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ESTIMATES OF ONE-DIMENSIONAL OSCILLATORY INTEGRALS

by Detlef MÜLLER

1. Introduction.

If U is an open domain in $\mathbf{R}^k$ and if f is a smooth, real valued function on U, one may define the associated oscillatory integral as

$$E_f(\vartheta) = \int_U \vartheta(x) e^{2\pi i f(x)}\, dx,$$

where ϑ belongs to $\mathscr{D}(U)$, the space of testfunctions on U.

When f has the form $f = \sum_{j=1}^{n} \eta_j \psi_j$, where the $\psi_j \in C^\infty(U)$ are real-valued functions and η_j are real parameters, one is interested in the asymptotic behaviour of $E_{\Sigma\eta_j\psi_j}(\vartheta)$ as $(\eta_1,\ldots,\eta_n)$ tends to infinity, for several reasons.

For example, if μ is a smooth measure on a smooth submanifold of $\mathbf{R}^m$, and if the support of μ is sufficiently small, then the Fourier-Stieltjes transform $\hat{\mu}(\eta_1,\ldots,\eta_n)$ may always be written as $E_{\Sigma\eta_j\psi_j}(\vartheta)$ for certain functions ψ_j and ϑ.

Good information about the asymptotic behaviour of such Fourier-Stieltjes transforms is needed to solve the synthesis problem for smooth submanifolds of $\mathbf{R}^m$ (see e.g. [7]). And, as Professor Y. Domar has pointed out to me, such knowledge would also yield information about the decay at infinity of solutions of partial differential equations (see e.g. [5]).

As far as I know, satisfactory aswers to the above problem have only been given for oscillatory integrals $E_{\Sigma\eta_j\psi_j}(\vartheta)$ with

$$\Sigma\eta_j\psi_j(x_1,\ldots,x_k) = \sum_{j=1}^{k} \eta_j x_j + \eta_{k+1}\psi_{k+1}(x_1,\ldots,x_k),$$

which correspond to surface carried measures (see [2], [4], [6]). In some sense, the other extreme is the case where $\Sigma\eta_j\psi_j$ is a function of only one real variable, which corresponds to measures on curves. For this case, we will prove some quite general results.

2.

Let $\psi \in C^\infty(I,\mathbf{R}^n)$, $\psi = (\psi_1,\ldots,\psi_n)$, where $I \neq \emptyset$ is some bounded open interval in $\mathbf{R}$. For ξ, $\eta \in \mathbf{R}^n$ let $\xi\cdot\eta$ denote the Euclidean inner product on $\mathbf{R}^n$, and correspondingly let

$$\eta\cdot\psi(x) = \sum_{j=1}^{n} \eta_j\psi_j(x).$$

Further let

$$|\eta| := \max_j |\eta_j| \qquad \text{for} \qquad \eta \in \mathbf{R}^n.$$

Define the *torsion* τ of ψ by

$$\tau(x) = \det\,(\psi_j^{(i+1)}(x))_{i,j=1,\ldots,n} = \det\,(\psi''(x)\psi'''(x)\ldots\psi^{(n+1)}(x)),$$

where ψ is regarded as a column vector and $\psi^{(k)}$ denotes the k – th derivative of ψ. At least for $n = 2$ we have $\tau(x) = k(x)|\psi''(x)|^2$, where k is the torsion of the curve $\gamma = \{(x,\psi(x)) : x \in I\}$ in $\mathbf{R}^{n+1}$. Let

$$e(t) = e^{2\pi i t} \qquad \text{for} \qquad t \in \mathbf{R}, \qquad \text{and} \qquad e(g) = e \circ g$$

for $g \in C^\infty(I,\mathbf{R})$. If $\psi_0(x) = x$ for $x \in \mathbf{R}$, then for $\vartheta \in \mathscr{D}(I)$, $\eta_0 \in \mathbf{R}$ and $\eta = (\eta_1,\ldots,\eta_n) \in \mathbf{R}^n$, we have

$$E_{\sum_0^n \eta_j\psi_j}(\vartheta) = (\vartheta e(\eta.\psi))\hat{}\,(-\eta_0).$$

So it will be slightly more general to study the behaviour of $|\vartheta e(\eta\cdot\psi)|_{PM}$ as $|\eta| \to \infty$, where

$$|\varphi|_{PM} = \sup_{t\in\mathbf{R}} |\hat{\varphi}(t)|$$

for every $\varphi \in \mathscr{D}(\mathbf{R})$.

For certain reasons (see [3]; [7], Th. 4.1), we will also study $|\vartheta e(\eta\cdot\psi)|_A$, where

$$|\varphi|_A = \int |\hat{\varphi}(t)|\, dt$$

for every $\varphi \in \mathscr{D}(\mathbf{R})$.

We will fiirst state our main results and prove some corollaries:

THEOREM 1. — *Let* $\vartheta \in \mathscr{D}(I)$. *Then*

(i) $|\vartheta e(\eta\cdot\psi)|_A = 0(|\eta|^{\frac{1}{2}})$, *as* $|\eta| \to \infty$.

(ii) *If for some subinterval* J *of* I *and some* $\sigma > 0$

$$|\vartheta(x)| \geqslant \sigma \quad \text{and} \quad |\vartheta(x) - \vartheta(y)| < \sigma/2 \quad \text{for all} \quad x, y \in J,$$

and if $\psi_1|_J, \ldots, \psi_n|_J$ *are linearly independent modulo affine linear functions, then there is a constant* $C > 0$, *such that*

$$|\vartheta e(\eta\cdot\psi)|_A \geqslant C(1+|\eta|)^{\frac{1}{2}}$$

for all $\eta \in \mathbf{R}^n$.

COROLLARY 1. — *The following two conditions are equivalent:*

(i) *For each* $\vartheta \in \mathscr{D}(\mathbf{R})$, $\vartheta \neq 0$, *there are constants* $c > 0$, $C > 0$, *such that for all* $\eta \in \mathbf{R}^n$

$$c(1+|\eta|)^{\frac{1}{2}} \leqslant |\vartheta e(\eta\cdot\psi)|_A \leqslant C(1+|\eta|)^{\frac{1}{2}}.$$

(ii) $\psi_1, \ldots, \psi_n$ *are linearly independent modulo affine linear functions on every non empty open subinterval of* I.

Proof of Corollary 1. — (i) follows directly from (ii) by Theorem 1. Now suppose that there exists a vector $\nu \in \mathbf{R}^n$, $\nu \neq 0$, such that $\nu\cdot\psi$ is affine linear on some open subinterval $\mathscr{J} \neq \emptyset$ of I. Then we have for any non-trivial $\vartheta \in \mathscr{D}(\mathscr{J})$

$$|\vartheta e(s\nu\cdot\psi)|_A = |\vartheta|_A \neq 0 \qquad \text{for all} \qquad s \in \mathbf{R},$$

since $e(s\nu\cdot\psi)$ is the product of a unimodular complex number and a unitary character of $\mathbf{R}$.

Thus (i) is not fulfilled, q.e.d.

Remark. — Condition (ii) of Corollary 1 is clearly satisfied if $\tau^{-1}(\{0\})$ has empty interior. As will be shown later (Lemma 3), this is always the case if $\psi_1, \ldots, \psi_n$ are real analytic and linearly independent modulo affine mappings. However one should notice that global linear independence does not in general imply local linear independence.

THEOREM 2. — *(i) If* $\tau^{-1}(\{0\}) = \emptyset$, *then for* $\vartheta \in \mathcal{D}(\mathrm{I})$

$$|\vartheta e(\eta \cdot \psi)|_{\mathrm{PM}} = 0(|\eta|^{-1/(n+1)}) \qquad as \qquad |\eta| \to \infty .$$

(ii) If $\vartheta \in \mathcal{D}(\mathrm{I})$, *and if there exists an* $x_0 \in \mathrm{I}$ *with* $\vartheta(x_0) \neq 0$ *and* $\tau(x_0) \neq 0$, *then there exists an* $\varepsilon > 0$ *and a function* $\xi \in \mathrm{C}^\infty((-\varepsilon,\varepsilon), \mathbf{R}^n)$ *with*

$$\det(\xi(y)\xi'(y) \ldots \xi^{(n-1)}(y)) \neq 0 \qquad for\ all \qquad y \in (-\varepsilon,\varepsilon),$$

such that, for some $\mathrm{C} > 0$,

$$|\vartheta e(s\xi(y) \cdot \psi)|_{\mathrm{PM}} \geqslant \mathrm{C}(1+|s|)^{-1/(n+1)}$$

for all $s \in \mathbf{R}$ *and* $y \in (-\varepsilon,\varepsilon)$.

Assume that $\tau^{-1}(\{0\})$ has empty interior. Then we have

COROLLARY 2. — *There exists a* $\vartheta \in \mathcal{D}(\mathrm{I})$, $\vartheta \neq 0$, *such that for all positive* $\alpha_1, \ldots, \alpha_n \in \mathbf{R}$ *with* $\sum_1^n \alpha_j \leqslant (n+1)^{-1}$, *there exists a constant* $\mathrm{C} = \mathrm{C}(\alpha_1, \ldots, \alpha_n) > 0$ *such that*

$$|\vartheta e(\eta \cdot \psi)|_{\mathrm{PM}} \leqslant \mathrm{C} \prod_{j=1}^n |\eta_j|^{-\alpha_j}. \tag{2.1}$$

Conversely, if $\alpha_1, \ldots, \alpha_n \in \mathbf{R}$ *are positive, and if there exists a* $\vartheta \in \mathcal{D}(\mathrm{I})$, $\vartheta \neq 0$, *and a* $\mathrm{C} > 0$ *such that* (2.1) *holds, then*

$$\sum_1^n \alpha_j \leqslant (n+1)^{-1}.$$

Proof of Corollary 2. — If $\tau^{-1}(\{0\})$ has empty interior, then there is of course an $x_0 \in \mathrm{I}$ with $\tau(x_0) \neq 0$, and so, for $\vartheta \in \mathcal{D}(\mathrm{I})$ with sufficiently small support near x_0,

$$|\vartheta e(\eta \cdot \psi)|_{\mathrm{PM}} \leqslant \mathrm{C}(1+|\eta|)^{-1/(n+1)}$$

by Theorem 2, (i).

If $\alpha_1, \ldots, \alpha_n$ are positive and $\Sigma\alpha_j \leqslant (n+1)^{-1}$, then

$$\prod_j |\eta_j|^{\alpha_j} \leqslant |\eta|^{1/(n+1)} \qquad \text{for} \qquad |\eta| \geqslant 1,$$

hence

$$|\vartheta e(\eta \cdot \psi)|_{\mathrm{PM}} \leqslant \mathrm{C} \prod_j |\eta_j|^{-\alpha_j} \qquad \text{for} \qquad |\eta| \geqslant 1,$$

and the same estimate holds for all η if one replaces C by $\mathrm{C} + |\vartheta|_{\mathrm{L}^1}$.

Conversely, let now $\vartheta \in \mathscr{D}(\mathrm{I})$, $\vartheta \neq 0$, such that (2.1) holds for some $\alpha_j \geqslant 0$, and assume

$$\Sigma\alpha_j = (n+1)^{-1} + \delta, \qquad \delta > 0.$$

Since $\tau^{-1}(\{0\})$ has empty interior, there is an $x_0 \in \mathrm{I}$ with $\vartheta(x_0) \neq 0$ and $\tau(x_0) \neq 0$. Choose $\varepsilon > 0$ and $\xi \in \mathrm{C}^\infty((-\varepsilon,\varepsilon),\mathbf{R}^n)$ as in Theorem 2 (ii). Since $\det(\xi(y)\xi'(y) \ldots \xi^{(n-1)}(y)) \neq 0$ for all $y \in (-\varepsilon,\varepsilon)$, there exists a $y_0 \in (-\varepsilon,\varepsilon)$ with

$$\xi_j(y_0) \neq 0 \qquad \text{for} \qquad j = 1, \ldots, n.$$

It follows

$$|\vartheta e(s\xi(y_0)\cdot\psi)|_{\mathrm{PM}} \geqslant \mathrm{C}'(1+|s|)^{-1/(n+1)}.$$

On the other hand, (2.1) yields

$$|\vartheta e(s\xi(y_0)\cdot\psi)|_{\mathrm{PM}} \leqslant \mathrm{C} \prod_j |s\xi_j(y_0)|^{-\alpha_j}$$
$$= \left(\mathrm{C} \prod_j |\xi_j(y_0)|^{-\alpha_j}\right)|s|^{-1/(n+1)}|s|^{-\delta}.$$

For $|s|$ sufficiently large this leads to a contradiction to (2.2), q.e.d.

Corollary 2 demonstrates that the result in Theorem 2 is in some sense best possible.

3.

Before we start to prove the theorems above we will state some lemmas. The first one is due to J.-E. Björk and is cited in [3], Lemma 1.6:

LEMMA 1. — *Let* $\mathrm{I} \neq \varnothing$ *be a bounded, open interval in* $\mathbf{R}$, *and let* $\varphi \in \mathscr{D}(\mathrm{I})$, $g \in \mathrm{C}^p(\mathrm{I})$ *with*

$$0 < \mathrm{C}_1 \leqslant |g'(x)| + |g''(x)| + \cdots + |g^{(p)}(x)| \leqslant \mathrm{C}_2$$

if $x \in \bar{\mathrm{I}}$, *where* C_1 *and* C_2 *are constants and* p *is a positive integer. Then there exists a constant* C *not depending on* g, *such that*

$$\left|\int \varphi(x)e^{2\pi i t g(x)}\,dx\right| \leqslant C(1+|t|)^{-1/p}$$

for every $t \in \mathbf{R}$.

The second lemma will be used to prove the remark following Corollary 1. I would like to thank Professor H. Leptin for pointing out to me a shorter proof than my original one. By « $\wedge$ » we denote the exterior product in the Grassmann algebra $\Lambda(\mathbf{R}^n)$.

LEMMA 2. – *Let* $\psi \in C^\infty(\mathrm{I},\mathbf{R}^n)$. *Then*

$$\psi(x) \wedge \psi'(x) \ldots \wedge \psi^{(n-1)}(x) = 0$$

for all $x \in \mathrm{I}$ *implies*

$$\psi^{(k_1)}(x) \wedge \psi^{(k_2)}(x) \wedge \ldots \wedge \psi^{(k_n)}(x) = 0$$

for all $x \in \mathrm{I}$ *and* $k_1, \ldots, k_n \in \mathbf{N}_0$.

Proof. – Fix $x_0 \in \mathrm{I}$, and assume first $\psi(x_0) \neq 0$. If $u \in C^\infty(\mathrm{I},\mathbf{R})$, then

$$(u\psi)^{(k)} = \sum_{j=0}^{k} \binom{k}{j} u^{(k-j)}\psi^{(j)},$$

so $\psi \wedge \psi' \wedge \ldots \wedge \psi^{(n-1)} \equiv 0$ implies

$$(u\psi) \wedge (u\psi)' \wedge \ldots \wedge (u\psi)^{(n-1)} \equiv 0.$$

So, it is no loss of generality to assume

$$\psi_n(x) = 1 \qquad \text{for} \qquad x \in \mathrm{I}.$$

If $\{e_j\}_j$ denotes the canonical basis of $\mathbf{R}^n$, we may thus write $\psi(x) = \sum_{j=1}^{n-1} \psi_j(x)e_j + e_n = \rho(x) + e_n$, where $\rho(x) \in \mathbf{R}^{n-1} \times \{0\} \subset \mathbf{R}^n$. This yields

$$0=\psi(x)\wedge \psi'(x) \wedge \ldots \wedge \psi^{n-1}(x)=\rho(x) \wedge \rho'(x) \wedge \ldots \wedge \rho^{(n-1)}(x) + e_n \wedge \rho'(x) \wedge \ldots \wedge \rho^{(n-1)}(x),$$

and since $\rho(x)$, $\rho'(x), \ldots, \rho^{(n-1)}(x)$ are clearly linearly dependent, we get

$$0 = \rho'(x) \wedge \rho''(x) \wedge \ldots \wedge \rho^{(n-1)}(x).$$

By induction over n, we now may assume

$$0 = \rho^{(k_2)}(x) \wedge \rho^{(k_3)}(x) \wedge \ldots \wedge \rho^{(k_n)}(x)$$

for $x \in \mathrm{I}$ and $k_j \geqslant 1$.

This implies

$$\psi^{(k_1)}(x) \wedge \ldots \wedge \psi^{(k_n)}(x) = e_n^{(k_1)}(x) \wedge \rho^{(k_2)}(x) \wedge \ldots \wedge \rho^{(k_n)}(x) = 0$$

for $0 \leqslant k_1 < k_2 < \cdots < k_n$, where we considered e_n as the function $e_n(x) = e_n$.

Thus we have proved

$$\psi^{(k_1)}(x_0) \wedge \psi^{(k_2)}(x_0) \wedge \ldots \wedge \psi^{(k_n)}(x_0) = 0$$

for all $x_0 \in \mathrm{I}_0 = \{x \in \mathrm{I} : \psi(x) \neq 0\}$ and $k_j \geqslant 0$. By continuity, the same holds true for $x_0 \in \overline{\mathrm{I}}_0 \wedge \mathrm{I}$, hence for all $x_0 \in \mathrm{I}$, since for $y \in \mathrm{I} \backslash \overline{\mathrm{I}}_0$ clearly $\psi^{(k)}(y) = 0$ for every $k \in \mathbf{N}_0$.

LEMMA 3. — *If* $\psi = (\psi_1, \ldots, \psi_n) \in \mathrm{C}^\infty(\mathrm{I}, \mathbf{R}^n)$ *is real analytic, and if* $\psi_1, \ldots, \psi_n$ *are linearly independent modulo affine mappings, then* $\tau^{-1}(\{0\})$ *has empty interior, where* τ *denotes the torsion of* ψ.

Proof. — Assume $\tau(x) = 0$ for every x in some nonempty open interval $\mathrm{J} \subset \mathrm{I}$. Fix $x_0 \in \mathrm{J}$. Then, passing to a possibly smaller interval, we may assume that ψ_j has an absolute convergent series expansion

$$\psi_j(x) = \sum_{k=0}^{\infty} a_k^j (x - x_0)^k, \qquad j = 1, \ldots, n, \qquad x \in \mathrm{J}.$$

Define vectors

$$a_k = (a_k^j)_{j=1,\ldots,n} \in \mathbf{R}^n$$

and

$$a^j = (a_k^j)_{k=2,\ldots,\infty} \in \mathbf{R}^{\mathbf{N}_1}, \qquad \mathbf{N}_1 = \mathbf{N} \backslash \{0,1\}.$$

By Lemma 2, $\psi^{(k_1)}(x_0), \ldots, \psi^{(k_n)}(x_0)$ are linearly dependent for any $k_j \in \mathbf{N}$ with $2 \leqslant k_1 < \ldots < k_n$, i.e. $a_{k_1}, \ldots, a_{k_n}$ are linearly dependent for $2 \leqslant k_1 < \ldots < k_n$. But this implies that $a^1, \ldots, a^n$ are linearly dependent, i.e. there exist $\nu_1, \ldots, \nu_n \in \mathbf{R}$, not all zero, with

$$0 = \sum_j \nu_j a^j, \quad \text{i.e.}$$

$$\sum_j \nu_j \psi_j(x) = \sum_j \nu_j a_0^j + \nu_j a_1^j (x - x_0) \qquad \text{for} \qquad x \in \mathrm{J}.$$

But, since ψ is real analytic, this equation holds for all $x \in \mathrm{I}$, i.e. $\sum_j \nu_j \psi_j$ is affine linear.

4.

Proof of Theorem 1. — It is well-known (see e.g. [1], [7]) that for $\varphi \in \mathscr{D}(\mathbf{R})$ one has the estimate

$$(4.1) \qquad |\varphi|_{\mathrm{A}} \leqslant \{2\,|\mathrm{supp}\,\varphi|\,|\varphi|_\infty\,|\varphi'|_\infty\}^{1/2},$$

where $|\mathrm{supp}\,\varphi|$ denotes the Lebesgue measure of the support of φ. From (4.1) one immediately gets (i) of Theorem 1.

Now, suppose there exists a subinterval J in I and a $\sigma > 0$ such that $|\vartheta(x)| \geqslant \sigma$ and $|\vartheta(x) - \vartheta(y)| < \sigma/2$ for $x, y \in \mathrm{J}$, and such that $\psi_1, \ldots, \psi_n$ are linearly independent modulo affine mappings on J. Then a simple compactness argument yields :

There are constants $\varepsilon > 0$, $\delta > 0$, such that for every $\eta \in \mathbf{R}^n$ with $|\eta| = 1$ there is an interval J_η of length 2ε in J with

$$(4.2) \qquad |\eta \cdot \psi''(x)| \geqslant \delta \qquad \text{for all} \qquad x \in \mathrm{J}_\eta.$$

Now choose $\varphi \in \mathscr{D}(-\varepsilon,\varepsilon)$, $\varphi \geqslant 0$, with $\int \varphi(x)\,dx = 1$. For fixed $\eta \in \mathbf{R}^n$, $\eta \neq 0$, set $\eta' = |\eta|^{-1}\eta$, and choose $\mathrm{J}_{\eta'}$ as in (4.2). Let $\tilde{\varphi}$ be a suitable translate of φ such that $\mathrm{supp}\,\tilde{\varphi} \subset \mathrm{J}_{\eta'}$. Then we get

$$(4.3) \quad 0 < \sigma/2 \leqslant \left| \int \vartheta(x)\tilde{\varphi}(x)\,dx \right|$$
$$= \left| \int \vartheta(x) e(\eta \cdot \psi)(x) \tilde{\varphi}(x) e(-\eta \cdot \psi)(x)\,dx \right|$$
$$\leqslant |\vartheta e(\eta \cdot \psi)|_{\mathrm{A}}\,|\tilde{\varphi} e(-\eta \cdot \psi)|_{\mathrm{PM}},$$

since $\mathrm{J}_{\eta'} \subset \mathrm{J}$.

For $\xi \in \mathbf{R}$ one has

$$\{\tilde{\varphi}e(\eta\cdot\psi)\}^{\wedge}(-\xi) = \int \tilde{\varphi}(x)e(-\xi x-\eta\cdot\psi(x))\,dx$$
$$= \int \varphi(x)e(-|\eta|g(x))\,dx,$$

where g is a function on $[-\varepsilon,\varepsilon]$ which is a certain translate of the function

$$x \mapsto \xi' x + \eta'\cdot\psi(x) \qquad \text{on} \qquad J_{\eta'},$$

where $\xi' = |\eta|^{-1}\xi$.

But (4.2) implies

$$\delta \leqslant |g''(x)| \qquad \text{for every} \qquad x \in [-\varepsilon,\varepsilon].$$

Moreover, if we set $A = 2\sup_{x\in J}|\psi'(x)|$, $B = \sup_{x\in J}|\psi''(x)|$, then for $|\xi| \leqslant A|\eta|$:

$$|g'(x)| + |g''(x)| \leqslant |\xi'| + |\eta'|(A+B)$$
$$\leqslant 2A + B$$

for every $x \in [-\varepsilon,\varepsilon]$.

Thus, by Lemma 1, there exists a $C > 0$, such that for $|\xi| \leqslant A|\eta|$

$$(4.4) \qquad \left|\int \tilde{\varphi}(x)e(-\xi x-\eta\cdot\psi(x))\,dx\right| \leqslant C(1+|\eta|)^{-1/2}.$$

And, if $|\xi| > A|\eta|$, then integration by parts yields

$$(4.5) \qquad \left|\int \tilde{\varphi}(x)e(-\xi x-\eta\cdot\psi(x))\,dx\right|$$
$$= \left|\int e(-|\eta|g(x))\left(\frac{\varphi}{2\pi i|\eta|g'}\right)'(x)\,dx\right|$$
$$\leqslant (2\pi|\eta|)^{-1}\int\left\{\frac{|\varphi'(x)|}{|g'(x)|}+\frac{|\varphi(x)|\,|g''(x)|}{|g'(x)|^2}\right\}dx$$
$$\leqslant C'|\eta|^{-1},$$

where C' is some constant depending on φ, ψ and A only, since for $x \in [-\varepsilon,\varepsilon]$ we have $|g''(x)| \leqslant B$ and $|g'(x)| = |\xi'+\eta'\psi'(y)| \geqslant A - A/2$ for some $y \in J$.

Now, by (4.4), (4.5),

$$|\tilde{\varphi}e(-\eta\cdot\psi)|_{PM} \leqslant (C+C')|\eta|^{-1/2} \qquad \text{if} \qquad |\eta| \geqslant 1,$$

which together with (4.3) proves Theorem 1 (ii).

Proof of Theorem 2. — Assume $\tau(x) \neq 0$ for every $x \in I$, and let $\vartheta \in \mathscr{D}(I)$, $\vartheta \neq 0$. Passing to a smaller interval, we may even assume that I is closed.

Set $A = 2\sup_{x\in I}|\psi'(x)|$, and for $\xi' \in \mathbf{R}$, $|\xi'| \leqslant A$, $\eta' \in \mathbf{R}^n$, $|\eta'| = 1$, $x \in I$ let

$$Q_{\xi',\eta'}(x) = \sum_{j=1}^{n+1} |(\xi' x + \eta'\cdot\psi(x))^{(j)}(x)|.$$

Since $\tau^{-1}(\{0\}) = \varnothing$, we have $Q_{\xi',\eta'}(x) \neq 0$ for every $x \in I$, and since $Q_{\xi',\eta'}(x)$ is continuous in ξ', η' and x on the compact space $[-A,A] \times \{\eta' \in \mathbf{R}^n : \ |\eta'|=1\} \times I$, there exist constants $C_1 > 0$, $C_2 > 0$, such that

$$C_1 \leqslant Q_{\xi',\eta'}(x) \leqslant C_2 \tag{4.6}$$

for all $x \in I$, ξ', η' with $|\xi'| \leqslant A, |\eta'| = 1$.

So, using quite the same arguments as in the proof of Theorem 1 (ii), we can deduce from (4.6) by Lemma 1 :

$$|\vartheta e(\eta\cdot\psi)|_{PM} \leqslant C(1+|\eta|)^{-1/(n+1)}$$

for some constant $C > 0$, which proves (i).

To prove (ii), we will assume, for convenience, $x_0 = 0$, i.e. $0 \in I$, and $\vartheta(0) \neq 0$, $\tau(0) \neq 0$.

Let $\varepsilon > 0$ such that $\tau(x) \neq 0$ for $x \in [-\varepsilon,\varepsilon]$.

Since $\psi''(x)$, $\psi'''(x), \ldots, \psi^{(n+1)}(x)$ are linearly independent for $x \in [-\varepsilon,\varepsilon]$, there exists a function $\xi \in C^\infty([-\varepsilon,\varepsilon], \mathbf{R}^n)$, such that for every $x \in [-\varepsilon,\varepsilon]$

$$\xi(x)\cdot\psi^{(j)}(x)=0, \qquad j=2, \ldots, n, \tag{4.7}$$

and

$$\xi(x)\cdot\hat{\psi}^{(n+1)}(x) = 1. \tag{4.8}$$

Differentiating (4.7) and inserting (4.8), we get

$$\xi'(x)\cdot\psi^{(j)}(x) = 0 \qquad \text{for} \qquad j = 2, \ldots, n-1,$$

and

$$\xi'(x)\psi^{(n)}(x) = -1.$$

Repeating this process, one inductively obtains for $k = 0, \ldots, n-1$

$$\text{(4.9)} \qquad \begin{cases} \xi^{(k)}(x)\cdot\psi^{(j)}(x) = 0 & \text{for} \quad j = 2, \ldots, n-k, \\ \xi^{(k)}(x)\cdot\psi^{(n+1-k)}(x) = (-1)^k. \end{cases}$$

So, if we define matrices

$$S(x) = (\xi_j^{(n-i)}(x))_{i,j=1,\ldots,n}, \qquad T(x) = (\psi_i^{(j+1)}(x))_{i,j=1,\ldots,n},$$

then (4.9) means that $S(x)T(x)$ is an upper triangular matrix with diagonal elements 1 or -1, which yields

$$\text{(4.10)} \quad |\det(\xi(x)\xi'(x)\ldots\xi^{(n-1)}(x))| = |\det S(x)| = |\tau(x)|^{-1} \neq 0$$

for all $x \in [-\varepsilon,\varepsilon]$.

We now claim:

There is a constant $C > 0$, such that for all $y \in (-\varepsilon,\varepsilon)$ and $s \in \mathbf{R}$

$$\text{(4.11)} \qquad |\vartheta e(s\xi(y)\cdot\psi)|_{PM} \geqslant C(1+|s|)^{-1/(n+1)}.$$

Choose $y \in (-\varepsilon,\varepsilon)$. Then by (4.7), $(\xi(y)\cdot\psi)^{(j)}(y) = \delta_{j,n+1}$ for $j = 2, \ldots, n+1$, and so a Taylor expansion of $\xi(y)\cdot\psi$ yields (for ε small enough)

$$\text{(4.12)} \quad (\xi(y)\cdot\psi)(x) = \alpha + \beta x + (x-y)^{n+1}g(x) \quad \text{for} \quad x \in (-2\varepsilon,2\varepsilon),$$

where g is some smooth function on $(-2\varepsilon,2\varepsilon)$ which depends on y, and where α and β are some real numbers.

Let us remark here that although $g = g_y$ depends on y, $\sup\limits_{|x|<2\varepsilon} |g_y'(x)|$ is uniformly bounded for $y \in (-\varepsilon,\varepsilon)$.

Now take $\rho \in \mathscr{D}(\mathbf{R})$ with $\operatorname{supp}\rho \subset (-\varepsilon,\varepsilon)$, $\rho \geqslant 0$ and $\int \rho(x)\,dx = 1$, and set $\tilde\rho(x) = \rho(|s|^{1/(n+1)}(x-y))$.

If we choose ε small enough such that

$$|\vartheta(0)-\vartheta(x)| < \frac{1}{2}|\vartheta(0)|$$

for $x \in (-2\varepsilon, 2\varepsilon)$, then we get

$$\left|\int \vartheta(x)\tilde{\rho}(x)\,dx\right| = \left|\int \vartheta(|s|^{-1/(n+1)}x + y)\rho(x)\,dx\right| |s|^{-1/(n+1)}$$
$$\geqslant \frac{1}{2}|\vartheta(0)|\,|s|^{-1/(n+1)}, \qquad \text{if } |s| \geqslant 1;$$

and since

$$\left|\int \vartheta(x)\tilde{\rho}(x)\,dx\right| = \left|\int \vartheta(x)e(s\xi(y)\cdot\psi)\tilde{\rho}(x)e(-s\xi(y)\cdot\psi)\,dx\right|$$
$$\leqslant |\vartheta e(s\xi(y)\cdot\psi|_{\mathrm{PM}}\,|\tilde{\rho}e(-s\xi(y)\cdot\psi)|_{\mathrm{A}},$$

(4.11) will follow if we can show that $|\tilde{\rho}e(-s\xi(y)\cdot\psi)|_{\mathrm{A}}$ is uniformly bounded for $y \in (-\varepsilon, \varepsilon)$ and $|s| \geqslant 1$.

Now, regular affine mappings of **R** induce isometries of the Fourier algebra $\mathrm{A} = \mathrm{A}(\mathbf{R})$, thus

$$|\tilde{\rho}e(-s\xi(y)\cdot\psi)|_{\mathrm{A}} = |\rho e(-s\xi(y)\cdot\tilde{\psi}|_{\mathrm{A}},$$

where $\tilde{\psi}(x) = \psi(|s|^{-1/(n+1)}x + y)$.

Since for $x \in \operatorname{supp}\rho$ and $|s| \geqslant 1$,

$$|s|^{-1/(n+1)}x + y \in (-2\varepsilon, 2\varepsilon),$$

(4.12) yields

$$\xi(y)\cdot\tilde{\psi}(x) = \alpha + \beta y + \beta|s|^{-1/(n+1)}x + |s|^{-1}x^{n+1}g(|s|^{-1/(n+1)}x + y).$$

Thus

$$|\tilde{\rho}e(-s\xi(y)\cdot\psi)|_{\mathrm{A}} = |\rho e(h)|_{\mathrm{A}},$$

where $h(x) = -s|s|^{-1}x^{n+1}g(|s|^{-1/(n+1)}x + y)$. If we again apply estimate (4.1), we easily see that $|\rho e(h)|_{\mathrm{A}}$ is uniformly bounded for $y \in (-\varepsilon, \varepsilon)$ and $|s| \geqslant 1$, q.e.d.

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