relative to some inner product on $\mathfrak{a}$ and are consequently semisimple.

Let

$$P : \mathfrak{a} \to \mathfrak{a}_{ss} \qquad \text{and} \qquad Q : \mathfrak{a} \to \mathfrak{s}$$

be the projections relative to the Levi decomposition $\mathfrak{a} = \mathfrak{a}_{ss} + \mathfrak{s}$. $P = \pi_{nc} + \pi_c$ where as before $\pi_{nc} : \mathfrak{a} \to \mathfrak{a}_{nc}$ and $\pi_c : \mathfrak{a} \to \mathfrak{a}_c$ are the projections relative to $\mathfrak{a} = \mathfrak{a}_{nc} + \mathfrak{a}_c + \mathfrak{s}$. By Theorem (2.2), $\mathfrak{a}_{nc} = \mathfrak{g}_{nc}$, so $\pi_{nc}(\mathfrak{n}) = \{0\}$ and $P(\mathfrak{n}) = \pi_c(\mathfrak{n})$. In particular, $\mathfrak{n}_1 = \mathfrak{n} \cap \mathfrak{a}_{ss} \subset \mathfrak{a}_c$ and $\mathrm{ad}_{\mathfrak{a}}\mathfrak{n}_1$ consists of semisimple operators. Hence the elements of $\mathrm{ad}_{\mathfrak{g}}\mathfrak{n}_1$ are semisimple as well as nilpotent, i.e. $\mathfrak{n}_1 \subset z(\mathfrak{g})$. Moreover

$$(1) \qquad P([\mathfrak{g},\mathfrak{n}]) = [P(\mathfrak{g}), P(\mathfrak{n})] = [P(\mathfrak{g}), \pi_c(\mathfrak{n})] = \{0\},$$

the last equality following from the proof of Proposition (2.4), so $\mathfrak{n}_1 \cap [\mathfrak{g},\mathfrak{n}] = \{0\}$. Letting $\mathfrak{n}_2$ denote any complement of $\mathfrak{n}_1$ in $\mathfrak{n}$ which contains $[\mathfrak{g},\mathfrak{n}]$; (*a*) follows.

Let

$$\mathfrak{g}_{nc} = \mathfrak{k} + \mathfrak{p}$$

be a Cartan decomposition with $\mathfrak{k}$ compactly imbedded in $\mathfrak{g}$. Since the connected subgroup of $\mathrm{Int}(\mathfrak{a})$ with Lie algebra $\mathrm{ad}_{\mathfrak{a}}\mathfrak{g}_{nc}$ is a semisimple matrix group, it has finite center and hence $\mathfrak{k}$ is compactly imbedded in $\mathfrak{a}$ (see Helgason [4], pp. 252-253). $\mathfrak{k} + \mathfrak{a}_c$ lies in a maximal compactly imbedded subalgebra $\mathfrak{w}$ of $\mathfrak{a}$. $P(\mathfrak{w}) = \mathfrak{k} + \mathfrak{a}_c$, $\mathfrak{k} + \mathfrak{a}_c$ being maximal compact in $\mathfrak{a}_{ss}$, so $\mathfrak{w} = (\mathfrak{k}+\mathfrak{a}_c) + (\mathfrak{w} \cap \mathfrak{s})$ with $(\mathfrak{w} \cap \mathfrak{s}) \subset z(\mathfrak{w})$. After replacing L by a conjugate subgroup of A, we may assume that $\mathfrak{l} \subset \mathfrak{w}$. Thus $\mathfrak{a} = \mathfrak{w} + \mathfrak{g}$ and $\mathfrak{s} = (\mathfrak{w} \cap \mathfrak{s}) + Q(\mathfrak{n})$. Let $\mathfrak{u}$ be a complement of

$\mathfrak{w} \cap Q(\mathfrak{n})$ in $\mathfrak{w} \cap \mathfrak{s}$ and set

$$(2) \qquad \mathfrak{v} = \mathfrak{u} + \mathfrak{k} + \mathfrak{a}_c .$$

Note that $\mathfrak{u} \subset z(\mathfrak{v})$. We have vector space direct sums

$$(3) \qquad \mathfrak{a} = \mathfrak{v} + \mathfrak{p} + \mathfrak{n}_2 \quad \text{and} \quad \mathfrak{s} = \mathfrak{u} + Q(\mathfrak{n}_2).$$

Denote by $\mathfrak{s}_0$ the 0-eigenspace in $\mathfrak{s}$ of $\mathrm{ad}_\mathfrak{a}\mathfrak{v}$. Since $\mathfrak{v}$ lies in the compactly imbedded subalgebra $\mathfrak{w}$, $\mathfrak{s} = \mathfrak{s}_0 + [\mathfrak{v},\mathfrak{s}]$.
$\mathfrak{s}_0 = \mathfrak{u} + (\mathfrak{s}_0 \cap Q(\mathfrak{n}_2))$. Set

$$(4) \qquad \mathfrak{n}_2' = [\mathfrak{v},\mathfrak{s}] + (\mathfrak{s}_0 \cap Q(\mathfrak{n}_2)).$$

Then $\mathfrak{s} = \mathfrak{u} + \mathfrak{n}_2'$ and $\mathfrak{v} \cap \mathfrak{n}_2' = \{0\}$.

$P(\mathfrak{n}_2) \subset \mathfrak{a}_c \subset \mathfrak{v}$, so (2) and (3) imply $\mathfrak{s} \subset \mathfrak{n}_2 + \mathfrak{v}$ with $\mathfrak{n}_2 \cap \mathfrak{v} = \{0\}$. For $X \in \mathfrak{s}$, write

$$X = X_\mathfrak{v} + X_\mathfrak{n} \quad X_\mathfrak{v} \in \mathfrak{v}, \quad X_\mathfrak{n} \in \mathfrak{n}_2 .$$

*Claim.* — For $X \in \mathfrak{n}_2'$, $[X_\mathfrak{v},\mathfrak{s}] = \{0\}$.

For $H \in \mathfrak{v}$, $Y \in \mathfrak{n}_2$, write

$$[H,Y] = \rho(H)Y - \varphi(Y)(H), \qquad \rho(H)Y \in \mathfrak{n}_2, \qquad \varphi(Y)H \in \mathfrak{v}.$$

To prove the claim, it suffices to show that $\rho(X_\mathfrak{v}) = 0$, since then

$$[X_\mathfrak{v},\mathfrak{s}] \subset \mathfrak{v} \cap [\mathfrak{v},\mathfrak{s}] \subset \mathfrak{v} \cap \mathfrak{n}_2' = \{0\}.$$

Let $\mathfrak{v}_0$ be the maximal $(\mathfrak{v}+\mathfrak{n}_2)$ — ideal in $\mathfrak{v}$ and

$$\pi : \mathfrak{v} + \mathfrak{n}_2 \to (\mathfrak{v}+\mathfrak{n}_2)/\mathfrak{v}_0$$

the projection. $\pi(\mathfrak{n}_2)$ is nilpotent, $\pi(\mathfrak{v})$ contains no ideals of $\pi(\mathfrak{v}+\mathfrak{n}_2)$ and $\pi(\mathfrak{n}_2) \cap \pi(\mathfrak{v}) = \{0\}$. Hence (Wilson [8]), $\pi(\mathfrak{n}_2)$ is an ideal in $\pi(\mathfrak{v} + \mathfrak{n}_2)$. i.e. for $Y \in \mathfrak{n}_2$, $\varphi(Y)(\mathfrak{v}) \subset \mathfrak{v}_0$ and

$$(5) \qquad \rho(\varphi(Y)H) = 0, \qquad H \in \mathfrak{v},\ Y \in \mathfrak{n}_2 .$$

We suppose first that $X \in \mathfrak{s}_0 \cap Q(\mathfrak{n}_2)$. Since $X \in \mathfrak{s}_0$, $[\mathfrak{a}_c, X] = \{0\}$ and for $H \in \mathfrak{a}_c$,

$$0 = [H,X]_\mathfrak{v} = [H,X_\mathfrak{v}] - \varphi(X_{\mathfrak{n}_2})H .$$

Thus by (5)

$$\rho([H,X_\mathfrak{v}]) = \{0\}, \qquad H \in \mathfrak{a}_c .$$

But $X_{\mathfrak{v}} = -P(X_{\mathfrak{n}}) \in \mathfrak{a}_c$ since $X \in Q(\mathfrak{n}_2)$. Noting that $\ker \rho|_{\mathfrak{a}_c}$ is an ideal in the semisimple algebra $\mathfrak{a}_c$, it follows that $\rho(X_{\mathfrak{v}}) = 0$.

Now let $\mathfrak{v}_1 = \{Y_{\mathfrak{v}} : Y \in [\mathfrak{v},\mathfrak{s}]\}$. Then

$$(6) \qquad [\mathfrak{v},\mathfrak{s}] = [\mathfrak{v}_1,\mathfrak{s}] + \{Y \in [\mathfrak{v},\mathfrak{s}] : [\mathfrak{v}_1,Y] = \{0\}\}.$$

Suppose $X = [H,Y]$ for some $H \in \mathfrak{v}_1$, $Y \in \mathfrak{s}$. Then

$$X_{\mathfrak{v}} = -\varphi(Y_{\mathfrak{n}})H + [H,Y_{\mathfrak{v}}].$$

$\mathfrak{v}_1 \subset P(\mathfrak{n}_2) + \mathfrak{u}$ by (3), $P(\mathfrak{n}_2)$ is abelian by (1), and $\mathfrak{u} \subset z(\mathfrak{v})$; hence $\mathfrak{v}_1$ is abelian and $[H,Y_{\mathfrak{v}}] = \{0\}$. Thus by (5), $\rho(X_{\mathfrak{v}}) = 0$.

In view of (4) and (6) it remains only to check the case $X \in [\mathfrak{v},\mathfrak{s}]$ while $[\mathfrak{v}_1,\mathfrak{s}] = \{0\}$. Since $[\mathfrak{v},\mathfrak{s}]$ is contained in the nil radical of $\mathfrak{a}$ (see Jacobson [5], p. 51), $\mathrm{ad}_{\mathfrak{a}}X$ is nilpotent. $X_{\mathfrak{v}} \in \mathfrak{v}_1$, so $[X_{\mathfrak{v}},X] = \{0\}$ and consequently $[X_{\mathfrak{n}},X] = 0$. Thus if we show that $\mathrm{ad}_{\mathfrak{a}}X_{\mathfrak{n}}|_{\mathfrak{s}}$ is nilpotent, it will follow that $\mathrm{ad}_{\mathfrak{a}}X_{\mathfrak{v}}|_{\mathfrak{s}} (= \mathrm{ad}_{\mathfrak{a}}(X - X_{\mathfrak{n}})|_{\mathfrak{s}})$ is nilpotent. Noting that $\mathrm{ad}_{\mathfrak{a}}X_{\mathfrak{v}}|_{\mathfrak{s}}$ is also semisimple since $X_{\mathfrak{v}} \in \mathfrak{w}$, the claim will be established.

For $Y \in \mathfrak{s}$,

$$(7) \qquad [X_{\mathfrak{n}},Y] = [X_{\mathfrak{n}},Y]_{\mathfrak{n}_2} + \varphi(X_{\mathfrak{n}})Y_{\mathfrak{v}}.$$

Setting $Z = [X_{\mathfrak{n}},Y]_{\mathfrak{n}_2}$, (5) and (7) inductively imply

$$(\mathrm{ad}_{\mathfrak{a}}X_{\mathfrak{n}})^m(Y) = (\mathrm{ad}_{\mathfrak{n}_2}X_{\mathfrak{n}})^{m-1}(Z) + (\varphi(X_{\mathfrak{n}}))^m(Y_{\mathfrak{v}}).$$

Since $\mathfrak{n}_2$ is nilpotent, $(\mathrm{ad}_{\mathfrak{n}_2}X_{\mathfrak{n}})^{k-1} = 0$ for some $k$, so

$$(\mathrm{ad}_{\mathfrak{a}}X_{\mathfrak{n}})^k(\mathfrak{s}) \subset \mathfrak{v} \cap \text{nil rad}(\mathfrak{a}).$$

But $\mathfrak{v} \cap \text{nil rad}(\mathfrak{a}) \subset z(\mathfrak{a})$ since $\mathfrak{v}$ lies in a compactly imbedded subalgebra of $\mathfrak{a}$, so $(\mathrm{ad}_{\mathfrak{a}}X_{\mathfrak{n}})^{k+1}_{|\mathfrak{s}} = 0$, i.e. $\mathrm{ad}_{\mathfrak{a}}X_{\mathfrak{n}}|_{\mathfrak{s}}$ is nilpotent. As noted above, the claim follows.

The claim implies

$$(8) \qquad [X,Y] = [X_{\mathfrak{n}},Y_{\mathfrak{n}}], \qquad X,Y \in \mathfrak{n}_2'.$$

Since $\mathfrak{s} = \mathfrak{u} + \mathfrak{n}_2'$ and $Q|_{\mathfrak{n}_2}$ is $1:1$, $\{X_{\mathfrak{n}} : X \in \mathfrak{n}_2'\} = \mathfrak{n}_2$. Thus (8) and part (*a*) together imply

$$(9) \qquad [\mathfrak{n}_2',\mathfrak{n}_2'] = [\mathfrak{n},\mathfrak{n}].$$

$[\mathfrak{v},\mathfrak{n}'_2] \subset \mathfrak{n}'_2$ by (4), so by (9)

$$(10) \qquad [\mathfrak{v},[\mathfrak{n},\mathfrak{n}]] \subset [\mathfrak{n},\mathfrak{n}].$$

For $X \in \mathfrak{n}'_2$, $[\mathfrak{g},X_{\mathfrak{n}}] \subset \mathfrak{s}$ by (1) and $[\mathfrak{g},X] \subset \mathfrak{s}$, so $[X_{\mathfrak{v}},\mathfrak{g}] \subset \mathfrak{s}$. But $[X_{\mathfrak{v}},\mathfrak{s}] = \{0\}$ by the claim, and $\mathrm{ad}_{\mathfrak{a}}X_{\mathfrak{v}}$ is a semisimple operator. Hence $[X_{\mathfrak{v}},\mathfrak{g}] = \{0\}$ and

$$(11) \qquad [Y,X] = [Y,X_{\mathfrak{n}}], \qquad Y \in \mathfrak{g}, \quad X \in \mathfrak{n}'_2.$$

In particular,

$$(12) \qquad [\mathfrak{k}+\mathfrak{g}_c, \mathfrak{s}] \subset \mathfrak{n}$$

since $[\mathfrak{k}+\mathfrak{g}_c,\mathfrak{u}] \subset [\mathfrak{v},\mathfrak{u}] = \{0\}$. Hence

$$[\mathfrak{p},\mathfrak{u}] = [[\mathfrak{k},\mathfrak{p}],\mathfrak{u}] = [\mathfrak{k},[\mathfrak{p},\mathfrak{u}]] \subset [\mathfrak{k},\mathfrak{s}] \subset \mathfrak{n}.$$

Thus $[\mathfrak{g}_{nc},\mathfrak{s}] \subset \mathfrak{n} \cap \mathfrak{s}$. Since $\mathfrak{g}_{nc}$ is semisimple and $[\mathfrak{g}_{nc},\mathfrak{n}] \subset \mathfrak{n} \cap \mathfrak{s}$ by (1),

$$(13) \qquad [\mathfrak{g}_{nc},\mathfrak{s}] = [\mathfrak{g}_{nc},\mathfrak{n} \cap \mathfrak{s}] = [\mathfrak{g}_{nc},\mathfrak{n}].$$

Similarly, using (12), we obtain $[\mathfrak{g}_c,\mathfrak{s}] = [\mathfrak{g}_c,\mathfrak{n}]$ and the second statement of (*c*) follows.

By Theorem (2.2) and (13),

$$(14) \qquad [\mathfrak{g}_{nc},\mathfrak{a}] = \mathfrak{g}_{nc} + [\mathfrak{g}_{nc},\mathfrak{n}].$$

Thus,

$$\begin{aligned}
[\mathfrak{v},[\mathfrak{g}_{nc},\mathfrak{s}]] &= [\mathfrak{v},[\mathfrak{g}_{nc},\mathfrak{n} \cap \mathfrak{s}]] \quad \text{by (13)} \\
&\subset [[\mathfrak{v},\mathfrak{g}_{nc}],\mathfrak{n} \cap \mathfrak{s}] + [\mathfrak{g}_{nc},[\mathfrak{v},\mathfrak{n}]] \\
&\subset [\mathfrak{g}_{nc}+\mathfrak{n},\mathfrak{n} \cap \mathfrak{s}] + [\mathfrak{g}_{nc},\mathfrak{s}] \quad \text{by (14)} \\
&\subset [\mathfrak{g}_{nc},\mathfrak{s}] + [\mathfrak{n},\mathfrak{n}].
\end{aligned}$$

Define

$$(15) \qquad \mathfrak{m} = [\mathfrak{g}_{nc},\mathfrak{s}] + [\mathfrak{n},\mathfrak{n}].$$

By (10) and the above computation, $\mathfrak{m}$ is an $\mathrm{ad}_{\mathfrak{a}}(\mathfrak{v})$-invariant subspace of $\mathfrak{n} \cap \mathfrak{s}$. Therefore

$$(16) \qquad \mathfrak{m} = [\mathfrak{v},\mathfrak{m}] + (\mathfrak{m} \cap \mathfrak{s}_0) \subset \mathfrak{n}'_2$$

by (4), so $[\mathfrak{g}_{nc},\mathfrak{s}] \subset \mathfrak{n}'_2$ by (15). Since

$$\mathfrak{s} = \mathfrak{u} + \mathfrak{n}'_2 \qquad \text{and} \qquad [\mathfrak{u},\mathfrak{n}'_2] \subset \mathfrak{n}'_2,$$

(9), (15), and (16) show that $[\mathfrak{n}'_2,\mathfrak{s}] \subset \mathfrak{n}'_2$. Noting that

$$\mathfrak{a} = \mathfrak{v} + \mathfrak{p} + \mathfrak{n}'_2,$$

we thus have $[\mathfrak{a},\mathfrak{s}] \subset \mathfrak{n}'_2$.

Finally define $\psi : \mathfrak{g}_{ss} + \mathfrak{n}_1 + \mathfrak{n}'_2 \to \mathfrak{g}$ by

$$\psi(Y+X) = Y + X_n, \qquad Y \in \mathfrak{g}_{ss} + \mathfrak{n}_1, \qquad X \in \mathfrak{n}'_2.$$

$\psi$ maps $\mathfrak{n}'_2$ injectively onto $\mathfrak{n}_2$ and by (8) and (11), $\psi$ is an isomorphism. □

COROLLARY (3.2). – *Under the hypothesis and notation of Theorem* (3.1), $[\mathfrak{n},\mathfrak{n}]$ *and* $[\mathfrak{g}_{nc}+\mathfrak{n},\mathfrak{g}_{nc}+\mathfrak{n}]$ *are ideals of* $\mathfrak{a}$.

*Proof.* – Both subalgebras are $\mathfrak{g}$-ideals. $\mathfrak{a} \subset \mathfrak{g} + \mathfrak{v}$ by (3), so the corollary follows from (10), (13) and Theorem (2.2). □

## 4. The simply transitive case.

Under the notation and hypotheses of Theorem (3.1), suppose that $G \cap L$ is trivial. Then $G$ intersects any conjugate of $L$ trivially, so the last statement of Proposition (2.4) implies $\mathfrak{n} \cap \mathfrak{a}_{ss} = \{0\}$, i.e. $\mathfrak{n}_1 = 0$ and $\mathfrak{n} = \mathfrak{n}_2 \simeq \mathfrak{n}'_2$.

THEOREM (4.1). – *Let the connected Lie group* $A$ *be a product of disjoint subgroups* $A = GL$ *with* $L$ *compact and* $G$ *connected with nilpotent radical. We use the notation of* (2.1) *and* (3.1) *but write* $\mathfrak{n}'$ *in place of* $\mathfrak{n}'_2$. *Then* $A = G'L$ *where* $G'$ *is a connected normal subgroup of* $A$ *with Lie algebra* $\mathfrak{g}'$ *satisfying :*

(i) $\mathfrak{g}' \cap \mathfrak{l} = \{0\}$;

(ii) $\mathfrak{g}' = \mathfrak{g}_{nc} + \mathfrak{g}'_c + \mathfrak{n}'$ *for some* $\mathfrak{a}_c$*-ideal* $\mathfrak{g}'_c$ *isomorphic to* $\mathfrak{g}_c$;

(iii) *if* $[\mathfrak{g}_c,\mathfrak{n}] = \{0\}$, *then* $\mathfrak{g}' \simeq \mathfrak{g}$.

*Proof.* – We will continue to use the notation developed in the proof of Theorem (3.1). In particular, recall the construction of the maximal compactly imbedded subalgebra $\mathfrak{w}$ of $\mathfrak{a}$. The conclusions of (4.1) are not affected when L is replaced by a conjugate subgroup of A, so we may assume that $\mathfrak{l} \subset \mathfrak{w}$. Then $\mathfrak{l}_{ss} \subset [\mathfrak{w},\mathfrak{w}] \subset \mathfrak{a}_{ss}$. Proposition (2.4) and Theorem (2.2) imply that

$$\mathfrak{a}_{ss} = \mathfrak{g}_{ss} + \mathfrak{l}_{ss}, \qquad \mathfrak{a}_c = \mathfrak{g}_c + \pi_c(\mathfrak{l}_{ss}),$$

and

$$\pi_c(\mathfrak{l}_{ss}) \subset \mathfrak{l}_{ss} + \mathfrak{g}_{nc}.$$

Thus $\pi_c(\mathfrak{l}_{ss}) \cap \mathfrak{g}_c = \{0\}$ since $\mathfrak{g} \cap \mathfrak{l} = \{0\}$. Let $\mathfrak{a}'_c$ be the minimal $\mathfrak{a}_c$-ideal containing $\mathfrak{g}_c$. $\mathfrak{a}'_c = \mathfrak{g}_c + (\mathfrak{a}'_c \cap \pi_c(\mathfrak{l}_{ss}))$, a vector space direct sum, so $\mathfrak{a}'_c$ contains an $\mathfrak{a}_c$-ideal $\mathfrak{g}'_c$ isomorphic to $\mathfrak{g}_c$ such that

$$\mathfrak{a}'_c = \mathfrak{g}'_c \dot{+} (\mathfrak{a}'_c \cap \pi_c(\mathfrak{l}_{ss})),$$

again a vector space direct sum (Ozeki [7]). Hence $\mathfrak{a}_c = \mathfrak{g}'_c + \pi_c(\mathfrak{l}_{ss})$ and

$$(1) \qquad \mathfrak{a}_{ss} = \mathfrak{g}_{nc} + \mathfrak{g}'_c + \mathfrak{l}_{ss} \quad \text{(vector space direct sum).}$$

Letting $\mathfrak{g}' = \mathfrak{g}_{nc} + \mathfrak{g}'_c + \mathfrak{n}'$, Theorems (2.2) and (3.1) imply that $\mathfrak{g}'$ is an $\mathfrak{a}$-ideal.

We now show that $\mathfrak{a} = \mathfrak{g}' + \mathfrak{l}$. Since $\mathfrak{a}_{ss} = \mathfrak{g}_{ss} + \mathfrak{l}_{ss}$,

$$\mathfrak{s} = Q(z(\mathfrak{l})) + Q(\mathfrak{n}). \quad Q(z(\mathfrak{l})) \subset Q(\mathfrak{w}) = \mathfrak{w} \cap \mathfrak{s}.$$

The subalgebra $\mathfrak{u}$ in (3.1) was defined to be any complement of $\mathfrak{w} \cap Q(\mathfrak{n})$ in $\mathfrak{w} \cap \mathfrak{s}$. We may therefore choose $\mathfrak{u}$ so that $\mathfrak{u} \subset Q(z(\mathfrak{l}))$. Then by (3.1),

$$\mathfrak{s} = \mathfrak{u} + \mathfrak{n}' = Q(z(\mathfrak{l})) + \mathfrak{n}' \subset \mathfrak{a}_{ss} + z(\mathfrak{l}) + \mathfrak{n}'.$$

Thus by (1), $\mathfrak{a} = \mathfrak{g}' + \mathfrak{l}$ and $A = G'L$, where $G'$ is the connected normal subgroup of A with Lie algebra $\mathfrak{g}'$. Since $\mathfrak{g}$ and $\mathfrak{g}'$ have the same dimension, $\mathfrak{g}' \cap \mathfrak{l} = \{0\}$.

Finally, suppose that $[\mathfrak{g}_c, \mathfrak{n}] = \{0\}$. Then Theorem (3.1) part (*c*) and the semisimplicity of $\mathfrak{g}_c$ imply $[\mathfrak{g}_c, \mathfrak{s}] = \{0\}$. Since $\mathfrak{a}'_c$ is the minimal $\mathfrak{a}_c$-ideal containing $\mathfrak{g}_c$, $[\mathfrak{a}'_c, \mathfrak{s}] = \{0\}$ and consequently $[\mathfrak{g}'_c, \mathfrak{n}'] \subset [\mathfrak{g}'_c, \mathfrak{s}] = \{0\}$. Since $\mathfrak{g}_{nc} + \mathfrak{n}' \simeq \mathfrak{g}_{nc} + \mathfrak{n}$ by (3.1), (iii) follows. □

## 5. A condition for normality of the transitive subgroup.

THEOREM (5.1). — *Let* M *be a connected homogeneous Riemannian manifold and* $I_0(M)$ *the connected component of the identity in the group of* *all isometries of* M. *Suppose that* G *is a connected transitive subgroup of* A *with Lie algebra* $\mathfrak{g}$ *satisfying* $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$ *and that some (hence every) Levi factor of* G *is of the noncompact type. Then* G *is normal in* A.

*Proof.* – The condition $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$ implies that the radical $\mathfrak{n}$ of $\mathfrak{g}$ is nilpotent and that $\mathfrak{g} = [\mathfrak{g}_{nc}+\mathfrak{n},\mathfrak{g}_{nc}+\mathfrak{n}]$, where $\mathfrak{g}_{nc}$ denotes a Levi factor of $\mathfrak{g}$. Thus Corollary (3.2) applies. □

The following proposition is a partial converse to Theorem (5.1).

PROPOSITION (5.2). – *Suppose that* G *is a connected simply-connected Lie group with Lie algebra* $\mathfrak{g}$ *satisfying* $[\mathfrak{g},\mathfrak{g}] \neq \mathfrak{g}$ *and that* G *is not solvable. Then there exists a Riemannian manifold* M *such that* G *acts simply transitively by isometries on* M *but is not normal in* $I_0(M)$.

*Proof.* – Let $\mathfrak{k}$ be a maximal compactly imbedded subalgebra of a Levi factor of $\mathfrak{g}$ and $\mathfrak{g}_1$ a codimension one ideal of $\mathfrak{g}$ containing $[\mathfrak{g},\mathfrak{g}]$. There exists a homomorphism $\lambda_1 : \mathfrak{g} \to \mathfrak{k}$ with kernel $\mathfrak{g}_1$. Denoting by K the connected subgroup of G with Lie algebra $\mathfrak{k}$, the simple-connectivity of G implies the existence of a homomorphism $\lambda : G \to K$ with $(d\lambda)_e = \lambda_1$. Denote the center of G by $G_z$ and set

$$D = \{(h,h) \in G \times K : h \in G_z \cap K\}.$$

Let

$$A = (G \times K)/D$$

with canonical projection $\pi : G \times K \to A$ and set

$$L = \{\pi((h,h)) : h \in K\}.$$

$L \simeq K/(G_z \cap K)$, hence is compact, and L contains no normal subgroups of A. M : = A/L may be given a left-invariant Riemannian metric, and A is then identified with a subgroup of $I_0(M)$. Define an imbedding $\eta : G \to A$ by $\eta(g) = \pi((g,\lambda(g)))$. $\lambda(K) = \{e\}$ since $\mathfrak{k} \subset [\mathfrak{g},\mathfrak{g}] \subset \mathfrak{g}$, so $\eta(G) \cap L$ is trivial. Under this imbedding G is a simply transitive subgroup of $I_0(M)$. However G is not normal in the subgroup A of $I_0(M)$. □

Suppose the group G in (5.2) has nilpotent radical so that A = GL satisfies the hypotheses of Theorem (3.1). In the notation of (3.1), $\mathfrak{a} \simeq \mathfrak{g} \oplus \mathfrak{k}$, where $\mathfrak{k}$ is the Lie algebra of K. However, $\mathfrak{g}$ is imbedded in

$\mathfrak{a}$ as $\{(X,\lambda_1(X)) : X \in \mathfrak{g}\}$. $\lambda_1|_{\mathfrak{n}}$ is non-trivial since $\mathfrak{g} = \mathfrak{g}_{ss} + \mathfrak{n}$ with $\mathfrak{g}_{ss} \subset [\mathfrak{g},\mathfrak{g}] \subset \ker \lambda_1$. Hence $\mathfrak{n}$ is not an $\mathfrak{a}$-ideal. But $\mathfrak{n} = \mathfrak{n}_2$ since $G \cap L = \{e\}$, so $\mathfrak{n}_2$ is not equal to the $\mathfrak{a}$-ideal $\mathfrak{n}'_2$. (See remark (3.2).)

## BIBLIOGRAPHY

[1] R. Azencott and E. N. Wilson, Homogeneous manifolds with negative curvature, Part I, *Trans. Amer. Math. Soc.*, 215 (1976), 323-362.

[2] R. Azencott and E. N. Wilson, Homogeneous manifolds with negative curvature, Part II, *Mem. Amer. Math. Soc.*, 8 (1976).

[3] C. Gordon, Riemannian isometry groups containing transitive reductive subgroups, *Math. Ann.*, 248 (1980), 185-192.

[4] S. Helgason, Differential geometry, Lie groups, and symmetric spaces, Academic Press, New York, 1978.

[5] N. Jacobson, Lie algebras, Wiley Interscience, New York, 1962.

[6] A. L. Oniščik, Inclusion relations among transitive compact transformation groups, *Amer. Math. Soc. Transl.*, 50 (1966), 5-58.

[7] H. Ozeki, On a transitive transformation group of a compact group manifold, *Osaka J. Math.*, 14 (1977), 519-531.

[8] E. N. Wilson, Isometry groups on homogeneous nilmanifolds, to appear in *Geometriae Dedicata*.

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