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ESTERLE'S PROOF OF THE TAUBERIAN THEOREM FOR BEURLING ALGEBRAS

by H. G. DALES and W. K. HAYMAN

1. Introduction.

In [5], J. Esterle gave a new proof of the Wiener Tauberian theorem for the algebra $L^1(\mathbf{R})$ by using some results from complex analysis and from the theory of radical Banach algebras. In this note, we show that a proof with the same idea also establishes the analogous result for Beurling algebras.

We first give the basic properties of the algebras of Beurling that we are considering.

Let φ be a non-negative, measurable function on $\mathbf{R}$, and set

$$L^1_\varphi = \{f : \|f\| = \int_{-\infty}^{\infty} |f(t)|e^{\varphi(t)}\,dt < \infty\}.$$

Then L^1_φ is a Banach space : as usual, we equate functions equal almost everywhere. If

$$\varphi(s+t) \leqslant \varphi(s) + \varphi(t) \quad (s,t \in \mathbf{R}), \tag{1}$$

then L^1_φ is a commutative Banach algebra with respect to convolution multiplication defined by the equation

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t-s)g(s)\,ds \quad (f,g \in L^1_\varphi).$$

These algebras were introduced by Beurling in 1938 [1].

Condition (1) ensures the existence of the finite limits $\alpha = \lim_{t\to\infty} \varphi(t)/t$ and $\beta = \lim_{t\to-\infty} \varphi(t)/t$. Let Π be the open strip $\{-\alpha < \mathrm{Re}\, z < -\beta\}$, and let $\bar{\Pi}$ be the closed strip $\{-\alpha \leqslant \mathrm{Re}\, z \leqslant -\beta\}$ of $\mathbf{C}$: if $\alpha = \beta$, then $\bar{\Pi}$ is a line. For $f \in L^1_\varphi$, we define the Laplace transform, $\hat{f}$, of f on $\bar{\Pi}$ by

$$\hat{f}(z) = \int_{-\infty}^{\infty} f(t)e^{-zt}\, dt \quad (z \in \bar{\Pi}).$$

The integral converges absolutely for $z \in \bar{\Pi}$. Let $A_0(\bar{\Pi})$ denote the uniform algebra of functions which are continuous on $\bar{\Pi}$, analytic on Π, and which converge uniformly to zero as $z \to \infty$ with $z \in \bar{\Pi}$. Then $\hat{f} \in A_0(\bar{\Pi})$. It is well known (for example, see [6], § 18) that the character space, or space of maximal modular ideals, of L^1_φ can be identified with $\bar{\Pi}$, and that the map $f \longmapsto \hat{f}$ is a monomorphism of L^1_φ into $A_0(\bar{\Pi})$.

Let I be a closed ideal of L^1_φ. We are interested in conditions on I which ensure that $I = L^1_\varphi$. Let

$$Z(I) = \{z \in \bar{\Pi} : \hat{f}(z) = 0 \quad (f \in I)\}.$$

Clearly, a necessary condition for the equality $I = L^1_\varphi$ is that $Z(I) = \varnothing$. Wiener posed the problem for the algebra $L^1(\mathbf{R})$ (for which $\varphi = 0$), and he proved that, if $Z(I) = \varnothing$, then $I = L^1(\mathbf{R})$. This is Wiener's Tauberian theorem; of course, the formulation of Wiener was different.

DEFINITION. — *Let* L^1_φ *be a Beurling algebra. Then spectral analysis holds for* L^1_φ *if each proper closed ideal of* L^1_φ *is contained in a maximal modular ideal of* L^1_φ.

Clearly, spectral analysis holds for L^1_φ if and only if $I = L^1_\varphi$ for each I with $Z(I) = \varnothing$, and Wiener's theorem is that spectral analysis holds for $L^1(\mathbf{R})$.

It was shown by Beurling in [1] that spectral analysis holds for the algebra L^1_φ provided that the weight φ satisfies (1) and the additional condition that

$$\int_{-\infty}^{\infty} \frac{\varphi(t)}{1+t^2}\, dt < \infty. \tag{2}$$

(Note that this condition implies that $\alpha = \beta = 0$, and so in this case we are identifying the character space of L^1_φ with the imaginary axis.)

Modern proofs of the theorem of Beurling use only the fact, ensured by (2), that the Banach algebra L^1_φ is regular, in the sense that, given $y_0 \in \mathbf{R}$ and a neighbourhood U of y_0, there exists $f \in L^1_\varphi$ with $\hat{f}(iy_0) = 1$ and $\hat{f}(iy) = 0$ $(y \notin U)$: see [6], § 40, for example, for a proof of the theorem given that L^1_φ is regular. Indeed, Gurarii ([7], page 24) states, « all proofs of Wiener's theorem known to us make essential use of this fact of regularity, and... it is hardly possible to manage without it. » Following the ideas of Esterle in [5], we shall prove Beurling's result without using the regularity of L^1_φ. It is not claimed that the present proof is any shorter than the usual one.

It is perhaps worth recalling how the regularity of L^1_φ follows from condition (2). The starting point is a result which is essentially Theorem XII of [10] : if φ is a non-negative, measurable function on $\mathbf{R}$, then a necessary and sufficient condition that there exists a function f which is bounded and analytic in the open upper half-plane Π^+ and which is such that $\lim_{y \to 0+} |f(x+iy)| = \exp(-\varphi(x))$ for almost all x is that φ satisfies (2). To show the sufficiency of (2), suppose that φ satisfies this condition, and define u on Π^+ by

$$u(x+iy) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\varphi(t)\, dt}{(x-t)^2 + y^2}.$$

Then u is harmonic on Π^+ and has non-tangential limits agreeing with φ at almost every point of $\mathbf{R}$. Let v be the harmonic conjugate of u, and set $f = \exp(-u-iv)$. This function f has the required properties.

To conclude the proof that L^1_φ is regular if φ satisfies condition (2), take $y_0 \in (a,b) \subset \mathbf{R}$. Construct a function f_0 which is analytic and bounded in Π^+ and which is such that

$$|f_0(x)| < \frac{e^{-\varphi(x)}}{1 + x^2} \quad (x \in \mathbf{R}).$$

Let $f_1(z) = f_0(z)/(z+i)$, so that $f_1|\mathbf{R} \in L^1_\varphi$. Also, $|f_1(z)| \to 0$ as $z \to \infty$ in Π^+, and so $\hat{f}_1(iy) = 0$ for $y \leqslant 0$. We can clearly choose $\alpha \in \mathbf{R}$ so that, if $g_1(x) = f_1(x)e^{i\alpha x}$, then $\hat{g}_1(iy_0) \neq 0$ and $\hat{g}_1(iy) = 0$ $(y < a)$. Similarly, there exists $g_2 \in L^1_\varphi$ with $\hat{g}_2(iy_0) \neq 0$ and $\hat{g}_2(iy) = 0$ $(y > b)$. If $h = g_1 * g_2$, then $h \in L^1_\varphi$, $h(iy_0) \neq 0$, and $h(iy) = 0$ $(y \notin (a,b))$. This shows that L^1_φ is regular.

In fact, the Banach algebra L^1_φ is regular if and only if condition (2) holds. The strongest result of this type is the famous theorem of Beurling and Malliavin [2] which shows that, if φ is a non-negative, measurable function on $\mathbf{R}$, then the following two conditions on φ are equivalent :

(i) for each $a > 0$, the Banach space L^1_φ contains a non-zero element whose Fourier transform has support in $[-ia,ia]$;

(ii) φ satisfies (2) and the condition that

$$\text{ess sup } \{|\varphi(s+t) - \varphi(s)| : \ s \in \mathbf{R}\} < \infty \quad (t \in \mathbf{R}).$$

Let φ be a function satisfying (1), and let α and β be the limits defined above. The algebra L^1_φ is termed *analytic* if $\beta > \alpha$. If $\alpha = \beta = 0$, then L^1_φ is *quasi-analytic* if the integral in (2) diverges, and L^1_φ is *non-quasi-analytic* if condition (2) holds. Thus, our theorem is that spectral analysis holds in the non-quasi-analytic case.

In fact, spectral analysis fails in both the analytic and in the quasi-analytic cases. This was first proved by Vretblad in [11] provided that φ satisfies some slight extra conditions. We are grateful to Professor Yngve Domar for pointing out that the proof of Theorem 4 in [4] implicitly shows this result without any extra conditions on φ. Thus, spectral analysis holds for the Beurling algebra L^1_φ if and only if φ satisfies condition (2).

In the special case that $\varphi(t) = \alpha|t|$ for a positive constant α, the family of all proper closed ideals of L^1_φ which are not contained in any maximal modular ideal was described by Korenblum ([9]). The family does not seem to have been fully described in more general cases : see [7] and [11] for the best partial results.

2. The proof.

THEOREM. — *Let φ be a non-negative, measurable function on $\mathbf{R}$ which satisfies* (1) *and* (2). *Then spectral analysis holds for the Banach algebra L^1_φ.*

The proof of this theorem depends heavily on a recent result given in [8] which we first describe. We write Δ for the open unit disc, and, for each $\sigma \in \mathbf{R}$, we write Π_σ for the open right half-plane $\{(x,y) : x > \sigma\}$.

LEMMA 1. — *Let k be a positive, continuous, increasing function on* $[0,1)$. *Let f be analytic on Δ and satisfy the condition that*

$$\log |f(re^{i\theta})| \leqslant k(r) \quad (re^{i\theta} \in \Delta). \tag{3}$$

If

$$\int_0^1 \left(\frac{k(r)}{1-r}\right)^{\frac{1}{2}} dr < \infty, \tag{4}$$

then either $f = 0$, *or* $\limsup_{r \to 1-} (1-r) \log |f(r)| > -\infty$.

Proof. – Theorem 5 of [8] shows that, under the hypotheses (3) and (4), there exists an analytic function g on Δ such that :

(i) g is real and increasing on $[0,1)$, with $g(r) \to 1$ as $r \to 1-$;

(ii) $g(\Delta) \subset \Delta$;

(iii) $\sup \{|1-g(r)|/|1-r| : r \in [0,1)\} < \infty$;

(iv) $f \circ g$ has bounded (Nevanlinna) characteristic in Δ.

It follows from (ii) and (iii) by the theory of the angular derivative that

$$\lim_{r \to 1-} \frac{1-g(r)}{1-r} \tag{5}$$

exists in $(0,\infty)$. (The existence of this limit can also be seen from the explicit construction of g in [8], pp. 192-193.)

Suppose that $f \neq 0$. By (iv), there exist bounded, non-zero, analytic functions, say h_1 and h_2, on Δ such that $f \circ g = h_1/h_2$ on Δ. If $\limsup_{r \to 1-} (1-r) \log |(f \circ g)(r)| = -\infty$, then $\limsup_{r \to 1-} (1-r) \log |h_1(r)| = -\infty$, and so, by a result of Phragmén-Lindelöf type ([3], 1.4.3, transferred from Π_0 to Δ), $h_1 = 0$, a contradiction. It follows that $\limsup_{r \to 1-} (1-r) \log |(f \circ g)(r)| > -\infty$.

The lemma follows from the existence of the finite non-zero limit given by (5).

Condition (4) in the above lemma is necessary in the sense that, if the integral in (4) diverges, then there exists a non-zero analytic function f on Δ satisfying (3) and such that $(1-r) \log |f(r)| \to -\infty$ as $r \to 1-$: see [8], Theorem 4.

We transform this result to the half-plane Π_1. Throughout, if K is a positive, continuous function on $[1,\infty)$, we set

$$J(K) = \int_1^\infty \left(\frac{K(R)}{R^3}\right)^{\frac{1}{2}} dR.$$

LEMMA 2. — *Let* K *be a positive, continuous, increasing function on* $[1,\infty)$ *such that* $J(K) < \infty$.

Let F *be analytic on* Π_1, *and let* F *satisfy the condition that*

$$\log |F(\rho e^{i\psi})| \leqslant K\left(\frac{\rho}{\cos \psi}\right) \quad (\rho e^{i\psi} \in \Pi_1).$$

Then either $F = 0$, *or* $\limsup_{\rho\to\infty} \rho^{-1} \log |F(\rho)| > -\infty$.

Proof. — Let $\zeta = \xi + i\eta = \rho e^{i\psi}$ belong to Π_1, and let $z = (\zeta-3)/(\zeta+1)$ define a conformal map of Π_1 onto Δ. Then $\zeta = (3+z)/(1-z)$. Let $f(z) = F(\zeta)$, so that f is an analytic function on Δ. If $|z| = r < 1$, then

$$r^2 = \left|\frac{\zeta-3}{\zeta+1}\right|^2 = 1 - \frac{8(\xi-1)}{(\xi+1)^2+\eta^2} > 1 - \frac{8\xi}{\xi^2+\eta^2},$$

so that

$$\frac{\rho}{\cos \psi} = \frac{\xi^2+\eta^2}{\xi} < \frac{8}{1-r^2} < \frac{8}{1-r}.$$

Hence, $\log |f(re^{i\theta})| \leqslant k(r)$ for $re^{i\theta} \in \Delta$, where

$$k(r) = K\left(\frac{8}{1-r}\right).$$

Then k is a positive, continuous, increasing function on $[0,1)$, and

$$\int_0^1 \left(\frac{k(r)}{1-r}\right)^{\frac{1}{2}} dr = 8^{\frac{1}{2}} \int_8^\infty \left(\frac{K(R)}{R^3}\right)^{\frac{1}{2}} dR,$$

and so k satisfies condition (4). By Lemma 1, either $f = 0$ or $\limsup_{r\to 1-} (1-r) \log |f(r)| > -\infty$. In the former case, $F = 0$, and in the latter case, $\limsup_{\rho\to\infty} \rho^{-1} \log |F(\rho)| > -\infty$, as required.

If F is an analytic function on Π_0 such that $\sup \{\exp(-|z|^\alpha)|F(z)|\} < \infty$ for some $\alpha < 1$, then, by applying Lemma 2 with $K(R) = R^\alpha$, we can deduce that either $F = 0$, or

$\limsup_{\rho\to\infty} \rho^{-1} \log |F(\rho)| > -\infty$. This is Corollary 2.2 of [5], and the theorem of Esterle followed from that Corollary. The present more general result will require the stronger Lemma 2.

Now, following [5], we introduce the functions a^ζ:

$$a^\zeta(t) = \frac{1}{\sqrt{\pi\zeta}} \exp\left(-\frac{t^2}{\zeta}\right) \qquad (\zeta \in \Pi_0,\ t \in \mathbf{R}).$$

Since $\varphi(t) = O(|t|)$ as $|t| \to \infty$, we have $a^\zeta \in L^1_\varphi$ for each $\zeta \in \Pi_0$. It is well known and straightforward to check that the map $\zeta \longmapsto a^\zeta$, $\Pi_0 \to L^1_\varphi$, is a semigroup monomorphism and an analytic map. We must calculate $\|a^\zeta\|$ in L^1_φ. We first give a technical lemma.

LEMMA 3. – *Let* φ *be a non-negative, measurable function on* $\mathbf{R}$ *satisfying* (1) *and such that* $\int_0^\infty (1+t^2)^{-1}\varphi(t)\,dt < \infty$.

(i) *If* $\varphi_1(t) = \max\{\varphi(s) : 0 \leqslant s \leqslant t\}$ $(t \in \mathbf{R}^+)$, *then* φ_1 *is monotone increasing on* $\mathbf{R}^+$, $\varphi_1(t) \geqslant \varphi(t)$ $(t \in \mathbf{R}^+)$, *and* $\int_1^\infty t^{-2}\varphi_1(t)\,dt < \infty$.

(ii) *If* $\varphi_2(t) = t \max\{s^{-1}\varphi_1(s) : s \geqslant t\}$ $(t \in \mathbf{R}^+)$, *then* $t^{-2}\varphi_2(t)$ *is a monotone decreasing function of* t *on* $\mathbf{R}^+$, $\varphi_2(t) \geqslant \varphi_1(t)$ $(t \in \mathbf{R}^+)$, *and* $\int_1^\infty t^{-2}\varphi_2(t)\,dt < \infty$.

Proof. – These results are obvious or are proved clearly in Lemmas 3.3 and 3.4 of [7]; they are originally due to Beurling.

LEMMA 4. – *Let* φ *be a non-negative, measurable function on* $\mathbf{R}$ *satisfying* (1) *and* (2). *Then there exists a positive, continuous, increasing function* K *on* $[1, \infty)$ *with* $J(K) < \infty$ *such that*

$$\log \|a^\zeta\| \leqslant K\left(\frac{\rho}{\cos\psi}\right) \qquad (\zeta = \rho e^{i\psi} \in \Pi_1). \tag{7}$$

Here, $\|a^\zeta\|$ *is calculated in* L^1_φ.

Proof. – Let $\zeta = \rho e^{i\psi} \in \Pi_1$. We have

$$\|a^\zeta\| = \frac{1}{\sqrt{\pi\rho}} \int_{-\infty}^{\infty} \exp\left(-\frac{t^2}{\rho}\cos\psi + \varphi(t)\right) dt.$$

Since $\rho \geqslant 1$,

$$\|a^{\zeta}\| \leqslant \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{t^2}{R} + \varphi(t)\right) dt$$
$$= \exp K(R), \text{ say,}$$

where $R = \rho/\cos\psi \geqslant 1$. Clearly, replacing K by $\sup\{K,0\}$, we can suppose that K is positive, continuous, and increasing on $[1,\infty)$. To show that $J(K) < \infty$, it suffices to show that $J(\log^+ \kappa) < \infty$, where

$$\kappa(R) = \int_0^{\infty} \exp\left(-\frac{t^2}{R} + \varphi(t)\right) dt = R^{\frac{1}{2}} \int_0^{\infty} \exp\,(-s^2 + \varphi(R^{\frac{1}{2}} s))\, ds.$$

Let φ_1 and φ_2 be as specified in Lemma 3. We can suppose that $\varphi_2(1) = 1$. For each $R \geqslant 1$, let

$$\mu(R) = \sup\{t : 2\varphi_2(t)R \geqslant t^2\}, \quad \nu(R) = R^{-\frac{1}{2}}\mu(R).$$

Then $\nu(R)$ is the supremum of the solutions of the inequality $\varphi_2(R^{\frac{1}{2}} s) \geqslant \frac{1}{2}s^2$. Since $\varphi(t) = O(t)$ as $t \to \infty$, $\mu(R) = O(R)$ as $R \to \infty$.

If $s \geqslant \nu(R)$, then $\varphi(R^{\frac{1}{2}} s) \leqslant \varphi_2(R^{\frac{1}{2}} s) \leqslant \frac{1}{2}s^2$, and so

$$\int_{\nu(R)}^{\infty} \exp\,(-s^2 + \varphi(R^{\frac{1}{2}} s))\, ds \leqslant \int_0^{\infty} \exp\,(-\tfrac{1}{2}s^2)\, ds < \infty.$$

If $s \leqslant \nu(R)$, then $\varphi(R^{\frac{1}{2}} s) \leqslant \varphi_1(R^{\frac{1}{2}} s) \leqslant \varphi_1(\mu(R)) \leqslant \varphi_2(\mu(R)) \leqslant \frac{1}{2}R^{-1}(\mu(R))^2$, and so

$$\int_0^{\nu(R)} \exp\,(-s^2 + \varphi(R^{\frac{1}{2}} s))\, ds \leqslant R^{-\frac{1}{2}}\mu(R) \exp\left[\frac{(\mu(R))^2}{2R}\right].$$

Thus, $\log \kappa(R) \leqslant \frac{1}{2}R^{-1}(\mu(R))^2 + O(\log R)$ as $R \to \infty$, and so

$$J(\log^+ \kappa) \leqslant \int_1^{\infty} \frac{\mu(R)}{R^2}\, dR + O(1) \quad \text{as } R \to \infty.$$

Using the definition of $\mu(R)$ and Lemma 3, we see that

$$\int_1^\infty \frac{\mu(R)}{R^2}\, dR - 1 = \int_1^\infty \frac{d\mu(R)}{R} = 2 \int_1^\infty \frac{\varphi_2(t)}{t^2}\, dt < \infty .$$

Thus, $J(\log^+ \kappa) < \infty$, as required.

LEMMA 5. — *If* A *is a radical Banach algebra, and if* (a^t) *is a continuous semigroup in* A *over* $\mathbf{R}^+$, *then* $\lim_{t\to\infty} t^{-1} \log \|a^t\| = -\infty$.

Proof. — This is [5], Lemma 2.3.

We now conclude the proof of the theorem.

Let I be a closed ideal of L^1_φ. We must show that, if I is not contained in a maximal modular ideal of L^1_φ, then $I = L^1_\varphi$. Let $A = L^1_\varphi/I$. Then the hypothesis is that A is a radical Banach algebra.

Let (a^ζ) be the analytic semigroup in L^1_φ given above, and let $[a^\zeta]$ be the coset of a^ζ in A. Let $\lambda \in A'$, the dual space of A, and set

$$\Phi(\zeta) = \langle [a^\zeta], \lambda \rangle \qquad (\zeta \in \Pi_0).$$

Then Φ is an analytic function over Π_0, and

$$|\Phi(\zeta)| \leqslant \|\lambda\|\,\|[a^\zeta]\| \leqslant \|\lambda\|\,\|a^\zeta\| \quad (\zeta \in \Pi_0).$$

By Lemma 4, there is a function K such that $J(K) < \infty$ and such that $\log |\Phi(\zeta)| \leqslant K(R)$ for $\zeta \in \Pi_1$, where $\zeta = \rho e^{i\psi}$ and $R = \rho/\cos \psi$. By Lemma 5, $\lim_{\rho\to\infty} \rho^{-1} \log |\Phi(\rho)| = -\infty$, and so, by Lemma 2, $\Phi = 0$. This shows that $[a^\zeta] = 0$ in A, and hence that $a^\zeta \in I$ for $\zeta \in \Pi_0$. However, for each $f \in L^1_\varphi$, $f = \lim_{\rho \to 0+} f * a^\rho$, and so $f \in \bar{I} = I$. Thus $I = L^1_\varphi$, as required.

The use of Lemma 2 in the above theorem seems to be necessary. For example, consider the case that $\varphi(t) = |t|^\beta$, where $0 < \beta < 1$, and take (a^ζ) as above. Then the best estimate of $\|a^\zeta\|$ in terms of $\rho = |\zeta|$ which we can obtain is that $\log \|a^\zeta\| = O(\rho^{2\beta/(2-\beta)})$ as $\rho \to \infty$ with $\zeta \in \Pi_1$: here we are using the fact that $1/\cos\theta \leqslant \rho$ for $\zeta \in \Pi_1$. We can thus apply [5], Corollary 2.2, only if $2\beta/(2-\beta) < 1$, that is, if $\beta < 2/3$, whereas the result holds if $\beta < 1$.

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