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Van Minh Nguyen

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# Type $A$ admissible cells are Kazhdan–Lusztig

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Abstract Admissible $W$-graphs were defined and combinatorially characterized by Stembridge in [13]. The theory of admissible $W$-graphs was motivated by the need to construct $W$-graphs for Kazhdan–Lusztig cells, which play an important role in the representation theory of Hecke algebras, without computing Kazhdan–Lusztig polynomials. In this paper, we shall show that type $A$-admissible $W$-cells are Kazhdan–Lusztig as conjectured by Stembridge in his original paper.

## 1. Introduction

Let $(W, S)$ be a Coxeter system and $\mathcal{H}(W)$ its Hecke algebra over $\mathbb{Z}[q, q^{-1}]$, the ring of Laurent polynomials in the indeterminate $q$. We are interested in representations of $W$ and $\mathcal{H}(W)$ that can be described by combinatorial objects, namely $W$-graphs. In particular, we are interested in $W$-graphs corresponding to Kazhdan–Lusztig left cells.

In principle, when computing left cells one encounters the problem of having to compute a large number of Kazhdan–Lusztig polynomials before any explicit description of their $W$-graphs can be given. In [13], Stembridge introduced *admissible* $W$-graphs; these can be described combinatorially and can be constructed without calculating Kazhdan–Lusztig polynomials. Moreover, the $W$-graphs corresponding to Kazhdan–Lusztig left cells are admissible. Stembridge showed in [15] that for any given finite $W$ there are only finitely many stongly connected admissible $W$-graphs. It was conjectured by Stembridge that in type $A$ all strongly connected admissible $W$-graphs are isomorphic to Kazhdan–Lusztig left cells. In this paper we complete the proof of Stembridge's conjecture.

We shall work with *$S$-coloured graphs* (as defined in Section 3 below), of which $W$-graphs are examples. These graphs have both edges (bi-directional) and arcs (uni-directional). A *cell* in such a graph $\Gamma$ is by definition a strongly connected component of $\Gamma$, and a *simple part* is a connected component of the graph obtained by removing all arcs that are not edges and edges of weight greater than 1. A *simple component* is the full subgraph spanned by a simple part. Stembridge introduced *$W$-molecular graphs*, defined by conditions that are weaker than those for admissible $W$-graphs. Simple components of molecular graphs are called *molecules*.

By an *admissible $W$-cell* we mean a cell in an admissible $W$-graph (rather than a cell in a $W$-molecular graph).

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In [4], Chmutov established the first step towards the proof of Stembridge's conjecture, showing that if $(W, S)$ is of type $A_{n-1}$ then the simple part of a $W$-molecule is isomorphic to the simple part of a Kazhdan–Lusztig left cell. The proof made use of the axiomatisation of dual equivalence graphs on standard tableaux generated by dual Knuth equivalence relations, given in an earlier paper by Assaf [1].

By Chmutov's result each molecule of an arbitrary admissible $W$-graph of type $A_{n-1}$ is associated with a partition of $n$, and the vertices within each molecule are parametrized by the set of standard tableaux of the corresponding shape. The Bruhat order on $W$ induces a partial order on these tableaux, and this extends naturally to a partial order on the set of all $n$-box standard tableaux (see Section 7 below), thus giving rise to a partial order on the vertex set of the graph. We are able to use the combinatorics of tableaux to show that type $A$ admissible $W$-graphs are *ordered*, in the sense of Definition 8.1: if an arc has tail corresponding to a standard tableau $t$ and head corresponding to a standard tableau $u$ then either $u < t$, or else head and tail belong to the same molecule and $u = st$ for some simple transposition $s$. This property of admissible $W$-graphs is the key to our proof of the conjecture of Stembridge.

The proof of Proposition 9.5 furnishes an algorithm for computation of $W$-graphs for left cells in type $A_{n-1}$ (avoiding the computation of Kazhdan–Lusztig polynomials). This has been implemented in Magma and checked for a variety of partitions $\lambda$ with $n \leqslant 16$, and module dimension up to 171600 (for $\lambda = (5, 5, 3, 3)$). The Magma code is available on request.

We organize the paper in the following sections. Section 2 and Section 3 deal with the background on Coxeter groups and the corresponding Hecke algebras. In Section 4 the definition and properties of $W$-graphs are recalled. In Section 5, we recall the definitions of admissible $W$-graphs and molecules and how these can be characterized combinatorially. Section 6 presents combinatorics of tableaux and the relationship between Kazhdan–Lusztig left cells, dual Knuth equivalence classes and admissible molecules. We introduce the paired dual Knuth equivalence relation in Section 7. In Section 8, we prove the first main result, namely that for type $A_{n-1}$, all admissible $W$-graphs are ordered. The proof that type $A$ admissible $W$-cells are isomorphic to Kazhdan–Lusztig left cells is completed in Section 9.

## 2. Coxeter groups

Let $(W, S)$ be a Coxeter system and $l$ the length function on $W$. The Coxeter group $W$ comes equipped with the left weak order, the right weak order and the Bruhat order, respectively denoted by $\leqslant_{\mathsf{L}}$, $\leqslant_{\mathsf{R}}$ and $\leqslant$, and defined as follows.

Definition 2.1. *The* left weak order *is the partial order on $W$ generated by the relations $x \leqslant_{\mathsf{L}} y$ for all $x$, $y \in W$ with $l(x) < l(y)$ and $yx^{-1} \in S$.*

*The* right weak order *is the partial order on $W$ generated by the relations $x \leqslant_{\mathsf{R}} y$ for all $x\, y \in W$ with $l(x) < l(y)$ and $x^{-1}y \in S$.*

*The* Bruhat order *is the partial order on $W$ generated by the relations $x \leqslant y$ for all $x$, $y \in W$ with $l(x) < l(y)$ and $yx^{-1}$ conjugate to an element of $S$.*

Observe that the weak orders are characterized by the property that $x \leqslant_{\mathsf{R}} xy$ and $y \leqslant_{\mathsf{L}} xy$ whenever $l(xy) = l(x) + l(y)$.

For each $J \subseteq S$ let $W_J$ be the (standard parabolic) subgroup of $W$ generated by $J$, and let $D_J$ the set of distinguished (or minimal) representatives of the left cosets of $W_J$ in $W$. Thus each $w \in W$ has a unique factorization $w = du$ with $d \in D_J$ and $u \in W_J$, and $l(du) = l(d) + l(u)$ holds for all $d \in D_J$ and $u \in W_J$. It is easily seen that $D_J$ is an ideal of $(W, \leqslant_{\mathsf{L}})$, in the sense that if $w \in D_J$ and $v \in W$ with $v \leqslant_{\mathsf{L}} w$ then $v \in D_J$.

If $W_J$ is finite then we denote the longest element of $W_J$ by $w_J$. By [8, Lemma 2.2.1], if $W$ is finite then $D_J = \{d \in W \mid d \leqslant_{\mathsf{L}} d_J\}$, where $d_J$ is the unique element in $D_J \cap w_S W_J$.

## 3. Hecke algebras

Let $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$, the ring of Laurent polynomials with integer coefficients in the indeterminate $q$, and let $\mathcal{A}^+ = \mathbb{Z}[q]$. The Hecke algebra of a Coxeter system $(W, S)$, denoted by $\mathcal{H}(W)$ or simply by $\mathcal{H}$, is an associative $\mathcal{A}$-algebra with $\mathcal{A}$-basis $\{ H_w \mid w \in W \}$ satisfying

$$\begin{aligned} H_s^2 &= 1 + (q - q^{-1})H_s \quad \text{for all } s \in S, \\ H_{xy} &= H_x H_y \quad \text{for all } x,\, y \in W \text{ with } l(xy) = l(x) + l(y). \end{aligned}$$

We let $a \mapsto \overline{a}$ be the involutory automorphism of $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$ defined by $\overline{q} = q^{-1}$. It is well known that this extends to an involutory automorphism of $\mathcal{H}$ satisfying

$$\overline{H_s} = H_s^{-1} = H_s - (q - q^{-1}) \quad \text{for all } s \in S.$$

If $J \subseteq S$ then $\mathcal{H}(W_J)$, the Hecke algebra associated with the Coxeter system $(W_J, J)$, is isomorphic to the subalgebra of $\mathcal{H}(W)$ generated by $\{ H_s \mid s \in J \}$. We shall identify $\mathcal{H}(W_J)$ with this subalgebra.

## 4. $W$-graphs

Given a set $S$, we define an *$S$-coloured graph* to be a triple $\Gamma = (V, \mu, \tau)$ consisting of a set $V$, a function $\mu\colon V \times V \to \mathbb{Z}[q, q^{-1}]$ and a function $\tau$ from $V$ to $\mathcal{P}(S)$, the power set of $S$. The elements of $V$ are the *vertices* of $\Gamma$, and if $v \in V$ then $\tau(v)$ is the *colour* of the vertex. To interpret $\Gamma$ as a (directed) graph, we adopt the convention that if $v,\, u \in V$ then $(v, u)$ is an arc from $v$ to $u$ of $\Gamma$ if and only if $\mu(u, v) \neq 0$ and $\tau(u) \nsubseteq \tau(v)$, and $\{v, u\}$ is an edge of $\Gamma$ if and only if $(v, u)$ and $(u, v)$ are both arcs. We call $\mu(u, v)$ the *weight* of the arc $(v, u)$. An edge $\{u, v\}$ is said to be *symmetric* if $\mu(u, v) = \mu(v, u)$, and *simple* if $\mu(u, v) = \mu(v, u) = 1$.

An $S$-coloured graph is *reduced* if $\mu(u, v) = 0$ whenever $\tau(u) \subseteq \tau(v)$. Except when stated otherwise, all $S$-coloured graphs we consider are assumed to be reduced.

If $(W, S)$ is a Coxeter system, then a *$W$-graph* is an $S$-coloured graph $\Gamma = (V, \mu, \tau)$ such that the free $\mathcal{A}$-module with basis $V$ admits an $\mathcal{H}$-module structure satisfying

$$H_s v = \begin{cases} -q^{-1} v & \text{if } s \in \tau(v) \\ qv + \sum_{\{u \in V \mid s \in \tau(u)\}} \mu(u, v) u & \text{if } s \notin \tau(v), \end{cases} \tag{1}$$

for all $s \in S$ and $v \in V$. We write $M_\Gamma$ for the $\mathcal{H}$-module given by the $W$-graph $\Gamma$ in this way.

Note that $\mu(u, v)$ appears in Eq. (1) only if there is an $s$ with $s \in \tau(u)$ and $s \notin \tau(v)$. Thus redefining $\mu(u, v) = 0$ whenever $\tau(u) \subseteq \tau(v)$ does not alter $M_\Gamma$. So a non-reduced $S$-coloured graph is a $W$-graph if and only if the corresponding reduced $S$-coloured graph is a $W$-graph.

Since $M_\Gamma$ is $\mathcal{A}$-free with basis $V$ it admits an $\mathcal{A}$-semilinear involution $\alpha \mapsto \overline{\alpha}$, uniquely determined by the condition that $\overline{v} = v$ for all $v \in V$. We call this the *bar involution* on $M_\Gamma$. It is a consequence of (1) that $\overline{h\alpha} = \overline{h}\overline{\alpha}$ for all $h \in \mathcal{H}$ and $\alpha \in M_\Gamma$.

We shall sometimes write $\Gamma(V)$ for the $W$-graph with vertex set $V$, if the functions $\mu$ and $\tau$ are clear from the context.

If $\Gamma = (V, \mu, \tau)$ is an $S$-coloured graph and $J \subseteq S$ then the $W_J$-restriction of $\Gamma$ is defined to be the $J$-coloured graph $\Gamma_J = (V, \mu_J, \tau_J)$ where $\tau_J(v) = \tau(v) \cap J$ for all

$v \in V$ and

$$\mu_J(u,v) = \begin{cases} \mu(u,v) & \text{if } \tau_J(u) \nsubseteq \tau_J(v), \\ 0 & \text{if } \tau_J(u) \subseteq \tau_J(v). \end{cases}$$

It is clear that if $J \subseteq S$ and $\Gamma = (V, \mu, \tau)$ is a $W$-graph then $\Gamma_J = (V, \mu_J, \tau_J)$ is a $W_J$-graph.

Following [9], define a preorder $\leqslant_\Gamma$ on $V$ as follows: $u \leqslant_\Gamma v$ if there exists a sequence of vertices $u = x_0, x_1, \ldots, x_m = v$ such that $\tau(x_{i-1}) \nsubseteq \tau(x_i)$ and $\mu(x_{i-1}, x_i) \neq 0$ for all $i \in [1, m]$. That is, $u \leqslant_\Gamma v$ if there is a directed path from $v$ to $u$ in $\Gamma$. Let $\sim_\Gamma$ be the equivalence relation determined by this preorder. The equivalence classes with respect to $\sim_\Gamma$ are called the *cells* of $\Gamma$. That is, the cells are the strongly connected components of the directed graph $\Gamma$. Each equivalence class, regarded as a full subgraph of $\Gamma$, is itself a $W$-graph, with the $\mu$ and $\tau$ functions being the restrictions of those for $\Gamma$. The preorder $\leqslant_\Gamma$ induces a partial order on the set of cells: if $\mathcal{C}$ and $\mathcal{C}'$ are cells, then $\mathcal{C} \leqslant_\Gamma \mathcal{C}'$ if $u \leqslant_\Gamma v$ for some $u \in \mathcal{C}$ and $v \in \mathcal{C}'$.

It follows readily from (1) that a subset of $V$ spans a $\mathcal{H}(W)$-submodule of $M_\Gamma$ if and only if it is $\Gamma$-closed, in the sense that for every vertex $v$ in the subset, each $u \in V$ satisfying $\mu(u,v) \neq 0$ and $\tau(u) \nsubseteq \tau(v)$ is also in the subset. Thus $U \subseteq V$ is a $\Gamma$-closed subset of $V$ if and only if $U = \bigcup_{v \in U} \{\, u \in V \mid u \leqslant_\Gamma v \,\}$. Clearly, a subset of $V$ is $\Gamma$-closed if and only if it is the union of cells that form an ideal with respect to the partial order $\leqslant_\Gamma$ on the set of cells.

Suppose that $U$ is a $\Gamma$-closed subset of $V$, and let $\Gamma(U)$ and $\Gamma(V \smallsetminus U)$ be the full subgraphs of $\Gamma$ induced by $U$ and $V \smallsetminus U$, with arc weights and vertex colours inherited from $\Gamma$. Then $\Gamma(U)$ and $\Gamma(V \smallsetminus U)$ are themselves $W$-graphs, and

$$M_{\Gamma(V \smallsetminus U)} \cong M_{\Gamma(V)}/M_{\Gamma(U)}$$

as $\mathcal{H}(W)$-modules.

We end this section by recalling the original Kazhdan–Lusztig $W$-graph for the regular representation of $\mathcal{H}(W)$. For each $w \in W$, define the sets

$$\mathcal{L}(w) = \{s \in S \mid l(sw) < l(w)\}$$

and

$$\mathcal{R}(w) = \{s \in S \mid l(ws) < l(w)\},$$

the elements of which are called the left descents of $w$ and the right descents of $w$, respectively. Kazhdan and Lusztig give a recursive procedure that defines polynomials $P_{y,w}$ whenever $y$, $w \in W$ and $y < w$. These polynomials satisfy $\deg P_{y,w} \leqslant \frac{1}{2}(l(w) - l(y) - 1)$, and $\mu_{y,w}$ is defined to be the leading coefficient of $P_{y,w}$ if the degree is $\frac{1}{2}(l(w) - l(y) - 1)$, or 0 otherwise.

Define $W^{\circ} = \{\, w^{\circ} \mid w \in W \,\}$ to be the group opposite to $W$, and observe that $(W \times W^{\circ}, S \sqcup S^{\circ})$ is a Coxeter system. Kazhdan and Lusztig show that if $\mu$ and $\bar{\tau}$ are defined by the formulas

$$\mu(w,y) = \mu(y,w) = \begin{cases} \mu_{y,w} & \text{if } y < w \\ \mu_{w,y} & \text{if } w < y \end{cases}$$
$$\bar{\tau}(w) = \mathcal{L}(w) \sqcup \mathcal{R}(w)^{\circ}$$

then $(W, \mu, \bar{\tau})$ is a $(W \times W^{\circ})$-graph (usually not reduced). Thus $M = \mathcal{A}W$ may be regarded as an $(\mathcal{H}, \mathcal{H})$-bimodule. Furthermore, the construction produces an explicit $(\mathcal{H}, \mathcal{H})$-bimodule isomorphism $M \cong \mathcal{H}$.

It follows easily from the definition of $\mu_{y,w}$ that $\mu(y,w) \neq 0$ only if $l(w) - l(y)$ is odd; thus $(W, \mu, \bar{\tau})$ is a bipartite graph. The non-negativity of all coefficients of the Kazhdan–Lusztig polynomials, conjectured in [9], has been proved by Elias and Williamson in [6].

Since $W$ and $W^{\mathrm{o}}$ are standard parabolic subgroups of $W \times W^{\mathrm{o}}$, it follows that $\Gamma = (W, \mu, \tau)$ is a $W$-graph and $\Gamma^{\mathrm{o}} = (W, \mu, \tau^{\mathrm{o}})$ is a $W^{\mathrm{o}}$-graph, where $\tau$ and $\tau^{\mathrm{o}}$ are defined by $\tau(w) = \mathcal{L}(w)$ and $\tau^{\mathrm{o}}(w) = \mathcal{R}(w)^{\mathrm{o}}$, for all $w \in W$.

In accordance with the theory described above, there are preorders on $W$ determined by the $(W \times W^{\mathrm{o}})$-graph structure, the $W$-graph structure and the $W^{\mathrm{o}}$-graph structure. We call these the *two-sided preorder* (denoted by $\preceq_{\mathsf{LR}}$), the *left preorder* ($\preceq_{\mathsf{L}}$) and the *right preorder* ($\preceq_{\mathsf{R}}$). The corresponding cells are the *two-sided cells*, the *left cells* and the *right cells*.

## 5. Admissible $W$-graphs

Let $(W, S)$ be a Coxeter system, not necessarily finite. For $s,\, t \in S$, let $m(s,t)$ be the order of $st$ in $W$. Thus $\{s,t\}$ is a bond in the Coxeter diagram if and only if $m(s,t) > 2$.

Definition 5.1 ([13, Definition 2.1]). *An $S$-coloured graph $\Gamma = (V, \mu, \tau)$ is* admissible *if the following three conditions are satisfied:*

(i) $\mu(V \times V) \subseteq \mathbb{N}$*;*
(ii) $\Gamma$ *is symmetric, that is,* $\mu(u,v) = \mu(v,u)$ *if* $\tau(u) \nsubseteq \tau(v)$ *and* $\tau(v) \nsubseteq \tau(u)$*;*
(iii) $\Gamma$ *has a bipartition.*

Remark 5.2. As we have seen in Sec. 4, the Kazhdan–Lusztig $W$-graph $\Gamma = \Gamma(W, \mu, \tau)$ is admissible. So its cells are admissible.

Let $(W, S)$ be a braid finite Coxeter system. (That is, $m(s,t) < \infty$ for all $s, t \in S$.)

Definition 5.3 ([14, Definition 2.1]). *An $S$-coloured graph $\Gamma = (V, \mu, \tau)$ is said to satisfy the* Compatibility Rule *if for all $u,\, v \in V$ with $\mu(u,v) \neq 0$, each $s \in \tau(u) \smallsetminus \tau(v)$ and each $t \in \tau(v) \smallsetminus \tau(u)$ are joined by a bond in the Coxeter diagram of $W$.*

By [13, Proposition 4.1], every $W$-graph satisfies the Compatibility Rule.

Definition 5.4 ([14, Definition 2.3]). *An $S$-coloured graph $\Gamma = (V, \mu, \tau)$ is said to satisfy the* Simplicity Rule *if for all $u,\, v \in V$ with $\mu(u,v) \neq 0$, either $\tau(v) \subsetneqq \tau(u)$ and $\mu(v,u) = 0$ or else $\tau(u)$ and $\tau(v)$ are not comparable and $\mu(u,v) = \mu(v,u) = 1$.*

The Simplicity Rule implies that if $\mu(u,v) \neq 0$ and $\mu(v,u) \neq 0$ then $\mu(u,v) = \mu(v,u) = 1$. That is, all edges are simple. Furthermore if $\{u,v\}$ is an edge then $\tau(u)$ and $\tau(v)$ are not comparable, so that there exist at least one $s \in \tau(u) \smallsetminus \tau(v)$ and at least one $t \in \tau(v) \smallsetminus \tau(u)$. If the Compatibility Rule is also satisfied, then $\{s,t\}$ must be a bond in the Coxeter diagram.

If $(W, S)$ is simply-laced then every $W$-graph with non-negative integer arc weights satisfies the Simplicity Rule, even if it fails to be admissible: see [13, Remark 4.3].

Definition 5.5 ([14, Definition 2.4]). *An admissible $S$-coloured graph $\Gamma = (V, \mu, \tau)$ satisfies the* Bonding Rule *if for all $s, t \in S$ with $m(s,t) > 2$, the vertices $v$ of $\Gamma$ satisfying either $s \in \tau(v)$ and $t \notin \tau(v)$ or $s \notin \tau(v)$ and $t \in \tau(v)$, together with edges of $\{u,v\}$ of $\Gamma$ such that $s \in \tau(u) \smallsetminus \tau(v)$ and $t \in \tau(v) \smallsetminus \tau(u)$, form a disjoint union of Dynkin diagrams of types $A$, $D$ or $E$ with Coxeter numbers that divide $m(s,t)$.*

Equivalently, an admissible $S$-coloured graph $\Gamma$ satisfies the Bonding Rule if and only if for all $s,\, t \in S$ with $m(s,t) > 2$ the graphs of the cells in the $W_{\{s,t\}}$-restriction of $\Gamma$ are all Dynkin diagrams of types $A$, $D$ or $E$ with Coxeter numbers that divide $m(s,t)$. (Note that in the $W_{\{s,t\}}$-restriction of $\Gamma$ each edge joins a vertex of colour $\{s\}$ to a vertex of colour $\{t\}$.)

Remark 5.6. In the case $m(s,t)=3$, the Bonding Rule becomes the *Simply-Laced Bonding Rule*: for every vertex $u$ such that $s\in\tau(u)$ and $t\notin\tau(u)$, there exists a unique adjacent vertex $v$ such that $t\in\tau(v)$ and $s\notin\tau(v)$.

By [13, Proposition 4.4], admissible $W$-graphs satisfy the Bonding Rule.

Let $\Gamma=(V,\mu,\tau)$ be an $S$-coloured graph. Let $s,t\in S$ with $m(s,t)=m\geqslant 2$. Suppose that $u,v\in V$ with $s,t\notin\tau(u)$ and $s,t\in\tau(v)$. For $2\leqslant k\leqslant m$, a directed path $(u,v_1,\dots,v_{k-1},v)$ in $\Gamma$ is said to be *alternating of type* $(s,t)$ if $s\in\tau(v_i)$ and $t\notin\tau(v_i)$ for odd $i$ and $t\in\tau(v_i)$ and $s\notin\tau(v_i)$ for even $i$. Define

$$N^k_{s,t}(\Gamma;u,v)=\sum_{v_1,\dots,v_{k-1}}\mu(v,v_{k-1})\mu(v_{k-1}v_{k-2})\cdots\mu(v_2,v_1)\mu(v_1,u), \tag{2}$$

where the sum extends over all paths $(u,v_1,\dots,v_{k-1},v)$ that are alternating of type $(s,t)$.

Note that if $\Gamma$ is admissible then all terms in (2) are positive.

Definition 5.7 ([14, Definition 2.9]). *An admissible $S$-coloured graph $\Gamma=(V,\mu,\tau)$ satisfies the* Polygon Rule *if for all $s,t\in S$ and all $u,v\in V$ such that $s,t\in\tau(v)\setminus\tau(u)$, we have*

$$N^k_{s,t}(\Gamma;u,v)=N^k_{t,s}(\Gamma;u,v)\quad \textit{for all } r \textit{ such that } 2\leqslant k\leqslant m(s,t).$$

By [13, Proposition 4.7], all $W$-graphs with integer arc weights satisfy the Polygon Rule.

The next result provides a necessary and sufficient condition for an admissible $S$-coloured graph to be a $W$-graph.

Theorem 5.8 ([13, Theorem 4.9]). *An admissible $S$-coloured graph $\Gamma=(V,\mu,\tau)$ is a $W$-graph if and only if it satisfies the Compatibility Rule, the Simplicity Rule, the Bonding Rule and the Polygon Rule.*

It is convenient to introduce a weakened version of the Polygon Rule.

Definition 5.9 ([14, Definition 2.9]). *An admissible $S$-coloured graph $\Gamma=(V,\mu,\tau)$ satisfies the* Local Polygon Rule *if for all $s,t\in S$, all $k$ such that $2\leqslant k\leqslant m(s,t)$, and all $u,v$ such that $s,t\in\tau(v)\setminus\tau(u)$, we have $N^k_{s,t}(\Gamma;u,v)=N^k_{t,s}(\Gamma;u,v)$ under any of the following conditions:*

(i) *$k=2$, and $\tau(u)\setminus\tau(v)\neq\varnothing$;*
(ii) *$k=3$, and there exist $r,r'\in\tau(u)\setminus\tau(v)$ (not necessarily distinct) such that $\{r,s\}$ and $\{r',t\}$ are not bonds in the Dynkin diagram of $W$;*
(iii) *$k\geqslant 4$, and there is $r\in\tau(u)\setminus\tau(v)$ such that $\{r,s\}$ and $\{r,t\}$ are not bonds in the Dynkin diagram of $W$.*

Definition 5.10 ([14, Definition 3.3]). *An admissible $S$-coloured graph is called a $W$-*molecular graph *if it satisfies the Compatibility Rule, the Simplicity Rule, the Bonding Rule and Local Polygon Rules.*

A *simple part* of an $S$-coloured graph $\Gamma$ is a connected component of the graph obtained by removing all arcs and all non-simple edges, and a *simple component* of $\Gamma$ is the full subgraph spanned by a simple part.

Definition 5.11. *A $W$-*molecule *is a $W$-molecular graph that has only one simple part.*

Remark 5.12. If $\Gamma$ is an admissible $W$-graph then its simple components are $W$-molecules, by [14, Fact 3.1]. More generally, by [14, Fact 3.2], the full subgraph of $\Gamma$ induced by any union of simple parts is a $W$-molecular graph.

It is easy to check that if $\Gamma = (V, \mu, \tau)$ is a $W$-molecular graph then its $W_J$-restriction $\Gamma_J$ is a $W_J$-molecular graph. The $W_J$-molecules of $\Gamma_J$ are called $W_J$-submolecules of $\Gamma$.

Proposition 5.13 ([4, Lemma 1]). *Let $(W, S)$ be a Coxeter system and $M = (V, \mu, \tau)$ a $W$-molecular graph, and let $J = \{r, s, t\} \subseteq S$ with $m(s,t) = 3$ and $r \notin \{s,t\}$. Suppose that $v, v', u, u' \in V$, and that $\{v, v'\}$ and $\{u, u'\}$ are simple edges with*

$$\begin{aligned} \tau(v) \cap J &= \{s\}, & \tau(u) \cap J &= \{s, r\}, \\ \tau(v') \cap J &= \{t\}, & \tau(u') \cap J &= \{t, r\}. \end{aligned}$$

*Then $\mu(u,v) = \mu(u',v')$.*

## 6. Tableaux, left cells and admissible molecules of type $A$

For the remainder of this paper we shall focus attention on Coxeter systems of type $A$. For each positive integer $n$ we write $W_n$ for the symmetric group on the set $\{1, 2, \ldots, n\}$, and let $S_n = \{s_i \mid i \in [1, n-1]\}$, where $s_i$ is the transposition that swaps $i$ and $i+1$. Then $(W_n, S_n)$ is a Coxeter system of type $A_{n-1}$. We write $\mathcal{H}_n$ for the Hecke algebra of $W_n$. If $1 \leqslant h \leqslant k \leqslant n$ then we write $W_{[h,k]}$ for the standard parabolic subgroup of $W_n$ generated by $\{\, s_i \mid i \in [h, k-1] \,\}$. We adopt a left operator convention for permutations, writing $wi$ for the image of $i$ under the permutation $w$. We also define $s_{ij} \in W_n$ to be the transposition that swaps $i$ and $j$.

A sequence of nonnegative integers $\alpha = (\alpha_1, \alpha_2 \ldots, \alpha_k)$ is called a *composition* of $n$ if $\sum_{i=1}^{k} \alpha_i = n$. The $\alpha_i$ are called the *parts* of $\alpha$. We adopt the convention that $\alpha_i = 0$ for all $i > k$. A composition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is called a *partition* of $n$ if $\lambda_1 \geqslant \cdots \geqslant \lambda_k > 0$. We define $C(n)$ and $P(n)$ to be the sets of all compositions of $n$ and all partitions of $n$, respectively.

Since some of the conventions we are about to adopt are slightly non-standard, it seems appropriate to first make the following motivational remarks.

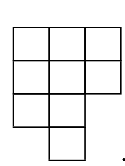
If $\Gamma = (V, \mu, \tau)$ is the $W_n$-graph associated with a Kazhdan–Lusztig left cell in $W_n$ then the $\mathcal{H}_n$-module $M(\Gamma)$ is irreducible, and specializing to $q = 1$ yields a $W_n$-module $M(\Gamma)_1$ that is isomorphic to a Specht module. It follows that $V$ is in bijective correspondence with the set of standard Young tableaux of some shape. For each $v \in V$ the set $\tau(v) \subseteq S_n$ generates a parabolic subgroup of $W_n$, which acts on the 1-dimensional subspace of $M(\Gamma)_1$ spanned by $v$ via the sign character. Now since a Young tableau is conventionally associated with the 1-character of its row group and the sign character of its column group, it becomes more natural for our purposes to focus on columns rather than rows. Hence we make the following definition.

Definition 6.1. *For each $\alpha = (\alpha_1, \ldots, \alpha_k) \in C(n)$ we define*

$$[\alpha] = \{\, (i,j) \mid 1 \leqslant i \leqslant \alpha_j \text{ and } 1 \leqslant j \leqslant k \,\},$$

*and call this as the* diagram *of $\alpha$.*

Pictorially $[\alpha]$ is represented by a top-justified array of boxes with $\alpha_j$ boxes in the $j$-th column; the pair $(i,j) \in [\alpha]$ corresponds to the $i$-th box in the $j$-th column. So the diagram of $\alpha = (3, 4, 2)$ is



.

Thus if $\lambda$ is a partition, the diagram $[\lambda]$ in our sense is the transpose of the usual Young diagram of $\lambda$.

If $\lambda \in P(n)$ then $\lambda^*$ denotes the *conjugate* of $\lambda$, defined to be the partition whose diagram is the transpose of $[\lambda]$; that is, $[\lambda^*] = \{(j,i) \mid (i,j) \in \lambda\}$ (which is the Young diagram of $\lambda$).

Let $\lambda \in P(n)$. If $(i,j) \in [\lambda]$ and $[\lambda] \smallsetminus \{(i,j)\}$ is still the diagram of a partition, we say that the box $(i,j)$ is $\lambda$-*removable*. Similarly, if $(i,j) \notin [\lambda]$ and $[\lambda] \cup \{(i,j)\}$ is again the diagram of a partition, we say that the box $(i,j)$ is $\lambda$-*addable*.

If $\alpha \in C(n)$ then an $\alpha$-*tableau* is a bijection $t\colon [\alpha] \to \mathcal{T}$, where $\mathcal{T}$ is a totally ordered set with $n$ elements. We call $\mathcal{T}$ the *target* of $t$. In this paper the target will always be an interval $[m+1, m+n]$, with $m = 0$ unless otherwise specified. The composition $\alpha$ is called the *shape* of $t$, and we write $\alpha = \mathrm{Shape}(t)$. For each $i \in [1,n]$ we define $\mathrm{row}_t(i)$ and $\mathrm{col}_t(i)$ to be the row index and column index of $i$ in $t$ (so that $t^{-1}(i) = (\mathrm{row}_t(i), \mathrm{col}_t(i))$). We define $\mathrm{Tab}_m(\alpha)$ to be the set of all $\alpha$-tableaux with target $\mathcal{T} = [m+1, m+n]$, and $\mathrm{Tab}(\alpha) = \mathrm{Tab}_0(\alpha)$. If $h \in \mathbb{Z}$ and $t \in \mathrm{Tab}_m(\alpha)$ then we define $t + h \in \mathrm{Tab}_{m+h}(\alpha)$ to be the tableau obtained by adding $h$ to all entries of $t$.

We define $\tau_\alpha \in \mathrm{Tab}(\alpha)$ to be the specific $\alpha$-tableau given by $\tau_\alpha(i,j) = i + \sum_{h=1}^{j-1} \alpha_h$ for all $(i,j) \in [\alpha]$. That is, in $\tau_\alpha$ the numbers 1, 2, …, $\alpha_1$ fill the first column of $[\alpha]$ in order from top to bottom, then the numbers $\alpha_1 + 1$, $\alpha_1 + 2$, …, $\alpha_1 + \alpha_2$ similarly fill the second column, and so on. If $\lambda \in P(n)$ then we also define $\tau^\lambda$ to be the $\lambda$-tableau that is the transpose of $\tau_{\lambda^*}$. Whenever $\lambda \in P(n)$ and $t \in \mathrm{Tab}_m(\lambda)$ we define $t^* \in \mathrm{Tab}_m(\lambda^*)$ to be the transpose of $t$. For example, if $\alpha = (3,2)$ then

$$\tau_\alpha = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1}\end{array}, \qquad \tau_\alpha^* = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array} \qquad \text{and} \qquad \tau^\alpha = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}.$$

We call $\tau_\alpha^{-1}(i)$ the $i$-th box of $[\alpha]$ in the top-to-bottom-left-to-right reading order, or TBLR order. (Below we shall also have occasion to make use of the BTLR order.)

Let $\alpha \in C(n)$ and $t \in \mathrm{Tab}(\alpha)$. We say that $t$ is *column standard* if the entries increase down each column. That is, $t$ is column standard if $t(i,j) < t(i+1,j)$ whenever $(i,j)$ and $(i+1,j)$ are both in $[\alpha]$. We define $\mathrm{CStd}(\alpha)$ to be the set of all column standard $\alpha$-tableaux. In the case $\lambda \in P(n)$ we say that $t$ is *row standard* if its transpose is column standard (so that $t(i,j) < t(i,j+1)$ whenever $(i,j)$ and $(i,j+1)$ are both in $[\lambda]$), and we say that $t$ is *standard* if it is both row standard and column standard. For example, if

$$t = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}, \qquad u = \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 4 & 3 \\ \hline 5 \\ \cline{1-1}\end{array}$$

then $t$ is standard, while $u$ is column standard but not row standard.

For each $\lambda \in P(n)$ we write $\mathrm{Std}(\lambda)$ for the set of all standard $\lambda$-tableaux (which is the set of all standard Young tableaux associated with $\lambda^*$). We also define $\mathrm{Std}(n) = \bigcup_{\lambda \in P(n)} \mathrm{Std}(\lambda)$.

It is clear that for any fixed composition $\alpha \in C(n)$ the group $W_n$ acts on $\mathrm{Tab}(\alpha)$, via $(wt)(i,j) = w(t(i,j))$ for all $(i,j) \in [\alpha]$, for all $\alpha$-tableaux $t$ and all $w \in W_n$. Moreover, the map from $W_n$ to $\mathrm{Tab}(\alpha)$ defined by $w \mapsto w\tau_\alpha$ for all $w \in W_n$ is bijective. We define the map $\mathrm{tblr}\colon \mathrm{Tab}(\alpha) \mapsto W_n$ to be the inverse of $w \mapsto w\tau_\alpha$, and use this to transfer the left weak order and the Bruhat order from $W_n$ to $\mathrm{Tab}(\alpha)$. Thus if $t_1$ and $t_2$ are arbitrary $\alpha$-tableaux, we write $t_1 \leqslant_{\mathsf{L}} t_2$ if and only if $\mathrm{tblr}(t_1) \leqslant_{\mathsf{L}} \mathrm{tblr}(t_2)$, and $t_1 \leqslant t_2$ if and only if $\mathrm{tblr}(t_1) \leqslant \mathrm{tblr}(t_2)$. Similarly, we define the length of $t \in \mathrm{Tab}(\alpha)$ by $l(t) = l(\mathrm{tblr}(t))$. Since the identity element of $W_n$ is the unique minimal element of $(W_n, \leqslant_{\mathsf{L}})$ and also of $(W_n, \leqslant)$, it follows that $\tau_\alpha$ is the unique minimal element of $(\mathrm{Tab}(\alpha), \leqslant_{\mathsf{L}})$ and of $(\mathrm{Tab}(\alpha), \leqslant)$.

REMARK 6.2. It follows from Definition 2.1 that the Bruhat order on $\mathrm{Tab}(\alpha)$ is generated by the requirement that $t \leqslant s_{ij}t$ whenever $i < j$ and $i$ precedes $j$ in $t$ the TBLR reading order.

We also define a bijection $\mathrm{btlr}\colon \mathrm{Tab}(\alpha) \to W_n$ by using the BTLR reading order instead of the TBLR order. Thus if $t \in \mathrm{Tab}(\alpha)$ and $b_1, \ldots, b_n$ is the sequence given by reading $t$ in bottom-to-top-left-to-right order, then $\mathrm{btlr}(t) \in W_n$ is given by $i \mapsto b_i$ for all $i \in \{1, \ldots, n\}$.

REMARK 6.3. It is easily seen that if $\alpha \in C(n)$ and $t \in \mathrm{Tab}(\alpha)$ then $\mathrm{tblr}(t) = \mathrm{btlr}(t)w_\alpha^{-1}$, where $w_\alpha = \mathrm{btlr}(\tau_\alpha)$.

Given $\alpha \in C(n)$ we define $J_\alpha$ to be the subset of $S$ consisting of those $s_i$ such that $i$ and $i+1$ lie in the same column of $\tau_\alpha$, and $W_\alpha$ to be the standard parabolic subgroup of $W_n$ generated by $J_\alpha$. Note that the longest element of $W_\alpha$ is the element $w_\alpha = \mathrm{btlr}(\tau_\alpha)$ defined in Remark 6.3 above. We write $D_\alpha$ for the set of minimal length representatives of the left cosets of $W_\alpha$ in $W_n$. Since $l(ds_i) > l(d)$ if and only if $di < d(i+1)$, it follows that $D_\alpha = \{\, d \in W_n \mid di < d(i+1) \text{ whenever } s_i \in W_\alpha \,\}$, and the set of column standard $\alpha$-tableaux is precisely $\{\, d\tau_\alpha \mid d \in D_\alpha \,\}$.

We shall also need to work with tableaux defined on skew diagrams.

DEFINITION 6.4. *If $m$ and $n$ are nonnegative inegers and $\lambda \in P(m+n)$ and $\pi \in P(m)$ are such that $[\pi] \subseteq [\lambda]$ then we define*

$$[\lambda/\pi] = [\lambda] \smallsetminus [\pi] = \{(i,j) \mid (i,j) \in [\lambda] \text{ and } (i,j) \notin [\pi]\}$$

*and call this a* skew diagram of shape $\lambda/\pi$*. We also write $\lambda/\pi \vdash n$ and call $\lambda/\pi$ a* skew partition of $n$*. In the case $m = 0$ we identify $\lambda/\pi$ with $\lambda$, and say that $\lambda/\pi$ is a* normal shape*.*

Thus the skew diagram corresponding to $\lambda/\pi$ is the transpose of the skew Young diagram corresponding to $\lambda/\pi$.

DEFINITION 6.5. *A skew tableau of shape $\lambda/\pi$, or $(\lambda/\pi)$-tableau, where $\lambda/\pi$ is a skew partition of $n$, is a bijective map $t\colon [\lambda/\pi] \to \mathcal{T}$, where $\mathcal{T}$ is a totally ordered set with $n$ elements. We write $\mathrm{Tab}_m(\lambda/\pi)$ for the set of all $(\lambda/\pi)$-tableaux for which the target set $\mathcal{T}$ is the interval $[m+1, m+n]$. We shall omit the subscript $m$ if $m = 0$.*

Let $\lambda/\pi$ be a skew partition of $n$. We define $\tau_{\lambda/\pi} \in \mathrm{Tab}(\lambda/\pi)$ by

$$\tau_{\lambda/\pi}(i,j) = i - \pi_j + \sum_{h=1}^{j-1}(\lambda_h - \pi_h) \tag{3}$$

for all $(i,j) \in [\lambda/\pi]$, and define $\tau^{\lambda/\pi} \in \mathrm{Tab}(\lambda/\pi)$ to be the transpose of $\tau_{\lambda^*/\pi^*}$.

If $\lambda/\pi \vdash n$ and $m \in \mathbb{Z}$ then $W_{[m+1,m+n]}$ acts naturally on $\mathrm{Tab}_m(\lambda/\pi)$, and as for normal shapes we define $\mathrm{tblr}\colon\ \mathrm{Tab}_m(\lambda/\pi) \to W_{[m+1,m+n]}$ to be the inverse of the map $w \mapsto w\tau_{\lambda/\pi}$. We transfer the Bruhat order and the left weak order from $W_{[m+1,m+n]}$ to $\mathrm{Tab}_m(\lambda/\pi)$ via the bijection tblr, just as for normal shapes.

All of our notation and terminology for partitions and tableaux extends naturally to skew partitions and tableaux, and will be used without further comment.

Let $\alpha \in C(n)$ and $t$ a column standard $\alpha$-tableau. For each $m \in \mathbb{Z}$ we define $t_{\leqslant m}$ to be the tableau obtained by removing from $t$ all boxes with entries greater than $m$. Thus if $\beta = \mathrm{Shape}(t_{\leqslant m})$ then $\beta \in C(m)$ and $[\beta] = \{\, b \in [\alpha] \mid t(b) \leqslant m \,\}$, and $t_{\leqslant m}\colon [\beta] \to [1,m]$ is the restriction of $t$. It is clear that $t_{\leqslant m}$ is column standard. Moreover, if $\alpha = \lambda \in P(n)$ and $t \in \mathrm{Std}(\lambda)$ then $\beta = \pi \in P(m)$ and $t_{\leqslant m} \in \mathrm{Std}(\pi)$.

Similarly, if $\lambda \in P(n)$ and $t \in \mathrm{Std}(\lambda)$ then for each $m \in \mathbb{Z}$ we define $t_{>m}$ to be the skew tableau obtained by removing from $t$ all boxes with entries less than or equal to $m$. Observe that $\{\, b \in [\lambda] \mid t(b) \leqslant m \,\}$ is the diagram of a partition $\nu \in P(n)$, and $\lambda/\nu$ is a skew partition of $n-m$. Clearly $t_{\,>m}$ is the restriction of $t$ to $[\lambda/\nu]$, and $t_{\,>m} \in \mathrm{Std}_m(\lambda/\nu)$.

We also let $t_{\,<m} = t_{\,\leqslant m-1}$ and $t_{\,\geqslant m} = t_{\,>m-1}$. For example, if $t = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1} \end{array} \in$ $\mathrm{Std}((3,2))$ then

$$t_{\,<5} = t_{\,\leqslant 4} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \in \mathrm{Std}((3,1)),$$

$$t_{\,\geqslant 2} = t_{\,>1} = \begin{array}{cc} \cline{2-2} & \multicolumn{1}{|c|}{2} \\ \hline \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{5} \\ \hline \multicolumn{1}{|c|}{4} \\ \cline{1-1} \end{array} \in \mathrm{Std}_1((3,2)/(1)).$$

For the next two definitions we follow the conventions of [10] (although we use columns rather than rows).

Definition 6.6. *Let $n$ be a nonnegative integer and $\alpha,\, \beta \in C(n)$. We say that $\alpha$* dominates *$\beta$, and write $\alpha \trianglerighteq \beta$, if $\sum_{i=1}^{k} \alpha_i \geqslant \sum_{i=1}^{k} \beta_i$ for each positive integer $k$.*

Definition 6.7. *Let $\alpha \in C(n)$ and $t,\, u \in \mathrm{CStd}(\alpha)$. We say that $t$* dominates *$u$, and write $t \trianglerighteq u$, if $\mathrm{Shape}(t_{\,\leqslant m}) \trianglerighteq \mathrm{Shape}(u_{\,\leqslant m})$ for all $m \in [1,n]$.*

In [10] Mathas defines a partial order $\trianglerighteq$ on $W_n$ as follows: if $v,\, w \in W_n$ then $v \trianglerighteq w$ if and only if $v$ has a reduced expression that is a subexpression of some reduced expression for $w$. By [5, Theorem 1.1] we see that this order is the reverse of the usual Bruhat order on $W$ (as defined in Definition 2.1), in that $v \trianglerighteq w$ if and only if $v \leqslant w$. Hence restating [10, Theorem 3.8] gives the following theorem.

Theorem 6.8. *Let $\alpha \in C(n)$, and let $t$ and $u$ be column standard $\alpha$-tableaux. Then $t \trianglerighteq u$ if and only if $\mathrm{tblr}(t) \leqslant \mathrm{tblr}(u)$.*

By our previous definitions, this says that if $t,\, u \in \mathrm{CStd}(\alpha)$ then $t \trianglerighteq u$ if and only if $t \leqslant u$. Accordingly, we make the following definition.

Definition 6.9. *If $\alpha,\, \beta \in C(n)$ we define $\alpha \leqslant \beta$ if and only if $\alpha \trianglerighteq \beta$. We call $\leqslant$ the* Bruhat order *on $C(n)$.*

We obtain the following variant of Definition 6.7.

Proposition 6.10. *Let $\alpha \in C(n)$, and let $t$ and $u$ be column standard $\alpha$-tableaux. Then $t \leqslant u$ if and only if $\mathrm{Shape}(t_{\,\leqslant m}) \leqslant \mathrm{Shape}(u_{\,\leqslant m})$ for all $m \in [1,n]$.*

Example. Let $t,\, u \in C(1,2,1,1)$ be given by

$$t = \begin{array}{|c|c|c|c|}\hline 2 & 1 & 4 & 5 \\ \hline \multicolumn{1}{c|}{} & 3 & \multicolumn{2}{c}{} \\ \cline{2-2} \end{array}, \qquad u = \begin{array}{|c|c|c|c|}\hline 2 & 3 & 4 & 1 \\ \hline \multicolumn{1}{c|}{} & 5 & \multicolumn{2}{c}{} \\ \cline{2-2} \end{array}$$

and consider $u_{\,\leqslant m}$ and $t_{\,\leqslant m}$ and their shapes for each $m \in [1,5]$.

$$\begin{array}{cccccc}
t_{\,\leqslant m}: & \boxed{1} & \begin{array}{|c|c|}\hline 2 & 1 \\ \hline \end{array} & \begin{array}{|c|c|}\hline 2 & 1 \\ \hline \multicolumn{1}{c|}{} & 3 \\ \cline{2-2} \end{array} & \begin{array}{|c|c|c|}\hline 2 & 1 & 4 \\ \hline \multicolumn{1}{c|}{} & 3 & \multicolumn{1}{c}{} \\ \cline{2-2} \end{array} & \begin{array}{|c|c|c|c|}\hline 2 & 1 & 4 & 5 \\ \hline \multicolumn{1}{c|}{} & 3 & \multicolumn{2}{c}{} \\ \cline{2-2} \end{array} \\
\mathrm{Shape}(t_{\,\leqslant m}): & (0,1,0,0) & (1,1,0,0) & (1,2,0,0) & (1,2,1,0) & (1,2,1,1) \\
u_{\,\leqslant m}: & \boxed{1} & \boxed{2}\quad\boxed{1} & \begin{array}{|c|c|}\hline 2 & 3 \\ \hline \end{array}\quad\boxed{1} & \begin{array}{|c|c|c|c|}\hline 2 & 3 & 4 & 1 \\ \hline \end{array} & \begin{array}{|c|c|c|c|}\hline 2 & 3 & 4 & 1 \\ \hline \multicolumn{1}{c|}{} & 5 & \multicolumn{2}{c}{} \\ \cline{2-2} \end{array} \\
\mathrm{Shape}(u_{\,\leqslant m}): & (0,0,0,1) & (1,0,0,1) & (1,1,0,1) & (1,1,1,1) & (1,2,1,1).
\end{array}$$

We find that $\mathrm{Shape}(t_{\,\leqslant m}) \leqslant \mathrm{Shape}(u_{\,\leqslant m})$ for all $m \in [1,5]$, and so $t \leqslant u$ by Proposition 6.10.

Let $\alpha$, $\beta \in C(n)$ with $\alpha \neq \beta$, and suppose that $\alpha \geqslant \beta$. Then $\sum_{i=1}^{k} \alpha_i \leqslant \sum_{i=1}^{k} \beta_i$ for all $k \in \mathbb{N}$, and so the least $k$ such that $\alpha_k \neq \beta_k$ must satisfy $\alpha_k < \beta_k$. We make the following definition.

Definition 6.11. *Let $\alpha$, $\beta \in C(n)$. We write $\alpha >_{\mathrm{lex}} \beta$ (or $\beta <_{\mathrm{lex}} \alpha$) if there exists a positive integer $k$ such that $\alpha_k < \beta_k$ and $\alpha_i = \beta_i$ for all $i < k$. We write $\alpha \geqslant_{\mathrm{lex}} \beta$ if $\alpha = \beta$ or $\alpha >_{\mathrm{lex}} \beta$.*

It is clear that $\geqslant_{\mathrm{lex}}$ is a total order on $C(n)$, and the remarks preceding Definition 6.11 have established the next proposition, which says that $\geqslant_{\mathrm{lex}}$ is a refinement of $\geqslant$.

Proposition 6.12. *If $\alpha, \beta \in C(n)$ with $\alpha \geqslant \beta$ then $\alpha \geqslant_{\mathrm{lex}} \beta$.*

We call $\geqslant_{\mathrm{lex}}$ the *lexicographic order* on $C(n)$. Note, however, that our lexicographic order is the reverse of the one defined in [10], which is defined as a refinement of $\trianglerighteq$ rather than $\geqslant$.

Remark 6.13. Let $\gamma \in C(n)$ and $t$, $u \in \mathrm{CStd}(\gamma)$ with $t \neq u$. Since $t_{\leqslant 0} = u_{\leqslant 0}$ and $t_{\leqslant n} \neq u_{\leqslant n}$, we can choose $i \in [0, n-1]$ satisfying $t_{\leqslant i} = u_{\leqslant i}$ and $t_{\leqslant (i+1)} \neq u_{\leqslant (i+1)}$. We shall show that if $t > u$ in the Bruhat order then $i+1$ occurs in a later column in $t$ than in $u$.

Note that $t_{\leqslant (i+1)}$ and $u_{\leqslant (i+1)}$ have different shapes, since $t_{\leqslant (i+1)}$ is obtained by adding the number $i+1$ to the bottom of some column of $t_{\leqslant i} = u_{\leqslant i}$, and clearly $u_{\leqslant (i+1)}$ must be obtained by adding $i+1$ to the bottom of a different column. Let $\alpha = \mathrm{Shape}(t_{\leqslant (i+1)})$ and $\beta = \mathrm{Shape}(u_{\leqslant (i+1)})$, and let $k = \mathrm{col}_t(i+1)$ and $l = \mathrm{col}_u(i+1)$. Then $k \neq l$, and $\alpha_j = \beta_j$ for all $j < m = \min(k, l)$. Furthermore, $\alpha_m = \beta_m + 1$ if $m = k$, and $\beta_m = \alpha_m + 1$ if $m = l$. Thus $\alpha_m < \beta_m$ if and only if $m = l$. So $\alpha >_{\mathrm{lex}} \beta$ if and only if $\min(k, l) = l$.

Now suppose that $t > u$. Since $t_{\leqslant i} = u_{\leqslant i}$ we have $\mathrm{Shape}(t_{\leqslant h}) = \mathrm{Shape}(u_{\leqslant h})$ for all $h \leqslant i$, and by Proposition 6.10, we must have $\mathrm{Shape}(t_{\leqslant (i+1)}) \geqslant \mathrm{Shape}(u_{\leqslant (i+1)})$. That is, $\alpha \geqslant \beta$. By Proposition 6.12 it follows that $\alpha \geqslant_{\mathrm{lex}} \beta$, and so $\mathrm{col}_t(i+1) = k > l = \mathrm{col}_u(i+1)$.

Let $\alpha = (\alpha_1, \ldots, \alpha_k) \in C(n)$. For each $t \in \mathrm{Tab}(\alpha)$, we define $\mathrm{cp}(t)$ to be the composition of the number $\sum_{i=1}^{k} i\alpha_i$ given by $\mathrm{cp}(t)_i = \mathrm{col}_t(n+1-i)$, the column index of $n+1-i$ in $t$, for all $i \in [1, n]$. Thus, for example, putting

$$t = \begin{array}{|c|c|c|c|} \hline 1 & 9 & 5 & 4 \\ \hline 6 & 2 & & 7 \\ \hline & 3 & & \\ \cline{2-2} & 8 & & \\ \cline{2-2} \end{array}, \qquad u = \begin{array}{|c|c|c|c|} \hline 1 & 9 & 5 & 4 \\ \hline 3 & 2 & & 7 \\ \hline & 6 & & \\ \cline{2-2} & 8 & & \\ \cline{2-2} \end{array}$$

gives $\mathrm{cp}(t) = (2, 2, 4, 1, 3, 4, 2, 2, 1)$ and $\mathrm{cp}(u) = (2, 2, 4, 2, 3, 4, 1, 2, 1)$.

Notation. Let $\alpha \in C(n)$ and $t$, $u \in \mathrm{Tab}(\alpha)$. We write $t \geqslant_{\mathrm{lex}} u$ if $\mathrm{cp}(t) \geqslant_{\mathrm{lex}} \mathrm{cp}(u)$, and we write $t >_{\mathrm{lex}} u$ if $\mathrm{cp}(t) >_{\mathrm{lex}} \mathrm{cp}(u)$.

In the example above, since $\mathrm{cp}(t)_4 < \mathrm{cp}(u)_4$ and $\mathrm{cp}(t)_i = \mathrm{cp}(u)_i$ for $i = 1, 2, 3$, it follows that $\mathrm{cp}(t) >_{\mathrm{lex}} \mathrm{cp}(u)$, whence $t >_{\mathrm{lex}} u$.

Observe that $\mathrm{cp}(t) = \mathrm{cp}(u)$ if and only if each column of $t$ contains the same numbers as the corresponding column of $u$. Thus $\geqslant_{\mathrm{lex}}$ is not a partial order on $\mathrm{Tab}(\alpha)$ but merely a preorder. It is, however, a total preorder, in the sense that any two elements of $\mathrm{Tab}(\alpha)$ are comparable, and it becomes a total order when restricted to $\mathrm{CStd}(\alpha)$.

Definition 6.14. *Given $\alpha \in C(n)$, we call the restriction of $\geqslant_{\mathrm{lex}}$ to* $\mathrm{CStd}(\alpha)$ *the* lexicographic order *on* $\mathrm{CStd}(\alpha)$.

Remark 6.15. If $t$, $u \in \mathrm{Tab}(\alpha)$ then $t >_{\mathrm{lex}} u$ if and only if there exists an integer $k \in [1,n]$ such that $\mathrm{cp}(t)_k < \mathrm{cp}(u)_k$ and $\mathrm{cp}(t)_l = \mathrm{cp}(u)_l$ for all $l < k$. That is, $t >_{\mathrm{lex}} u$ if and only if there exists $k \in [1,n]$ such that $\mathrm{col}_t(n+1-k) < \mathrm{col}_u(n+1-k)$ and $\mathrm{col}_t(n{+}1{-}l) = \mathrm{col}_u(n{+}1{-}l)$ for all $l < k$. Writing $j = n{+}1{-}k$, this says that $t >_{\mathrm{lex}} u$ if and only if there exists $j \in [1,n]$ such that $\mathrm{col}_t(j) < \mathrm{col}_u(j)$ and $\mathrm{col}_t(h) = \mathrm{col}_u(h)$ for all $h \in [j+1,n]$. Observe that if $u$ and $t$ are column standard then the latter condition is equivalent to $t_{>j} = u_{>j}$.

Lemma 6.16. *Let $\alpha \in C(n)$, and let $t$, $u \in \mathrm{Tab}(\alpha)$. If $t \geqslant u$ then $t \geqslant_{\mathrm{lex}} u$.*

*Proof.* By Remark 6.2 it suffices to show that if $1 \leqslant i < j \leqslant n$ and $i$ precedes $j$ in $u$ in the TBLR order, then $t = (i,j)u \geqslant_{\mathrm{lex}} u$. We may assume that $\mathrm{col}_u(i) \neq \mathrm{col}_u(j)$, since otherwise $\mathrm{cp}(t) = \mathrm{cp}(u)$ and $t \geqslant_{\mathrm{lex}} u$ certainly holds. Since $i < j$ and $i$ precedes $j$ in $u$, we see that $\mathrm{col}_t(j) = \mathrm{col}_u(i) < \mathrm{col}_u(j)$ and $j$ is the maximum element of $\{\, k \mid \mathrm{col}_t(k) \neq \mathrm{col}_u(k)\,\} = \{i,j\}$, and it follows from Remark 6.15 that $t >_{\mathrm{lex}} u$, as required. ☐

Corollary 6.17. *For each $\alpha \in C(n)$, the lexicographic order on* $\mathrm{CStd}(\alpha)$ *is a total order that refines the Bruhat order.*

Definition 6.18. *Let $\lambda \in P(n)$. For each $t \in \mathrm{Std}(\lambda)$ we define*

$$\begin{aligned}
\mathrm{SA}(t) &= \{i \in [1,n-1] \mid \mathrm{row}_t(i) > \mathrm{row}_t(i+1)\,\},\\
\mathrm{SD}(t) &= \{i \in [1,n-1] \mid \mathrm{col}_t(i) > \mathrm{col}_t(i+1)\},\\
\mathrm{WA}(t) &= \{i \in [1,n-1] \mid \mathrm{row}_t(i) = \mathrm{row}_t(i+1)\},\\
\mathrm{WD}(t) &= \{i \in [1,n-1] \mid \mathrm{col}_t(i) = \mathrm{col}_t(i+1)\},
\end{aligned}$$

*and call the elements of these (respectively) the* strong ascents, strong descents, weak ascents *and* weak descents *of $t$. We also define* $\mathrm{A}(t) = \mathrm{SA}(t) \cup \mathrm{WA}(t)$ *and* $\mathrm{D}(t) = \mathrm{SD}(t) \cup \mathrm{WD}(t)$.

Remark 6.19. It is easily checked that $i \in \mathrm{SA}(t)$ if and only if $s_it \in \mathrm{Std}(\lambda)$ and $s_it > t$, while $i \in \mathrm{SD}(t)$ if and only if $s_it < t$ (which implies that $s_it \in \mathrm{Std}(\lambda)$). Note also that if $w = \mathrm{tblr}(t)$ then $i \in \mathrm{D}(t)$ if and only if $s_i \in \mathcal{L}(ww_\lambda)$; this is proved in [11, Lemma 5.2].

Lemma 6.20. *Let $\lambda \in P(n)$ and $t \in \mathrm{Std}(\lambda)$. If $i \in \mathrm{SA}(t)$ then $\mathrm{D}(s_it) \setminus \mathrm{D}(t) = \{i\}$.*

*Proof.* By Remark 6.19 the condition $i \in \mathrm{SA}(t)$ implies $i \in \mathrm{SD}(s_it)$. Thus $i \in \mathrm{D}(s_it) \setminus \mathrm{D}(t)$.

Since $t$ and $s_it$ differ only in the positions of $i$ and $i+1$, it is clear that if $j < i-1$ or $j > i+1$ then $j$ and $j+1$ occupy the same positions in $t$ and in $s_it$, and it is immediate from Definition 6.18 that $j \in \mathrm{D}(s_it)$ if and only if $j \in \mathrm{D}(t)$. So it remains to check that if $j \in \{i-1,i+1\}$ and $j \in \mathrm{D}(s_it)$ then $j \in \mathrm{D}(t)$. If $i-1 \in \mathrm{D}(s_it)$ then $\mathrm{col}_{s_it}(i{-}1) \geqslant \mathrm{col}_{s_it}(i)$, and $\mathrm{col}_t(i{-}1) = \mathrm{col}_{s_it}(i{-}1) \geqslant \mathrm{col}_{s_it}(i) > \mathrm{col}_{s_it}(i{+}1) = \mathrm{col}_t(i)$. If $\mathrm{col}_{s_it}(i{+}1) \geqslant \mathrm{col}_{s_it}(i{+}2)$ then $\mathrm{col}_t(i{+}1) = \mathrm{col}_{s_it}(i) > \mathrm{col}_{s_it}(i{+}1) \geqslant \mathrm{col}_{s_it}(i{+}2) = \mathrm{col}_t(i+2)$, and the result follows. ☐

Remark 6.21. It is clear that if $\lambda/\pi \vdash n$ and $m \in \mathbb{Z}$ then $m + \tau_{\lambda/\pi}$ is the unique minimal element of $\mathrm{Std}_m(\lambda/\pi)$ with respect to the Bruhat order and the left weak order. Accordingly, we call $m + \tau_{\lambda/\pi}$ the *minimal element of* $\mathrm{Std}_m(\lambda/\pi)$. It is easily shown that if $t \in \mathrm{Std}_m(\lambda/\pi)$ then $t = m + \tau_{\lambda/\pi}$ if and only if $\mathrm{SD}(t) = \varnothing$. That is, $t$ is minimal if and only if $\mathrm{D}(t) = \mathrm{WD}(t)$.

For technical reasons it is convenient to make the following definition.

Definition 6.22. *Let $\lambda/\pi \vdash n > 1$ and $m \in \mathbb{Z}$. Let $i$ be minimal such that $\lambda_i > \pi_i$, and assume that $\lambda_{i+1} > \pi_{i+1}$. The $m$-critical tableau of shape $\lambda/\pi$ is the tableau $t \in \mathrm{Std}_{m-1}(\lambda/\pi)$ such that $\mathrm{col}_t(m) = i$ and $\mathrm{col}_t(m+1) = i+1$, and $t_{>(m+1)}$ is the minimal tableau of its shape.*

For example, if we put

$$u = \begin{array}{|c|c|c|c|} \cline{3-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 5 & 7 \\ \cline{2-4} \multicolumn{1}{c|}{} & 4 & 6 & 8 \\ \cline{2-4} \multicolumn{4}{c}{} \\ \cline{1-1} 3 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}, \qquad v = \begin{array}{|c|c|c|c|} \cline{2-4} \multicolumn{1}{c|}{} & 5 & 8 & 9 \\ \cline{2-4} \multicolumn{1}{c|}{} & 7 & \multicolumn{2}{c}{} \\ \cline{1-2} 4 & \multicolumn{3}{c}{} \\ \cline{1-1} 6 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

then $u$ is the 3-critical tableau of shape $(4,4,2,2,2)/(4,3,1)$ and $v$ is the 4-critical tableau of shape $(4,2,1,1)/(2)$. (Note that the first column of $u$ is empty, as is its third row.)

If $t$ is the $m$-critical tableau of shape $\lambda/\pi$ then, with $i$ as in the definition, column $i$ of $t$ is the first nonempty column, the number $m$ goes at the top of column $i$ and $m+1$ goes at the top of column $i+1$, after which the numbers $m+2$, $m+3$, $\dots$, $m+n-1$ are inserted into the remaining places, in TBLR order. Thus $\mathrm{col}_t(m+2) = i$ if and only if $\lambda_i - \pi_i > 1$.

Lemma 6.23. *Let $\lambda \in P(n)$ and $m \in \mathbb{Z}$, and let $t \in \mathrm{Std}(\lambda)$ satisfy $\mathrm{col}_t(m+1) = \mathrm{col}_t(m) + 1$. Then $t_{\geqslant m}$ is $m$-critical if and only if the following two conditions both hold:*

(1) *either $\mathrm{col}_t(m) = \mathrm{col}_t(m+2)$ or $m+1 \notin \mathrm{SD}(t)$,*
(2) *every $j \in \mathrm{D}(t)$ with $j > m+1$ is in $\mathrm{WD}(t)$.*

*Proof.* Let $\mathrm{Shape}(t_{\geqslant m}) = \lambda/\pi$, and put $i = \mathrm{col}_t(m)$. Note that since $m+1$ is in column $i+1$ of $t_{\geqslant m}$, it follows that $\lambda_{i+1} > \pi_{i+1}$.

Given that $\mathrm{col}_t(m+1) = \mathrm{col}_t(m)+1$, the second alternative in condition (1) is equivalent to $\mathrm{col}_t(m)+1 \leqslant \mathrm{col}_t(m+2)$. Hence condition (1) is equivalent to $\mathrm{col}_t(m) \leqslant \mathrm{col}_t(m+2)$. But by Remark 6.21, condition (2) holds if and only if $t_{>(m+1)}$ is minimal, which in turn is equivalent to $\mathrm{col}_t(m+2) \leqslant \mathrm{col}_t(m+3) \leqslant \cdots \leqslant \mathrm{col}_t(n)$. So (1) and (2) both hold if and only if $t_{>(m+1)}$ is minimal and $\mathrm{col}_t(j) \geqslant \mathrm{col}_t(m)$ for all $j \geqslant m$.

Since $\mathrm{col}_t(m+1) = i+1$, it follows from the definition that $t_{\geqslant m}$ is $m$-critical if and only if $t_{>(m+1)}$ is minimal and $i = \mathrm{col}_t(m)$ is equal to $\min\{\, j \mid \lambda_j > \pi_j \,\}$. But this last condition holds if and only if $m$ is in the first nonempty column of $t_{\geqslant m}$, and since this holds if and only if $\mathrm{col}_t(j) \geqslant \mathrm{col}_t(m)$ for all $j \geqslant m$, the result is established. ☐

Recall that if $w \in W_n$ then applying the Robinson–Schensted algorithm to the sequence $(w1, w2, \dots, wn)$ produces a pair $\mathrm{RS}(w) = (\mathrm{P}(w), \mathrm{Q}(w))$, where $\mathrm{P}(w), \mathrm{Q}(w) \in \mathrm{Std}(\lambda)$ for some $\lambda \in P(n)$. Details of the algorithm can be found (for example) in [12, Section 3.1]. The first component of $\mathrm{RS}(w)$ is called the *insertion* tableau and the second component is called the *recording* tableau.

The next two results are well-known.

Theorem 6.24 ([12, Theorem 3.1.1]). *The map $\mathrm{RS}\colon W_n \to \bigcup_{\lambda\in P(n)} \mathrm{Std}(\lambda)^2$ is bijective.*

Theorem 6.25 ([12, Theorem 3.6.6]). *Let $w \in W_n$. If $\mathrm{RS}(w) = (t, x)$ then $\mathrm{RS}(w^{-1}) = (x, t)$.*

The next lemma will be used below in the discussion of dual Knuth equivalence classes.

Lemma 6.26 ([11, Lemma 6.3]). *Let $\lambda \in P(n)$ and let $w \in W_n$. Then $\mathrm{RS}(w) = (t, \tau_\lambda)$ for some $t \in \mathrm{Std}(\lambda)$ if and only if $w = vw_\lambda$ for some $v \in W_n$ such that $v\tau_\lambda \in \mathrm{Std}(\lambda)$. When these conditions hold, $t = v\tau_\lambda$.*

Definition 6.27. *The* dual Knuth equivalence relation *is the equivalence relation* $\approx$ *on* $W_n$ *generated by the requirements that for all* $x \in W_n$ *and* $k \in [1, n-2]$,

(1) $x \approx s_{k+1}x$ *whenever* $\mathcal{L}(x) \cap \{s_k, s_{k+1}\} = \{s_k\}$ *and* $\mathcal{L}(s_{k+1}x) \cap \{s_k, s_{k+1}\} = \{s_{k+1}\}$,

(2) $x \approx s_k x$ *whenever* $\mathcal{L}(x) \cap \{s_k, s_{k+1}\} = \{s_{k+1}\}$ *and* $\mathcal{L}(s_k x) \cap \{s_k, s_{k+1}\} = \{s_k\}$.

The relations (1) and (2) above are known as the dual Knuth relations of the first kind and second kind, respectively.

Remark 6.28. It is not hard to check that (1) and (2) above can be combined to give an alternative formulation of Definition 6.27, as follows: $\approx$ is the equivalence relation on $W_n$ generated by the requirement that $x \approx sx$ for all $x \in W_n$ and $s \in S_n$ such that $x < sx$ and $\mathcal{L}(x) \nsubseteq \mathcal{L}(sx)$. In [9] Kazhdan and Lusztig show that whenever this holds then $x$ and $sx$ are joined by a simple edge in the Kazhdan–Lusztig $W$-graph $\Gamma = \Gamma(W_n)$. Furthermore, they show that the dual Knuth equivalence classes coincide with the left cells in $\Gamma(W_n)$.

The following result is well-known.

Theorem 6.29 ([12, Theorem 3.6.10]). *Let* $x, y \in W_n$. *Then* $x \approx y$ *if and only if* $\mathrm{Q}(x) = \mathrm{Q}(y)$.

Let $\lambda \in P(n)$, and for each $t \in \mathrm{Std}(\lambda)$ define $C(t) = \{\, w \in W_n \mid \mathrm{Q}(w) = t \,\}$. Theorem 6.29 says that these sets are the dual Knuth equivalence classes in $W_n$. It follows from Lemma 6.26 that $C(\tau_\lambda) = \{\, v w_\lambda \mid v\tau_\lambda \in \mathrm{Std}(\lambda) \,\} = \{\, \mathrm{tblr}(t) w_\lambda \mid t \in \mathrm{Std}(\lambda) \,\} = \{\, \mathrm{btlr}(t) \mid t \in \mathrm{Std}(\lambda) \,\}$.

Let $t$, $u \in \mathrm{Std}(\lambda)$, and suppose that $t = s_k u$ for some $k \in [2, n-1]$. By Remark 6.19 above, if $x = \mathrm{btlr}(u)$ then $\mathcal{L}(x) \cap \{s_{k-1}, s_k\} = \{s_{k-1}\}$ and $\mathcal{L}(s_k x) \cap \{s_{k-1}, s_k\} = \{s_k\}$ if and only if $\mathrm{D}(u) \cap \{k-1, k\} = \{k-1\}$ and $\mathrm{D}(t) \cap \{k-1, k\} = \{k\}$. Under these circumstances we write $u \to^{*1} t$, and say that there is a *dual Knuth move of the first kind* from $u$ to $t$. Similarly, if $t = s_k u$ for some $k \in [1, n-2]$ such that $\mathrm{D}(u) \cap \{k, k+1\} = \{k+1\}$ and $\mathrm{D}(t) \cap \{k, k+1\} = \{k\}$ then we write $u \to^{*2} t$, and say that there is a *dual Knuth move of the second kind* from $u$ to $t$. Since $C(\tau_\lambda)$ is a single dual Knuth equivalence class, we obtain the following result.

Proposition 6.30. *Let* $\lambda \in P(n)$ *and* $t$, $u \in \mathrm{Std}(\lambda)$. *Then* $t$ *can be transformed into* $u$ *by a sequence of dual Knuth moves or inverse dual Knuth moves.*

We call the integer $k$ above the *index* of the corresponding dual Knuth move, and denote it by $\mathrm{ind}(u, t)$. For convenience, we shall abbreviate "dual Knuth Move" to "DKM".

Remark 6.31. DKMs are also defined for standard skew tableaux; the definitions are exactly the same as for tableaux of normal shape. If $\lambda/\pi \vdash n$ and $u$, $t \in \mathrm{Std}(\lambda/\pi)$ then we write $u \approx t$ if and only if $u$ and $t$ are related by a sequence of DKMs.

Definition 6.32. *For each* $J \subseteq S_n$ *let* $\approx_J$ *be the equivalence relation on* $W_n$ *generated by the requirement that* $x \approx_J sx$ *for all* $s \in J$ *and* $x \in W_n$ *such that* $x < sx$ *and* $\mathcal{L}(x) \cap J \nsubseteq \mathcal{L}(sx)$.

Remark 6.33. Let $J \subseteq S_n$, let $(W, S) = (W_n, S_n)$ and let $\Gamma$ be the regular Kazhdan–Lusztig $W$-graph. By the results of Section 4 we know that a simple edge $\{x, y\}$ of $\Gamma$ remains a simple edge of $\Gamma_J$ provided that $\mathcal{L}(x) \cap J \nsubseteq \mathcal{L}(y) \cap J$ and $\mathcal{L}(y) \cap J \nsubseteq \mathcal{L}(x) \cap J$. Recall that the simple edges of $\Gamma$ all have the form $\{x, sx\}$, where $s \in S$ and $x < sx \in W$. Given that $x < sx$, the condition $\mathcal{L}(sx) \cap J \nsubseteq \mathcal{L}(x) \cap J$ holds if and only if $s \in J$, and so $\{x, sx\}$ is a simple edge of $\Gamma_J$ if and only if $s \in J$ and $\mathcal{L}(x) \cap J \nsubseteq \mathcal{L}(sx)$. Thus $\approx_J$ is the equivalence relation on $W$ generated by the requirement that $x \approx_J y$ whenever $\{x, y\}$ is a simple edge of $\Gamma_J$.

Definition 6.34. *Let $\lambda \in P(n)$ and $1 \leqslant m \leqslant n$. Let $\approx_m$ be the equivalence relation on* $\mathrm{Std}(\lambda)$ *defined by the requirement that $u \approx_m t$ whenever there is a DKM of index at most $m-1$ from $u$ to $t$ and $\mathrm{D}(u) \cap [1, m-1] \nsubseteq \mathrm{D}(t)$. We shall call such a DKM a $(\leqslant m)$*-DKM. *The $\approx_m$ equivalence classes in* $\mathrm{Std}(\lambda)$ *will be called the $(\leqslant m)$-*subclasses *of* $\mathrm{Std}(\lambda)$*, and we shall say that $u, t \in \mathrm{Std}(\lambda)$ are $(\leqslant m)$* dual Knuth equivalent *whenever $u \approx_m t$.*

Remark 6.35. Assume that $\lambda \in P(n)$ and $1 \leqslant m \leqslant n$, and let $u, t \in \mathrm{Std}(\lambda)$. If $u \to^{*2} t$ and $\mathrm{ind}(u, t) \leqslant m-1$ then $\mathrm{D}(u) \cap [1, m-1] \nsubseteq \mathrm{D}(t)$ if and only if $\mathrm{ind}(u, t) \in [1, m-2]$. Clearly this holds if and only if $u_{\,>m} = t_{\,>m}$ and $u_{\,\leqslant m} \to^{*2} t_{\,\leqslant m}$. If $u \to^{*1} t$ and $\mathrm{ind}(u, t) \leqslant m-1$ then $\mathrm{ind}(u, t) \in [2, m-1]$, and $\mathrm{D}(u) \cap [1, m-1] \nsubseteq \mathrm{D}(t)$ is automatically satisfied. Clearly this holds if and only if $u_{\,>m} = t_{\,>m}$ and $u_{\,\leqslant m} \to^{*1} t_{\,\leqslant m}$. It follows that $u \approx_m t$ if and only if $u_{\,>m} = t_{\,>m}$, since $\mathrm{Shape}(u_{\,\leqslant m}) = \mathrm{Shape}(t_{\,\leqslant m})$ guarantees that $u_{\,\leqslant m}$ and $t_{\,\leqslant m}$ are related by a sequence of DKMs. So in fact $u \approx_m t$ if and only if $t = wu$ for some $w \in W_m$.

It is a consequence of Definitions 6.32 and 6.34 that if $u, t \in \mathrm{Std}(\lambda)$ then $u \approx_m t$ if and only if $\mathrm{btlr}(u) \approx_J \mathrm{btlr}(t)$, where $J = S_m$. The set of all $(\leqslant m)$-subclasses of $\mathrm{Std}(\lambda)$ is in bijective correspondence with the set $\{\, v \in \mathrm{Std}_m(\lambda/\pi) \mid \pi \in P(m) \text{ and } [\pi] \subseteq [\lambda] \,\}$, and each $(\leqslant m)$-subclass of $\mathrm{Std}(\lambda)$ is in bijective correspondence with $\mathrm{Std}(\pi)$ for some $\pi \in P(m)$ with $[\pi] \subseteq [\lambda]$. If $t \in \mathrm{Std}(\lambda)$ then the $(\leqslant m)$-subclass that contains $t$ is denoted by $C_m(t)$ and is given by $C_m(t) = \{\, u \in \mathrm{Std}(\lambda) \mid u_{\,>m} = t_{\,>m} \,\}$.

In view of Remark 6.28 and Theorem 6.29, the following theorem follows from the results of Kazhdan and Lusztig [9, §5].

Theorem 6.36. *With $\Gamma$ as in Remark 6.33, if $t, t' \in \mathrm{Std}(n)$ then the $W_n$-graphs $\Gamma(C(t))$ and $\Gamma(C(t'))$ are isomorphic if and only if* $\mathrm{Shape}(t) = \mathrm{Shape}(t')$*. In particular, if $\lambda \in P(n)$ then $\Gamma(C(t)) \cong \Gamma(C(\tau_\lambda))$ for every $t \in \mathrm{Std}(\lambda)$.*

Corollary 6.37. *Let $\Gamma$ be the $W_n$-graph of a Kazhdan–Lusztig left cell of $W_n$. Then $\Gamma$ is isomorphic to $\Gamma(C(\tau_\lambda))$ for some $\lambda \in P(n)$.*

Clearly for each $\lambda \in P(n)$ the bijection $t \mapsto \mathrm{btlr}(t)$ from $\mathrm{Std}(\lambda)$ to $C(\tau_\lambda)$ can be used to create a $W_n$-graph isomorphic to $\Gamma(C(\tau_\lambda))$ with $\mathrm{Std}(\lambda)$ as the vertex set.

Notation 6.38. For each $\lambda \in P(n)$ we write $\Gamma_\lambda = \Gamma(\mathrm{Std}(\lambda), \mu^{(\lambda)}, \tau^{(\lambda)})$ for the $W_n$-graph just described.

Remark 6.39. Let $\lambda \in P(n)$ and let $J = S_m \subseteq S_n$. It follows from Remark 6.33 and Definition 6.34 that the $J$-submolecules of $\Gamma_\lambda$ are spanned by the $(\leqslant m)$-subclasses of $\mathrm{Std}(\lambda)$.

Now let $\lambda \in P(n)$ and $1 \leqslant m \leqslant n$, and put $J = S_n \setminus S_m$. The $J$-submolecules of $\Gamma_\lambda$ can be determined by an analysis similar to that used above. We define $\approx^m$ to be the equivalence relation on $\mathrm{Std}(\lambda)$ generated by the requirement that $u \approx^m t$ whenever there is a DKM of index at least $m$ from $u$ to $t$ and $\mathrm{D}(u) \cap [m, n-1] \nsubseteq \mathrm{D}(t)$. The $\approx^m$ equivalence classes in $\mathrm{Std}(\lambda)$ will be called the $(\geqslant m)$*-subclasses* of $\mathrm{Std}(\lambda)$. If $u, t \in \mathrm{Std}(\lambda)$ then $u \approx^m t$ if and only if $\mathrm{btlr}(u) \approx_J \mathrm{btlr}(t)$, with $J = S_n \setminus S_m$. An equivalent condition is that $u_{\,<m} = t_{\,<m}$ and $u_{\,\geqslant m} \approx t_{\,\geqslant m}$. It follows that if $t \in \mathrm{Std}(\lambda)$ then the $(\geqslant m)$-subclass that contains $t$ is the set $C^m(t) = \{\, u \in \mathrm{Std}(\lambda) \mid u_{\,<m} = t_{\,<m} \text{ and } u_{\,\geqslant m} \approx t_{\,\geqslant m} \,\}$.

Remark 6.40. Let $\lambda \in P(n)$ and $m \in [1, n]$, and put $J = S_n \setminus S_m$. By the discussion above, the $J$-submolecules of $\Gamma_\lambda$ are spanned by the $(\geqslant m)$-subclasses of $\mathrm{Std}(\lambda)$.

We shall need to use some properties of the well-known “jeu-de-taquin” operation on skew tableaux, which we now describe.

Fix a positive integer $n$ and a target set $\mathcal{T} = [m+1, m+n]$. It is convenient to define a *partial tableau* to be a bijection $t$ from a subset of $\{\,(i,j) \mid i,\, j \in \mathbb{Z}^+\,\}$ to $\mathcal{T}$. We shall also assume that the domain of $t$ is always of the form $[\kappa/\xi] \smallsetminus \{(i,j)\}$, where $\kappa/\xi$ is a skew partition of $n+1$ and $(i,j) \in [\kappa/\xi]$. If $(i,j)$ is $\xi$-addable then $t$ is a $(\kappa/\pi)$-tableau, with $[\pi] = [\xi] \cup \{(i,j)\}$, and if $(i,j)$ is $\kappa$-removable then $t$ is a $(\lambda/\xi)$-tableau, with $[\lambda] = [\kappa] \smallsetminus \{(i,j)\}$.

Now suppose that $\lambda/\pi$ is a skew partition of $n$ and $t \in \mathrm{Std}(\lambda/\pi)$, and suppose also that $c = (i,j)$ is a $\pi$-removable box. Note that $t$ may be regarded as a partial tableau, since $[\lambda/\pi] = [\kappa/\xi] \smallsetminus \{(i,j)\}$, where $[\kappa] = [\lambda]$ and $[\xi] = [\pi] \smallsetminus \{(i,j)\}$. The (forward) *jeu-de-taquin slide on $t$ into $c$* is the process $\mathrm{jdt}(c,t)$ given as follows.

Start by defining $t_0 = t$ and $b_0 = (i,j)$. Proceeding recursively, suppose that $k \geqslant 0$ and that $t_k$ and $b_k$ are defined, with $t_k$ a partial tableau whose domain is $[\kappa/\xi] \smallsetminus \{b_k\}$. If $b_k$ is $\lambda$-removable then the process terminates, we define $t' = t_k$ and put $m = k$. If $b_k = (g,h)$ is not $\lambda$-removable we put $x = \min(t_k(g+1,h), t_k(g,h+1))$, define $b_{k+1} = t_k^{-1}(x)$, and define $t_{k+1}$ to be the partial tableau with domain $[\kappa/\xi] \smallsetminus \{b_{k+1}\}$ given by

$$t_{k+1}(b) = \begin{cases} t_k(b) & \text{whenever } b \text{ is in the domain of } t_k \text{ and } b \neq b_{k+1}, \\ x & \text{if } b = b_k. \end{cases}$$

(We say that $x$ slides from $b_{k+1}$ into $b_k$.) The tableau $t'$ obtained by the above process is denoted by $\mathrm{jdt}^{(c)}(t)$. The sequence of boxes $[b_0, b_1, \ldots, b_m]$ is called the *slide path* of $\mathrm{jdt}(c,t)$, and we say that $\mathrm{jdt}(c,t)$ *vacates* the box $d = b_m$ and *produces* the tableau $\mathrm{jdt}^{(c)}(t)$.

Backward jeu-de-taquin slides are defined by similar rules: the slide path of a backward slide is the reverse of the slide path of a forward slide. The backward slide $\mathrm{jdt}(c,t)$ is defined whenever $c$ is a $\lambda$-addable box, its slide path terminates with a $\pi$-addable box $d$, and we write $\mathrm{jdt}_{(c)}(t)$ for the tableau produced. We have the following well-known result.

Proposition 6.41. *Let $\lambda/\pi \vdash n$, let $c$ be a $\pi$-removable box and $d$ a $\lambda$-removable box, and define $\pi'$ and $\lambda'$ by $[\pi'] = [\pi] \smallsetminus \{c\}$ and $[\lambda'] = [\lambda] \smallsetminus \{d\}$. Let $t \in \mathrm{Std}(\lambda/\pi)$ and $t' \in \mathrm{Std}(\lambda'/\pi')$. Then* $\mathrm{jdt}(c,t)$ *vacates $d$ and produces $t'$ if and only if* $\mathrm{jdt}(d,t')$ *vacates $c$ and produces $t$.*

Example 6.42. Suppose that $t \in \mathrm{Std}((3,3,2)/(2,1))$ is given by

$$t = \begin{array}{ccc} & & \boxed{2} \\ & \boxed{3} & \boxed{4} \\ \boxed{1} & \boxed{5} & \end{array},$$

and note that the box $c = (1,2)$ is $(2,1)$-removable. The jeu-de-taquin slide on $t$ into $c$ is

$$\begin{array}{ccc} & & \boxed{2} \\ & \boxed{3} & \boxed{4} \\ \boxed{1} & \boxed{5} & \end{array} \longrightarrow \begin{array}{ccc} & \boxed{2} & \\ & \boxed{3} & \boxed{4} \\ \boxed{1} & \boxed{5} & \end{array} \longrightarrow \begin{array}{ccc} & \boxed{2} & \boxed{4} \\ & \boxed{3} & \\ \boxed{1} & \boxed{5} & \end{array},$$

terminating here since the $(3,3,2)$-removable box $(2,3)$ has been vacated. Thus

$$\mathrm{jdt}^{(c)}(t) = \begin{array}{ccc} & \boxed{2} & \boxed{4} \\ & \boxed{3} & \\ \boxed{1} & \boxed{5} & \end{array},$$

the slide path of $\mathrm{jdt}(c,t)$ is $[(1,2),(1,3),(2,3)]$, and the box vacated by $\mathrm{jdt}(c,t)$ is $(2,3)$.

The following observation follows immediately from the definition of a slide path.

Lemma 6.43. *Let $[(i_0, j_0), (i_1, j_1), \dots, (i_m, j_m)]$ be the slide path of a jeu-de-taquin slide, as described above. Then $i_0 \leqslant i_1 \leqslant \cdots \leqslant i_m$ and $j_0 \leqslant j_1 \leqslant \cdots \leqslant j_m$.*

It is straightforward to check the following result.

Lemma 6.44. *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l) \in P(n)$ and choose $m$ so that $\lambda_1 = \lambda_2 = \cdots = \lambda_m > \lambda_{m+1}$. Let $\pi = (1) \in P(1)$ and put $t = \tau_{\lambda/\pi} = (\tau_\lambda)_{>1} - 1$. Then the slide* $\mathrm{jdt}((1,1), t)$ *vacates the box $(\lambda_1, m)$. Similarly, if $u = \tau^{\lambda/\pi} = (\tau^\lambda)_{>1} - 1$ and $k$ satisfies $\lambda_1^* = \lambda_2^* = \cdots = \lambda_k^* > \lambda_{k+1}^*$ then the slide* $\mathrm{jdt}((1,1), u)$ *vacates the box $(k, \lambda_1^*)$.*

A sequence of boxes $\beta = (b_1, \dots, b_l)$ is called a *slide sequence* for a standard skew tableau $t$ if there exists a sequence of standard skew tableaux $t = t_0, t_1, \dots, t_l$ such that the jeu-de-taquin slide $\mathrm{jdt}(b_i, t_{i-1})$ is defined for each $i \in [1, l]$, and $t_i = \mathrm{jdt}^{(b_i)}(t_{i-1})$. We write $t_l = \mathrm{jdt}_\beta(t)$. Note that the slide sequence $\beta = (b_1, \dots, b_l)$ can be extended to a longer slide sequence $b_1, \dots, b_{l+1}$ if and only if the skew tableau $t_l$ is not of normal shape. Indeed, if $\mathrm{Shape}(t_l) = \lambda/\pi$ and $\pi$ is not the empty partition then $b_{l+1}$ may be chosen to be any $\pi$-removable box. If $t_l$ is of normal shape, so that the slide sequence $\beta$ is maximal, then Theorem 6.45 below shows that the tableau $t_l$ depends only on the initial skew tableau $t$ and not on the particular choice of maximal slide sequence $\beta$. Accordingly, we write $t_l = \mathrm{jdt}(t)$ whenever $\beta$ is maximal.

Example. With $t$ as in Example 6.42 above, there are two maximal slide sequences for $t$, namely $(1,2), (2,1), (1,1)$ and $(2,1), (1,2), (1,1)$. They produce the following sequences of standard skew tableaux, ending at the same standard tableau of normal shape:

$$\begin{array}{|c|c|c|}\hline & & 2 \\ \hline & 3 & 4 \\ \hline 1 & 5 & \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|c|}\hline & 2 & 4 \\ \hline & 3 & \\ \hline 1 & 5 & \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|c|}\hline & 2 & 4 \\ \hline 1 & 3 & \\ \hline 5 & & \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & & \\ \hline 5 & & \\ \hline\end{array},$$

$$\begin{array}{|c|c|c|}\hline & & 2 \\ \hline & 3 & 4 \\ \hline 1 & 5 & \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|c|}\hline & & 2 \\ \hline 1 & 3 & 4 \\ \hline 5 & & \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|c|}\hline & 2 & 4 \\ \hline 1 & 3 & \\ \hline 5 & & \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & & \\ \hline 5 & & \\ \hline\end{array}.$$

In fact, the standard tableau $\mathrm{jdt}(t)$ produced by applying a maximal slide sequence to a standard skew tableau $t$ is the Robinson–Schensted insertion tableau of $\mathrm{btlr}(t)$.

Theorem 6.45 ([12, Theorem 3.7.7]). *Let $\lambda/\pi$ be a skew partition of $n$ and $t \in \mathrm{Std}(\lambda/\pi)$. If $\beta$ is any maximal length slide sequence for $t$ then* $\mathrm{jdt}_\beta(t) = \mathrm{P}(\mathrm{btlr}(t))$.

Skew tableaux $u$ and $t$ are said to be *dual equivalent* if the skew tableaux $\mathrm{jdt}_\beta(u)$ and $\mathrm{jdt}_\beta(t)$ are of the same shape whenever $\beta$ is a slide sequence for both $u$ and $t$. Dual equivalent skew tableaux are necessarily of the same shape, since $\beta$ is allowed to be the empty sequence. It is easily shown that if $u$ and $t$ are dual equivalent then every slide sequence for $u$ is also a slide sequence for $t$; so dual equivalence is indeed an equivalence relation. Theorem 6.46 below says that this equivalence relation coincides with dual Knuth equivalence. It is easily shown that $\mathrm{D}(t) = \mathrm{D}(\mathrm{jdt}(t))$ holds for all $t \in \mathrm{Std}(\lambda/\pi)$; indeed, if $i \in [1, n-1]$ and $u$ is any partial tableau used in the construction of $\mathrm{jdt}(t)$, then $\mathrm{col}_t(i+1) \leqslant \mathrm{col}_t(i)$ if and only if $\mathrm{col}_u(i+1) \leqslant \mathrm{col}_u(i)$.

Theorem 6.46 ([12, Theorem 3.8.8]). *Let $\lambda/\pi$ be a skew partition, and let $u$ and $t$ be standard $(\lambda/\pi)$-tableaux. Then $u$ is dual equivalent to $t$ if and only if $u \approx t$.*

Note that Theorem 6.46 generalizes the fact that the set of standard tableaux of a given normal shape form a single dual Knuth equivalence class.

If $\lambda/\pi$ is a skew partition of $n$ then the *dual equivalence graph* $\mathcal{G}_{\lambda/\pi}$ has vertex set $\mathrm{Std}(\lambda/\pi)$ and edge set $\bigl\{\{u, t\} \mid u, t \in \mathrm{Std}(\lambda/\pi) \text{ and } u \to^{*1} t \text{ or } u \to^{*2} t\bigr\}$. By

Theorem 6.46, the connected components of $\mathcal{G}_{\lambda/\pi}$ correspond to the dual equivalence classes in $\mathrm{Std}(\lambda/\pi)$; if $C$ is a dual equivalence class we write $\mathcal{G}_{\lambda/\pi}(C)$ for component with vertex set $C$. It follows from Proposition 6.30 that if $\pi$ is the empty partition then $\mathcal{G}_\lambda = \mathcal{G}_{\lambda/\pi}$ is connected. We call $\mathcal{G}_\lambda$ the *standard dual equivalence graph* corresponding to $\lambda \in P(n)$.

Proposition 6.47. *Let $\lambda/\pi \vdash n$ and $C \subseteq \mathrm{Std}(\lambda/\pi)$ a dual equivalence class. Then there exists a $\xi \in P(n)$ such that $u \mapsto \mathrm{jdt}(u)$ is a bijection $C \to \mathrm{Std}(\xi)$ inducing a graph isomorphism from $\mathcal{G}_{\lambda/\pi}(C)$ to $\mathcal{G}_\xi$. Furthermore, this isomorphism preserves descent sets of vertices.*

*Proof.* We use induction on the cardinality of $[\pi]$, the result being trivial if $[\pi] = \varnothing$. Now suppose that $[\pi] \neq \varnothing$, let $c \in [\pi]$ be $\pi$-removable, and let $C' = \{\mathrm{jdt}^{(c)}(t) \mid t \in C\}$. It follows from the definition of dual equivalence all elements of $C'$ have the same shape $\lambda'/\pi'$, where $[\pi'] = [\pi] \setminus \{c\}$ and $[\lambda'] = [\lambda] \setminus \{d\}$ for some $\lambda$-removable $d \in [\lambda]$. Furthermore, since it is also clear from the definition that if $t_1, t_2 \in C$ then $\mathrm{jdt}^{(c)}(t_1)$ and $\mathrm{jdt}^{(c)}(t_2)$ are dual equivalent, there is a dual equivalence class containing all elements of $C'$. Now choose some $t \in C$ and let $t' = \mathrm{jdt}^{(c)}(t)$. Let $u' \in \mathrm{Std}(\lambda'/\pi')$ and put $u = \mathrm{jdt}_{(d)}(u')$. If $u'$ is dual equivalent to $t'$ then $u' \approx t'$ by [12, Theorem 3.8.8], that is, $u' = u_0 \to^{*i_1} u_1 \to^{*i_2} \cdots \to^{*i_{l-1}} u_{l-1} \to^{*i_l} u_l = t'$, for some integer $l$ and some $i_1, \ldots, i_l \in \{1, 2\}$. The proof of [12, Theorem 3.8.8] now shows that $u = \mathrm{jdt}_{(d)}(u_0) \to^{*j_1} \mathrm{jdt}_{(d)}(u_1) \to^{*j_2} \cdots \to^{*j_{l-1}} \mathrm{jdt}(u_{l-1}) \to^{*j_l} t$ for some $j_1, \ldots, j_l \in \{1, 2\}$, and so $u$ is dual equivalent to $t$ (by [12, Theorem 3.8.8]). Hence $u \in C$, and so $u' = \mathrm{jdt}^{(c)}(u) \in C'$. Thus $C'$ is a dual equivalence class in $\mathrm{Std}(\lambda'/\pi')$ and $\mathrm{jdt}^{(c)} \colon C \mapsto C'$ is surjective. Since $\mathrm{jdt}_{(d)}(\mathrm{jdt}^{(c)}(t)) = t$ for all $t \in C$, the map $\mathrm{jdt}^{(c)}$ is also injective. Moreover, the proof of [12, Theorem 3.8.8] shows that $u$ and $t$ in $C$ are related by a DKM if and only if $\mathrm{jdt}^{(c)}(u)$ and $\mathrm{jdt}^{(c)}(t)$ in $C'$are related by a DKM. It follows that $\mathrm{jdt}^{(c)}$ is an edge preserving bijection from $C$ to a dual equivalence class $C'$ in $\mathrm{Std}(\lambda'/\pi')$. Since the jeu-de-taquin process also preserves descent sets, as we observed above, the result now follows simply by induction. □

If $k \in [1, n-2]$ then each $v \in \mathrm{Std}(\lambda/\pi)$ with $\mathrm{D}(v) \cap \{k, k+1\} = \{k\}$ is adjacent in the dual equivalence graph to a unique $v'$ with $\mathrm{D}(v') \cap \{k, k+1\} = \{k+1\}$, and each $v$ with $\mathrm{D}(v) \cap \{k, k+1\} = \{k+1\}$ is adjacent to a unique $v'$ with $\mathrm{D}(v') \cap \{k, k+1\} = \{k\}$. In fact, $v' = s_k v$ if the box $v^{-1}(k+2)$ is between the boxes $v^{-1}(k)$ and $v^{-1}(k+1)$, in the sense that $\mathrm{col}_v(k) < \mathrm{col}_v(k+2) \leqslant \mathrm{col}_v(k+1)$ or $\mathrm{col}_v(k+1) < \mathrm{col}_v(k+2) \leqslant \mathrm{col}_v(k)$, while $v' = s_{k+1} v$ if $v^{-1}(k)$ is between $v^{-1}(k+1)$ and $v^{-1}(k+2)$ (meaning that $\mathrm{col}_v(k+1) \leqslant \mathrm{col}_v(k) < \mathrm{col}_v(k+2)$ or $\mathrm{col}_v(k+2) \leqslant \mathrm{col}_v(k) < \mathrm{col}_v(k+1)$).

Definition 6.48. *We call the above tableau $v'$ the* $k$-neighbour *of $v$, and write $v' = k\text{-}\mathrm{neb}(v)$.*

It follows from Remark 6.28 that the standard dual equivalence graph $\mathcal{G}_\lambda$ is isomorphic to the simple part of each Kazhdan–Lusztig left cell $\Gamma(C(t))$ for $t \in \mathrm{Std}(\lambda)$. Extending earlier work of Assaf [1], Chmutov showed in [4] that the simple part of an admissible $W_n$-molecule is isomorphic to a standard dual equivalence graph. (Recall that $W$-molecules need not be $W$-graphs, since the Polygon Rule may not be satisfied.) The following result is the main theorem of [4].

Theorem 6.49 ([4, Theorem 2]). *The simple part of an admissible molecule of type $A_{n-1}$ is isomorphic to the simple part of a Kazhdan–Lusztig left cell.*

Remark 6.50. It follows that if $M = (V, \mu, \tau)$ is a molecule then there exists $\lambda \in P(n)$ and a bijection $t \mapsto c_t$ from $\mathrm{Std}(\lambda)$ to $V$ such that the simple edges of $M$ are the pairs

$\{c_u, c_t\}$ such that $u, t \in \mathrm{Std}(\lambda)$ and there is a DKM from $u$ to $t$ or from $t$ to $u$, and $\tau(c_t) = \{\, s_j \mid j \in \mathrm{D}(t)\,\}$. The molecule $M$ is said to be of type $\lambda$.

Let $M = (V, \mu, \tau)$ be an arbitrary $S_n$-coloured molecular graph, and for each $\lambda \in P(n)$ let $m_\lambda$ be the number of molecules of type $\lambda$ in $M$. For each $\lambda$ such that $m_\lambda \neq 0$ let $\mathcal{I}_\lambda$ be some indexing set of cardinality $m_\lambda$. Then we can write

$$V = \bigsqcup_{\lambda\in\Lambda} \bigsqcup_{\alpha\in\mathcal{I}_\lambda} V_{\alpha,\lambda}, \tag{4}$$

where $\Lambda = \{\, \lambda \in P(n) \mid m_\lambda \neq 0\,\}$, each $V_{\alpha,\lambda} = \{\, c_{\alpha,t} \mid t \in \mathrm{Std}(\lambda)\,\}$ is the vertex set of a molecule of type $\lambda$, the simple edges of $M$ are the pairs $\{c_{\alpha,u}, c_{\beta,t}\}$ such that $\alpha = \beta \in \mathcal{I}_\lambda$ for some $\lambda \in \Lambda$ and $u, t \in \mathrm{Std}(\lambda)$ are related by a DKM, and $\tau(c_{\alpha,t}) = \{\, s_j \mid j \in \mathrm{D}(t)\,\}$ for all $(\alpha, t) \in \mathcal{I}_\lambda \times \mathrm{Std}(\lambda)$. We call $\Lambda$ the set of *molecule types* for $M$.

Note that if $\Gamma = (V, \mu, \tau)$ is an admissible $W_n$-graph then $\Gamma$ is an $S_n$-coloured molecular graph, by Remark 5.12, and hence Eq. (4) can be used to describe the vertex set of $\Gamma$.

Remark 6.51. We know from Remark 5.2 and Corollary 6.37 that, for each $\lambda \in P(n)$, the $W_n$-graph $\Gamma_\lambda = (\mathrm{Std}(\lambda), \mu^{(\lambda)}, \tau^{(\lambda)})$ is admissible. Since $\{u, t\}$ is a simple edge in $\Gamma_\lambda$ when $u$, $t \in \mathrm{Std}(\lambda)$ are related by a DKM, and $\mathrm{Std}(\lambda)$ is a single dual Knuth equivalence class, we see that $\Gamma_\lambda$ consists of a single molecule (of type $\lambda$).

Remark 6.52. Let $\Gamma = (V, \mu, \tau)$ be an admissible $W_n$-graph, and continue with the notation and terminology of Remark 6.50 above. Let $m \in [1, n]$, and let $K = S_m$ and $L = S_n \setminus S_m$.

Let $\lambda \in \Lambda$ and $\alpha \in \mathcal{I}_\lambda$, and let $\Theta$ be the molecule of $\Gamma$ whose vertex set is $V_{\alpha,\lambda}$. By Remark 6.50 applied to $\Theta_K$, the $W_K$-restriction of $\Theta$ (as defined in Section 4 above), we may write

$$V_{\alpha,\lambda} = \bigsqcup_{\kappa\in\Lambda_{K,\alpha,\lambda}} \bigsqcup_{\beta\in\mathcal{I}_{K,\alpha,\lambda,\kappa}} V_{\alpha,\lambda,\beta,\kappa},$$

where $\Lambda_{K,\alpha,\lambda}$ is the set of all $\kappa \in P(m)$ such that $\Theta$ contains a $K$-submolecule of type $\kappa$, and $\mathcal{I}_{K,\alpha,\lambda,\kappa}$ is an indexing set whose size is the number of such $K$-submolecules. Each $V_{\alpha,\lambda,\beta,\kappa}$ is the vertex set of a $K$-submolecule of $\Theta$ of type $\kappa$. Writing $V_{\alpha,\lambda,\beta,\kappa} = \{\, c'_{\beta,u} \mid u \in \mathrm{Std}(\kappa)\,\}$, we see that each $c_{\alpha,t} \in V_{\alpha,\lambda}$ coincides with some $c'_{\beta,v}$ with $\beta \in \mathcal{I}_{K,\alpha,\lambda,\kappa}$ and $v \in \mathrm{Std}(\kappa)$. It follows from Remark 6.39 above that the $K$-submolecule of $\Theta$ containing a given vertex $c_{\alpha,t}$ is spanned by the $(\leqslant m)$-subclass $C_m(t) = \{\, u \in \mathrm{Std}(\lambda) \mid u_{>m} = t_{>m}\,\}$. Thus when we write $c_{\alpha,t} = c'_{\beta,v}$ as above, we can identify $v$ with $t_{\leqslant k}$.

Similarly, applying Remark 6.50 to $\Theta_L$, we may write

$$V_{\alpha,\lambda} = \bigsqcup_{\theta\in\Lambda_{L,\alpha,\lambda}} \bigsqcup_{\gamma\in\mathcal{I}_{L,\alpha,\lambda,\theta}} V_{\alpha,\lambda,\gamma,\theta},$$

where $\Lambda_{L,\alpha,\lambda}$ is the set of all $\theta \in P(n-m+1)$ such that $\Theta$ contains an $L$-submolecule of type $\theta$, each $V_{\alpha,\lambda,\gamma,\theta}$ is the vertex set of an $L$-submolecule of type $\theta$, and the set $\mathcal{I}_{L,\alpha,\lambda,\theta}$ indexes these submolecules. Writing $V_{\alpha,\lambda,\gamma,\theta} = \{\, c''_{\gamma,v} \mid v \in \mathrm{Std}_{m-1}(\theta)\,\}$ (where $\mathrm{Std}_{m-1}(\theta)$ is the set of standard $\theta$-tableaux with target $[m, n]$), we see that each $c_{\alpha,t} \in V_{\alpha,\lambda}$ coincides with some $c''_{\gamma,v}$ with $\gamma \in \mathcal{I}_{L,\alpha,\lambda,\theta}$ and $v \in \mathrm{Std}_{m-1}(\theta)$. By Remark 6.40 above we see that the $L$-submolecule of $\Theta$ containing a given vertex $c_{\alpha,t}$ is spanned by the $(\geqslant m)$-subclass $C^m(t) = \{\, u \in \mathrm{Std}(\lambda) \mid u_{<m} = t_{<m}$ and $u_{\geqslant m} \approx t_{\geqslant m}\,\}$. Since the condition $u_{\geqslant m} \approx t_{\geqslant m}$ is equivalent to $1 - m + (u_{\geqslant m}) \approx 1 - m + (t_{\geqslant m})$, it follows from Proposition 6.47 that when we write $c_{\alpha,t} = c''_{\beta,v}$ as above we can identify $v$ with $\mathrm{jdt}(t_{\geqslant m}) = m - 1 + \mathrm{jdt}(1 - m + (t_{\geqslant m}))$.

## 7. Extended Bruhat order on $\mathrm{Std}(n)$ and paired dual Knuth equivalence relation

Let $n \geqslant 1$ and let $(W_n, S_n)$ be the Coxeter group of type $A_{n-1}$ Recall from Proposition 6.10 that if $\lambda \in P(n)$ and $u$, $t \in \mathrm{Std}(\lambda)$ then $u \leqslant t$ if and only if $\mathrm{Shape}(u_{\leqslant m}) \leqslant \mathrm{Shape}(t_{\leqslant m})$ for all $m \in [1, n]$. Hence it is natural to make the following definition.

Definition 7.1. *Let $\lambda$, $\pi \in P(n)$, and let $u \in \mathrm{Std}(\lambda)$ and $t \in \mathrm{Std}(\pi)$. We write $u \leqslant t$ if $\mathrm{Shape}(u_{\leqslant m}) \leqslant \mathrm{Shape}(t_{\leqslant m})$ for all $m \in [1, n]$.*

It is obvious that this is a partial order on $\mathrm{Std}(n) = \bigcup_{\lambda \in P(n)} \mathrm{Std}(\lambda)$ extending the Bruhat order on each $\mathrm{Std}(\lambda)$.

Example. For $n = 2$ and $n = 3$ we obtain

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} < \begin{array}{|c|c|}\hline 1&2\\\hline\end{array} \qquad \text{and} \qquad \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array} < \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} < \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} < \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}.$$

Observe that $u \leqslant t$ if and only if $\mathrm{Shape}(u) \leqslant \mathrm{Shape}(t)$ and $u_{\leqslant (n-1)} \leqslant t_{\leqslant (n-1)}$.

We remark that in [2] this order was used in the context of the representation theory of symmetric groups, while in [3] it was used in the context of combinatorics of permutations.

Lemma 7.2. *Let $\pi$, $\lambda \in P(n)$, let $u \in \mathrm{Std}(\pi)$ and $t \in \mathrm{Std}(\lambda)$, and let $\sigma = \mathrm{Shape}(u_{<n})$ and $\theta = \mathrm{Shape}(t_{<n})$. Suppose that $\sigma \leqslant \theta$ and $\mathrm{col}_u(n) \leqslant \mathrm{col}_t(n)$. Then $\pi \leqslant \lambda$.*

*Proof.* Let $\mathrm{col}_u(n) = p$ and $\mathrm{col}_t(n) = r$, and assume that $p \leqslant r$. Recall that $\sigma \leqslant \theta$ is equivalent to $\sigma \trianglerighteq \theta$, and so $\sum_{m=1}^{l} \sigma_m \geqslant \sum_{m=1}^{l} \theta_m$ for all $l \in [1, r-1]$. Hence for all $l \in [1, p-1]$ we have

$$\sum_{m=1}^{l} \pi_m = \sum_{m=1}^{l} \sigma_m \geqslant \sum_{m=1}^{l} \theta_m = \sum_{m=1}^{l} \lambda_m,$$

while for all $l \in [p, r-1]$ we have

$$\sum_{m=1}^{l} \pi_m = (\sigma_p + 1) + \sum_{\substack{m=1\\ m\neq p}}^{l} \sigma_m > \sum_{m=1}^{l} \theta_m = \sum_{m=1}^{l} \lambda_m,$$

and for all $l > r$ we have

$$\sum_{m=1}^{l} \pi_m = (\sigma_p + 1) + \sum_{\substack{m=1\\ m\neq p}}^{l} \sigma_m \geqslant (\theta_r + 1) + \sum_{\substack{m=1\\ m\neq r}}^{l} \theta_m = \sum_{m=1}^{l} \lambda_m.$$

Hence $\pi \leqslant \lambda$. $\square$

Lemma 7.3. *Let $\lambda \in P(n)$ and $t \in \mathrm{Std}(\lambda)$. Suppose that $i \in \mathrm{SD}(t)$, and let $p = \mathrm{col}_t(i)$ and $j = \mathrm{col}_t(i+1)$. For all $h \in [1, n-1]$ let $\lambda^{(h)} = \mathrm{Shape}(t_{\leqslant h})$ and $\theta^{(h)} = \mathrm{Shape}(s_i t_{\leqslant h})$. Then*

$$\sum_{m=1}^{l} \theta_m^{(i)} = \begin{cases} \sum_{m=1}^{l} \lambda_m^{(i)} = \sum_{m=1}^{l} \lambda_m^{(i+1)} = \sum_{m=1}^{l} \lambda_m^{(i-1)} & \text{if } l < j \\ \sum_{m=1}^{l} \lambda_m^{(i)} + 1 = \sum_{m=1}^{l} \lambda_m^{(i+1)} = \sum_{m=1}^{l} \lambda_m^{(i-1)} + 1 & \text{if } j \leqslant l < p \\ \sum_{m=1}^{l} \lambda_m^{(i)} = \sum_{m=1}^{l} \lambda_m^{(i+1)} - 1 = \sum_{m=1}^{l} \lambda_m^{(i-1)} + 1 & \text{if } p < l \end{cases} \tag{5}$$

*and*

$$\sum_{m=1}^{l} \lambda_m^{(i)} = \begin{cases} \sum_{m=1}^{l} \theta_m^{(i)} = \sum_{m=1}^{l} \theta_m^{(i+1)} = \sum_{m=1}^{l} \theta_m^{(i-1)} & \text{if } l < j \\ \sum_{m=1}^{l} \theta_m^{(i)} - 1 = \sum_{m=1}^{l} \theta_m^{(i+1)} - 1 = \sum_{m=1}^{l} \theta_m^{(i-1)} & \text{if } j \leqslant l < p \\ \sum_{m=1}^{l} \theta_m^{(i)} = \sum_{m=1}^{l} \theta_m^{(i+1)} - 1 = \sum_{m=1}^{l} \theta_m^{(i-1)} + 1 & \text{if } p < l. \end{cases} \tag{6}$$

*Proof.* The results given by Eq. (5) and Eq. (6) are readily obtained from the following formulae

$$\theta_m^{(i)} = \begin{cases} \lambda_m^{(i)} + 1 = \lambda_m^{(i+1)} = \lambda_m^{(i-1)} + 1 & \text{if } m = j \\ \lambda_m^{(i)} - 1 = \lambda_m^{(i+1)} - 1 = \lambda_m^{(i-1)} & \text{if } m = p \\ \lambda_m^{(i)} = \lambda_m^{(i+1)} = \lambda_m^{(i-1)} & \text{if } m \neq j, p \end{cases} \tag{7}$$

and

$$\lambda_m^{(i)} = \begin{cases} \theta_m^{(i)} - 1 = \theta_m^{(i+1)} - 1 = \theta_m^{(i-1)} & \text{if } m = j \\ \theta_m^{(i)} + 1 = \theta_m^{(i+1)} = \theta_m^{(i-1)} + 1 & \text{if } m = p \\ \theta_m^{(i)} = \theta_m^{(i+1)} = \theta_m^{(i-1)} & \text{if } m \neq j, p, \end{cases} \tag{8}$$

respectively. □

Lemma 7.4. *Let $\pi$, $\lambda \in P(n)$, let $u \in \mathrm{Std}(\pi)$ and $t \in \mathrm{Std}(\lambda)$. Suppose that $i \in \mathrm{SD}(u) \cap \mathrm{SD}(t)$. Then $u \leqslant t$ if and only if $s_i u \leqslant s_i t$.*

*Proof.* Let $j = \mathrm{col}_t(i+1)$ and let $k = \mathrm{col}_u(i+1)$. For all $h \in [1, n]$ let $\lambda^{(h)} = \mathrm{Shape}(t_{\leqslant h})$, let $\theta^{(h)} = \mathrm{Shape}(s_i t_{\leqslant h})$, let $\pi^{(h)} = \mathrm{Shape}(u_{\leqslant h})$ and let $\sigma^{(h)} = \mathrm{Shape}(s_i u_{\leqslant h})$.

Suppose that $u \leqslant t$. Since $s_i u$ and $s_i t$ differ from $u$ and $t$ respectively only in the positions of $i$ and $i+1$, we have $\pi^{(h)} = \sigma^{(h)}$ and $\lambda^{(h)} = \theta^{(h)}$ for all $h \neq i$. But since $\pi^{(h)} \leqslant \lambda^{(h)}$ for all $h$ by our assumption, it follows that $\sigma^{(h)} \leqslant \theta^{(h)}$ for all $h \neq i$. Hence to show that $s_i u \leqslant s_i t$ it suffices to show that $\sigma^{(i)} \leqslant \theta^{(i)}$. Let $l \in \mathbb{Z}^+$ be arbitrary.

*Case 1.*
Suppose that $l \geqslant k$. By Lemma 7.3 applied to $u$, we have $\sum_{m=1}^{l} \sigma_m^{(i)} = \sum_{m=1}^{l} \pi_m^{(i-1)} + 1$, by the last two formulae of Eq.(5). Since $\pi^{(i-1)} \leqslant \lambda^{(i-1)}$ gives $\sum_{m=1}^{l} \pi_m^{(i-1)} \geqslant \sum_{m=1}^{l} \lambda_m^{(i-1)}$, it follows that $\sum_{m=1}^{l} \sigma_m^{(i)} \geqslant \sum_{m=1}^{l} \lambda_m^{(i-1)} + 1$. But by Lemma 7.3 applied to $t$, in each case in Eq.(5) we have $\sum_{m=1}^{l} \lambda_m^{(i-1)} + 1 \geqslant \sum_{m=1}^{l} \theta_m^{(i)}$. Hence $\sum_{m=1}^{l} \sigma_m^{(i)} \geqslant \sum_{m=1}^{l} \theta_m^{(i)}$.

*Case 2.*
Suppose that $l < k$. By Lemma 7.3 applied to $u$, we have $\sum_{m=1}^{l} \sigma_m^{(i)} = \sum_{m=1}^{l} \pi_m^{(i)} = \sum_{m=1}^{l} \pi_m^{(i+1)}$, by the first formula of Eq. (5). Since $\pi^{(i)} \leqslant \lambda^{(i)}$ and $\pi^{(i+1)} \leqslant \lambda^{(i+1)}$, for each $h \in \{i, i+1\}$ we obtain $\sum_{m=1}^{l} \pi_m^{(h)} \geqslant \sum_{m=1}^{l} \lambda_m^{(h)}$, and hence $\sum_{m=1}^{l} \sigma_m^{(i)} \geqslant \sum_{m=1}^{l} \lambda_m^{(h)}$. By Lemma 7.3 applied to $t$, in each case in Eq.(5) there exists $h \in \{i, i+1\}$ such that $\sum_{m=1}^{l} \lambda_m^{(h)} = \sum_{m=1}^{l} \theta_m^{(i)}$. Hence $\sum_{m=1}^{l} \sigma_m^{(i)} \geqslant \sum_{m=1}^{l} \theta_m^{(i)}$.

Conversely, suppose that $s_i u \leqslant s_i t$. As above, it suffices to show that $\pi^{(i)} \leqslant \lambda^{(i)}$. Let $l \in \mathbb{Z}^+$ be arbitrary.

*Case 1.*
Suppose that $l \geqslant j$. By Lemma 7.3 applied to $t$, we have $\sum_{m=1}^{l} \lambda_m^{(i)} = \sum_{m=1}^{l} \theta_m^{(i+1)} - 1$, by the last two formulae of Eq.(6). Since $\sigma^{(i+1)} \leqslant \theta^{(i+1)}$ gives $\sum_{m=1}^{l} \sigma_m^{(i+1)} \geqslant \sum_{m=1}^{l} \theta_m^{(i+1)}$, it follows that $\sum_{m=1}^{l} \sigma_m^{(i+1)} - 1 \geqslant \sum_{m=1}^{l} \lambda_m^{(i)}$. But by Lemma 7.3 applied to $u$, in each case in Eq.(6) we have $\sum_{m=1}^{l} \pi_m^{(i)} \geqslant \sum_{m=1}^{l} \sigma^{(i+1)} - 1$. Hence $\sum_{m=1}^{l} \pi_m^{(i)} \geqslant \sum_{m=1}^{l} \lambda_m^{(i)}$.

*Case 2.*
Suppose that $l < j$. By Lemma 7.3 applied to $t$, we have $\sum_{m=1}^{l} \lambda_m^{(i)} = \sum_{m=1}^{l} \theta_m^{(i-1)} = \sum_{m=1}^{l} \theta_m^{(i)}$, by the first formula of Eq. (6). Since $\theta^{(i-1)} \geqslant \sigma^{(i-1)}$ and $\theta^{(i)} \geqslant \sigma^{(i)}$, for each $h \in \{i-1, i\}$ we obtain $\sum_{m=1}^{l} \theta_m^{(h)} \leqslant \sum_{m=1}^{l} \sigma_m^{(h)}$, and hence $\sum_{m=1}^{l} \lambda_m^{(i)} \leqslant$

$\sum_{m=1}^{l}\sigma_m^{(h)}$. By Lemma 7.3 applied to $u$, in each case in Eq. 6) there exists $h\in\{i-1,i\}$ such that $\sum_{m=1}^{l}\sigma_m^{(h)}=\sum_{m=1}^{l}\pi_m^{(i)}$. Hence $\sum_{m=1}^{l}\lambda_m^{(i)}\leqslant\sum_{m=1}^{l}\pi_m^{(i)}$. $\square$

Definition 7.5. *Let $\lambda$, $\pi\in P(n)$ and let $1\leqslant m\leqslant n$. Let $u$, $v\in\operatorname{Std}(\pi)$ and $t$, $x\in\operatorname{Std}(\lambda)$, and let $i\in\{1,2\}$. We say that there is a* paired ($\leqslant m$)-DKM of the $i$-th kind from $(u,t)$ to $(v,x)$ *if there exists $k\leqslant m-1$ such that $u\to^{*i}v$ and $t\to^{*i}x$ are ($\leqslant m$)-DKMs of index $k$. When this holds we write $(u,t)\to^{*i}(v,x)$, and call $k$ the index of the paired move.*

Definition 7.6. *Let $\lambda$, $\pi\in P(n)$. The* paired ($\leqslant m$) dual Knuth equivalence relation *is the equivalence relation $\approx_m$ on $\operatorname{Std}(\pi)\times\operatorname{Std}(\lambda)$ generated by paired ($\leqslant m$)-DKMs. When $m=n$ we write $\approx$ for $\approx_n$, and call it the paired dual Knuth equivalence relation.*

We denote by $C_m(u,t)$ the $\approx_m$ equivalence class that contains $(u,t)$.

Remark 7.7. It is clear that $(u,t)\approx_m(v,x)$ implies $(u,t)\approx_{m'}(v,x)$ whenever $m\leqslant m'$. In particular, $(u,t)\approx_m(v,x)$ implies $(u,t)\approx(v,x)$.

Remark 7.8. By Remark 6.35, if $(v,x)\in C_m(u,t)$ then $(v,x)=(wu,wt)$ for some $w\in W_m$. Thus $v_{>m}=u_{>m}$ and $x_{>m}=t_{>m}$. Now suppose that $u_{\leqslant m}=t_{\leqslant m}$, and write $\xi=\operatorname{Shape}(u_{\leqslant m})$. It is clear that every $(v,x)\in C_m(u,t)$ satifies $v_{\leqslant m}=x_{\leqslant m}$. Furthermore, since $\operatorname{Std}(\xi)$ is a single dual Knuth equivalence class, for every $y\in\operatorname{Std}(\xi)$ there is a sequence of ($\leqslant m$)-DKMs taking $u_{\leqslant m}$ to $y$. This same sequence of DKMs takes $(u,t)$ to $(v,x)$, where $v$ satisfies $v_{\leqslant m}=y$ and $v_{>m}=u_{>m}$ and $x$ satisfies $x_{\leqslant m}=y$ and $x_{>m}=t_{>m}$. So it follows that if $u_{\leqslant m}=t_{\leqslant m}$ then $C_m(u,t)=\{\,(v,x)\in\operatorname{Std}(\pi)\times\operatorname{Std}(\lambda)\mid(v,x)=(wu,wt)\text{ for some }w\in W_m\,\}$.

Example. Suppose that $\pi=(3,1)$ and $\lambda=(2,1,1)$. Then the set $\operatorname{Std}(\pi)\times\operatorname{Std}(\lambda)$ has 9 elements. It is easily verified that there are seven paired dual Knuth equivalence classes, of which two classes have 2 elements and five classes have 1 element only. The two non-trivial classes are

$$\left\{\left(\begin{array}{|c|c|}\hline 1&4\\\hline 2\\\cline{1-1}3\\\cline{1-1}\end{array},\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1}\end{array}\right),\left(\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}4\\\cline{1-1}\end{array},\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4\\\cline{1-1}\end{array}\right)\right\}$$

and

$$\left\{\left(\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}4\\\cline{1-1}\end{array},\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2\\\cline{1-1}\end{array}\right),\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}4\\\cline{1-1}\end{array},\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1}\end{array}\right)\right\}.$$

Let $\pi$, $\lambda\in P(n)$ and $(u,t)$, $(v,x)\in\operatorname{Std}(\pi)\times\operatorname{Std}(\lambda)$, and suppose that $(v,x)=(s_iu,s_it)$ for some $i\in[1,n-1]$. If $i\in\operatorname{SD}(u)\cap SD(t)$ then $u\leqslant t$ if and only if $v\leqslant x$, by Lemma 7.4, and it follows by interchanging the roles of $(u,t)$ and $(v,x)$ that the same is true if $i\in\operatorname{SA}(u)\cap SA(t)$. In particular, if there is a paired DKM from $(u,t)$ to $(v,x)$ or from $(v,x)$ to $(u,t)$ then $u\leqslant t$ if and only if $v\leqslant x$. An obvious induction now yields the following result.

Proposition 7.9. *Let $\pi$, $\lambda\in P(n)$. Let $(u,t)$, $(v,x)\in\operatorname{Std}(\pi)\times\operatorname{Std}(\lambda)$ and suppose that $(u,t)\approx(v,x)$. Then $u\leqslant t$ if and only if $v\leqslant x$.*

Let $\pi$, $\lambda\in P(n)$ and $(u,t)$, $(v,x)\in\operatorname{Std}(\pi)\times\operatorname{Std}(\lambda)$. and suppose that $(v,x)=(s_iu,s_it)$ for some $i\in[1,n-1]$. If $i\in\operatorname{SD}(u)\cap SD(t)$ then $l(v)-l(x)=(l(u)-1)-(l(t)-1)=l(u)-l(t)$, and if $i\in\operatorname{SA}(u)\cap\operatorname{SA}(t)$ then $l(v)-l(x)=(l(u)+1)-(l(t)+1)=l(u)-l(t)$. In particular, $l(v)-l(x)=l(u)-l(t)$ if there is a paired DKM from $(u,t)$ to $(v,x)$ or from $(v,x)$ to $(u,t)$. It follows that $l(x)-l(v)$ is constant for all $(v,x)\in C(u,t)$. Hence we obtain the following result.

Proposition 7.10. *Let $\pi$, $\lambda \in P(n)$. Let $(u,t)$, $(v,x) \in \mathrm{Std}(\pi) \times \mathrm{Std}(\lambda)$ and suppose that $(u,t) \approx (v,x)$. Then $u \leqslant_{\mathsf{L}} v$ if and only if $t \leqslant_{\mathsf{L}} x$.*

*Proof.* Since $(u,t) \approx (v,x)$ there exists $w \in W_m$ such that $v = wu$ and $x = wt$. Now $u \leqslant_{\mathsf{L}} v$ if and only if $l(v) - l(u) = l(w)$, and $t \leqslant_{\mathsf{L}} x$ if and only if $l(x) - l(t) = l(w)$, by the definition of the left weak order. Since $(u,t) \approx (v,x)$ implies that $l(v) - l(u) = l(x) - l(t)$, the result follows. □

Definition 7.11. *Let $\pi$, $\lambda \in P(n)$ and $(u,t) \in \mathrm{Std}(\pi) \times \mathrm{Std}(\lambda)$. If $j \in [1,n]$ and $u_{\leqslant j} = t_{\leqslant j}$ then we say that the pair $(u,t)$ is* $j$-restrictable*.*

Remark 7.12. It is clear that the set $R(u,t) = \{\, j \in [1,n] \mid (u,t) \text{ is } j\text{-restrictable} \,\}$ is always nonempty, since $1 \in R(u,t)$. Moreover, $R(u,t) = [1,k]$ for some $k \in [1,n]$.

Definition 7.13. *Let $\pi$, $\lambda \in P(n)$ and $(u,t) \in \mathrm{Std}(\pi) \times \mathrm{Std}(\lambda)$. We shall call the number $k$ satisfying $R(u,t) = [1,k]$ the* restriction number *of the pair $(u,t)$. If $k$ is the restriction number of $(u,t)$ then we say that $(u,t)$ is* $k$-restricted*.*

Remark 7.14. With $(u,t)$ as above, the restriction number of $(u,t)$ is at least 1 and at most $n$. If $k \in [1,n]$ then $(u,t)$ is $k$-restricted if and only if it is $k$-restrictable and not $(k+1)$-restrictable. If $(u,t)$ is $k$-restricted then $k = n$ if and only if $u = t$, and if $k < n$ then $\mathrm{col}_u(k+1) \neq \mathrm{col}_t(k+1)$ and $\mathrm{row}_u(k+1) \neq \mathrm{row}_t(k+1)$.

Lemma 7.15. *Let $\pi$, $\lambda \in P(n)$, and let $u \in \mathrm{Std}(\pi)$ and $t \in \mathrm{Std}(\lambda)$. If $n < 4$ then $\mathrm{D}(u) = \mathrm{D}(t)$ implies $u = t$.*

*Proof.* This is trivially proved by listing all the standard tableaux. □

Definition 7.16. *Let $\pi$, $\lambda \in P(n)$ and $(u,t) \in \mathrm{Std}(\pi) \times \mathrm{Std}(\lambda)$. We say that the pair $(u,t)$ is* favourable *if the restriction number of $(u,t)$ lies in $\mathrm{D}(u) \oplus \mathrm{D}(t)$, the symmetric difference of the descent sets of $u$ and $t$.*

Note that $k \in \mathrm{D}(u) \oplus \mathrm{D}(t)$ if in one of the two tableaux the column number of $k+1$ is less than or equal to the column number of $k$, and in the other the row number of $k+1$ is less than or equal to the row number of $k$.

Example 7.17. The pair

$$(u,t) = \left( \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \\ \cline{1-1} \end{array}\, , \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & \\ \cline{1-2} \end{array} \right)$$

is 3-restricted, since 1, 2 and 3 occupy the same boxes in $u$ and $t$ but 4 does not, and is not favourable since 3 is not a descent of either $u$ or $t$.

Remark 7.18. Let $\pi$, $\lambda \in P(n)$, and suppose that $(u,t) \in \mathrm{Std}(\pi) \times \mathrm{Std}(\lambda)$ is $k$-restricted. Then no element of $[1,k-1]$ can belong to $\mathrm{D}(u) \oplus \mathrm{D}(t)$, since the fact that $u_{\leqslant k} = t_{\leqslant k}$ means that $\mathrm{D}(u) \cap [1,k-1] = \mathrm{D}(t) \cap [1,k-1]$. So if $(u,t)$ is favourable then $k = \min(\mathrm{D}(u) \oplus \mathrm{D}(t))$, and if $(u,t)$ is not favourable and $\mathrm{D}(u) \oplus \mathrm{D}(t)$ is nonempty then $k < \min(\mathrm{D}(u) \oplus \mathrm{D}(t))$.

Let $\pi$, $\lambda \in P(n)$, and let $(u,t) \in \mathrm{Std}(\pi) \times \mathrm{Std}(\lambda)$. Let $i$ be the restriction number of $(u,t)$, and suppose that $i \neq n$. Let $w = u_{\leqslant i} = t_{\leqslant i} \in \mathrm{Std}(\xi)$, where $\xi = \mathrm{Shape}(w)$, and let also $(g,p) = u^{-1}(i+1)$ and $(h,r) = t^{-1}(i+1)$, the boxes of $u$ and $t$ that contain $i+1$. Thus $(g,p)$ and $(h,r)$ are $\xi$-addable, and $(g,p) \neq (h,r)$ since $(u,t)$ is not $(i+1)$-restrictable. Clearly there is at least one $\xi$-removable box $(d,m)$ that lies between $(g,p)$ and $(h,r)$ (in the sense that either $g > d \geqslant h$ and $p \leqslant m < r$, or $h > d \geqslant g$ and $r \leqslant m < p$), and note that $i \in \mathrm{D}(u) \oplus \mathrm{D}(t)$ if and only if the $\xi$-removable box $w^{-1}(i)$ is such a box.

With $(d,m)$ as above, suppose that $w' \in \mathrm{Std}(\xi)$ satisfies $w'(d,m) = i$. Then there exist unique $v \in \mathrm{Std}(\pi)$ and $x \in \mathrm{Std}(\lambda)$ such that $v$ satisfies $v_{\leqslant i} = w'$ and $v_{>i} = u_{>i}$ and $x$ satisfies $x_{\leqslant i} = w'$ and $x_{>i} = t_{>i}$. We see that $(v,x)$ is $i$-restricted and favourable. Furthermore, it follows from Remark 7.8 that $(v,x) \approx_i (u,t)$.

We denote by $F(u,t)$ the set of all $(v,x)$ obtained as above as $(d,m)$ and $w'$ vary. Thus

$$F(u,t) = \{(v,x) \in C_i(u,t) \mid v^{-1}(i) = x^{-1}(i) \text{ lies between } u^{-1}(i+1) \text{ and } t^{-1}(i+1)\}$$

where $i$ is the restriction number of $(u,t)$. Note that $(u,t) \in F(u,t)$ if and only if $(u,t)$ is favourable.

Example. Let $u$ and $t$ be as in Example 7.17 above, so that $i = 3$ and $w = u_{\leqslant 3} = t_{\leqslant 3} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$.

The shape of $w$ is $\xi = (2,1)$, and the boxes of $u$ and $t$ that contain 4 are $(g,p) = (2,2)$ and $(h,r) = (1,3)$. Observe that these are indeed both $\xi$-addable. There are two $\xi$-removable boxes, namely $(2,1)$ and $(1,2)$; the fact that $(u,t)$ is not favourable corresponds to the fact that $w^{-1}(3) = (2,1)$ does not lie between $(g,p)$ and $(h,r)$. Putting $(d,m) = (1,2)$, which (necessarily) does lie between $(g,p)$ and $(h,r)$, we find that (in this small example) there is only one standard $\xi$-tableau $w'$ satisfying $w'(d,m) = 3$, namely $w' = w^*$. Hence the set $F(u,t)$ consists of a single pair $(v,x)$:

$$F(u,t) = \{(v,x)\} = \left\{ \left( \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array} \right) \right\}.$$

Since $w$ and $w'$ are the only standard $\xi$-tableaux, we see that $C_3(u,t) = \{(u,t),(v,x)\}$.

Lemma 7.19. *Let $\pi$, $\lambda \in P(n)$ and let $(u,t) \in \mathrm{Std}(\pi) \times \mathrm{Std}(\lambda)$ with $u \neq t$. Let $i$ be the restriction number of $(u,t)$, and assume that $i \notin \mathrm{D}(u) \oplus \mathrm{D}(t)$. Let $(v,x) \in F(u,t)$. Then either $\mathrm{D}(x) \smallsetminus \mathrm{D}(v) = \mathrm{D}(t) \smallsetminus \mathrm{D}(u)$ and $\mathrm{D}(v) \smallsetminus \mathrm{D}(x) = \{i\} \cup (\mathrm{D}(u) \smallsetminus \mathrm{D}(t))$, this alternative occurring in the case that $\mathrm{col}_u(i+1) < \mathrm{col}_t(i+1)$, or else $\mathrm{D}(x) \smallsetminus \mathrm{D}(v) = \{i\} \cup (\mathrm{D}(t) \smallsetminus \mathrm{D}(u))$ and $\mathrm{D}(v) \smallsetminus \mathrm{D}(x) = \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$, this occurring in the case that $\mathrm{col}_t(i+1) < \mathrm{col}_u(i+1)$.*

*Proof.* The construction of $(v,x)$ is given in the discussion above. Since $(v,x)$ and $(u,t)$ are both $i$-restricted, $\mathrm{D}(v) \cap [1,i-1] = \mathrm{D}(x) \cap [1,i-1]$ and $\mathrm{D}(u) \cap [1,i-1] = \mathrm{D}(t) \cap [1,i-1]$. That is, $(\mathrm{D}(v) \oplus \mathrm{D}(x)) \cap [1,i-1] = (\mathrm{D}(u) \oplus \mathrm{D}(t)) \cap [1,i-1] = \varnothing$. Furthermore, since $v_{>i} = u_{>i}$ and $x_{>i} = t_{>i}$ it follows that $(\mathrm{D}(v) \smallsetminus \mathrm{D}(x)) \cap [i+1,n-1] = (\mathrm{D}(u) \smallsetminus \mathrm{D}(t)) \cap [i+1,n-1]$ and $(\mathrm{D}(x) \smallsetminus \mathrm{D}(v)) \cap [i+1,n-1] = (\mathrm{D}(t) \smallsetminus \mathrm{D}(u)) \cap [i+1,n-1]$. It remains to observe that if $p = \mathrm{col}_v(i+1) \leqslant m = \mathrm{col}_v(i) = \mathrm{col}_x(i) < r = \mathrm{col}_x(i+1)$ then $i \in \mathrm{D}(v) \smallsetminus \mathrm{D}(x)$, while if $r \leqslant m < p$ then $i \in \mathrm{D}(x) \smallsetminus \mathrm{D}(v)$. □

Lemma 7.20. *Let $\pi$, $\lambda \in P(n)$ and let $(u,t) \in \mathrm{Std}(\pi) \times \mathrm{Std}(\lambda)$. Assume that the restriction number of $(u,t)$ lies in $\mathrm{D}(u) \oplus \mathrm{D}(t)$, and let $(v,x) \in F(u,t)$. Then $\mathrm{D}(v) \smallsetminus \mathrm{D}(x) = \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$ and $\mathrm{D}(x) \smallsetminus \mathrm{D}(v) = \mathrm{D}(t) \smallsetminus \mathrm{D}(u)$.*

*Proof.* The proof is the same as the proof of Lemma 7.19, except that it can be seen now that $i \in \mathrm{D}(v) \smallsetminus \mathrm{D}(x)$ if $i \in \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$ and $i \in \mathrm{D}(x) \smallsetminus \mathrm{D}(v)$ if $i \in \mathrm{D}(t) \smallsetminus \mathrm{D}(u)$. □

Lemma 7.21. *Let $\pi$, $\lambda \in P(n)$ and $(u,t) \in \mathrm{Std}(\pi) \times \mathrm{Std}(\lambda)$, and $i$ the restriction number of $(u,t)$. Suppose that $\mathrm{D}(t) \subsetneq \mathrm{D}(u)$ and $i < j$, where $j = \min(\mathrm{D}(u) \smallsetminus \mathrm{D}(t))$. Let $(v,x) \in F(u,t)$. If $\mathrm{col}_u(i+1) < \mathrm{col}_t(i+1)$ then $\mathrm{D}(v) \smallsetminus \mathrm{D}(x) = \{i\} \cup (\mathrm{D}(u) \smallsetminus \mathrm{D}(t))$ and $\mathrm{D}(x) \smallsetminus \mathrm{D}(v) = \varnothing$, while if $\mathrm{col}_t(i+1) < \mathrm{col}_u(i+1)$ then $\mathrm{D}(v) \smallsetminus \mathrm{D}(x) = \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$ and $\mathrm{D}(x) \smallsetminus \mathrm{D}(v) = \{i\}$. In the former case $\mathrm{D}(v) \cap \{i,j\} = \{i,j\}$ and $\mathrm{D}(x) \cap \{i,j\} = \varnothing$, while in the latter case $\mathrm{D}(v) \cap \{i,j\} = \{j\}$ and $\mathrm{D}(x) \cap \{i,j\} = \{i\}$.*

*Proof.* Since $\mathrm{D}(t) \subsetneq \mathrm{D}(u)$ we have $\mathrm{D}(u) \oplus \mathrm{D}(t) = \mathrm{D}(u) \smallsetminus \mathrm{D}(t) \neq \varnothing$. So $j = \min(\mathrm{D}(u) \oplus \mathrm{D}(t))$, and since $j > i$ we have $i \notin \mathrm{D}(u) \oplus \mathrm{D}(t)$. Hence $(v, x) \in F(u, t)$ satisfies the extra properties specified in Lemma 7.19.

If $\mathrm{col}_u(i+1) < \mathrm{col}_t(i+1)$ then Lemma 7.19 gives $\mathrm{D}(v) \smallsetminus \mathrm{D}(x) = \{i\} \cup (\mathrm{D}(u) \smallsetminus \mathrm{D}(t))$ and $\mathrm{D}(x) \smallsetminus \mathrm{D}(v) = \varnothing$, since $\mathrm{D}(t) \smallsetminus \mathrm{D}(u) = \varnothing$ by hypothesis. In particular, since $j \in \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$, we see that $\mathrm{D}(v) \smallsetminus \mathrm{D}(x)$ contains both $i$ and $j$.

If $\mathrm{col}_t(i+1) < \mathrm{col}_u(i+1)$ then Lemma 7.19 combined together with $\mathrm{D}(t) \smallsetminus \mathrm{D}(u) = \varnothing$ gives $\mathrm{D}(v) \smallsetminus \mathrm{D}(x) = \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$ and $\mathrm{D}(x) \smallsetminus \mathrm{D}(v) = \{i\}$. In particular it follows that $j \in \mathrm{D}(v) \smallsetminus \mathrm{D}(x)$ and $i \in \mathrm{D}(x) \smallsetminus \mathrm{D}(v)$. □

Let $\Gamma = \Gamma(C, \mu, \tau)$ be a $W_n$-molecular graph, and let $\Lambda$ be the set of molecule types for $\Gamma$. For each $\lambda \in \Lambda$ let $m_\lambda$ be the number of molecules of type $\lambda$ in $\Gamma$, and $\mathcal{I}_\lambda$ some indexing set of cardinality $m_\lambda$. As in Remark 6.50, the vertex set of $\Gamma$ can be expressed in the form

$$C = \bigsqcup_{\lambda \in \Lambda} \bigsqcup_{\alpha \in \mathcal{I}_\lambda} C_{\alpha,\lambda},$$

where $C_{\alpha,\lambda} = \{c_{\alpha,t} \mid t \in \mathrm{Std}(\lambda)\}$ for each $\alpha \in \mathcal{I}_\lambda$, and the simple edges of $\Gamma$ are the pairs $\{c_{\beta,u}, c_{\alpha,t}\}$ such that $\alpha = \beta \in \mathcal{I}_\lambda$ for some $\lambda \in \Lambda$ and $u$, $t \in \mathrm{Std}(\lambda)$ are related by a DKM. Furthermore, $\tau(c_{\alpha,t}) = \{\, s_j \in S_n \mid j \in \mathrm{D}(t)\,\}$, whenever $\lambda \in \Lambda$ and $(\alpha, t) \in \mathcal{I}_\lambda \times \mathrm{Std}(\lambda)$.

Now let $\lambda$, $\pi \in \Lambda$, and let $(\alpha, t) \in \mathcal{I}_\lambda \times \mathrm{Std}(\lambda)$ and $(\beta, u) \in \mathcal{I}_\pi \times \mathrm{Std}(\pi)$, so that $c_{\alpha,t}$ and $c_{\beta,u}$ are vertices of $\Gamma$. Suppose that $\mathrm{D}(u) \smallsetminus \mathrm{D}(t) \neq \varnothing$, and let $j \in \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$.

Suppose that there exist $i < j$ and $(v, x) \in \mathrm{Std}(\pi) \times \mathrm{Std}(\lambda)$ such that $(u, t)$ and $(v, x)$ are related by a paired $(\leqslant i)$-DKM. Then $j \in \mathrm{D}(u_{\,>\,i}) \smallsetminus \mathrm{D}(t_{\,>\,i})$, since $j \in \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$ and $j > i$. Thus $j \in \mathrm{D}(v_{\,>\,i}) \smallsetminus \mathrm{D}(x_{\,>\,i})$, since $(v, x) \approx_i (u, t)$ gives $v_{\,>\,i} = u_{\,>\,i}$ and $x_{\,>\,i} = t_{\,>\,i}$. Hence $j \in \mathrm{D}(v) \smallsetminus \mathrm{D}(x)$. Moreover, since $(u, t)$ and $(v, x)$ are related by a paired $(\leqslant i)$-DKM, there are $k, l \leqslant i - 1$ with $|k - l| = 1$ such that

$$\begin{aligned} \mathrm{D}(x) \cap \{k, l, j\} &= \{k\}, \qquad & \mathrm{D}(v) \cap \{k, l, j\} &= \{k, j\}, \\ \mathrm{D}(t) \cap \{k, l, j\} &= \{l\}, \qquad & \mathrm{D}(u) \cap \{k, l, j\} &= \{l, j\}, \end{aligned}$$

and it follows from Proposition 5.13 that $\mu(c_{\beta,v}, c_{\alpha,x}) = \mu(c_{\beta,u}, c_{\alpha,t})$.

More generally, suppose that $i < j$ and $(v, x) \in \mathrm{Std}(\pi) \times \mathrm{Std}(\lambda)$ satisfy $(v, x) \approx_i (u, t)$, so that for some $m \in \mathbb{N}$ there exist $(u_0, t_0)$, $(u_1, t_1)$, $\ldots$, $(u_m, t_m)$ in $\mathrm{Std}(\pi) \times \mathrm{Std}(\lambda)$, with $(u_{h-1}, t_{h-1})$ and $(u_h, t_h)$ related by a paired $(\leqslant i)$-DKM for each $h \in [1, m]$, and $(u_0, t_0) = (u, t)$ and $(u_m, t_m) = (v, x)$. Applying the argument in the preceding paragraph and a trivial induction, we deduce that $j \in \mathrm{D}(u_h) \smallsetminus \mathrm{D}(t_h)$ and $\mu(c_{\beta,u_h}, c_{\alpha,t_h}) = \mu(c_{\beta,u}, c_{\alpha,t})$ for all $h \in [0, m]$. Thus we obtain the following result.

Lemma 7.22. *Let $\Gamma$ be a $W_n$-molecular graph. Using the notation as above, let $\lambda$, $\pi \in \Lambda$, and let $(\alpha, t) \in \mathcal{I}_\lambda \times \mathrm{Std}(\lambda)$ and $(\beta, u) \in \mathcal{I}_\pi \times \mathrm{Std}(\pi)$. Suppose that $\mathrm{D}(u) \smallsetminus \mathrm{D}(t) \neq \varnothing$, and let $j \in \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$. Then for all $i < j$ and all $(v, x) \in C_i(u, t)$ we have $j \in \mathrm{D}(v) \smallsetminus \mathrm{D}(x)$ and $\mu(c_{\beta,v}, c_{\alpha,x}) = \mu(c_{\beta,u}, c_{\alpha,t})$.*

Corollary 7.23. *Continuing with the same notation, let $\lambda \in \Lambda$ and $u$, $t \in \mathrm{Std}(\lambda)$, and suppose that $u = s_j t > t$ for some $j \in [1, n-1]$. Then $\mu(c_{\alpha,u}, c_{\alpha,t}) = 1$ for all $\alpha \in \mathcal{I}_\lambda$.*

*Proof.* Since $t < s_j t = u$, it follows from Remark 6.28 that if $\mathrm{D}(t) \nsubseteq \mathrm{D}(u)$ then there is a DKM from $t$ to $u$, and $\{c_{\alpha,u}, c_{\alpha,t}\}$ is a simple edge. Thus $\mu(c_{\alpha,u}, c_{\alpha,t}) = 1$ in this case, and so we may assume that $\mathrm{D}(t) \subsetneq \mathrm{D}(u)$.

Since $u$ is obtained from $t$ by interchanging $j$ and $j+1$, it is clear that $j-1$ is the restriction number of $(u, t)$, and since $\mathrm{D}(u) \smallsetminus \mathrm{D}(t) = \{j\}$, by Lemma 6.20, we see that the hypotheses of Lemma 7.21 are satisfied with $i = j-1$. Since $\mathrm{col}_t(i+1) < \mathrm{col}_u(i+1)$, it follows that $(u,t) \approx_i (v,x)$, for some $(v,x) \in F(u,t)$ satisfying $\mathrm{D}(v) \cap \{i,j\} = \{j\}$ and $\mathrm{D}(x) \cap \{i,j\} = \{i\}$. Since $(u,t) \approx_i (v,x)$ there exists $w \in W_i$ with $v = wu$ and $x = wt$, and $s_j w = w s_j$ since$j > i$. Thus $s_j x = s_j wt = w s_j t = wu = v$. Furthermore $s_j x > x$, since $j \notin \mathrm{D}(x)$, and $\mathrm{D}(x) \nsubseteq \mathrm{D}(v)$ since $i \in \mathrm{D}(x) \setminus \mathrm{D}(v)$. So there is a DKM indexed by $j$ from $x$ to $v$, and so $\{c_{\alpha,v}, c_{\alpha,x}\}$ is a simple edge. Thus $\mu(c_{\alpha,v}, c_{\alpha,x}) = 1$, and so $\mu(c_{\alpha,u}, c_{\alpha,t}) = 1$ by Lemma 7.22. ☐

Lemma 7.24. *Continue with the notation used in Lemma 7.22 above. Let $\pi$, $\lambda \in \Lambda$ and let $(u,t) \in \mathrm{Std}(\pi) \times \mathrm{Std}(\lambda)$. Suppose that $\mathrm{D}(u) = \{n-1\} \cup \mathrm{D}(t)$ and that $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$ for some $\beta \in \mathcal{I}_\pi$ and $\alpha \in \mathcal{I}_\lambda$. Suppose also that the restriction number of $(u,t)$ is $i < n-2$. Then $\mathrm{col}_u(i+1) < \mathrm{col}_t(i+1)$, and $(u,t) \approx_i (v,x)$ for some $(v,x) \in \mathrm{Std}(\pi) \times \mathrm{Std}(\lambda)$ such that $\mathrm{D}(v) = \mathrm{D}(x) \cup \{i, n-1\}$ and $\mu(c_{\beta,v}, c_{\alpha,x}) = \mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$.*

*Proof.* Since $u \neq t$, the set $F(u,t)$ is defined and nonempty. Let $(v,x) \in F(u,t)$. Then it follows by Lemmas 7.21 and 7.22 that $(u,t) \approx_i (v,x)$ and $\mu(c_{\beta,v}, c_{\alpha,x}) = \mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$. Moreover, if $\mathrm{col}_u(i+1) < \mathrm{col}_t(i+1)$ then Lemma 7.21 gives $\mathrm{D}(v) = \mathrm{D}(x) \cup \{i, n-1\}$. Thus it remains to show that $\mathrm{col}_u(i+1) < \mathrm{col}_t(i+1)$. Suppose otherwise. Then Lemma 7.21 shows that $n-1 \in \mathrm{D}(v) \setminus \mathrm{D}(x)$ and $i \in \mathrm{D}(x) \setminus \mathrm{D}(v)$, and now the Compatibility Rule says that $i$ and $n-1$ must be joined by a bond in the Coxeter diagram of $W_n$. But this contradicts $i < n-2$. ☐

Lemma 7.25. *Suppose that $u$, $t \in \mathrm{Std}(n)$ are such that the restriction number of $(u,t)$ is $n-1$ and $\mathrm{D}(u) = \{n-1\} \cup \mathrm{D}(t)$. Then $\mathrm{col}_u(n) < \mathrm{col}_t(n)$. Thus $\mathrm{Shape}(u) < \mathrm{Shape}(t)$ and $u < t$.*

*Proof.* Clearly $n \geqslant 2$. Since $u_{\leqslant (n-1)} = t_{\leqslant (n-1)}$ we have $\mathrm{Shape}(u_{<n}) = \mathrm{Shape}(t_{<n})$, and since $n-1 \in \mathrm{D}(u) \setminus \mathrm{D}(t)$ we have $\mathrm{col}_u(n) \leqslant \mathrm{col}_u(n-1) = \mathrm{col}_t(n-1) < \mathrm{col}_t(n)$. Hence $\mathrm{Shape}(u) < \mathrm{Shape}(t)$ by Lemma 7.2, and $u < t$ by Definition 7.1. ☐

Lemma 7.26. *Continue with the notation used in Lemma 7.22 above. Let $\pi$, $\lambda \in \Lambda$, let $u \in \mathrm{Std}(\pi)$, and let $t \in \mathrm{Std}(\lambda)$. Suppose that $\mathrm{D}(u) = \{n-1\} \cup \mathrm{D}(t)$ and that $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$ for some $\beta \in \mathcal{I}_\pi$ and $\alpha \in \mathcal{I}_\lambda$, and suppose that the restriction number of $(u,t)$ is $n-2$. Then $(u,t) \approx_{n-2} (v,x)$ for some $(v,x) \in \mathrm{Std}(\pi) \times \mathrm{Std}(\lambda)$ with $\mu(c_{\beta,v}, c_{\alpha,x}) = \mu(c_{\beta,u}, c_{\alpha,t})$, and either $u < t$ and $\mathrm{D}(v) = \{n-2, n-1\} \cup \mathrm{D}(x)$ (in the case $\mathrm{col}_u(n-1) < \mathrm{col}_t(n-1)$), or else $(\lambda, \alpha) = (\pi, \beta)$ and $u = s_{n-1}t > t$, and $\mu(c_{\beta,u}, c_{\alpha,t}) = 1$ (in the case $\mathrm{col}_t(n-1) < \mathrm{col}_u(n-1)$).*

*Proof.* Clearly $n \geqslant 3$. We have $u_{\leqslant (n-2)} = t_{\leqslant (n-2)}$ and $\mathrm{col}_u(n-1) \neq \mathrm{col}_t(n-1)$, since $(u,t)$ is $(n-2)$-restricted. Let $(v,x) \in F(u,t)$ be arbitrary, and note that the hypotheses of Lemma 7.21 hold with $i = n-2$ and $j = n-1$. Moreover, since $(u,t) \approx_{n-2} (v,x)$, we may apply Lemma 7.22 with $i = n-2$ and $j = n-1$, and deduce that $\mu(c_{\beta,v}, c_{\alpha,x}) = \mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$.

*Case 1.*
Suppose that $\mathrm{col}_u(n-1) < \mathrm{col}_t(n-1)$. Since $\mathrm{Shape}(u_{\leqslant (n-2)}) = \mathrm{Shape}(t_{\leqslant (n-2)})$ it follows from Lemma 7.2 that $\mathrm{Shape}(u_{\leqslant (n-1)}) < \mathrm{Shape}(t_{\leqslant (n-1)})$. Moreover, since $n-1 \in \mathrm{D}(u) \setminus \mathrm{D}(t)$, it follows that $\mathrm{col}_u(n) \leqslant \mathrm{col}_u(n-1) < \mathrm{col}_t(n-1) < \mathrm{col}_t(n)$. Hence $\pi < \lambda$ by Lemma 7.2, and $u < t$ by Definition 7.1. Moreover, since $\mathrm{col}_u(n-1) < \mathrm{col}_t(n-1)$ and $\mathrm{D}(u) \setminus \mathrm{D}(t) = \{n-1\}$, it follows from Lemma 7.19 that $\mathrm{D}(v) = \mathrm{D}(x) \cup \{n-2, n-1\}$.

*Case 2.*
Suppose that $\mathrm{col}_t(n-1) < \mathrm{col}_u(n-1)$. Lemma 7.19 gives $\mathrm{D}(x) \cap \{n-2, n-1\} = \{n-2\}$ and $\mathrm{D}(v) \cap \{n-2, n-1\} = \{n-1\}$, and since $\mu(c_{\beta,v}, c_{\alpha,x}) \neq 0$ it follows from Simplicity Rule that $\mu(c_{\beta,v}, c_{\alpha,x}) = 1$. Hence $\mu(c_{\beta,u}, c_{\alpha,t}) = 1$. Moreover, since $\{c_{\beta,v}, c_{\alpha,x}\}$ is a simple edge, it follows that from Theorem 6.49 and Remark 6.50 that

$\lambda = \pi$ and $\alpha = \beta$. Hence $u = s_{n-1}t$, since $u_{\leqslant(n-2)} = t_{\leqslant(n-2)}$, and $u > t$ since $\mathrm{col}_t(n-1) < \mathrm{col}_u(n-1)$. □

Remark 7.27. Continue with the notation used in Lemma 7.22 above. Let $\pi$, $\lambda \in \Lambda$, let $u \in \mathrm{Std}(\pi)$, and let $t \in \mathrm{Std}(\lambda)$. Suppose that $\mathrm{D}(u) = \{n-1\} \cup \mathrm{D}(t)$ and that $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$ for some $\beta \in \mathcal{I}_\pi$ and $\alpha \in \mathcal{I}_\lambda$. Let $i$ be the restriction number of $(u,t)$, and note that $i \leqslant n-1$. If $i < n-2$ then $\mathrm{col}_u(i+1) < \mathrm{col}_t(i+1)$ by Lemma 7.24, and if $i = n-1$ then $\mathrm{col}_u(n) < \mathrm{col}_t(n)$ by Lemma 7.25. In the remaining case, namely $i = n-2$, if $\mathrm{col}_u(n-1) > \mathrm{col}_t(n-1)$ then $u = s_{n-1}t > t$ by Lemma 7.26. Thus $\mathrm{col}_u(i+1) < \mathrm{col}_t(i+1)$ unless $i = n-2$ and $u = s_{n-1}t > t$.

Remark 7.28. Continue with the notation used in Lemma 7.22 above. Suppose that $\pi$, $\lambda \in \Lambda$, and let $(\beta, u) \in \mathcal{I}_\pi \times \mathrm{Std}(\pi)$ and $(\alpha, t) \in \mathcal{I}_\lambda \times \mathrm{Std}(\lambda)$ satisfy $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$ and $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$. Let $j = \min(\mathrm{D}(u) \smallsetminus \mathrm{D}(t))$, and $i$ the restriction number of $(u,t)$. Note that $i \leqslant j$.

Let $K = \{s_1, s_2, \dots, s_j\} \subseteq S_n$, and $\Gamma_K$ the $W_K$-restriction of $\Gamma$. As in Remark 6.52, for each $\lambda \in \Lambda$ and $\alpha \in \mathcal{I}_\lambda$ we define $\Lambda_{K,\alpha,\lambda}$ to be the set of all $\kappa \in P(j+1)$ such that the molecule of $\Gamma$ with the vertex set $C_{\alpha,\lambda}$ contains a $K$-submolecule of type $\kappa$, and let $\mathcal{I}_{K,\alpha,\lambda,\kappa}$ index these submolecules. Let $\Lambda_K = \bigcup_{\alpha,\lambda} \Lambda_{K,\alpha,\lambda}$, the set of molecule types for $\Gamma_K$, and for each $\kappa \in \Lambda_K$ let $\mathcal{I}_{K,\kappa} = \bigsqcup_{\{(\alpha,\lambda) \mid \kappa \in \Lambda_{K,\alpha,\lambda}\}} \mathcal{I}_{K,\alpha,\lambda,\kappa}$. For each $\beta \in \mathcal{I}_{K,\kappa}$ we write $\{c'_{\beta,u} \mid u \in \mathrm{Std}(\kappa)\}$ for the vertex set of the corresponding $K$-submolecule of $\Gamma$.

Let $v = u_{\leqslant(j+1)}$ and $x = t_{\leqslant(j+1)}$, and write $\sigma = \mathrm{Shape}(v)$ and $\theta = \mathrm{Shape}(x)$. Then by Remark 6.52, we can identify the vertex $c_{\beta,u}$ of $\Gamma_K$ with $c'_{\delta,v}$ for some $\delta \in \mathcal{I}_{K,\beta,u,\sigma}$, and the vertex $c_{\alpha,t}$ of $\Gamma_K$ with $c'_{\gamma,x}$ for some $\gamma \in \mathcal{I}_{K,\alpha,\lambda,\theta}$. It is clear that $\mathrm{D}(v) = \mathrm{D}(x) \cup \{j\}$, and it follows that $\mu(c'_{\delta,v}, c'_{\gamma,x}) = \mu_K(c_{\beta,u}, c_{\alpha,t}) \neq 0$. Moreover, since $i \leqslant j$, the restriction number of $(v,x)$ is also $i$. We can now apply Remark 7.27 with $j+1$ in place of $n$ and $\Gamma_K$ in place of $\Gamma$, and with $(v,x)$ in place of the $(u,t)$ of Remark 7.27. The conclusion is that $\mathrm{col}_v(i+1) < \mathrm{col}_x(i+1)$ unless $v = s_jx > x$, in which case $j = i+1$. So $\mathrm{col}_u(i+1) < \mathrm{col}_t(i+1)$ unless $u = s_{i+1}t > t$.

We end this section with two technical lemmas that will be used throughout Sections 8 and 9. They will be useful when the Polygon Rule is to be applied.

Recall that if $t \in \mathrm{Std}(n)$ and $i \in [1, n-1]$ then $s_it \in \mathrm{Std}(n)$ if and only if $i \in \mathrm{SA}(t) \cup \mathrm{SD}(t)$.

Lemma 7.29. *Let $t \in \mathrm{Std}(n)$ and let $i \in \mathrm{A}(t)$ and $j \in \mathrm{SD}(t)$. Put $v = s_jt$.*

(i) *Suppose that $i < j-1$. Then $i \notin \mathrm{D}(v)$ and $j \notin \mathrm{D}(v)$.*
*Additionally, if $i \in \mathrm{SA}(v)$ then $i \in \mathrm{D}(s_iv)$ and $j \notin \mathrm{D}(s_iv)$.*

(ii) *Suppose that $i = j-1$ and $\mathrm{col}_t(j+1) > \mathrm{col}_t(j-1)$. Then $j-1 \notin \mathrm{D}(v)$ and $j \notin \mathrm{D}(v)$.*
*Additionally, if $j-1 \in \mathrm{SA}(v)$ then $j-1 \in \mathrm{D}(s_{j-1}v)$ and $j \notin \mathrm{D}(s_{j-1}v)$.*

(iii) *Suppose that $i = j-1$ and $\mathrm{col}_t(j+1) < \mathrm{col}_t(j-1)$. Then $j-1 \in \mathrm{SD}(v)$. Writing $w = s_{j-1}v$, we have $j-1 \in \mathrm{D}(v)$ and $j \notin \mathrm{D}(v)$, and $j-1 \notin \mathrm{D}(w)$ and $j \notin \mathrm{D}(w)$.*
*Additionally, if $j \in \mathrm{SA}(w)$, then $j-1 \in \mathrm{SA}(s_jw)$, and we have $j \in \mathrm{D}(s_jw)$ and $j-1 \notin \mathrm{D}(s_jw)$, and $j-1 \in \mathrm{D}(s_{j-1}s_jw)$ and $j \notin \mathrm{D}(s_{j-1}s_jw)$.*

*Proof.*

(i) Since $v = s_jt$ and $j \in \mathrm{SD}(t)$, it follows that $j \in \mathrm{SA}(v)$, whence $j \notin \mathrm{D}(v)$. Since $v$ is obtained from $t$ by switching the positions of $j$ and $j+1$, and since $i+1 < j$, it follows that $i$ and $i+1$ have the same row and column index in $v$ as they have in $t$. Since $i \notin \mathrm{D}(t)$, this shows that $i \notin \mathrm{D}(v)$.

If $i \in \mathrm{SA}(v)$ then $s_i v$ is standard and $i \in \mathrm{D}(s_i v)$. Since $s_i v$ is obtained from $v$ by switching $i$ and $i+1$, and since $j > i+1$, it follows that $j$ and $j+1$ have the same row and column index in $s_i v$ as in $v$. Since $j \notin \mathrm{D}(v)$ it follows that $j \notin \mathrm{D}(s_i v)$.

(ii) Since $v = s_j t$ and $j \in \mathrm{SD}(t)$, it follows that $j \in \mathrm{SA}(v)$, whence $j \notin \mathrm{D}(v)$. Now since $\mathrm{col}_v(j-1) = \mathrm{col}_t(j-1)$ and $\mathrm{col}_v(j) = \mathrm{col}_t(j+1)$, and $\mathrm{col}_t(j-1) < \mathrm{col}_t(j+1)$ by assumption, it follows that $\mathrm{col}_v(j-1) < \mathrm{col}_v(j)$. That is, $j-1 \notin \mathrm{D}(v)$.

If $j-1 \in \mathrm{SA}(v)$ then $s_{j-1}v$ is standard and $j-1 \in \mathrm{D}(s_{j-1}v)$. Since $j-1$ and $j$ are both ascents of $v$, we have $\mathrm{col}_v(j-1) < \mathrm{col}_v(j) < \mathrm{col}_v(j+1)$, and since $s_{j-1}v$ is obtained from $v$ by switching $j-1$ and $j$, we have $\mathrm{col}_{s_{j-1}v}(j) = \mathrm{col}_v(j-1)$ and $\mathrm{col}_{s_{j-1}v}(j+1) = \mathrm{col}_v(j+1)$, and it follows that $\mathrm{col}_{s_{j-1}v}(j) < \mathrm{col}_{s_{j-1}v}(j+1)$. Thus $j \notin \mathrm{D}(s_{j-1}v)$.

(iii) As in (i) and (ii) we have $j \notin \mathrm{D}(v)$. The assumption $\mathrm{col}_t(j+1) < \mathrm{col}_t(j-1)$ gives $\mathrm{col}_v(j) < \mathrm{col}_v(j-1)$, and so $j-1 \in \mathrm{SD}(v)$. Hence $w = s_{j-1}v$ is standard, and $j-1 \in \mathrm{SA}(w)$. Since $\mathrm{col}_w(j+1) = \mathrm{col}_v(j+1) = \mathrm{col}_t(j)$ and $\mathrm{col}_w(j) = \mathrm{col}_v(j-1) = \mathrm{col}_t(j-1)$, and since $j-1 \in \mathrm{A}(t)$ by assumption, it follows that $j \in \mathrm{A}(w)$. Thus $j-1 \in \mathrm{D}(v)$ and $j \notin \mathrm{D}(v)$, and $j-1 \notin \mathrm{D}(w)$ and $j \notin \mathrm{D}(w)$, as required.

If $j \in \mathrm{SA}(w)$ then $s_j w \in \mathrm{Std}(\lambda)$. Since $j-1$ and $j$ are both strong ascents of $w$, we have $\mathrm{row}_w(j-1) > \mathrm{row}_w(j) > \mathrm{row}_w(j+1)$, and since $s_j w$ is obtained from $w$ by switching $j$ and $j+1$, we have $\mathrm{row}_{s_j w}(j-1) = \mathrm{row}_w(j-1)$ and $\mathrm{row}_{s_j w}(j) = \mathrm{row}_w(j+1)$, and it follows that $\mathrm{row}_{s_j w}(j-1) > \mathrm{row}_{s_j w}(j)$. Thus $j-1 \in \mathrm{SA}(s_j w)$.

Now $j-1 \in \mathrm{SA}(s_j w)$ gives $j-1 \notin \mathrm{D}(s_j w)$, and gives $j-1 \in \mathrm{D}(s_{j-1}s_j w)$. Similarly, $j \in \mathrm{SA}(w)$ gives $j \in \mathrm{D}(s_j w)$. Finally, the assumption $\mathrm{col}_t(j+1) < \mathrm{col}_t(j-1)$ gives $\mathrm{col}_{s_{j-1}s_j w}(j) = \mathrm{col}_{s_j w}(j-1) = \mathrm{col}_t(j+1) < \mathrm{col}_t(j-1) = \mathrm{col}_{s_j w}(j+1) = \mathrm{col}_{s_{j-1}s_j w}(j+1)$, and $j \notin \mathrm{D}(s_{j-1}s_j w)$. □

Recall from Remark 6.15 that if $\lambda \in P(n)$ and $u, t \in \mathrm{Std}(\lambda)$ then $t >_{\mathrm{lex}} u$ if and only if there exists $l \in [1,n]$ such that $\mathrm{col}_t(l) < \mathrm{col}_u(l)$ and $t_{>l} = u_{>l}$.

Lemma 7.30. *Let $\lambda \in P(n)$ and $0 \leqslant i \leqslant n-1$. Let $t, t' \in \mathrm{Std}(\lambda)$ satisfy $t_{>i} = (t')_{>i}$. Let $j \in \mathrm{SD}(t)$ and put $v = s_j t$, and suppose that $i \in \mathrm{A}(t)$ and $i < j$. Then $v <_{\mathrm{lex}} t'$, and the following all hold.*

(i) *If $i \in \mathrm{SA}(v)$ then $s_i v \in \mathrm{Std}(\lambda)$ and $s_i v <_{\mathrm{lex}} t'$.*
(ii) *If $y \in \mathrm{Std}(\lambda)$ and $y < v$ then $y <_{\mathrm{lex}} t'$.*
(iii) *Suppose that $i = j-1$ and that $\mathrm{col}_t(j+1) < \mathrm{col}_t(j-1)$, and let $w = s_{j-1}v$. Then $w \in \mathrm{Std}(\lambda)$ and $w <_{\mathrm{lex}} t'$. If $j \in \mathrm{SA}(w)$ then $s_{j-1}s_j w \in \mathrm{Std}(\lambda)$ and $s_{j-1}s_j w <_{\mathrm{lex}} t'$.*
(iv) *Suppose that $i = j-1$ and that $\mathrm{col}_t(j+1) < \mathrm{col}_t(j-1)$, and let $w = s_{j-1}v$. Let $x \in \mathrm{Std}(\lambda)$ be such that $x < w$ and $\mathrm{D}(x)$ contains exactly one of $j-1$ or $j$, and let $y$ be the $(j-1)$-neighbour of $x$ (see Definition 6.48). Then $y <_{\mathrm{lex}} t'$.*

*Proof.* Since $j \in \mathrm{SD}(t)$ we have $t > s_j t = v$, and hence $t >_{\mathrm{lex}} v$ by Corollary 6.17. Indeed, $\mathrm{col}_t(j+1) < \mathrm{col}_t(j) = \mathrm{col}_v(j+1)$ and $t_{>(j+1)} = v_{>(j+1)}$. Since $t_{>i} = (t')_{>i}$ and $j+1 > i$ it follows that $\mathrm{col}_{t'}(j+1) < \mathrm{col}_v(j+1)$ and $(t')_{>(j+1)} = v_{>(j+1)}$, giving $t' >_{\mathrm{lex}} v$.

(i) The assumption $i \in \mathrm{SA}(v)$ gives $s_i v \in \mathrm{Std}(\lambda)$, and since $j+1 > i+1$ it follows that $\mathrm{col}_{t'}(j+1) < \mathrm{col}_v(j+1) = \mathrm{col}_{s_i v}(j+1)$ and $(t')_{>(j+1)} = (s_i v)_{>(j+1)}$. So $t' >_{\mathrm{lex}} s_i v$.

(ii) If $y < v$ then $y <_{\mathrm{lex}} v$, by Corollary 6.17, and since $v <_{\mathrm{lex}} t'$ this gives $y <_{\mathrm{lex}} t'$.

(iii) Since $\mathrm{col}_v(j) = \mathrm{col}_t(j+1) < \mathrm{col}_t(j-1) = \mathrm{col}_v(j-1)$, we have $j-1 \in \mathrm{SD}(v)$, and since this gives $s_{j-1}v \in \mathrm{Std}(\lambda)$, an argument similar to that for (i) yields $w <_{\mathrm{lex}} t'$.

If $j \in \mathrm{SA}(w)$ then $s_j w \in \mathrm{Std}(\lambda)$. Since $j-1 \in \mathrm{SA}(s_j w)$ by Lemma 7.29 (iii), we also see that $s_{j-1}s_j w \in \mathrm{Std}(\lambda)$. Since $\mathrm{col}_t(j+1) < \mathrm{col}_t(j-1) = \mathrm{col}_{s_{j-1}s_j w}(j+1)$, and since $j+1 > i+1$, it follows that $\mathrm{col}_{t'}(j+1) < \mathrm{col}_{s_{j-1}s_j w}(j+1)$ and $(t')_{>(j+1)} = (s_{j-1}s_j w)_{>(j+1)}$. Hence $t' >_{\mathrm{lex}} s_{j-1}s_j w$.

(iv) There are two cases to consider.

*Case 1.*
Suppose that $\mathrm{D}(x) \cap \{j-1, j\} = \{j-1\}$ and $\mathrm{D}(y) \cap \{j-1, j\} = \{j\}$. Then either $y = s_j x > x$ or $y = s_{j-1}x < x$.

Suppose first that $y = s_j x > x$. Since $x < w$ and $w = s_{j-1}v < v$ by the proof of (iii), it follows that $x < v$. Since $v < s_j v = t$ and $x < s_j x = y$, it follows by Lemma 7.4 that $y < t$. Hence $y <_{\mathrm{lex}} t$ by Corollary 6.17. That is, there exists $l \in [1, n]$ such that $\mathrm{col}_t(l) < \mathrm{col}_y(l)$ and $t_{>l} = y_{>l}$. Suppose, for a contradiction, that $l \leqslant j-1$. Then $t_{>(j-1)} = y_{>(j-1)}$, giving $(s_j t)_{>(j-1)} = (s_j y)_{>(j-1)}$, that is, $v_{>(j-1)} = x_{>(j-1)}$. Therefore $v_{>j} = x_{>j}$, which gives $w_{>j} = (s_{j-1}v)_{>j} = v_{>j} = x_{>j}$. Now since $\mathrm{col}_x(j) = \mathrm{col}_v(j) < \mathrm{col}_v(j-1) = \mathrm{col}_w(j)$ (using again $s_{j-1}v < v$), it follows that $w <_{\mathrm{lex}} x$, contradicting the assumption that $x < w$. Thus $l \geqslant j$. Since $t_{>(j-1)} = t'_{>(j-1)}$, it follows that $\mathrm{col}_{t'}(l) = \mathrm{col}_t(l) < \mathrm{col}_y(l)$ and $t'_{>l} = y_{>l}$, and hence $y <_{\mathrm{lex}} t'$, as required.

Suppose now that $y = s_{j-1}x < x$. Since $x < w$, we have $y < w$, and by Corollary 6.17 this gives $y <_{\mathrm{lex}} w$. But since $w <_{\mathrm{lex}} t'$ by (iii), this yields $y <_{\mathrm{lex}} t'$.

*Case 2.*
Suppose that $\mathrm{D}(x) \cap \{j-1, j\} = \{j\}$ and $\mathrm{D}(y) \cap \{j-1, j\} = \{j-1\}$. Then either $y = s_j x < x$ or $y = s_{j-1}x > x$.

Suppose first that $y = s_{j-1}x > x$. Since $x < s_{j-1}x = y$ and $w < s_{j-1}w = v$, the assumption $x < w$ gives $y < v$ by Lemma 7.4. Thus $y <_{\mathrm{lex}} t'$ by (ii).

Suppose now that $y = s_j x < x$. Since $x < w$, we have $y < w$, and by Corollary 6.17 this gives $y <_{\mathrm{lex}} w$. But since $w <_{\mathrm{lex}} t'$ by (iii), this yields $y <_{\mathrm{lex}} t'$. ☐

## 8. Ordered admissible $W$-graphs in type $A$

Let $\Gamma = \Gamma(C, \mu, \tau)$ be an admissible $W_n$-graph, and let $\Lambda \subseteq P(n)$ be the set of molecule types for $\Gamma$. As in Remark 6.50 we write

$$C = \bigsqcup_{\lambda \in \Lambda} \bigsqcup_{\alpha \in \mathcal{I}_\lambda} C_{\alpha,\lambda},$$

where for each $\lambda \in \Lambda$ the set $\mathcal{I}_\lambda$ indexes the molecules of $\Gamma$ of type $\lambda$, and for each $\lambda \in \Lambda$ and $\alpha \in \mathcal{I}_\lambda$ the set $C_{\alpha,\lambda} = \{c_{\alpha,t} \mid t \in \mathrm{Std}(\lambda)\}$ is the vertex set of a molecule of type $\lambda$.

Recall that, by Theorem 5.8, $\Gamma$ satisfies the Compatibility Rule, the Simplicity Rule, the Bonding Rule and the Polygon Rule. In particular, in view of Definition 5.4, it follows that whenever vertices $c_{\alpha,t}$ and $c_{\beta,u}$ belong to different molecules and $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$, we must have $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$ and $\mu(c_{\alpha,t}, c_{\beta,u}) = 0$.

We make the following definition.

Definition 8.1. *Let $\Gamma = \Gamma(C, \mu, \tau)$ be an admissible $W_n$-graph, and let*

$$C = \bigsqcup_{\lambda \in \Lambda} \bigsqcup_{\alpha \in \mathcal{I}_\lambda} C_{\alpha,\lambda},$$

*as above. Then $\Gamma$ is said to be* ordered *if for all vertices $c_{\alpha,t}$ and $c_{\beta,u}$ with $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$, either $u < t$ (in the extended Bruhat order) or else $\alpha = \beta$ and $u = st > t$ for some $s \in S_n$.*

Note that $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$ implies that $\mathrm{D}(u) \nsubseteq \mathrm{D}(t)$. In particular, since $S_1 = \varnothing$, the condition $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$ can never be satisfied in the case $n = 1$. Thus it is vacuously true that every $W_1$-graph is ordered.

Our main objective in this section is to prove the next theorem, which is one of the main results of this paper.

Theorem 8.2. *All admissible $W_n$-graphs are ordered.*

The proof of Theorem 8.2 will proceed by induction on $n$. Accordingly, we assume now that $n$ is a positive integer and that all admissible $W_m$-graphs are ordered for $1 \leqslant m < n$. We let $\Gamma = \Gamma(C, \mu, \tau)$ be an admissible $W_n$-graph, and use the notation introduced in the preamble to this section: $\Lambda$ is the set of molecule types of $\Gamma$, and for each $\lambda \in \Lambda$ the set $\mathcal{I}_\lambda$ indexes the molecules of type $\lambda$. We fix $K = S_n \smallsetminus \{s_{n-1}\}$ and $L = S_n \smallsetminus \{s_1\}$, and we let $\Gamma_K$ and $\Gamma_L$ be the $W_K$-graph and $W_L$-graph obtained by restricting $\Gamma$ to $W_K$ and $W_L$. Since $|K| = |L| = n-1$, the inductive hypothesis tells us that $\Gamma_K$ and $\Gamma_L$ are ordered.

For $\lambda \in \Lambda$ and $\alpha \in \mathcal{I}_\lambda$ let $\Theta_{\alpha,\lambda}$ be the molecule of $\Gamma$ with vertex set $C_{\alpha,\lambda}$, and consider the $W_K$-restriction of $\Theta_{\alpha,\lambda}$, as in Remark 6.52. Write $\Lambda_{K,\alpha,\lambda}$ for the set of all $\kappa \in P(n-1)$ such that $\Theta_{\alpha,\lambda}$ contains a $K$-submolecule of type $\kappa$, and for each $\kappa \in \Lambda_{K,\alpha,\lambda}$ let $\mathcal{I}_{K,\alpha,\lambda,\kappa}$ be a set that indexes the $K$-submolecules of $\Theta_{\alpha,\lambda}$ of type $\kappa$. For each $\gamma \in \mathcal{I}_{K,\alpha,\lambda,\kappa}$ let $\{\, c'_{\gamma,x} \mid x \in \mathrm{Std}(\kappa) \,\}$ be the vertex set of the corresponding $K$-submolecule of $\Theta_{\alpha,\lambda}$. Now $\Lambda_K = \bigcup_{\alpha,\lambda} \Lambda_{K,\alpha,\lambda}$ is the set of molecule types for $\Gamma_K$, for each $\kappa \in \Lambda_K$ the set $\mathcal{I}_{K,\kappa} = \bigsqcup_{\{\alpha,\lambda \mid \kappa \in \Lambda_{K,\alpha,\lambda}\}} \mathcal{I}_{K,\alpha,\lambda,\kappa}$ indexes the $K$-submolecules of $\Gamma$ of type $\kappa$, and since the vertex set of $\Gamma_K$ is $C$ we deduce that

$$C = \bigsqcup_{\kappa \in \Lambda_K} \{c'_{\gamma,x} \mid (\gamma, x) \in \mathcal{I}_{K,\kappa} \times \mathrm{Std}(\kappa)\}. \tag{9}$$

Similarly, by Remark 6.52, the set of molecule types for $\Gamma_L$ is $\Lambda_L = \bigcup_{\alpha,\lambda} \Lambda_{L,\alpha,\lambda}$, where $\Lambda_{L,\alpha,\lambda}$ is the set of all $\theta \in P(n-1)$ such that $\Theta_{\alpha,\lambda}$ has an $L$-submolecule of type $\theta$, and for each $\theta \in \Lambda_L$ the $L$-submolecules of $\Gamma$ of type $\theta$ are indexed by $\mathcal{I}_{L,\theta} = \bigsqcup_{\{\alpha,\lambda \mid \theta \in \Lambda_{L,\alpha,\lambda}\}} \mathcal{I}_{L,\alpha,\lambda,\theta}$, where $\mathcal{I}_{L,\alpha,\lambda,\theta}$ indexes the $L$-submolecules of type $\theta$ in $\Theta_{\alpha,\lambda}$. The vertex set of $\Gamma_L$ is

$$C = \bigsqcup_{\theta \in \Lambda_L} \{c''_{\epsilon,y} \mid (\epsilon, y) \in \mathcal{I}_{L,\theta} \times \mathrm{Std}(\theta)\}. \tag{10}$$

Lemma 8.3. *Let $\pi$, $\lambda \in \Lambda$ with $\pi \leqslant \lambda$, and let $(\beta, u) \in \mathcal{I}_\pi \times \mathrm{Std}(\pi)$ and $(\alpha, t) \in \mathcal{I}_\lambda \times \mathrm{Std}(\lambda)$ satisfy the condition $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$ and $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$. Let $j = \min(\mathrm{D}(u) \smallsetminus \mathrm{D}(t))$ and assume that $j < n-1$. Then $u < t$ unless $\alpha = \beta$ and $u = s_j t > t$.*

*Proof.* Since $j$ is at least 1, the requirement that $n - 1 > j$ implies that $n \geqslant 3$. Let $v = u_{\leqslant(n-1)}$ and $x = t_{\leqslant(n-1)}$, and write $\sigma = \mathrm{Shape}(v)$ and $\theta = \mathrm{Shape}(x)$. By Eq. (9) we can identify $c_{\beta,u} \in C$ with $c'_{\delta,v}$ for some $\delta \in \mathcal{I}_{K,\beta,\pi,\sigma}$, and $c_{\alpha,t} \in C$ with $c'_{\gamma,x}$ for some $\gamma \in \mathcal{I}_{K,\alpha,\lambda,\theta}$.

Since $\mathrm{D}(v) \smallsetminus \mathrm{D}(x) = \mathrm{D}(u) \cap [1, n-2]) \smallsetminus (\mathrm{D}(t) \cap [1, n-2])$, and since we are given that $j \in \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$ and $j < n-1$, it follows that $\mu(c'_{\delta,v}, c'_{\gamma,x}) = \mu_K(c_{\beta,u}, c_{\alpha,t}) \neq 0$. Since $\Gamma_K$ is ordered, we have either $v < x$ or $\gamma = \delta$ and $v = s_i x > x$ for some $i \in [1, n-2]$. In the former case, since $\mathrm{Shape}(u) = \pi \leqslant \lambda = \mathrm{Shape}(t)$ by hypothesis, and since $u_{\leqslant(n-1)} = v < x = t_{\leqslant(n-1)}$, we have $u < t$ by the remark following Definition 7.1 In the latter case, the fact that $\gamma = \delta$ implies that $\mathcal{I}_{K,\alpha,\lambda,\theta} = \mathcal{I}_{K,\beta,\pi,\sigma}$ and hence that $\alpha = \beta$. Moreover, since $v = s_i x > x$ it follows from Lemma 6.20 that $i$ is the unique element of $\mathrm{D}(v) \smallsetminus \mathrm{D}(x)$. So $i = j$, and $u = s_j t > t$. $\square$

Proposition 8.4. *Let $\pi$, $\lambda \in \Lambda$ with $\pi \leqslant \lambda$, and suppose that $(\beta, u) \in \mathcal{I}_\pi \times \mathrm{Std}(\pi)$ and $(\alpha, t) \in \mathcal{I}_\lambda \times \mathrm{Std}(\lambda)$ satisfy $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$. Then $u < t$ unless $\alpha = \beta$ and $u = s_i t > t$ for some $i \in [1, n-1]$.*

*Proof.* Since $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$, it follows that $\mathrm{D}(u) \nsubseteq \mathrm{D}(t)$. If $\mathrm{D}(t) \nsubseteq \mathrm{D}(u)$ also holds then the Simplicity Rule shows that $\{c_{\beta,u}, c_{\alpha,t}\}$ is a simple edge, so that $\alpha = \beta$ and $u = s_i t$ for some $i \in [1, n-1]$. Thus we may assume that $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$. If $\min(\mathrm{D}(u) \smallsetminus \mathrm{D}(t)) < n-1$ then the result is given by Lemma 8.3.

It remains to consider the case $\mathrm{D}(u) = \mathrm{D}(t) \cup \{n-1\}$. Let $i$ be the restriction number of the pair $(u,t)$ and note that $i < n$ by Remark 7.14. If $i = n-1$ or $i = n-2$ then the results are given by Lemma 7.25 and Lemma 7.26, respectively. So we may assume that $i < n-2$, and it follows by Lemma 7.24 that $(u,t) \approx_i (v,x)$ for some $(v,x) \in \mathrm{Std}(\pi) \times \mathrm{Std}(\lambda)$ satisfying the condition $\mu(c_{\beta,u}, c_{\alpha,t}) = \mu(c_{\beta,v}, c_{\alpha,x}) \neq 0$ and $\mathrm{D}(x) \subsetneqq \mathrm{D}(v) = \mathrm{D}(x) \cup \{i, n-1\}$. We can now apply Lemma 8.3 with $c_{\beta,v}$ and $c_{\alpha,x}$ in place of $c_{\beta,u}$ and $c_{\alpha,t}$. Since $\mathrm{D}(v) = \mathrm{D}(x) \cup \{i, n-1\}$ it follows from Lemma 6.20 that $v \neq s_i x$, and so we must have $v < x$. So $u < t$, by Proposition 7.9. $\square$

Notation 8.5. Given $\lambda \in \Lambda$, let $C'_\lambda = C \smallsetminus \left(\bigsqcup_{\alpha \in \mathcal{I}_\lambda} C_{\alpha,\lambda}\right)$, the set of vertices of $\Gamma$ belonging to molecules of type different from $\lambda$. We define $\mathrm{Ini}_\lambda(\Gamma)$ to be the set of $(\alpha, t) \in \mathcal{I}_\lambda \times \mathrm{Std}(\lambda)$ such that there exists an arc from $c_{\alpha,t}$ to some vertex in $C'_\lambda$. That is,

$$\mathrm{Ini}_\lambda(\Gamma) = \left\{ (\alpha,t) \in \mathcal{I}_\lambda \times \mathrm{Std}(\lambda) \;\middle|\; \mu(c_{\beta,u}, c_{\alpha,t}) \neq 0 \text{ for some } (\beta,u) \in \bigsqcup_{\pi \in \Lambda \smallsetminus \{\lambda\}} (\mathcal{I}_\pi \times \mathrm{Std}(\pi)) \right\}.$$

For each $\alpha \in \mathcal{I}_\lambda$ we define $\mathrm{Ini}_\Gamma(\alpha,\lambda) = \{\, t \in \mathrm{Std}(\lambda) \mid (\alpha,t) \in \mathrm{Ini}_\lambda(\Gamma) \,\}$, and we also define $\mathrm{Ini}_\Gamma(\lambda) = \bigcup_{\alpha \in \mathcal{I}_\lambda} \mathrm{Ini}_\Gamma(\alpha,\lambda)$. If $\mathrm{Ini}_\lambda(\Gamma) \neq \varnothing$, so that $\mathrm{Ini}_\Gamma(\lambda)$ is a nonempty subset of $\mathrm{Std}(\lambda)$, we define $t_\lambda$ to be the element of $\mathrm{Ini}_\Gamma(\lambda)$ that is minimal in the lexicographic order on $\mathrm{Std}(\lambda)$.

It is clear that whenever $\lambda \in \Lambda$ and $\mathrm{Ini}_\lambda(\Gamma) \neq \varnothing$ there must be at least one $\alpha \in \mathcal{I}_\lambda$ having the property that $t_\lambda \in \mathrm{Ini}_\Gamma(\alpha,\lambda)$, but for an arbitrary $\alpha \in \mathcal{I}_\lambda$ it may or may not be the case that $t_\lambda \in \mathrm{Ini}_\Gamma(\alpha,\lambda)$.

The following definitions will be used in results dealing with $t_\lambda$.

Definition 8.6. *Let $\pi, \lambda \in P(n)$. Let $(u,t) \in \mathrm{Std}(\pi) \times \mathrm{Std}(\lambda)$, and let $k$ be the restriction number of $(u,t)$. The pair $(u,t)$ is said to be* $k$-minimal*, and $t$ is said to be* $k$-minimal with respect to $u$*, if $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$ and $t_{\geqslant k}$ is $k$-critical, and $t_{<k}$ is the minimal tableau of its shape.*

Example. Consider the following three pairs $(u_i, t_i)$:

$$(u_1,t_1) = \left( \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array} \, , \, \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} \end{array} \right),$$

$$(u_2,t_2) = \left( \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \, , \, \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} \end{array} \right),$$

$$(u_3,t_3) = \left( \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \, , \, \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} \end{array} \right).$$

The pair $(u_1,t_1)$ is 4-restricted, $(t_1)_{<4} = (u_1)_{<4}$ is the minimal tableau of shape $(1,1,1)$, and from Definition 6.22 it is readily checked that $(t_1)_{\geqslant 4}$ is the 4-critical tableau of shape $(2,2,1)/(1,1,1)$. Since $\mathrm{D}(t_1) = \{3\} \subsetneqq \{3,4\} = \mathrm{D}(u_1)$, it follows that $(u_1,t_1)$ is 4-minimal.

The pair $(u_2, t_2)$ is 2-restricted, $(t_2)_{<2}$ is minimal, and $\mathrm{D}(t_2) = \{3\} \subsetneq \{2,3\} = \mathrm{D}(u_2)$. But $(t_2)_{\geqslant 2}$ is not 2-critical, since 2 is not in its first nonempty column. So $(u_2, t_2)$ is not 2-minimal.

The pair $(u_3, t_3)$ is 3-restricted, $(t_3)_{<3}$ is minimal, $(t_3)_{\geqslant 3}$ is the 3-critical tableau of shape $(2,2,1)/(1,1,1)$, and $\mathrm{D}(t_3) = \{2\} \subsetneq \{2,3\} = \mathrm{D}(u_3)$. So $(u_3, t_3)$ is 3-minimal.

Let $\pi$, $\lambda \in P(n)$, and let $(u,t) \in \mathrm{Std}(\pi) \times \mathrm{Std}(\lambda)$. Let $k$ be the restriction number of $(u,t)$, and assume that $k \in [1, n-1]$ (or, equivalently, $u \neq t$). Recall that

$$F(u,t) = \{(v,x) \in C_k(u,t) \mid v^{-1}(k) = x^{-1}(k) \text{ lies between } u^{-1}(k+1) \text{ and } t^{-1}(k+1)\}.$$

Definition 8.7. *Let $\pi, \lambda \in P(n)$ and $(u,t) \in \mathrm{Std}(\pi) \times \mathrm{Std}(\lambda)$ with $u \neq t$, and let $k$ be the restriction number of $(u,t)$. We define $A(u,t) = \{(v,x) \in F(u,t) \mid \mathrm{col}_x(k) = \mathrm{col}_t(k+1) - 1\}$ and call any element of $A(u,t)$ an* approximate *of $(u,t)$.*

Since every pair $(v,x) \in F(u,t)$ must satisfy either $\mathrm{col}_u(k+1) \leqslant \mathrm{col}_x(k) < \mathrm{col}_t(k+1)$ or $\mathrm{col}_t(k+1) \leqslant \mathrm{col}_x(k) < \mathrm{col}_u(k+1)$, it is clear that $A(u,t) \neq \varnothing$ only if $\mathrm{col}_u(k+1) < \mathrm{col}_t(k+1)$. Conversely, suppose that $\mathrm{col}_u(k+1) < \mathrm{col}_t(k+1)$. Let $r = \mathrm{col}_t(k+1)$ and $\xi = \mathrm{Shape}(t_{\leqslant k})$, and note that $\xi_{r-1} > \xi_r$, since $t^{-1}(k+1)$ is $\xi$-addable and in column $r$. Hence $(\xi_{r-1}, r-1)$ is $\xi$-removable, and we can choose a tableau $w' \in \mathrm{Std}(\xi)$ with $w'(\xi_{r-1}, r-1) = k$. The unique pair $(v,x) \in C_k(u,t)$ with $v_{\leqslant k} = x_{\leqslant k} = w'$ is then an element of $A(u,t)$, and so $A(u,t) \neq \varnothing$.

Example 8.8. Let

$$(u,t) = \left( \begin{ytableau} 1 & 2 & 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{ytableau}, \begin{ytableau} 1 & 2 & 3 & 6 \\ 4 & 7 \\ 5 \end{ytableau} \right).$$

We see that $(u,t)$ is 5-restricted and $\mathrm{col}_u(6) = 1 < 4 = \mathrm{col}_t(6)$. By Definition 8.7, the set $A(u,t)$ consists of all $(v,x) \in F(u,t)$ with $\mathrm{col}_x(5) = \mathrm{col}_t(6) - 1 = 3$. So as well as being 5-restricted and satisfying the conditions $v_{>5} = u_{>5}$ and $x_{>5} = t_{>5}$, each $(v,x) \in F(u,t)$ satisfies $v^{-1}(5) = x^{-1}(5) = (1,3)$. The tableau $v_{\leqslant 4} = x_{\leqslant 4}$ can be any element of $\mathrm{Std}(3,1)$. So

$$A(u,t) = \left\{ \left( \begin{ytableau} 1 & 4 & 5 \\ 2 \\ 3 \\ 6 \\ 7 \end{ytableau}, \begin{ytableau} 1 & 4 & 5 & 6 \\ 2 & 7 \\ 3 \end{ytableau} \right), \left( \begin{ytableau} 1 & 3 & 5 \\ 2 \\ 4 \\ 6 \\ 7 \end{ytableau}, \begin{ytableau} 1 & 3 & 5 & 6 \\ 2 & 7 \\ 4 \end{ytableau} \right), \left( \begin{ytableau} 1 & 2 & 5 \\ 3 \\ 4 \\ 6 \\ 7 \end{ytableau}, \begin{ytableau} 1 & 2 & 5 & 6 \\ 3 & 7 \\ 4 \end{ytableau} \right) \right\}.$$

Remark 8.9. Let $u$, $t$ be as in Definition 8.7, and assume that $A(u,t) \neq \varnothing$. It is immediate from the definition that every approximate $(v,x)$ of $(u,t)$ is $k$-restricted and satisfies $(v,x) \approx_k (u,t)$. Furthermore, if $r = \mathrm{col}_t(k+1)$ then $v^{-1}(k) = x^{-1}(k) = (d,m)$, where $m = r-1$ and $d$ is the $(r-1)$-th part of $\xi = \mathrm{Shape}(t_{\leqslant k})$. So $A(u,t)$ is a nonempty $(k-1)$-subclass of $C_k(u,t)$. Furthermore, if we define $\kappa \in P(k-1)$ by $[\kappa] = [\xi] \setminus \{(d,m)\}$, then there is a bijection from $\mathrm{Std}(\kappa)$ to $A(u,t)$ such that $w \mapsto (v,x)$ if and only if $v_{\leqslant (k-1)} = x_{\leqslant (k-1)} = w$. We use this bijection to transfer the partial order $\leqslant$ from $\mathrm{Std}(\kappa)$ to $A(u,t)$, and observe that $A(u,t)$ has a unique minimal element, given by $w = \tau_\kappa$, and a unique maximal element, given by $w = \tau^\kappa$. We call these elements of $A(u,t)$ the *minimal approximate* of $(u,t)$ and the *maximal approximate* of $(u,t)$, respectively.

In Example 8.8, the first and the last are the minimal and maximal approximates of $(u,t)$.

Remark 8.10. Let $\pi$, $\lambda \in \Lambda$, and let $(\beta, u) \in \mathcal{I}_\pi \times \mathrm{Std}(\pi)$ and $(\alpha, t) \in \mathcal{I}_\lambda \times \mathrm{Std}(\lambda)$ satisfy the condition $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$ and $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$. Let $k \in [1, n-1]$ be the restriction number of the pair $(u,t)$. Let $l = \min(\mathrm{D}(u) \smallsetminus \mathrm{D}(t))$, and let $H = \{s_1, \ldots, s_l\}$. Remark 7.28 applied to $\Gamma_H$, the $W_H$-restriction of $\Gamma$, shows that $\mathrm{col}_u(k+1) < \mathrm{col}_t(k+1)$ unless $u = s_{k+1}t > t$. It follows that $A(u,t) \neq \varnothing$ unless $u = s_{k+1}t > t$.

Since $u = s_{k+1}t$ forces $\pi = \lambda$, Proposition 8.4 shows that $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$ and $u = s_{k+1}t > t$ occurs only if $\alpha = \beta$. Moreover, if $\alpha \neq \beta$ then $c_{\beta,u}$ and $c_{\alpha,t}$ are in different molecules, and $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$ implies that $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$. So when $\alpha \neq \beta$ the condition $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$ suffices to ensure that $A(u,t) \neq \varnothing$.

Lemma 8.11. *Let $\pi, \lambda \in P(n)$ and $(u,t) \in \mathrm{Std}(\pi) \times \mathrm{Std}(\lambda)$ with $u \neq t$, and let $k$ be the restriction number of $(u,t)$. Assume that $A(u,t) \neq \varnothing$, and let $(v,x) \in A(u,t)$. Then $(v,x)$ is $k$-restricted and satisfies $(v,x) \approx_k (u,t)$. Moreover, if $D(t) \subsetneqq \mathrm{D}(u)$ then we have $\mathrm{D}(x) \subsetneqq \mathrm{D}(v)$ and $k = \min(\mathrm{D}(v) \smallsetminus \mathrm{D}(x))$.*

*Proof.* It follows from Remark 8.9 that $(v,x)$ is $k$-restricted and satisfies $(v,x) \approx_k (u,t)$. So it remains to show that if $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$ then $\mathrm{D}(x) \subsetneqq \mathrm{D}(v)$ and $k = \min(\mathrm{D}(v) \smallsetminus \mathrm{D}(x))$. So assume that $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$.

Since $\mathrm{col}_u(k+1) < \mathrm{col}_t(k+1)$ (since $A(u,t) \neq \varnothing$), Lemma 7.19 and Lemma 7.20 show that $\mathrm{D}(x) \smallsetminus \mathrm{D}(v) = \mathrm{D}(t) \smallsetminus \mathrm{D}(u)$ and $\mathrm{D}(v) \smallsetminus \mathrm{D}(x) \supseteq \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$, and since $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$ it follows that $\mathrm{D}(x) \subsetneqq \mathrm{D}(v)$. Since $(v,x)$ is favourable, we have $k = \min(\mathrm{D}(v) \oplus \mathrm{D}(x)$ by Remark 7.18, and it follows that $k = \min(\mathrm{D}(v) \smallsetminus \mathrm{D}(x))$. ☐

Lemma 8.12. *Let $\pi, \lambda \in \Lambda$ with $\pi \neq \lambda$, and let $(\beta, u) \in \mathcal{I}_\pi \times \mathrm{Std}(\pi)$ and $(\alpha, t) \in \mathcal{I}_\lambda \times \mathrm{Std}(\lambda)$ satisfy $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$. Let $k$ be the restriction number of $(u,t)$. Then $A(u,t) \neq \varnothing$, and for all $(v,x) \in A(u,t)$ the following three conditions hold:*

(i) *$(v,x) \approx (u,t)$,*
(ii) *$\mathrm{D}(x) \subsetneqq \mathrm{D}(v)$ and $k = \min(\mathrm{D}(v) \smallsetminus \mathrm{D}(x))$,*
(iii) *$\mu(c_{\beta,v}, c_{\alpha,x}) = \mu(c_{\beta,u}, c_{\alpha,t})$.*

*Proof.* Remark 8.10 gives $A(u,t) \neq \varnothing$. Let $(v,x) \in A(u,t)$ be arbitrary. By Lemma 8.11, we have $(u,t) \approx_k (v,x)$, whence $(u,t) \approx (v,x)$. Since $c_{\beta,u}$ and $c_{\alpha,t}$ are in distinct molecules (since $\pi \neq \lambda$), and since $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$, we have $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$, and it follows by Lemma 8.11 that $\mathrm{D}(x) \subsetneqq \mathrm{D}(v)$ and $k = \min(\mathrm{D}(v) \smallsetminus \mathrm{D}(x))$. It remains to show that $\mu(c_{\beta,v}, c_{\alpha,x}) = \mu(c_{\beta,u}, c_{\alpha,t})$. Let $l = \min(\mathrm{D}(u) \smallsetminus \mathrm{D}(t))$, and note that $k \leqslant l$ since $(u,t)$ is $k$-restricted.

Suppose first that $k < l$. Since $(u,t) \approx_k (v,x)$ and $k < l \in \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$, the result follows from Lemma 7.22.

Suppose now that $k = l = \min(\mathrm{D}(u) \smallsetminus \mathrm{D}(t))$. In particular, this means that $k \in \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$. Let $w = t_{\leqslant k} = u_{\leqslant k}$ and $\xi = \mathrm{Shape}(w)$, and let $(h,r) = t^{-1}(k+1)$ and $(g,p) = t^{-1}(k)$, the boxes of $t$ that contain $k+1$ and $k$. Note that $\xi_{r-1} > \xi_r$, since $(h,r)$ is $\xi$-addable. Since $k \notin \mathrm{D}(t)$, it follows that $g \geqslant h$ and $p < r$. If $p = r-1$ then $(u,t) \in A(u,t)$, and then $(v,x) \in C_{k-1}(u,t)$, since Remark 8.9 tells us that $A(u,t)$ is a single $(k-1)$-subclass. The desired conclusion $\mu(c_{\beta,v}, c_{\alpha,x}) = \mu(c_{\beta,u}, c_{\alpha,t})$ then follows from Lemma 7.22, since $k-1 < k \in \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$. Thus we can assume that $p < r-1$.

Let $(d,m) = (\xi_{r-1}, r-1)$, noting that $(g,p)$ and $(d,m)$ are distinct $\xi$-removable boxes. Let $\rho = \mathrm{Shape}(w_{<k}) \in P(k-1)$, so that $[\rho] = [\xi] \smallsetminus \{(g,p)\}$, and let $\rho' \in P(k-2)$ satisfy $[\rho'] = [\xi] \smallsetminus \{(g,p),(d,m)\}$. Let $(i,j)$ be a $\rho'$-removable box that lies between $(g,p)$ and $(d,m)$ (in the sense that $g > i \geqslant d$ and $p \leqslant j < m$), and observe that we can choose $w' \in \mathrm{Std}(\rho)$ satisfying $w'(i,j) = k-2$ and $w'(d,m) = k-1$. Since $\mathrm{Shape}(w) = \mathrm{Shape}(t_{<k}) = \mathrm{Shape}(u_{<k})$, we can define $(u_1, t_1) \in C_{k-1}(u,t)$ by the conditions that $(u_1)_{<k} = w'$ and $(u_1)_{\geqslant k} = u_{\geqslant k}$, and $(t_1)_{<k} = w'$ and

$(t_1)_{\geqslant k} = t_{\geqslant k}$. Since $(u_1, t_1) \in C_{k-1}(u, t)$ and $k - 1 < k \in \mathrm{D}(u) \setminus \mathrm{D}(t)$, it follows from Lemma 7.22 that $\mu(c_{\beta,u_1}, c_{\alpha,t_1}) = \mu(c_{\beta,u}, c_{\alpha,t})$ and $k \in \mathrm{D}(u_1) \setminus \mathrm{D}(t_1)$.

Note that $u_1^{-1}(k) = t_1^{-1}(k) = (g, p)$ and $u_1^{-1}(k-1) = t_1^{-1}(k-1) = (d, m)$. Since $p < m$ we have that $k - 1 \in \mathrm{SD}(u_1) \cap \mathrm{SD}(t_1)$, and therefore we can define $(u_2, t_2) \in \mathrm{Std}(\pi) \times \mathrm{Std}(\lambda)$ by $(u_2, t_2) = (s_{k-1}u_1, s_{k-1}t_1)$. Since $(u_1)_{\leqslant k} = (t_1)_{\leqslant k}$, clearly also $(u_2)_{\leqslant k} = (t_2)_{\leqslant k}$. Furthermore, $(u_2)_{>k} = (u_1)_{>k} = u_{>k}$ and $(t_2)_{>k} = (t_1)_{>k} = t_{>k}$. Thus it follows that $(u_2, t_2) \in C_k(u, t)$. We check that in fact $(u_2, t_2) \in A(u, t)$. One of the requirements is that $\mathrm{col}_{t_2}(k) = \mathrm{col}_t(k+1) - 1$, which is satisfied since

$$t_2^{-1}(k) = t_1^{-1}(k-1) = (w')^{-1}(k-1) = (d, m) = (\xi_{r-1}, r-1),$$

while $t^{-1}(k+1) = (h, r)$. The other requirement is that $t_2^{-1}(k)$ lies between $u^{-1}(k+1)$ and $t^{-1}(k+1)$, and this holds since

$$\mathrm{col}_t(k+1) = r > \mathrm{col}_{t_2}(k) = r - 1 \geqslant p = \mathrm{col}_t(k) = \mathrm{col}_u(k) \geqslant \mathrm{col}_u(k+1),$$

the last inequality because $k \in \mathrm{D}(u)$. Since we also have $(v, x) \in A(u, t)$, it follows from Remark 8.9 that $(u_2, t_2) \in C_{k-1}(v, x)$, and since $k - 1 < k \in \mathrm{D}(v) \setminus \mathrm{D}(x)$ it follows from Lemma 7.22 that $\mu(c_{\beta,u_2}, c_{\alpha,t_2}) = \mu(c_{\beta,v}, c_{\alpha,x})$ and $k \in \mathrm{D}(u_2) \setminus \mathrm{D}(t_2)$.

Our task is now reduced to proving that $\mu(c_{\beta,u_1}, c_{\alpha,t_1}) = \mu(c_{\beta,u_2}, c_{\alpha,t_2})$, and for this we apply Proposition 5.13 with $\{c_{\beta,u_1}, c_{\beta,u_2}\}$ in place of $\{v, v'\}$ and $\{c_{\alpha,t_1}, c_{\alpha,t_2}\}$ in place of $\{u, u'\}$, and with $\{s_{k-2}, s_{k-1}\}$ in place of $\{s, t\}$ and $s_k$ in place of $r$.

We must first check that $\{c_{\beta,u_1}, c_{\beta,u_2}\}$ and $\{c_{\alpha,t_1}, c_{\alpha,t_2}\}$ are simple edges of $\Gamma$. For this it suffices to show that there are DKMs taking $u_2$ to $u_1$ and taking $t_2$ to $t_1$. Observe first that $\mathrm{col}_{u_1}(k-2) = \mathrm{col}_{t_1}(k-2) = j$, which gives $\mathrm{col}_{u_1}(k) \leqslant \mathrm{col}_{u_1}(k-2) < \mathrm{col}_{u_1}(k-1)$ and $\mathrm{col}_{t_1}(k) \leqslant \mathrm{col}_{t_1}(k-2) < \mathrm{col}_{t_1}(k-1)$, since $p \leqslant j < m$. Applying $s_{k-1}$ interchanges $k-1$ and $k$, and so $\mathrm{col}_{u_2}(k-1) \leqslant \mathrm{col}_{u_2}(k-2) < \mathrm{col}_{u_1}(k)$ and $\mathrm{col}_{t_2}(k-1) \leqslant \mathrm{col}_{t_2}(k-2) < \mathrm{col}_{t_2}(k)$. Now since $\mathrm{col}_{u_2}(k-1) \leqslant \mathrm{col}_{u_2}(k-2)$ and $\mathrm{col}_{u_2}(k-1) < \mathrm{col}_{u_2}(k)$, it follows from Definition 6.18 that $k-2 \in \mathrm{D}(u_2)$ and $k-1 \notin \mathrm{D}(u_2)$. So $\mathrm{D}(u_2) \cap \{k-2, k-1\} = \{k-2\}$. In a similar way, $\mathrm{D}(u_1) \cap \{k-2, k-1\} = \{k-1\}$. Hence, since $u_1 = s_{k-1}u_2$, it follows from the definition that $u_2 \to u_1$ is a DKM (of the first kind) of index $k-1$. Completely analogous reasoning shows that $t_2 \to t_1$ is also a DKM (of the first kind) of index $k-1$. Hence $\{c_{\beta,u_1}, c_{\beta,u_2}\}$ and $\{c_{\alpha,t_1}, c_{\alpha,t_2}\}$ are simple edges, as required.

To show that the hypotheses of Proposition 5.13 are satisfied, it remains to check that

$$\mathrm{D}(t_1) \cap \{k-2, k-1, k\} = \{k-1\}, \qquad \mathrm{D}(u_1) \cap \{k-2, k-1, k\} = \{k-1, k\},$$
$$\mathrm{D}(t_2) \cap \{k-2, k-1, k\} = \{k-2\}, \qquad \mathrm{D}(u_2) \cap \{k-2, k-1, k\} = \{k-2, k\}.$$

But these are just the DKM conditions established in the previous paragraph together with $k \in \mathrm{D}(u_1) \setminus \mathrm{D}(t_1)$ and $k \in \mathrm{D}(u_2) \setminus \mathrm{D}(t_2)$, both of which have been established above. Hence Proposition 5.13 applies, and $\mu(c_{\beta,u_2}, c_{\alpha,t_2}) = \mu(c_{\beta,u_1}, c_{\alpha,t_1})$, as required. □

Proposition 8.13. *Let $\lambda$, $\pi \in \Lambda$ with $\mathrm{Ini}_\lambda(\Gamma) \neq \varnothing$ and $\pi \in \Lambda \setminus \{\lambda\}$, and let $\alpha \in \mathcal{I}_\lambda$ and $(\beta, u') \in \mathcal{I}_\pi \times \mathrm{Std}(\pi)$ satisfy $\mu(c_{\beta,u'}, c_{\alpha,t_\lambda}) \neq 0$. Let $k$ be the restriction number of $(u', t_\lambda)$, and let $(u, t) \in A(u', t_\lambda)$. Then $t_{\geqslant k}$ is $k$-critical. In particular, if $(u, t)$ is the minimal approximate of $(u', t_\lambda)$ then $t$ is $k$-minimal with respect to $u$.*

*Proof.* Lemma 8.12 tells us that $(u, t) \approx (u', t_\lambda)$, that $\mathrm{D}(t) \subsetneq \mathrm{D}(u)$ and $k = \min(\mathrm{D}(u) \setminus \mathrm{D}(t))$, and that $\mu(c_{\beta,u}, c_{\alpha,t}) = \mu(c_{\beta,u'}, c_{\alpha,t_\lambda}) \neq 0$. Note that $\mathrm{col}_t(k+1) = \mathrm{col}_t(k) + 1$, since $(u, t)$ is an approximate of $(u', t_\lambda)$ (see Definition 8.7). Thus, by Lemma 6.23, to show that $t_{\geqslant k}$ is $k$-critical it will suffice to show that every $j \in \mathrm{D}(t)$ with $j > k+1$ is in $\mathrm{WD}(t)$, and that either $\mathrm{col}_t(k+2) = \mathrm{col}_t(k)$ or $k+1 \notin \mathrm{SD}(t)$. We do both parts of this by contradiction.

For the first part, suppose that $j > k+1$ and $j \in \mathrm{SD}(t)$. Let $v = s_j t$, which is standard since $j \in \mathrm{SD}(t)$. Observe that the conditions of Lemma 7.30 are satisfied with $k$ in place of $i$ and $t_\lambda$ in place of $t'$. Since $k+1 < j$, the conditions $0 \leqslant k \leqslant n-1$ and $k < j$ are certainly satisfied, and we also have $j \in \mathrm{SD}(t)$ and $v = s_j t$. The condition $t_{>k} = (t_\lambda)_{>k}$ holds since $(u,t) \approx_k (u', t_\lambda)$ (see Remark 7.8), and $k \in \mathrm{A}(t)$ holds since $k \in \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$ (and $\mathrm{A}(t)$ is the complement of $\mathrm{D}(t)$). So Lemma 7.30 applies, and in particular shows that $v <_{\mathrm{lex}} t_\lambda$.

Note that $(c_{\alpha,t}, c_{\beta,u})$ is an arc in $\Gamma$, since $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$, and note that $(c_{\alpha,v}, c_{\alpha,t})$ is also an arc, since $\mu(c_{\alpha,t}, c_{\alpha,v}) = 1$ by Corollary 7.23. We claim that the directed path $(c_{\alpha,v}, c_{\alpha,t}, c_{\beta,u})$ is of alternating type $(s_j, s_k)$. That is, we claim that $k, j \notin \mathrm{D}(v)$, that $j \in \mathrm{D}(t)$ and $k \notin \mathrm{D}(t)$, and that $k, j \in \mathrm{D}(u)$. The first of these follows from Lemma 7.29 (i), since $v = s_j t$ and $k < j-1$, and we have $k \in \mathrm{A}(t)$ and $j \in \mathrm{SD}(t)$. The others follow from facts established above, namely that $k \in \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$ and $j \in \mathrm{SD}(t) \subseteq \mathrm{D}(t) \subsetneqq \mathrm{D}(u)$.

To simplify the notation slightly, if $c, d \in C$ we write $N^h_{j,k}(c,d)$ for $N^h_{s_j,s_k}(\Gamma; c, d)$ (defined in Eq. (2) in Section 5 above). Since $\Gamma$ is admissible the quantity $N^2_{j,k}(c_{\alpha,v}, c_{\beta,u})$ is a sum of positive terms, and there is at least one term since $(c_{\alpha,v}, c_{\alpha,t}, c_{\beta,u})$ is an alternating path of type $(j,k)$. Hence $N^2_{j,k}(c_{\alpha,v}, c_{\beta,u}) \neq 0$, and since $\Gamma$ satisfies the Polygon Rule it follows from Definition 5.7 that $N^2_{k,j}(c_{\alpha,v}, c_{\beta,u}) \neq 0$. So there must exist at least one $\nu \in \Lambda$ and one $(\gamma, y) \in \mathcal{I}_\nu \times \mathrm{Std}(\nu)$ such that $(c_{\alpha,v}, c_{\gamma,y}, c_{\beta,u})$ is an alternating directed path of type $(k,j)$. If $\nu \neq \lambda$ this implies that $(\alpha, v) \in \mathrm{Ini}_\lambda(\Gamma)$, and $v \in \mathrm{Ini}_\Gamma(\alpha, \lambda) \subseteq \mathrm{Ini}_\Gamma(\lambda)$. But this is impossible since $t_\lambda$ is lexicographically minimal in $\mathrm{Ini}_\Gamma(\lambda)$, and we have shown above that $v <_{\mathrm{lex}} t_\lambda$. So $\nu = \lambda$. Hence, by Proposition 8.4, we must have either $y < v$, or $\gamma = \alpha$ and $y = s_k v > v$.

If $y = s_k v > v$ then Lemma 7.30 (i) applies (since the condition $k \in \mathrm{SA}(v)$ is equivalent to $v < s_k v \in \mathrm{Std}(\lambda)$), giving $y = s_k v <_{\mathrm{lex}} t_\lambda$. On the other hand, if $y < v$ then Lemma 7.30 (ii) applies, again giving $y <_{\mathrm{lex}} t_\lambda$. But since $(c_{\gamma,y}, c_{\beta,u})$ is an arc from a molecule of type $\nu = \lambda$ to a molecule of type $\pi \neq \lambda$ it follows that $y \in \mathrm{Ini}_\Gamma(\lambda)$, and $y <_{\mathrm{lex}} t_\lambda$ contradicts the minimality of $t_\lambda$. This completes the proof of the first part.

For the second part, suppose that $k+1 \in \mathrm{SD}(t)$ and $\mathrm{col}_t(k+2) \neq \mathrm{col}_t(k)$.

Since $(u,t) \in A(u', t_\lambda)$, it follows from Definition 8.7 that $\mathrm{col}_t(k) = \mathrm{col}_{t_\lambda}(k+1) - 1$, and therefore $\mathrm{col}_t(k) = \mathrm{col}_t(k+1) - 1$, since $t_{>k} = (t_\lambda)_{>k}$. But $\mathrm{col}_t(k+1) > \mathrm{col}_t(k+2)$ since $k+1 \in \mathrm{SD}(t)$, and so $\mathrm{col}_t(k) \geqslant \mathrm{col}_t(k+2)$. Thus the assumption $\mathrm{col}_t(k+2) \neq \mathrm{col}_t(k)$ in fact means that $\mathrm{col}_t(k+2) < \mathrm{col}_t(k)$.

Let $v = s_{k+1} t$, noting that $v \in \mathrm{Std}(\lambda)$ since $k+1 \in \mathrm{SD}(t)$. We now apply Lemma 7.29 (iii) with $k$ and $k+1$ in place of $i$ and $j$. The hypotheses of Lemma 7.29 are that $k \in \mathrm{A}(t)$, which holds since $k \in \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$, and $k+1 \in \mathrm{SD}(t)$, which is given here, and the additional hypothesis for 7.29 (iii) is $\mathrm{col}_t(k+2) < \mathrm{col}_t(k)$, which is also given. The conclusions are that $k \in \mathrm{SD}(v)$ and $k+1 \notin \mathrm{D}(v)$, and also that $k \notin \mathrm{D}(w)$ and $k+1 \notin \mathrm{D}(w)$, where $w = s_k v$.

We can now check that $(c_{\alpha,w}, c_{\alpha,v}, c_{\alpha,t}, c_{\beta,u})$ is an alternating directed path of type $(k, k{+}1)$. It is certainly a path, since $\mu(c_{\alpha,v}, c_{\alpha,w}) = \mu(c_{\alpha,t}, c_{\alpha,v}) = 1$ by Corollary 7.23, and we have already seen that $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$. To show that the path is alternating of type $(k, k+1)$ we must show that $k,\ k+1 \notin \mathrm{D}(w)$, that $k \in \mathrm{D}(v)$ and $k+1 \notin \mathrm{D}(v)$, that $k \notin \mathrm{D}(t)$ and $k+1 \in \mathrm{D}(t)$, and that $k,\ k+1 \in \mathrm{D}(u)$. These all appear explicitly in the last paragraph above (given that $\mathrm{SD}(t) \subseteq \mathrm{D}(t)$ and $\mathrm{SD}(v) \subseteq \mathrm{D}(v)$), apart from the statement that $k+1 \in \mathrm{D}(u)$. But this also holds, since $k+1 \in \mathrm{D}(t)$ and $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$.

Admissibility of $\Gamma$ tells us that all arc weights are positive, and so it follows that $N^3_{k,k+1}(c_{\alpha,w}, c_{\beta,u}) > 0$. By the Polygon Rule, $N^3_{k+1,k}(c_{\alpha,w}, c_{\beta,u})$ is also nonzero. So there must exist $\xi \in \Lambda$ and $(\delta, x) \in \mathcal{I}_\xi \times \mathrm{Std}(\xi)$, and $\nu \in \Lambda$ and $(\gamma, y) \in \mathcal{I}_\nu \times \mathrm{Std}(\nu)$, such that $(c_{\alpha,w}, c_{\delta,x}, c_{\gamma,y}, c_{\beta,u})$ is an alternating directed path of type $(k+1, k)$.

Lemma 7.30 applies, as in the first part above, with $k$ in place of $i$ and $t_\lambda$ in place of $t'$, but now with $j = k+1$. The conditions $0 \leqslant k \leqslant n-1$ and $k < j$ are still satisfied, and we have $j \in \mathrm{SD}(t)$ and $v = s_j t$. The conditions $t_{>k} = (t_\lambda)_{>k}$ and $k \in \mathrm{A}(t)$ still hold, as before, and the extra condition needed for 7.30 (iii) is $\mathrm{col}_t(k+2) < \mathrm{col}_t(k)$, which we also have. Hence $w <_{\mathrm{lex}} t_\lambda$. Now by the definition of $t_\lambda$ it follows that $w \notin \mathrm{Ini}_\Gamma(\lambda)$, and since $(c_{\alpha,w}, c_{\delta,x})$ is an arc we conclude that the molecules containing $c_{\alpha,w}$ and $c_{\delta,x}$ are of the same type. Thus $\xi = \lambda$, and $x \in \mathrm{Std}(\lambda)$. Furthermore, by Proposition 8.4, either $x < w$ or else $\delta = \alpha$ and $x = s_l w > w$ for some $l \in [1, n-1]$. In the latter case Lemma 6.20 tells us that $\mathrm{D}(x) \setminus \mathrm{D}(w) = \{l\}$, but the fact that $(c_{\alpha,w}, c_{\delta,x}, c_{\gamma,y}, c_{\beta,u})$ is alternating of type $(k+1, k)$ means that $k+1 \in \mathrm{D}(x) \setminus \mathrm{D}(w)$, and it follows that $l = k+1$. So either $x < w$ or $x = s_{k+1} w > w$.

Since $\mathrm{D}(x) \cap \{k, k+1\} = \{k+1\}$ and $\mathrm{D}(y) \cap \{k, k+1\} = \{k\}$, and since $\mu(c_{\gamma,y}, c_{\delta,x}) \neq 0$, it follows that $\{c_{\delta,x}, c_{\gamma,y}\}$ is a simple edge of $\Gamma$, by the Simplicity Rule. Thus $\nu = \lambda$ and $\gamma = \delta$, and $y$ and $x$ are related by a dual Knuth move. Note that $y = k\text{-neb}(x)$ (see Definition 6.48). Note also that $y \in \mathrm{Ini}_\Gamma(\lambda)$, since $(c_{\gamma,y}, c_{\beta,u})$ is an arc whose head $c_{\gamma,y}$ is in a molecule of type $\nu = \lambda$ and whose tail $c_{\beta,u}$ is in a molecule of type $\pi \neq \lambda$.

If $x < w$ then it follows from Lemma 7.30 (iv) that $y <_{\mathrm{lex}} t_\lambda$, contradicting the fact that $t_\lambda$ is lexicographically minimal in $\mathrm{Ini}_\Gamma(\lambda)$. So we must have $x = s_{k+1} w > w$. Since this says that $k+1 \in \mathrm{SA}(w)$, it follows from Lemma 7.29 (iii) that $s_k s_{k+1} w = k\text{-neb}(s_{k+1} w)$. That is, $s_k x = y$. But it follows from Lemma 7.30 (iii) that $s_k s_{k+1} w <_{\mathrm{lex}} t_\lambda$. So $y <_{\mathrm{lex}} t_\lambda$, contradicting the minimality of $t_\lambda$, and completing the proof that $t_{\geqslant k}$ is $k$-critical.

The minimal approximate of $(u', t_\lambda)$ is by definition the element $(u, t) \in A(u', t_\lambda)$ such that $u_{<k} = t_{<k} = \tau_\kappa$, where $\kappa = \mathrm{Shape}(u_{<k})$. By the proof above, it must also have the property that $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$ and $t_{\geqslant k}$ is $k$-critical, and by Definition 8.6 this means $t$ is $k$-minimal with respect to $u$. ☐

Corollary 8.14. *Let $\lambda$, $\pi \in \Lambda$ with $\mathrm{Ini}_\lambda(\Gamma) \neq \varnothing$ and $\pi \in \Lambda \setminus \{\lambda\}$, and let $\alpha \in \mathcal{I}_\lambda$ and $(\beta, u') \in \mathcal{I}_\pi \times \mathrm{Std}(\pi)$ satisfy $\mu(c_{\beta,u'}, c_{\alpha,t_\lambda}) \neq 0$. Let $k$ be the restriction number of $(u', t_\lambda)$. Then $(t_\lambda)_{>(k+1)}$ is minimal, and if $k+1 \in \mathrm{SD}(t_\lambda)$ then $\mathrm{col}_{t_\lambda}(k+1) = 1 + \mathrm{col}_{t_\lambda}(k+2)$.*

*Proof.* Let $(u, t) \in A(u', t_\lambda)$. Then $t_{\geqslant k}$ is $k$-critical, by Proposition 8.13. So $t_{>(k+1)}$ is minimal, and if $k+1 \in \mathrm{SD}(t)$ then $\mathrm{col}_t(k+1) = 1 + \mathrm{col}_t(k+2)$. Since $(t_\lambda)_{>k} = t_{>k}$, the result follows. ☐

Lemma 8.15. *Let $n \geqslant 2$, and let $\pi$, $\lambda \in P(n)$. Let $t \in \mathrm{Std}(\lambda)$ and $u \in \mathrm{Std}(\pi)$ and suppose that $t$ is 1-minimal with respect to $u$. Then $\pi < \lambda$.*

*Proof.* By Definition 8.6 the tableau $t$ is 1-critical and the pair $(u, t)$ is not 2-restrictable. It follows that $t(1,1) = u(1,1) = 1$ and $t(1,2) = u(2,1) = 2$. Thus $1 \in \mathrm{D}(u) \setminus \mathrm{D}(t)$. Furthermore, if $n = 2$ then $\pi = (2)$ and $\lambda = (1,1)$, giving $\pi < \lambda$ by Definitions 6.6 and 6.9.

We proceed inductively on $n \geqslant 3$. If $\lambda = (1, 1, \ldots, 1)$ then it follows readily from Definitions 6.6 and 6.9 that $\nu < \lambda$ holds for all $\nu \in P(n) \setminus \{\lambda\}$, and so $\pi < \lambda$. So we may assume that $\lambda_1 \geqslant 2$, and since $t$ is 1-critical we have $t(i, 1) = i+1$ for all $i \in [2, \lambda_1]$. In particular, $\mathrm{col}_t(3) = 1 < 2 = \mathrm{col}_t(2)$, and $\mathrm{col}_t(i+1) = \mathrm{col}_t(i) = 1$ for all $i \in [3, \lambda_1]$. It follows that $i \in \mathrm{D}(t)$ for all $i \in [2, \lambda_1]$. But Definition 8.6 requires that $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$, and so for all $i \in [2, \lambda_1]$ we must have $\mathrm{col}_u(i+1) \leqslant \mathrm{col}_u(i)$. So all the numbers from 1 to $\lambda_1 + 1$ are in the first column of $u$, and so $\pi_1 \geqslant \lambda_1 + 1$. In particular, $\pi \neq \lambda$.

Put $(u', t') = (u_{\leqslant(n-1}, t_{\leqslant(n-1)})$, and let $\sigma = \mathrm{Shape}(u')$ and $\theta = \mathrm{Shape}(t')$. Since 1 is the restriction number of $(u, t)$ it is clear that 1 is also the restriction number of $(u', t')$, and since $t$ is 1-critical and $n \geqslant 3$ it is clear that $t'$ is 1-critical. Since $\mathrm{D}(t)$ is a subset of $\mathrm{D}(u)$, and since $\mathrm{D}(u') = \mathrm{D}(u) \smallsetminus \{n-1\}$ and $\mathrm{D}(t') = \mathrm{D}(t) \smallsetminus \{n-1\}$, it follows that $\mathrm{D}(t')$ is a subset of $\mathrm{D}(u')$. Hence $\mathrm{D}(t') \subsetneqq \mathrm{D}(u')$, since $1 \in \mathrm{D}(u') \smallsetminus \mathrm{D}(t')$. So $t'$ is 1-minimal with respect to $u'$, whence $\sigma < \theta$ by the inductive hypothesis. In view of Lemma 7.2 it now suffices to show that $\mathrm{col}_u(n) \leqslant \mathrm{col}_t(n)$. Since this obviously holds if $\mathrm{col}_u(n) = 1$, we may assume that $\mathrm{col}_u(n) > 1$.

Let $r = \max\{\, i \mid \theta_i > 0\,\}$, the number of parts of $\theta$, and note that $\sum_{i=1}^{r} \theta_i = n-1$. Since $\sigma < \theta$ it follows that $\sum_{i=1}^{r} \sigma_i \geqslant n-1$, and since $\sigma$ is a partition of $n-1$ this means that $\sigma$ has at most $r$ nonzero parts. In other words, all the numbers from 1 to $n-1$ are in the first $r$ columns of $u$. Thus $\mathrm{col}_u(n) \leqslant r+1$. We may now assume that $\mathrm{col}_t(n) \leqslant r$, since otherwise the desired conclusion $\mathrm{col}_u(n) \leqslant \mathrm{col}_t(n)$ obviously holds.

Since $t$ and $t'$ are 1-critical, $t_{>2}$ and $(t')_{>2}$ are minimal of their respective shapes, by Definition 6.22. Thus $\mathrm{col}_t(n-1) \leqslant \mathrm{col}_t(n)$, and $n-1$ is in the last column of $(t')_{>2}$. Note that unless $(t')_{>2}$ has only one column, it has the same number of columns as $t'$, namely $r$ columns.

Consider first the case that $(t')_{>2}$ has $r$ columns. Then $r = \mathrm{col}_t(n-1) \leqslant \mathrm{col}_t(n) \leqslant r$, so that $\mathrm{col}_t(n) = \mathrm{col}_t(n-1) = r$, and $n-1$ is a (weak) descent of $t$. Since $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$ it follows that $n-1 \in \mathrm{D}(u)$, and so $\mathrm{col}_u(n) \leqslant \mathrm{col}_u(n-1) \leqslant r = \mathrm{col}_t(n)$, as desired.

It remains to consider the case that $(t')_{>2}$ has only one column. Then $\theta = (n-2, 1)$, and $n-1 = \theta_1 + 1 \leqslant \lambda_1 + 1 \leqslant \pi_1$. Thus $\pi = (n-1, 1)$, since we have assumed that $\mathrm{col}_u(n) \neq 1$. So $\mathrm{col}_u(n-1) = 1$ and $\mathrm{col}_u(n) = 2$, which means that $n-1 \notin \mathrm{D}(u)$. So $n-1 \notin \mathrm{D}(t)$, and so $\mathrm{col}_t(n) \geqslant \mathrm{col}_t(n-1) + 1 = 2 = \mathrm{col}_u(n)$, as desired. $\square$

Lemma 8.16. *Let $\lambda, \pi \in \Lambda$ with $\mathrm{Ini}_\lambda(\Gamma) \neq \varnothing$ and $\pi \in \Lambda \smallsetminus \{\lambda\}$, and suppose that $\alpha \in \mathcal{I}_\lambda$ and $(\beta, u') \in \mathcal{I}_\pi \times \mathrm{Std}(\pi)$ satisfy $\mu(c_{\beta,u'}, c_{\alpha,t_\lambda}) \neq 0$. Let $k$ be the restriction number of $(u', t_\lambda)$, and assume that $k \geqslant 3$ and $(2, k-1) \notin [\pi]$. Then there is no $(u,t) \in A(u', t_\lambda)$ having the properties that $u(1,k) = n$ and $\mathrm{col}_t(n) = k-1$, and satisfying*

$$t_{\leqslant k} = u_{\leqslant k} = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & \cdots & k-2 & k-1 \\ \hline k \\ \cline{1-1}\end{array}\ .$$

*Proof.* Assume to the contrary that $(u,t) \in A(u', t_\lambda)$ has the stated properties. By Remark 8.9 the pair $(u,t)$ has the same restriction number as $(u', t_\lambda)$, namely $k$, and $A(u', t_\lambda) = C_{k-1}(u,t)$ is in bijective correspondence with the set of standard tableaux of the same shape as $t_{<k}$. Since there is only one standard tableau of shape $(1^{k-1})$, it follows that $A(u', t_\lambda) = \{(u,t)\}$.

It follows from Lemma 8.12 that $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$ and $k = \min(\mathrm{D}(u) \smallsetminus \mathrm{D}(t))$, and also that $\mu(c_{\beta,u}, c_{\alpha,t}) = \mu(c_{\beta,u'}, c_{\alpha,t_\lambda}) \neq 0$. Furthermore, $t_{\geqslant k}$ is $k$-critical, by Lemma 8.13. Hence $\mathrm{col}_t(k+1) = \mathrm{col}_t(k) + 1$, and so $t(2,2) = k+1$. Since $k \in \mathrm{D}(u)$ we have $\mathrm{col}_u(k+1) \leqslant \mathrm{col}_u(k)$. Hence $\mathrm{col}_u(k+1) = 1$, and it follows that $u(3,1) = k+1$. Thus $k+1 \neq n$, since it is given that $u(1,k) = n$.

*Case 1.*

Suppose that $(u,t) = (u', t_\lambda)$.

Since $k \geqslant 3$, we have $\mathrm{col}_u(k) = 1 < k-1 = \mathrm{col}_u(k-1)$, and so $k-1 \in \mathrm{SD}(u) \subseteq \mathrm{D}(u)$. Let $v = s_{k-1}u$. Since $k-1 \in \mathrm{SD}(u)$, it follows that $v \in \mathrm{Std}(\pi)$ and $k-1 \notin \mathrm{D}(v)$. Since $\mathrm{col}_u(k-2) = k-2 < k-1 = \mathrm{col}_u(k-1)$, it follows that $k-2 \notin \mathrm{D}(u)$. Moreover, since $v$ is obtained from $u$ by switching the positions of $k-1$ and $k$, and since $k \geqslant 3$, we have $\mathrm{col}_v(k-1) = \mathrm{col}_u(k) = 1 \leqslant k-2 = \mathrm{col}_u(k-2) = \mathrm{col}_v(k-2)$, and so $k-2 \in \mathrm{D}(v)$. So $v \to u$ is a DKM (of the first kind) of index $k-1$, and so $\{c_{\beta,u}, c_{\beta,v}\}$ is a simple edge in $\Gamma$.

Since $k \geqslant 3$, we have $1 \leqslant k-2 = \mathrm{col}_t(k-2) < k-1 = \mathrm{col}_t(k-1)$, and it follows that $k-2 \notin \mathrm{D}(t)$. Since $k \in \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$, we also have $k \notin \mathrm{D}(t)$. Similarly, since $u_{\leqslant k} = t_{\leqslant k}$, we have $k-2 \notin \mathrm{D}(u)$, but since $k \in \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$, we have $k \in \mathrm{D}(u)$. We showed above that $k-2 \in \mathrm{D}(v)$. Since $\mathrm{col}_v(k+1) = \mathrm{col}_u(k+1) = 1 < k-1 = \mathrm{col}_u(k-1) = \mathrm{col}_v(k)$, as $v$ is obtained from $u$ by switching the positions of $k-1$ and $k$, we also have $k \in \mathrm{D}(v)$. Moreover, since $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$ and $\mu(c_{\beta,v}, c_{\beta,u}) = 1$ (as $\{c_{\beta,u}, c_{\beta,v}\}$ is a simple edge), it follows that $(c_{\alpha,t}, c_{\beta,u}, c_{\beta,v})$ is an alternating directed path of type $(k, k-2)$. So $N^2_{k,k-2}(c_{\alpha,t}, c_{\beta,v}) > 0$.

Since $\Gamma$ satisfies the Polygon Rule, $N^2_{k-2,k}(c_{\alpha,t}, c_{\beta,v}) = N^2_{k,k-2}(c_{\alpha,t}, c_{\beta,v}) \neq 0$, and it follows that there are $\xi \in \Lambda$ and $(\gamma, x) \in \mathcal{I}_\xi \times \mathrm{Std}(\xi)$ such that $(c_{\alpha,t}, c_{\gamma,x}, c_{\beta,v})$ is an alternating directed path of type $(k-2, k)$.

Suppose that $k-1 \notin \mathrm{D}(x)$. Since $\mathrm{col}_t(k-1) = k-1 > 1 = \mathrm{col}_t(k)$ we see that $k-1 \in \mathrm{D}(t)$, and so $\mathrm{D}(t) \nsubseteq \mathrm{D}(x)$. Since $\mu(c_{\gamma,x}, c_{\alpha,t}) \neq 0$, it follows from the Simplicity Rule that $\{c_{\alpha,t}, c_{\gamma,x}\}$ is a simple edge. So $\xi = \lambda$ and $\gamma = \alpha$, and $x$ and $t$ are related by a DKM. By Lemma 6.20 the index of this DKM is either the unique element of $\mathrm{D}(x) \smallsetminus \mathrm{D}(t)$ or the unique element of $\mathrm{D}(t) \smallsetminus \mathrm{D}(x)$. Now $k-2 \in \mathrm{D}(x) \smallsetminus \mathrm{D}(t)$, since $(c_{\alpha,t}, c_{\gamma,x}, c_{\beta,v})$ is alternating of type $(k-2, k)$, but it is not the case that $x = s_{k-2}t$ since $s_{k-2}t$ is not standard. So the index of the DKM is the unique element of $\mathrm{D}(t) \smallsetminus \mathrm{D}(x)$, which must be $k-1$ since $k-1 \in \mathrm{D}(t) \smallsetminus \mathrm{D}(x)$. So $x = s_{k-1}t$, and $\mathrm{col}_x(k+1) = \mathrm{col}_t(k+1) = 2 \leqslant k-1 = \mathrm{col}_t(k-1) = \mathrm{col}_x(k)$. But this means that $k \in \mathrm{D}(x)$, which is not allowed since $(c_{\alpha,t}, c_{\gamma,x}, c_{\beta,v})$ is alternating of type $(k-2, k)$. So we must have $k-1 \in \mathrm{D}(x)$.

Since $\mu(c_{\beta,v}, c_{\gamma,x}) \neq 0$, and since $\mathrm{D}(x) \cap \{k-1, k\} = \{k-1\}$ and $\mathrm{D}(v) \cap \{k-1, k\} = \{k\}$, it follows from the Simplicity Rule that $\{c_{\beta,v}, c_{\gamma,x}\}$ is a simple edge. Thus $\xi = \pi$ and $\gamma = \beta$, and $x$ and $v$ are related by a DKM. Furthermore, $x = (k-1)\text{-neb}(v)$ (see Definition 6.48). But since by definition $v = s_{k-1}u$, and $\mathrm{col}_u(k+1) = \mathrm{col}_u(k) = 1 < k-1 = \mathrm{col}_u(k-1)$, we see that $\mathrm{col}_v(k+1) = \mathrm{col}_v(k-1) < \mathrm{col}_v(k)$, and so $(k-1)\text{-neb}(v) = s_k v$. Therefore, $x = s_k v$.

Since $t = t_\lambda$ we have $\mu(c_{\beta,x}, c_{\alpha,t_\lambda}) \neq 0$, and we can apply Corollary 8.14 with $x$ in place of $u'$. Since $x = s_k s_{k-1} u$ and $(u, t)$ is $k$-restricted, we see that $(x, t)$ is $(k-2)$-restricted, and so $k-2$ will replace the $k$ of 8.14. Since $k-1 \in \mathrm{SD}(t)$, it follows from Corollary 8.14 that $\mathrm{col}_t(k-1) = \mathrm{col}_t(k) + 1$. But $\mathrm{col}_t(k-1) = k-1$ and $\mathrm{col}_t(k) = 1$; hence $k = 3$. We were given that $\mathrm{col}_t(n) = k-1$; so $\mathrm{col}_t(n) = 2$. Recall also that $t(2,1) = k = 3$ and $t(2,2) = k+1 = 4$. Since Corollary 8.14 also tells us that $t_{>2}$ is the minimal tableau of its shape, it follows that $(3,1) \notin [\lambda]$ (since otherwise minimality of $t_{>2}$ would require that $t(3,1) = t(2,1) + 1 = 4$). So $\lambda_1 = 2$, and so $\lambda_2 \leqslant 2$. Hence $(2,2) = t^{-1}(4)$ is the last box in the second column of $[\lambda]$, and since $\mathrm{col}_t(n) = 2$ this forces $n = 4$. This contradicts $n > k+1$, which was proved above.

*Case 2.*
Suppose that $(u, t) \neq (u', t_\lambda)$.

Since $(u, t) \approx_k (u', t_\lambda)$ by Lemma 8.11, it follows that $(t_\lambda)_{\leqslant k} = (u')_{\leqslant k}$ is a standard tableau of the same shape as $t_{\leqslant k} = u_{\leqslant k}$. Hence there is an $i \in [1, k-2]$ such that $u'$ and $t_\lambda$ satisfy

$$w' = (t_\lambda)_{\leqslant k} = (u')_{\leqslant k} = \begin{array}{|c|c|c|c|c|c|} \hline 1 & \cdots & i & i+2 & \cdots & k \\ \hline i+1 \\ \cline{1-1} \end{array}\,.$$

Furthermore, $(t_\lambda)_{>k} = t_{>k}$ and $(u')_{>k} = u_{>k}$ must also hold.

Recall that $t(2,2) = k+1$ and $u(3,1) = k+1$ have been proved above. Recall also that $(2, k-1) \notin [\pi]$ is given, as are $u(1,k) = n$ and $\mathrm{col}_t(n) = k-1$.

Suppose first that $k = 3$. Since $u^{-1}(n)$ is necessarily the last box in its row and the last box in its column, and $u^{-1}(n) = (1,k) = (1,3)$, it follows that $\pi_3 = 1$ and that $\pi$ has only three parts. And $\pi_2 = 1$, since $(2,2) = (2, k-1) \notin [\pi]$. So $\pi = (n-2, 1, 1)$, the first row of $u$ is $\begin{array}{|c|c|c|} \hline 1 & 2 & n \\ \hline \end{array}$, and $u(j,1) = j+1$ for all $j \in [2, n-2]$. Observe that

$n-1 \notin \mathrm{D}(u)$, since $\mathrm{col}_u(n-1) = 1 < 3 = \mathrm{col}_u(n)$, and since $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$ it follows that $n-1 \notin \mathrm{D}(t)$. That is, $\mathrm{col}_t(n-1) < \mathrm{col}_t(n) = k-1 = 2$. Since $t_{>4}$ is minimal of its shape, by Corollary 8.14,

$$1 \leqslant \mathrm{col}_t(5) \leqslant \mathrm{col}_t(6) \leqslant \cdots \leqslant \mathrm{col}_t(n-1) < \mathrm{col}_t(n) = 2,$$

and so $\mathrm{col}_t(j) = 1$ for all $j \in [5, n-1]$ and $\mathrm{col}_t(n) = 2$. Since we also have $t(2,2) = 4$, it follows that $\lambda = (n-3, 3)$, the first three rows of $t$ are $\boxed{1 \mid 2}$, $\boxed{3 \mid 4}$ and $\boxed{5 \mid n}$, and $t(j,1) = j+2$ for $j \in [4, n-3]$. Now since there are only two standard tableaux of shape $(2,1)$ we see that $C_k(u,t) = \{(u,t), (s_2u, s_2t)\}$, and hence $(u', t_\lambda) = (s_2u, s_2t)$. But it is easily checked that $\mathrm{D}(s_2u) = \mathrm{D}(s_2t) = \{1\} \cup [3, n-2]$, and it follows that $\mu(c_{\beta,s_2u}, c_{\alpha,s_2t}) = 0$. Since this contradicts the hypothesis that $\mu(c_{\beta,u'}, c_{\alpha,t_\lambda}) \neq 0$, it follows that $k \geqslant 4$.

Since $(t_\lambda)_{\leqslant k} = (u')_{\leqslant k}$, we have $\mathrm{D}(t_\lambda) \cap [1, k-1] = \mathrm{D}(u') \cap [1, k-1]$. Since $k \geqslant 4$ and $(t_\lambda)_{>k} = t_{>k}$, we have $\mathrm{col}_{t_\lambda}(k+1) = \mathrm{col}_t(k+1) = 2 < k-1 = \mathrm{col}_t(k-1) = \mathrm{col}_{t_\lambda}(k)$, and similarly we have $\mathrm{col}_{u'}(k+1) = \mathrm{col}_u(k+1) = 1 < k-1 = \mathrm{col}_u(k-1) = \mathrm{col}_{u'}(k)$. Thus $k \in \mathrm{D}(t_\lambda) \cap \mathrm{D}(u')$, and so $\mathrm{D}(t_\lambda) \cap [1,k] = \mathrm{D}(u') \cap [1,k]$. So every element $l \in \mathrm{D}(u') \smallsetminus \mathrm{D}(t_\lambda)$ satisfies $l > k$. But $\mathrm{D}(t_\lambda) \subsetneqq \mathrm{D}(u')$, since $\mu(c_{\beta,u'}, c_{\alpha,t_\lambda}) \neq 0$ and $\alpha \neq \beta$, and so it follows that $\mathrm{D}((t_\lambda)_{>k}) \subsetneqq \mathrm{D}((u')_{>k})$.

Suppose that $i > 1$. Then $\mathrm{col}_{w'}(i+1) = 1 \leqslant i-1 = \mathrm{col}_{w'}(i-1) < \mathrm{col}_{w'}(i-1)+1 = \mathrm{col}_{w'}(i)$, and so $i \in \mathrm{SD}(w') \subseteq \mathrm{D}(w')$ and $i-1 \notin \mathrm{D}(w')$. Thus $\mathrm{D}(w') \cap \{i-1, i\} = \{i\}$. It follows from $i \in \mathrm{SD}(w')$, or by inspection, that $s_iw'$ is standard and $i \notin \mathrm{D}(s_iw')$. Furthermore, since $\mathrm{col}_{s_iw'}(i-1) = \mathrm{col}_{w'}(i) > \mathrm{col}_{w'}(i+1) = \mathrm{col}_{s_iw'}(i)$, it follows that $i-1 \in \mathrm{D}(s_iw')$, and hence $\mathrm{D}(s_iw') \cap \{i-1, i\} = \{i-1\}$. Thus $s_iw' \to^{*1} w'$, a DKM of index $i \leqslant k-1$. Since the same DKM takes $(s_iu', s_it_\lambda)$ to $(u', t_\lambda)$, we have $(s_iu', s_it_\lambda) \approx_k (u', t_\lambda)$. It follows by Lemma 7.22 that $\mu(c_{\beta,s_iu'}, c_{\alpha,s_it_\lambda}) = \mu(c_{\beta,u'}, c_{\alpha,t_\lambda}) \neq 0$. Since $\pi \neq \lambda$ the vertices $c_{\alpha,s_it_\lambda}$ and $c_{\beta,s_iu'}$ lie in molecules of different types, and so it follows that $(\alpha, s_it_\lambda) \in \mathrm{Ini}_\lambda(\Gamma)$, and $s_it_\lambda \in \mathrm{Ini}_\Gamma(\lambda)$. But it follows from Remark 6.19 that $s_it_\lambda < t_\lambda$, and hence $s_it_\lambda <_{\mathrm{lex}} t_\lambda$ by Corollary 6.17. Since this contradicts the minimality of $t_\lambda$, we conclude that $i = 1$.

We make use of $\Gamma_L$, the $W_L$-restriction of $\Gamma$, using the notation of Remark 6.52 and Eq. (10). Let $v = \mathrm{jdt}((u')_{>1})$ and $x = \mathrm{jdt}((t_\lambda)_{>1})$, and write $\rho = \mathrm{Shape}(v)$ and $\xi = \mathrm{Shape}(x)$.

By Remark 6.52, we can identify the vertex $c_{\beta,u'}$ of $\Gamma_L$ with $c''_{\delta,v}$ for some $\delta \in \mathcal{I}_{L,\beta,\pi,\rho}$, and the vertex $c_{\alpha,t_\lambda}$ of $\Gamma_L$ with $c''_{\gamma,x}$ for some $\gamma \in \mathcal{I}_{L,\alpha,\lambda,\xi}$. Note that $\mathcal{I}_{L,\beta,\pi,\xi} \cap \mathcal{I}_{L,\alpha,\lambda,\rho} = \varnothing$, since $\pi \neq \lambda$, and so it follows that $\delta \neq \gamma$. Now $\mathrm{D}(v) \nsubseteq \mathrm{D}(x)$, since $\mathrm{D}((u')_{>1}) \nsubseteq \mathrm{D}((t_\lambda)_{>1})$, and so it follows that $\mu_L(c''_{\delta,v}, c''_{\gamma,x}) = \mu(c_{\beta,u'}, c_{\alpha,t_\lambda}) \neq 0$. Since $\Gamma_L$ is ordered, and since $\delta \neq \gamma$, we deduce that $v < x$. Thus $\rho \leqslant \xi$.

Observe that $\mathrm{col}_{t_\lambda}(j) = \mathrm{col}_{w'}(j) \leqslant k-1$ for all $j \in [1,k]$. Since $(t_\lambda)_{>k} = t_{>k}$, we have $\mathrm{col}_{t_\lambda}(k+1) = 2$ and $\mathrm{col}_{t_\lambda}(n) = k-1$. Furthermore, $t_{\lambda\,>(k+1)}$ is the minimal tableau of its shape, by Corollary 8.14, and so $\mathrm{col}_{t_\lambda}(k+2) \leqslant \mathrm{col}_{t_\lambda}(k+3) \leqslant \cdots \leqslant \mathrm{col}_{t_\lambda}(n)$. It follows that $\mathrm{col}_{t_\lambda}(j) \leqslant \mathrm{col}_{t_\lambda}(n) = k-1$ for all $j \in [1,n]$, showing that the partition $\pi$ has exactly $k-1$ parts. Since $u(1,k) = n$ the partition $\pi$ has exactly $k$ parts, and since also $(2, k-1) \notin \pi$ it follows that $\pi_{k-1} = \pi_k = 1$. Now let $(g,p)$ and $(h,r)$ be the boxes vacated in $\mathrm{jdt}((u')_{>1})$ and $\mathrm{jdt}((t_\lambda)_{>1})$ respectively. Recalling that $u'(2,1) = t_\lambda(2,1) = 2$, since $i = 1$, we see that the box $(2,1)$ is in both slide paths, and so $g \geqslant 2$ and $h \geqslant 2$. In particular, $(g,p) \neq (1,k)$, and so $\rho_k = \pi_k = 1$. On the other hand, $\xi$ has only $k-1$ parts, since $\lambda$ has only $k-1$ parts. Hence we find that $\sum_{m=1}^{k-1} \xi_m = n-1 = \sum_{m=1}^{k} \rho_m = 1 + \sum_{m=1}^{k-1} \rho_m > \sum_{m=1}^{k-1} \rho_m$, contradicting $\rho \leqslant \xi$. $\square$

Lemma 8.17. *Let $\lambda$, $\pi \in \Lambda$ with $\mathrm{Ini}_\lambda(\Gamma) \neq \varnothing$ and $\pi \in \Lambda \setminus \{\lambda\}$, and suppose that $\alpha \in \mathcal{I}_\lambda$ and $(\beta, u') \in \mathcal{I}_\pi \times \mathrm{Std}(\pi)$ satisfy $\mu(c_{\beta,u'}, c_{\alpha,t_\lambda}) \neq 0$. Then $\pi < \lambda$.*

*Proof.* Clearly $n$ is at least 2. Since the vertices $c_{\beta,u'}$ and $c_{\alpha,t_\lambda}$ belong to different molecules we must have $\mathrm{D}(t_\lambda) \subsetneqq \mathrm{D}(u')$ and $\mu(c_{\alpha,t_\lambda}, c_{\beta,u'}) = 0$. Let $k$ be the restriction number of $(u', t_\lambda)$, noting that $1 \leqslant k \leqslant n-1$ since $\pi \neq \lambda$. We write $\nu = \mathrm{Shape}((u')_{\leqslant k}) = \mathrm{Shape}((t_\lambda)_{\leqslant k})$, and let $(u,t)$ be an arbitrary element of $A(u', t_\lambda)$ (which is nonempty, by Lemma 8.12).

Lemma 8.11 shows that $(u,t)$ is $k$-restricted and $(u,t) \approx_k (u', t_\lambda)$, and Proposition 8.13 shows that $t_{\geqslant k}$ is $k$-critical. Furthermore, we have $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$ and $k = \min(\mathrm{D}(u) \setminus \mathrm{D}(t))$, and $\mu(c_{\beta,u}, c_{\alpha,t}) = \mu(c_{\beta,u'}, c_{\alpha,t_\lambda}) \neq 0$, by Lemma 8.12. Since $t_{\geqslant k}$ is $k$-critical, it follows from Definition 6.22 that the following all hold:

$$\mathrm{col}_t(k) = \min\{\mathrm{col}_t(i) \mid i \in [k,n]\}, \tag{11}$$

$$\mathrm{col}_t(k+1) = \mathrm{col}_t(k) + 1, \tag{12}$$

$$\mathrm{col}_t(k+2) \leqslant \mathrm{col}_t(k+3) \leqslant \cdots \leqslant \mathrm{col}_t(n-1) \leqslant \mathrm{col}_t(n). \tag{13}$$

Note that $\mathrm{col}_t(i) = \mathrm{col}_{t_\lambda}(i)$ for all $i \in [k+1, n]$, since $t_{>k} = (t_\lambda)_{>k}$. Let $l = \mathrm{col}_t(n) = \mathrm{col}_{t_\lambda}(n)$.

If $k = 1$ then since $t$ is 1-minimal with respect to $u$, it follows by Lemma 8.15 that $\pi < \lambda$, while if $k = n-1$ then we must have $\mathrm{D}(u) = \{n-1\} \cup \mathrm{D}(t)$, and $\pi < \lambda$ follows by Lemma 7.25. So may assume that $1 < k < n-1$.

We make use of $\Gamma_K$ and $\Gamma_L$, the restrictions of $\Gamma$ to $W_K$ and $W_L$, using the notation from Remark 6.52 and Eqs (9) and (10) above. Looking at $\Gamma_K$ first, let $v = u_{\leqslant (n-1)}$ and $x = t_{\leqslant (n-1)}$, and let $\sigma = \mathrm{Shape}(v)$ and $\theta = \mathrm{Shape}(x)$. Note that $\mathrm{Shape}(v_{>k}) = \mathrm{Shape}((u_{\leqslant n-1})_{>k}) = \sigma/\nu$, and similarly $\mathrm{Shape}(x_{>k}) = \theta/\nu$. So $\sigma/\nu \vdash (n-1-k)$ and $\theta/\nu \vdash (n-1-k)$.

By Eq. (9) we have $c_{\beta,u} = c'_{\delta,v}$ and $c_{\alpha,t} = c'_{\gamma,x}$ for some $\delta \in \mathcal{I}_{K,\sigma}$ and $\gamma \in \mathcal{I}_{K,\theta}$. Since $c_{\beta,u}$ and $c_{\alpha,t}$ lie in different molecules of $\Gamma$ they lie in different $K$-submolecules; that is, $\delta \neq \gamma$. By the definition of the $W_K$-restriction of $\Gamma$ we have $\mathrm{D}(v) \setminus \mathrm{D}(x) = (\mathrm{D}(u) \setminus \mathrm{D}(t)) \cap [1, n-2]$, and so we see that $k \in \mathrm{D}(v) \setminus \mathrm{D}(x)$. Hence $\mu_K(c'_{\delta,v}, c'_{\gamma,x}) = \mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$. Since $\Gamma_K$ is ordered and $\gamma \neq \delta$ it follows that $v < x$, and hence $\sigma \leqslant \theta$, by Definition 7.1.

If $\mathrm{col}_u(n) \leqslant \mathrm{col}_t(n)$ then, since $\sigma \leqslant \theta$, Lemma 7.2 yields $\pi \leqslant \lambda$. Since $\pi \neq \lambda$, the desired conclusion $\pi < \lambda$ holds in this case. So we may assume henceforth that $\mathrm{col}_u(n) > \mathrm{col}_t(n) = l$.

Observe that $\mathrm{col}_t(k) \leqslant l = \mathrm{col}_t(n)$, by Eq. (11). Suppose first that $\mathrm{col}_t(k) = l$. Then it follows from Eqs (11) and (13) that $l \leqslant \mathrm{col}_t(k+2) \leqslant \mathrm{col}_t(k+3) \leqslant \cdots \leqslant \mathrm{col}_t(n-1) \leqslant l$, and so $\mathrm{col}_t(i) = l$ for all $i \in [k+2, n]$. Hence $i \in \mathrm{WD}(t)$ for all $i \in [k+2, n-1]$, and $k+1 \in \mathrm{SD}(t)$ since $\mathrm{col}_t(k+1) = l+1$ by Eq. (12). Since $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$ and also $k \in \mathrm{D}(u) \setminus \mathrm{D}(t)$, it follows that $i \in \mathrm{D}(u)$ for all $i \in [k, n-1]$. Thus $\mathrm{col}_u(n) \leqslant \mathrm{col}_u(n-1) \leqslant \cdots \leqslant \mathrm{col}_u(k)$. But since $(u,t)$ is $k$-restricted, $\mathrm{col}_u(k) = \mathrm{col}_t(k) = l$, and we deduce that $\mathrm{col}_u(n) \leqslant l = \mathrm{col}_t(n)$. Since we have already shown that this gives $\pi < \lambda$, for the rest of the proof we may (and do) assume that

$$\mathrm{col}_t(k) < \mathrm{col}_t(n) = l < \mathrm{col}_u(n). \tag{14}$$

In fact we shall show that Eq. (14) leads to a contradiction, by showing that $(u,t)$ satisfies the conditions of Lemma 8.16.

Since $\sigma \leqslant \theta$ (equivalent to $\sigma \trianglerighteq \theta$ by Definition 6.9) we have

$$\sum_{m=1}^{l} \theta_m \leqslant \sum_{m=1}^{l} \sigma_m. \tag{15}$$

It follows from Eqs (12) and (14) that $\mathrm{col}_t(k+1) \leqslant l$, and by Eq. (13) we deduce that

$$[k+1, n-1] \subseteq \{\, i \mid \mathrm{col}_t(i) \leqslant l \,\}. \tag{16}$$

So all elements of $[k+1, n-1]$ appear in the first $l$ columns of the tableau $(t_{\leqslant n-1})_{>k} = x_{>k}$, and since $x_{>k} \in \mathrm{Std}_k(\theta/\nu)$ it follows that $\sum_{i=1}^{l}(\theta_m - \nu_m) = n-1-k$. So Eq. (15) can be expressed in the form

$$n-1-k+\sum_{m=1}^{l}\nu_m \leqslant \sum_{m=1}^{l}(\sigma_m-\nu_m)+\sum_{m=1}^{l}\nu_m.$$

Thus $\sum_{m=1}^{l}(\sigma_m - \nu_m) \geqslant n-1-k$. Now since $\sigma/\nu \vdash (n-1-k)$ it follows that $\sigma_m - \nu_m = 0$ for all $m > l$ and $\sum_{m=1}^{l}(\sigma_m - \nu_m) = n-1-k$. Since $(u_{\leqslant n-1})_{>k} \in \mathrm{Std}_k(\sigma/\nu)$ it follows that

$$[k+1, n-1] \subseteq \{\, i \mid \mathrm{col}_u(i) \leqslant l \,\}. \tag{17}$$

Eq. (16) and $\mathrm{col}_t(n) = l$ combined give $[k+1, n] \subseteq \{\, i \mid \mathrm{col}_t(i) \leqslant l \,\}$, and we see that column $l$ is the last nonempty column of $t_{>k}$. Since $\mathrm{Shape}(t_{>k}) = \lambda/\nu \vdash (n-k)$, it follows that $\sum_{m=1}^{l}(\lambda_m - \nu_m) = n-k$. Now $\mathrm{Shape}(u_{>k}) = \pi/\nu \vdash (n-k)$, but although the first $l$ columns of $(u_{\leqslant n-1})_{>k}$ include all elements of $[k+1, n-1]$ (by Eq. (17)) they do not include all elements of $[k+1, n]$, since $\mathrm{col}_u(n) > l$ (by Eq. (14)). Thus $[\pi/\nu]$ is the disjoint union of $[\sigma/\nu]$ and $\{u^{-1}(n)\}$, and $\sum_{m=1}^{l}(\pi_m - \nu_m) = n-k-1$. We deduce that

$$\sum_{m=1}^{l}\pi_m = n-k-1+\sum_{m=1}^{l}\nu_m = \left(\sum_{m=1}^{l}\lambda_m\right)-1. \tag{18}$$

Looking now at $\Gamma_L$, we put $w = \mathrm{jdt}(u_{>1})$ and $y = \mathrm{jdt}(t_{>1})$, and define $\rho = \mathrm{Shape}(w)$ and $\xi = \mathrm{Shape}(y)$. By Eq. (10) we have $c_{\beta,u} = c''_{\zeta,w}$ and $c_{\alpha,t} = c''_{\epsilon,y}$ for some $\zeta \in \mathcal{I}_{L,\rho}$ and $\epsilon \in \mathcal{I}_{L,\xi}$. Since $c_{\beta,u}$ and $c_{\alpha,t}$ lie in different molecules of $\Gamma$ they must also lie in different $L$-submolecules; that is, $\zeta \neq \epsilon$. By Proposition 6.47 and the definition of the $W_L$-restriction of $\Gamma$ we have $\mathrm{D}(w) \setminus \mathrm{D}(y) = \mathrm{D}(u_{>1}) \setminus \mathrm{D}(t_{>1}) = (\mathrm{D}(u)) \setminus \mathrm{D}(t)) \cap [2, n-1]$, and so it follows that $k \in \mathrm{D}(w) \setminus \mathrm{D}(y)$. Hence $\mu_L(c''_{\zeta,w}, c''_{\epsilon,y}) = \mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$. Since $\Gamma_L$ is ordered and $\zeta \neq \epsilon$ it follows that $w < y$, and hence $\rho \leqslant \xi$, by Definition 7.1.

Let $(g,p)$ and $(h,r)$ be the boxes vacated by $\mathrm{jdt}((1,1), u_{>1})$ and $\mathrm{jdt}((1,1), t_{>1})$. Note that $\rho_p = \pi_p - 1$ and $\rho_m = \pi_m$ for all $m \neq p$. Similarly, $\xi_r = \lambda_r - 1$ and $\xi_m = \lambda_m$ for all $m \neq r$. We claim that

$$r \leqslant l < p. \tag{19}$$

If $p \leqslant l$ then $\sum_{m=1}^{l}\rho_m = \left(\sum_{m=1}^{l}\pi_m\right) - 1 < \sum_{m=1}^{l}\pi_m$, and since $\left(\sum_{m=1}^{l}\lambda_m\right) - 1 \leqslant \sum_{m=1}^{l}\xi_m$, it follows by Eq. (18) that $\sum_{m=1}^{l}\rho_m < \sum_{m=1}^{l}\xi_m$, which contradicts $\rho \leqslant \xi$. Similarly, if $l < r$ then $\sum_{m=1}^{l}\xi_m = \sum_{m=1}^{l}\lambda_m$, and since $\sum_{m=1}^{l}\pi_m \geqslant \sum_{m=1}^{l}\rho_m$, it again follows by Eq.(18) that $\sum_{m=1}^{l}\xi_m > \sum_{m=1}^{l}\rho_m$, contradicting $\rho \leqslant \xi$. Hence $r \leqslant l < p$, as claimed.

We shall now show that $u(g,p) = n$. If $k+1 \leqslant u(g,p) < n$ then $p = \mathrm{col}_u(u(g,p)) \leqslant l$ by Eq. (17), contradicting Eq. (19). Now suppose that $u(g,p) \leqslant k$. Then clearly $u(b) \leqslant k$ for all boxes $b$ in the slide path of $\mathrm{jdt}((1,1), u_{>1})$ and hence the slide path of $\mathrm{jdt}((1,1), u_{>1})$ coincides with the slide path of $\mathrm{jdt}((1,1), (u_{\leqslant k})_{>1})$. Furthermore, since $u_{\leqslant k} = t_{\leqslant k}$, and the slide path of $\mathrm{jdt}((1,1), t_{>1})$ extends (or equals) that of $\mathrm{jdt}((1,1), (t_{\leqslant k})_{>1})$, it follows that $(g,p)$ is in the slide path of $\mathrm{jdt}((1,1), t_{>1})$. By Lemma 6.43 it follows that $h \geqslant g$ and $r \geqslant p$. Since the latter inequality contradicts Eq.( 19) we conclude that $u(g,p) < n$ is impossible, and so $u(g,p) = n$, as claimed.

We claim that

$$[k+1, n-1] \subseteq \{\, i \mid \mathrm{col}_u(i) < p-1 \,\}. \tag{20}$$

By Eqs (17) and (14) we see that $\mathrm{col}_u(n-1) < \mathrm{col}_u(n)$. Thus $n-1 \in \mathrm{A}(u)$, and hence $n-1 \in \mathrm{A}(t)$, since $\mathrm{D}(t) \subsetneq \mathrm{D}(u)$. So $\mathrm{col}_t(n-1) < \mathrm{col}_t(n) = l$. Note that, by Eq. (16), we have $\mathrm{col}_t(k+1) \leqslant l$.

Suppose first that $\mathrm{col}_t(k+1) = l$. Then $\mathrm{col}_t(n-1) < \mathrm{col}_t(k+1) = \mathrm{col}_t(k)+1$, by Eq. (12), and since Eq. (11) gives $\mathrm{col}_t(k) \leqslant \mathrm{col}_t(n-1)$ it follows that $\mathrm{col}_t(n-1) = \mathrm{col}_t(k)$. By Eqs (11) and (13) we deduce that $\mathrm{col}_t(k) = \mathrm{col}_t(k+2) = \mathrm{col}_t(k+3) = \cdots = \mathrm{col}_t(n-1)$. Thus $[k+2, n-2] \subseteq \mathrm{WD}(t)$. Moreover, $k+1 \in \mathrm{SD}(t)$, since $\mathrm{col}_t(k+1) > \mathrm{col}_t(k) = \mathrm{col}_t(k+2)$. Hence $[k+1, n-2] \subseteq \mathrm{D}(t)$. Since $\mathrm{D}(t) \subsetneq \mathrm{D}(u)$, and we also have $k = \min(\mathrm{D}(u) \smallsetminus \mathrm{D}(t))$, it follows that $[k, n-2] \subseteq \mathrm{D}(u)$. Thus $\mathrm{col}_u(n-1) \leqslant \mathrm{col}_u(n-2) \leqslant \cdots \leqslant \mathrm{col}_u(k+1) \leqslant \mathrm{col}_u(k)$. But $\mathrm{col}_u(k) = \mathrm{col}_t(k) = l-1$, since $(u,t)$ is $k$-restricted and $l = \mathrm{col}_t(k+1) = \mathrm{col}_t(k)+1$. So $\mathrm{col}_u(k) < p-1$ by Eq. (19), and so Eq. (20) holds in this case.

Suppose now that $\mathrm{col}_t(k+1) < l$. Since we have shown above that $\mathrm{col}_t(n-1) < l$, and since we also have $\mathrm{col}_t(k+2) \leqslant \mathrm{col}_t(k+3) \leqslant \cdots \leqslant \mathrm{col}_t(n-1)$ by Eq. (13), it follows that $[k+1, n-1] \subseteq \{\, i \mid \mathrm{col}_t(i) \leqslant l-1 \,\}$ (a strengthening of Eq. (16)). So all elements of $[k+1, n-1]$ appear in the first $l-1$ columns of $(t_{\leqslant (n-1)})_{>k}$, and so $\sum_{m=1}^{l-1}(\theta_m - \nu_m) = n-k-1$. Since $\sigma \leqslant \theta$, we have $\sum_{m=1}^{l-1}\theta_m \leqslant \sum_{m=1}^{l-1}\sigma_m$, which can be written as

$$n-1-k+\sum_{m=1}^{l-1}\nu_m \leqslant \sum_{m=1}^{l-1}(\sigma_m-\nu_m)+\sum_{m=1}^{l-1}\nu_m,$$

giving $n-1-k \leqslant \sum_{m=1}^{l-1}(\sigma_m-\nu_m)$. Since $\sigma/\nu \vdash (n-1-k)$ it follows that $\sigma_m - \nu_m = 0$ for all $m > l-1$ and $\sum_{m=1}^{l-1}(\sigma_m - \nu_m) = n-1-k$. Since $(u_{\leqslant (n-1)})_{>k} \in \mathrm{Std}_k(\sigma/\nu)$ it follows that $[k+1, n-1] \subseteq \{\, i \mid \mathrm{col}_u(i) \leqslant l-1 \,\}$, and since $l < p$ by Eq. (19) it follows that Eq. (20) also holds in this case. This completes the proof of our claim.

Recall that $(g,p)$ is vacated by $\mathrm{jdt}((1,1), u_{>1})$ and that $u(g,p) = n$. Let $b$ be the box that $n$ slides into, so that either $b = (g-1, p)$ or $b = (g, p-1)$. Obviously $u(b) \notin \{\, i \mid \mathrm{col}_u(i) < p-1 \}$, and so Eq. (20) shows that $u(b) \notin [k+1, n]$. So $b$ is in the diagram of $\mathrm{Shape}(u_{\leqslant k})$, and must be the box vacated by $\mathrm{jdt}((1,1), (u_{\leqslant k})_{>1})$. The slide path of $\mathrm{jdt}((1,1), u_{>1})$ is the slide path of $\mathrm{jdt}((1,1), (u_{\leqslant k})_{>1})$ extended by the additional box $(g,p)$. Since $u_{\leqslant k} = t_{\leqslant k}$, the slide path of $\mathrm{jdt}((1,1), t_{>1})$ also extends the slide path of $\mathrm{jdt}((1,1), (u_{\leqslant k})_{>1})$, and, in particular, includes $b$. If $b = (g-1, p)$ then Lemma 6.43 gives $p \leqslant r$, contradicting Eq. (19). So $b = (g, p-1)$, and Lemma 6.43 gives $p-1 \leqslant r$. But since $r \leqslant l < p$ by Eq. (19), this shows that

$$p-1 = r = l = \mathrm{col}_t(n). \tag{21}$$

Recall that $(u,t)$ was chosen as an arbitrary element of $A(u', t_\lambda)$, and so all that we have proved thus far applies for all elements of $A(u', t_\lambda)$. However, since $n > k$ and $u_{>k} = (u')_{>k}$ for all $(u,t) \in A(u', t_\lambda)$, it follows that $(g,p) = u^{-1}(n) = (u')^{-1}(n)$ is independent of the choice of $(u,t)$. So $(g, p-1)$ in independent of the choice of $(u,t)$. Since we have just shown that $(g, p-1)$ is the box vacated by $\mathrm{jdt}((1,1), (u_{\leqslant k})_{>1})$, this must be true for all choices of $(u,t)$. Furthermore, $(g, p-1) \neq u^{-1}(k)$, by Eqs (21) and (14), and so $(g, p-1)$ is in the diagram of $\mathrm{Shape}(u_{<k})$, and is the box vacated by $\mathrm{jdt}((1,1), (u_{<k})_{>1})$, for all choices of $(u,t)$.

Recall from Remark 8.9 that there exists $\kappa \in P(k-1)$ and a bijection $A(u', t_\lambda) \to \mathrm{Std}(\kappa)$ given by $(u,t) \mapsto w = u_{<k} = t_{<k}$. So the previous paragraph tells us that for every $w \in \mathrm{Std}(\kappa)$ the jeu-de-taquin slide $\mathrm{jdt}((1,1), w_{>1})$ vacates the box $(g, p-1)$.

Considering in particular the cases $w = \tau_\kappa$ and $w = \tau^\kappa$ and applying Lemma 6.44, we deduce that $\kappa$ has exactly $p-1$ parts, all equal to $g$.

Observe that since $\kappa = (g^{p-1})$ there are only two $\kappa$-addable boxes, namely $(g+1,1)$ and $(1,p)$. But since $[\mathrm{Shape}(t_{\leqslant k})] = \{t^{-1}(k)\} \cup [\mathrm{Shape}(t_{<k})] = \{t^{-1}(k)\} \cup [\kappa]$, it is clear that $t^{-1}(k)$ is $\kappa$-addable. And $t^{-1}(k) \neq (1,p)$, by Eq. (14) and the fact that $\mathrm{col}_u(n) = p$. So $t^{-1}(k) = (g+1,1)$, and since $(u,t)$ is $k$-restricted it follows that $u^{-1}(k) = (g+1,1)$ also.

Since $\mathrm{Shape}(u_{<k}) = (g^{p-1})$ it follows that $\mathrm{col}_u(i) < p$ for all $i \in [1,k-1]$. But we have just shown that $\mathrm{col}_u(k) = 1$, and Eq. (20) gives $\mathrm{col}_u(i) < p-1$ for all $i \in [k+1,n-1]$. So $n$ is the only number in column $p$ of $u$, and so clearly it must be in the first row. That is, $g = 1$.

Since $\kappa = \mathrm{Shape}(t_{<k})$ is a partition of $k-1$, and since $\kappa = (1^{p-1})$, it follows that $k = p$. It follows from Eq. (14) that $1 = \mathrm{col}_t(k) < \mathrm{col}_t(n) < \mathrm{col}_u(n) = k$, and so $k \geqslant 3$. Moreover, since $\mathrm{row}_u(i) = 1$ for $i \in [1,k-1] \cup \{n\}$, while $\mathrm{col}_u(k) = 1$ and $\mathrm{col}_u(i) < k-1$ for $i \in [k+1,n-1]$ (by Eq. (20)), we see that $(2,k-1) \notin [\pi]$. We have now shown that $(u,t)$ satisfies all the conditions in Lemma 8.16, contradicting $(u,t) \notin A(u',t')$, and completing our proof. $\square$

Lemma 8.18. *Suppose that $\Gamma$ is strongly connected. Then $\Lambda$ has only one element.*

*Proof.* Assume to the contrary that $\Lambda$ has more than one element, and choose $\lambda \in \Lambda$ to be minimal, in the sense that there is no $\pi \in \Lambda \setminus \{\lambda\}$ such that $\pi < \lambda$. Note that $\mathrm{Ini}_\lambda(\Gamma) \neq \varnothing$, since $\Gamma$ is strongly connected and $\Lambda \setminus \{\lambda\} \neq \varnothing$. Hence there exists $\pi \in \Lambda \setminus \{\lambda\}$ such that $\mu(c_{\beta,u}, c_{\alpha,t_\lambda}) \neq 0$, for some $(\beta,u) \in \mathcal{I}_\pi \times \mathrm{Std}(\pi)$ and $\alpha \in \mathcal{I}_\lambda$. But now Lemma 8.17 gives $\pi < \lambda$, contradicting the choice of $\lambda$. $\square$

Lemma 8.18 says that if $\Gamma$ is an admissible $W_n$-cell then all molecules of $\Gamma$ have the same type $\lambda$, for some $\lambda \in P(n)$. In general whenever $\Lambda = \{\lambda\}$ we may call $\lambda$ the type of $\Gamma$.

Lemma 8.19. *Suppose that $n \geqslant 2$. Let $D$ and $D'$ be cells of $\Gamma$, of types $\lambda$ and $\pi$ respectively. Let $\leqslant_\Gamma$ be the partial order on the set of cells of $\Gamma$ (as defined in Section 4 above). Then $D' \leqslant_\Gamma D$ implies $\pi \leqslant \lambda$. In particular, $\pi \leqslant \lambda$ holds if there exist vertices $c \in D$ and $c' \in D'$ such that $\mu(c',c) \neq 0$.*

*Proof.* We can assume that $\pi \neq \lambda$, since the result is trivial otherwise. Write $\mathcal{C}$ for the set of cells of $\Gamma$, and observe that $|\mathcal{C}| \geqslant 2$, since $\pi \neq \lambda$ implies that $D \neq D'$.

Suppose first that $\mathcal{C} = \{D, D'\}$. Since $D' \leqslant_\Gamma D$ it is immediate from the definition of the partial order $\leqslant_\Gamma$ that there exist $c \in D$ and $c' \in D'$ with $\mu(c',c) \neq 0$, and since $\pi \neq \lambda$ it follows that $\mathrm{Ini}_\Gamma(\lambda) \neq \varnothing$. So $t_\lambda$ exists, and by the definition of $\mathrm{Ini}_\Gamma(\lambda)$ there exist $\alpha \in \mathcal{I}_\lambda$ and $(\beta,u) \in \mathcal{I}_\pi \times \mathrm{Std}(\pi)$ with $\mu(c_{\beta,u}, c_{\alpha,t_\lambda}) \neq 0$. So Lemma 8.17 gives $\pi < \lambda$, as required.

Proceeding inductively, suppose now that $|\mathcal{C}| > 2$ and that the result holds for all admissible $W_n$-graphs with fewer than $|\mathcal{C}|$ cells. Let $C_0, C_1 \in \mathcal{C}$ be such that $C_0$ is minimal and $C_1$ is maximal in the partial order $\leqslant_\Gamma$. Then $C_0$ and $C \setminus C_1$ are subsets of $C$ that are closed in the sense defined in Section 4. Writing $\Gamma_0$ and $\Gamma_1$ for the full subgraphs of $\Gamma$ spanned by $C \setminus C_0$ and $C \setminus C_1$, we see that $\Gamma_0$ and $\Gamma_1$ are admissible $W_n$-graphs, with arc weights and vertex colours inherited from $\Gamma$, and with cells that are cells of $\Gamma$. It follows that if $C_0$ and $C_1$ can be chosen so that $D$ and $D'$ are both contained in $C \setminus C_0$ or both contained in $C \setminus C_1$ then the result follows from the inductive hypothesis. Since $D' \leqslant_\Gamma D$ by assumption, it remains to consider the possibility that $C_0 = D'$ is the unique minimal cell and $C_1 = D$ is the unique maximal cell.

Since $|\mathcal{C}| > 2$ we can choose $C' \in \mathcal{C} \setminus \{C_0, C_1\}$. By Lemma 8.18 the cell $C'$ has type $\{\nu\}$ for some $\nu \in \Lambda$. Moreover, $C_0 \leqslant_\Gamma C'$ since $C_0$ is the unique minimal cell, and $C' \leqslant_\Gamma C_1$ since $C_1$ is the unique maximal cell. Since $D'$ and $C'$ are cells of $\Gamma_1$, we have $\pi \leqslant \nu$ by the inductive hypothesis, and since $C'$ and $D$ are cells of $\Gamma_0$, we have $\nu \leqslant \lambda$ by the inductive hypothesis. So it follows that $\pi \leqslant \lambda$, as required.

Recall that if $\mu(c', c) \neq 0$ then it follows as a consequence that $\tau(c') \nsubseteq \tau(c)$ (since $\Gamma$ is reduced, in the terminology of Section 4). So, by the definition of $\leqslant_\Gamma$, if $\mu(c', c) \neq 0$ then $c' \leqslant_\Gamma c$, and so the last assertion of the lemma follows from the rest. ☐

We are now able to complete the proof of Theorem 8.2.

*Proof.* Suppose that $(\alpha, t) \in \mathcal{I}_\lambda \times \mathrm{Std}(\lambda)$ and $(\beta, u) \in \mathcal{I}_\pi$ satisfy $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$. It follows from Lemma 8.19 that $\pi \leqslant \lambda$. Now Proposition 8.4 says that $u < t$ unless $\alpha = \beta$ and $u = s_i t > t$ for some $i \in [1, n-1]$. That is, $\Gamma$ is ordered. ☐

Remark 8.20. In particular, it follows from Theorem 8.2 that the Kazhdan–Lusztig $W_n$-graph corresponding to the regular representation of $\mathcal{H}(W_n)$ is ordered in the sense of Definition 8.1. In this case the vertex set of $\Gamma = (C, \mu, \tau)$ is $C = W_n$, the set of molecule types is $\Lambda = P(n)$, for each $\lambda \in P(n)$ the set of molecules of type $\lambda$ is indexed by $\mathcal{I}_\lambda = \mathrm{Std}(\lambda)$, and for each $\lambda \in \Lambda$ and $x \in \mathcal{I}_\lambda$ the set $C_{x,\lambda}$ consists of those $w \in W_n$ such that $\mathrm{Q}(w) = x$, where $\mathrm{Q}(w)$ is the recording tableau in the Robinson–Schensted process.

Now let $y$, $w \in W_n$ and put $\mathrm{RS}(w) = (t, x) \in \mathrm{Std}(\lambda)^2$ and $\mathrm{RS}(y) = (u, v) \in \mathrm{Std}(\pi)^2$, where $\lambda$, $\pi \in P(n)$. The conclusion of Theorem 8.2, applied in this case, is that if $\mu(y, w) \neq 0$ and $\tau(y) \nsubseteq \tau(w)$ then either $u < t$ or else $\pi = \lambda$ and $(u, v) = (st, x)$ for some $s \in S_n$.

If $\Gamma$ is replaced by $\Gamma^{\mathrm{o}} = (C, \mu, \tau^{\mathrm{o}})$, then since $\mathrm{RS}(w^{-1}) = (x, t)$ and $\mathrm{RS}(y^{-1}) = (v, u)$ by Theorem 6.25, the conclusion of Theorem 8.2 is that if $\mu(y, w) \neq 0$ and $\tau^{\mathrm{o}}(y) \nsubseteq \tau^{\mathrm{o}}(w)$ then either $v < x$ or else $\pi = \lambda$ and $(u, v) = (t, sx)$ for some $s \in S_n$.

Thus, in particular, if $\mu(y, w) \neq 0$ and $\tau(y) \nsubseteq \tau(w)$ or $\tau^{\mathrm{o}}(y) \nsubseteq \tau^{\mathrm{o}}(w)$ then $\pi \leqslant \lambda$.

It follows from the definition of the preorder $\preceq_{\mathsf{LR}}$ (in Section 4 above) that if $y \preceq_{\mathsf{LR}} w$ then there is a sequence of elements $y = z_0, z_1, \ldots, z_{m-1}, z_m = w$ such that $\mu(z_{i-1}, z_i) \neq 0$ and $\bar{\tau}(z_{i-1}) \nsubseteq \bar{\tau}(z_i)$ for each $i \in [1, m]$. Since $\bar{\tau}(z_{i-1}) \nsubseteq \bar{\tau}(z_i)$ is equivalent to $\tau(z_{i-1}) \nsubseteq \tau(z_i)$ or $\tau^{\mathrm{o}}(z_{i-1}) \nsubseteq \tau^{\mathrm{o}}(z_i)$, it follows that $\pi \leqslant \lambda$ whenever $y \preceq_{\mathsf{LR}} w$.

Remark 8.21. Let $y, w \in W_n$, and let $RS(y) = (u, v)$ and $RS(w) = (t, x)$. Remark 8.20 says that if $y \preceq_{\mathsf{LR}} w$ then $\pi \leqslant \lambda$, where $\pi = \mathrm{Shape}(x) = \mathrm{Shape}(u)$ and $\lambda = \mathrm{Shape}(y) = \mathrm{Shape}(v)$. This gives an alternative approach to the "only if" part of the following well-known result. (See, for example, [7, Theorem 5.1].)

Theorem 8.22. *Let $y, w \in W_n$ and $\pi, \lambda \in P(n)$, and suppose that* $\mathrm{RS}(y) \in \mathrm{Std}(\pi) \times \mathrm{Std}(\pi)$ *and* $\mathrm{RS}(w) \in \mathrm{Std}(\lambda) \times \mathrm{Std}(\lambda)$. *Then $y \preceq_{\mathsf{LR}} w$ if and only if $\pi \leqslant \lambda$. In particular, the sets $D(\lambda) = \{w \in W_n \mid \mathrm{RS}(w) \in \mathrm{Std}(\lambda)^2\}$, where $\lambda \in P(n)$, are precisely the Kazhdan–Lusztig two-sided cells.*

Let $\lambda \in P(n)$. Recall that for each $t \in \mathrm{Std}(\lambda)$ the set $C(t) = \{w \in W_n \mid \mathrm{Q}(w) = t\}$ is a left cell, and $\Gamma(C(t))$ is isomorphic to $\Gamma_\lambda$ (defined in 6.38 above). Thus $C(t)$ is of type $\lambda$, by Remark 6.51, and the set $D(\lambda) = \bigsqcup_{t \in \mathrm{Std}(\lambda)} C(t)$ is a union of $|\mathrm{Std}(\lambda)|$ left cells of type $\lambda$.

## 9. $W$-graphs for admissible cells in type $A$

Definition 9.1. *Let $\lambda \in P(n)$. A pair of standard $\lambda$-tableaux $(u, t)$ is said to be a* probable pair *if $u < t$ and* $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$.

It is readily checked that there are no probable pairs in $P(n)$ unless $n \geqslant 5$. Note that if $(u,t)$ is a probable pair then $u \neq t$, and so the set $F(u,t)$ is defined and nonempty.

Recall Definition 7.16: if $\pi$, $\lambda \in P(n)$ then a pair $(u,t) \in \mathrm{Std}(\pi) \times \mathrm{Std}(\lambda)$ is said to be favourable if the restriction number of $(u,t)$ lies in $\mathrm{D}(u) \oplus \mathrm{D}(t)$.

Lemma 9.2. *Let $\lambda \in P(n)$ and $u,t \in \mathrm{Std}(\lambda)$, and let $i$ be the restriction number of $(u,t)$. Suppose that $(u,t)$ is favourable and $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$. Then there exists $j \in \mathrm{SD}(t)$ such that $j > i$.*

*Proof.* Suppose that there is no such $j$, so that $\mathrm{D}(t_{>i}) \cap [i+1, n-1] = \mathrm{WD}(t_{>i}) \cap [i+1, n-1]$. Thus $t_{>i}$ is the minimal tableau of its shape, by Remark 6.21, and so $\mathrm{col}_t(k) \geqslant \mathrm{col}_t(i+1)$ for all $k \in [i+1, n]$. Furthermore, $\mathrm{D}(u) \oplus \mathrm{D}(t) = \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$, since $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$, and so $i \in \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$, since $(u,t)$ is favourable. Thus $\mathrm{col}_t(i+1) > \mathrm{col}_t(i)$, since $i \notin \mathrm{D}(t)$, and so $\mathrm{col}_t(k) > \mathrm{col}_t(i)$ for all $k > i$. In other words, for all $m \in [1, \mathrm{col}_t(i)]$, column $m$ of $t$ is entirely filled by numbers from the set $[1,i]$. But now since $u$ and $t$ have the same shape and $u_{\leqslant i} = t_{\leqslant i}$, it follows that the same holds for $u$: for all $m \in [1, \mathrm{col}_u(i)]$, column $m$ of $u$ is entirely filled by numbers from the set $[1,i]$. In particular, $\mathrm{col}_u(i+1) > \mathrm{col}_u(i)$, contradicting $i \in \mathrm{D}(u)$. □

Lemma 9.3. *Let $\lambda \in P(n)$ and $u,t \in \mathrm{Std}(\lambda)$ with $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$, and suppose that $(u,t)$ is favourable. Let $i$ be the restriction number of $(u,t)$, and suppose that $i+1 = \max(\mathrm{SD}(t))$. Then $\mathrm{col}_t(i+2) \neq \mathrm{col}_t(i)$.*

*Proof.* Suppose to the contrary that $\mathrm{col}_t(i+2) = \mathrm{col}_t(i)$. We have that $u_{\leqslant i} = t_{\leqslant i}$, since $(u,t)$ is $i$-restricted. Since $i+1 = \max(\mathrm{SD}(t))$, it follows $\mathrm{SD}(t_{>(i+1)}) = \varnothing$, and thus $t_{>(i+1)}$ is minimal, by Remark 6.21. Hence $\mathrm{col}_t(i) = \mathrm{col}_t(i+2) \leqslant \mathrm{col}_t(k)$ for all $k > i+2$. Furthermore, $i \in \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$, since $(u,t)$ is favourable, and so $\mathrm{col}_t(i) < \mathrm{col}_t(i+1)$. So $\mathrm{col}_t(k) \geqslant \mathrm{col}_t(i)$ for all $k > i$. Now since $u$ and $t$ are of the same shape and $u_{\leqslant i} = t_{\leqslant i}$, we deduce that $\mathrm{col}_u(k) \geqslant \mathrm{col}_u(i)$ for all $k > i$. In particular, $\mathrm{col}_u(i+1) \geqslant \mathrm{col}_u(i)$. But $\mathrm{col}_u(i+1) \leqslant \mathrm{col}_u(i)$, since $i \in \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$, and so $\mathrm{col}_u(i+1) = \mathrm{col}_u(i)$.

Let $(g,p) = t^{-1}(i) = u^{-1}(i)$. Since $i+1$ is in the same column of $u$ as $i$, it follows that $u(g+1,p) = i+1$. Since $i+2$ is in the same column of $t$ as $i$, and $i+1$ is in a different column, it follows that $t(g+1,p) = i+2$. Now let $m$ be the maximal positive integer such that $\mathrm{col}_u(i+l) = p$ for all $l \in [1,m]$, so that $u(g+l,p) = i+l$ for all $l \in [1,m]$. Since $t$ and $u$ have the same shape and $t_{>(i+1)}$ is minimal, we see that $t(g+l,p) = i+l+1$ for all $l \in [1,m]$. If $m > 1$ then $\mathrm{col}_t(i+m+1) = p = \mathrm{col}_t(i+m)$, and it follows that $i+m \in \mathrm{WD}(t)$, while if $m = 1$ then $i+m = i+1 \in \mathrm{SD}(t)$ (since $i+1 = \max(\mathrm{SD}(t))$ is given). In either case it follows that $i+m \in \mathrm{D}(u)$, since $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$. So $\mathrm{col}_u(i{+}m{+}1) \leqslant \mathrm{col}_u(i{+}m) = p = \mathrm{col}_u(i)$, whence $\mathrm{col}_u(i{+}m{+}1) = p$, since it was shown above that $\mathrm{col}_u(k) \geqslant \mathrm{col}_u(i)$ for all $k > i$. But this contradicts the choice of $m$. □

Suppose that $\Gamma = \Gamma(C,\mu,\tau)$ is an admissible $W_n$-graph whose molecules are all of type $\lambda$, for some $\lambda \in P(n)$, and let $\mathcal{I}$ index the molecules. By Remark 6.50, the vertex set of $\Gamma$ can be written as $C = \bigsqcup_{\alpha \in \mathcal{I}} C_\alpha$, where for each $\alpha \in \mathcal{I}$ the set $C_\alpha = \{c_{\alpha,t} \mid t \in \mathrm{Std}(\lambda)\}$ spans a molecule, $\tau(c_{\alpha,t}) = \{\, s_j \mid j \in \mathrm{D}(t)\,\}$ for all $\alpha \in \mathcal{I}$ and $t \in \mathrm{Std}(\lambda)$, and the simple edges of $\Gamma$ are the pairs $\{c_{\beta,u}, c_{\alpha,t}\}$ such that $\alpha = \beta$ and $u$ and $t$ are related by a DKM.

Lemma 9.4. *Let $u,t \in \mathrm{Std}(\lambda)$, and suppose that $(u,t)$ is a probable pair. Let $(v,x)$ be an arbitrary element of the set $F(u,t)$. Then $(v,x)$ is a probable pair, $\max(\mathrm{SD}(x)) = \max(\mathrm{SD}(t))$, and $\mu(c_{\beta,v}, c_{\alpha,x}) = \mu(c_{\beta,u}, c_{\alpha,t})$.*

*Proof.* Note that $t$ is not minimal, since $u < t$. Hence $\mathrm{SD}(t) \neq \varnothing$. Let $i$ be the restriction number of $(u,t)$. Recall that, by the way $F(u,t)$ is defined, $(v,x)$ is $i$-restricted and favourable.

Suppose first that $(u,t)$ is favourable. Then $i < \max(\mathrm{SD}(t))$, by Lemma 9.2, and so $\max(\mathrm{SD}(t)) = \max(\mathrm{SD}(t_{>i}))$. But $\max(\mathrm{SD}(t_{>i})) = \max(\mathrm{SD}(x_{>i}))$, since $t_{>i} = x_{>i}$, and so $\max(\mathrm{SD}(t)) = \max(\mathrm{SD}(x))$. Moreover, $\mathrm{col}_u(i+1) \leqslant \mathrm{col}_u(i) = \mathrm{col}_t(i) < \mathrm{col}_t(i+1)$, since $i \in \mathrm{D}(u) \oplus \mathrm{D}(t) = \mathrm{D}(u) \setminus \mathrm{D}(t)$, and hence $\mathrm{col}_v(i+1) \leqslant \mathrm{col}_v(i) = \mathrm{col}_x(i) < \mathrm{col}_x(i+1)$. Thus $i \in \mathrm{D}(v) \setminus \mathrm{D}(x)$. Clearly $\mathrm{D}(v) \cap [1, i-1] = \mathrm{D}(x) \cap [1, i-1]$, since $v_{\leqslant i} = x_{\leqslant i}$, and since $\mathrm{D}(v) \cap [i+1, n-1] = \mathrm{D}(u) \cap [i+1, n-1]$ and $\mathrm{D}(x) \cap [i+1, n-1] = \mathrm{D}(t) \cap [i+1, n-1]$, it follows from $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$ that $\mathrm{D}(x) \subsetneqq \mathrm{D}(v)$. Since $u < t$, Proposition 7.9 gives $v < x$, and it follows that $(v,x)$ is a probable pair. Finally, by Lemma 7.22 applied with $j = \max(\mathrm{SD}(t))$, we see that $\mu(c_{\beta,v}, c_{\alpha,x}) = \mu(c_{\beta,u}, c_{\alpha,t})$, as required.

Now suppose that $(u,t)$ is not favourable. Again it follows from Proposition 7.9 that $v < x$. Moreover, $u < t$ gives $u_{\leqslant (i+1)} \leqslant t_{\leqslant (i+1)}$, so that $u_{\leqslant (i+1)} < t_{\leqslant (i+1)}$ (since $u_{\leqslant (i+1)} \neq t_{\leqslant (i+1)}$). It follows by Remark 6.13 that $\mathrm{col}_u(i+1) < \mathrm{col}_t(i+1)$. Since $\mathrm{D}(u) \oplus \mathrm{D}(t) = \mathrm{D}(u) \setminus \mathrm{D}(t)$, we have $i < \min(\mathrm{D}(u) \setminus \mathrm{D}(t))$ by Remark 7.18. Thus $(u,t)$ satisfies the hypothesis of Lemma 7.21, and the conclusion is that $\mathrm{D}(v) \setminus \mathrm{D}(x) = \{i\} \cup (\mathrm{D}(u) \setminus \mathrm{D}(t))$ and $\mathrm{D}(x) \setminus \mathrm{D}(v) = \varnothing$. Since $\mathrm{D}(x) \setminus \mathrm{D}(v) = \varnothing$ while $\mathrm{D}(v) \setminus \mathrm{D}(x) \neq \varnothing$, we have $\mathrm{D}(x) \subsetneqq \mathrm{D}(v)$. Hence $(v,x)$ is probable.

Let $j = \max(\mathrm{SD}(x))$, and note that $j > i$ by Lemma 9.2, Thus $j = \max(\mathrm{SD}(x_{>i}))$, and since $t_{>i} = x_{>i}$, it follows that $j = \max(\mathrm{SD}(t_{>i})) = \max(\mathrm{SD}(t))$. Finally, since it was shown above that $i < \min(\mathrm{D}(u) \setminus \mathrm{D}(t))$, Lemma 7.22 again gives $\mu(c_{\beta,v}, c_{\alpha,x}) = \mu(c_{\beta,u}, c_{\alpha,t})$. □

Proposition 9.5. *Monomolecular admissible cells of type $A_{n-1}$ are Kazhdan–Lusztig.*

*Proof.* Suppose that $\Gamma = \Gamma(C, \mu, \tau)$ is a monomolecular admissible $W_n$-cell. Then there is a partition $\lambda$ of $n$ such that $C = \{c_t \mid t \in \mathrm{Std}(\lambda)\}$, and $\{c_u, c_t\}$ is a simple edge of $\Gamma$ if and only if $u,\, t \in \mathrm{Std}(\lambda)$ are related by a DKM. In view of Corollary 6.37, our task is to show that $\Gamma \cong \Gamma_\lambda = \Gamma(\mathrm{Std}(\lambda), \mu^{(\lambda)}, \tau^{(\lambda)})$. Recall that $\Gamma_\lambda$ is an admissible $W_n$-graph consisting of a single molecule of type $\lambda$ (by Remark 6.51). Clearly $t \mapsto c_t$ is a bijection from the vertex set of $\Gamma_\lambda$ to the vertex set of $\Gamma$. Since it follows from Remark 6.50 that $\tau(c_t) = \{\, s_j \mid j \in \mathrm{D}(t)\,\} = \tau^{(\lambda)}(t)$ for all $t \in \mathrm{Std}(\lambda)$, it remains to show that $\mu(c_u, c_t) = \mu^{(\lambda)}(u,t)$ for all $u,\, t \in \mathrm{Std}(\lambda)$. Note that, by Theorem 5.8, both $\Gamma$ and $\Gamma_\lambda$ satisfy the Compatibility Rule, the Simplicity Rule, the Bonding Rule and the Polygon Rule.

We have shown in Theorem 8.2 that $\Gamma$ and $\Gamma_\lambda$ are both ordered. Thus if $u, t \in \mathrm{Std}(\lambda)$ then $\mu(c_u, c_t) = \mu^{(\lambda)}(u,t) = 0$ unless $u < t$ or $u = s_i t > t$ for some $i \in [1, n-1]$. If $u = s_i t > t$ for some $i \in [1, n-1]$ then we have $\mu(c_u, c_t) = \mu^{(\lambda)}(u,t) = 1$ by Corollary 7.23. Now suppose that $u < t$ and $\mathrm{D}(t) \nsubseteq \mathrm{D}(u)$. If one or other of $\mu(c_u, c_t)$ and $\mu^{(\lambda)}(u,t)$ is nonzero then, by the Simplicity Rule, one or other of $\{c_u, c_t\}$ and $\{u, t\}$ is a simple edge, whence $u$ and $t$ are related by a DKM (by Remark 6.50), and both $\{c_u, c_t\}$ and $\{u, t\}$ are simple edges. So $\mu(c_u, c_t) = \mu^{(\lambda)}(u,t) = 1$ in this case. Obviously there is nothing to show if $\mu(c_u, c_t)$ are $\mu^{(\lambda)}(u,t)$ both zero, and so all that remains is to show that $\mu(c_u, c_t) = \mu^{(\lambda)}(u,t)$ whenever $u < t$ and $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$. That is, it remains to show that $\mu(c_u, c_t) = \mu^{(\lambda)}(u,t)$ for all probable pairs of standard $\lambda$-tableaux. In other words we must show that for all $t \in \mathrm{Std}(\lambda)$ we have $\mu(c_u, c_t) = \mu^{(\lambda)}(u,t)$ whenever $u \in \mathrm{Std}(\lambda)$ and $(u,t)$ is a probable pair.

If $t = \tau_\lambda$ then there is nothing to prove. Proceeding inductively on the lexicographic order, let $t' \in \mathrm{Std}(\lambda) \setminus \{\tau_\lambda\}$, and assume that the result holds for all $t \in \mathrm{Std}(\lambda)$ such that $t <_{\mathrm{lex}} t'$.

Suppose that $(u',t')$ is a probable pair. Let $i$ be the restriction number of $(u',t')$, and let $j = \max(\mathrm{SD}(t'))$. Let $(u,t) \in F(u',t')$, and note that $(u,t)$ is $i$-restricted and favourable, and satisfies $t_{>i} = t'_{>i}$ and $u_{>i} = u'_{>i}$. Moreover, Lemma 9.4 shows that $(u,t)$ is probable, $\mu(c_u, c_t) = \mu(c_{u'}, c_{t'})$ and $\mu^{(\lambda)}(u,t) = \mu^{(\lambda)}(u',t')$, and $\max(\mathrm{SD}(t)) = \max(\mathrm{SD}(t')) = j$. Thus the conclusion $\mu(c_{u'}, c_{t'}) = \mu^{(\lambda)}(u',t')$ will follow if we can show that $\mu(c_u, c_t) = \mu^{(\lambda)}(u,t)$.

Note that $i \in \mathrm{A}(t)$ and $i < j$, by Lemma 9.2. Let $v = s_j t$. Then $v \in \mathrm{Std}(\lambda)$ and $v < t$, since $j \in \mathrm{SD}(t)$, and $v <_{\mathrm{lex}} t'$ by Lemma 7.30. Since $(u,t)$ is favourable, and since $\mathrm{D}(t) \subsetneq \mathrm{D}(u)$ (since $(u,t)$ is probable), we have $i \in \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$ and $j \in D(u) \cap \mathrm{D}(t)$. That is, $i \notin \mathrm{D}(t)$ and $j \in \mathrm{D}(t)$, and $i, j \in \mathrm{D}(u)$. Since $i < j$ we have either $j - i > 1$ or $j - i = 1$.

*Case 1.*
Suppose that $j - i > 1$, so that $m(s_i, s_j) = 2$. Lemma 7.29 (i) tells us that $i, j \notin \mathrm{D}(v)$. We set $X = \{\, x \in \mathrm{Std}(\lambda) \mid i \in \mathrm{D}(x) \text{ and } j \notin \mathrm{D}(x) \,\}$ and $Y = \{\, y \in \mathrm{Std}(\lambda) \mid j \in \mathrm{D}(y) \text{ and } i \notin \mathrm{D}(y) \,\}$.

If $(c_v, c_{y_1}, c_u)$ is any alternating directed path of type $(j,i)$, then, since $\Gamma$ is ordered, it follows that either $y_1 = s_j v = t > v$ or $y_1 < v$. Similarly, if $(c_v, c_{x_1}, c_u)$ is any alternating directed path of type $(i,j)$, then it follows that either $x_1 = s_i v > v$ or $x_1 < v$. Note that if $x_1 = s_i v > v$, then since $x_1 \in \mathrm{Std}(\lambda)$, it follows that $i \in \mathrm{SA}(v)$. Thus, if $x_1 = s_i v > v$, then $i \in \mathrm{D}(s_i v)$ and $j \notin \mathrm{D}(s_i v)$ by Lemma 7.29 (i). That is, $s_i v \in X$. Now since $\Gamma$ satisfies the Polygon Rule, we have $N^2_{j,i}(c_v, c_u) = N^2_{i,j}(c_v, c_u)$, and it follows that

$$\begin{aligned}(22)\quad \mu(c_u,c_t)\mu(c_t,c_v) \;+ \sum_{y_1\in Y,\ y_1<v} \mu(c_u,c_{y_1})\mu(c_{y_1},c_v) \\ = \mu(c_u,c_{s_iv})\mu(c_{s_iv},c_v) \;+ \sum_{x_1\in X,\ x_1<v} \mu(c_u,c_{x_1})\mu(c_{x_1},c_v),\end{aligned}$$

where the term $\mu(c_u, c_{s_i v})\mu(c_{s_i v}, c_v)$ on the right hand side of Eq. (22) should be omitted if $i \notin \mathrm{SA}(v)$. Note that if $i \in \mathrm{SA}(v)$ then $(c_v, c_{s_i v}, c_u)$ is not necessarily a directed path, since there need not be an arc from $s_i v$ to $u$, but in this case $\mu(c_u, c_{s_i v})\mu(c_{s_i v}, c_v) = 0$ since $\mu(c_u, c_{s_i v}) = 0$. Similarly, $(c_v, c_t, c_u)$ is not necessarily a directed path, since there need not be an arc from $t$ to $u$, but $\mu(c_u, c_t)\mu(c_t, c_v) = 0$ in this case. So Eq. (22) still holds in these cases.

Since Corollary 7.23 gives $\mu(c_t, c_v) = 1$, and $\mu(c_{s_i v}, c_v) = 1$ if $i \in \mathrm{SA}(v)$, Eq. (22) yields the following formula for $\mu(c_u, c_t)$:

$$\mu(c_u,c_t) = \mu(c_u,c_{s_iv}) \;+ \sum_{x_1\in X,\ x_1<v} \mu(c_u,c_{x_1})\mu(c_{x_1},c_v) \;- \sum_{y_1\in Y,\ y_1<v} \mu(c_u,c_{y_1})\mu(c_{y_1},c_v),$$

where $\mu(c_u, c_{s_i v})$ should be interpreted as 0 if $s_i v \notin \mathrm{Std}(\lambda)$.

Working similarly on $\Gamma_\lambda$ yields the following formula for $\mu^{(\lambda)}(u,t)$:

$$\mu^{(\lambda)}(u,t) = \mu^{(\lambda)}(u,s_iv) + \sum_{x_1\in X,\ x_1<v} \mu^{(\lambda)}(u,x_1)\mu^{(\lambda)}(x_1,v) - \sum_{y_1\in Y,\ y_1<v} \mu^{(\lambda)}(u,y_1)\mu^{(\lambda)}(y_1,v).$$

Now $v <_{\mathrm{lex}} t'$ by Lemma 7.30 and $s_i v <_{\mathrm{lex}} t'$ (if $i \in \mathrm{SA}(v)$) by Lemma 7.30 (i). Moreover, all $x_1$ and $y_1$ appearing above satisfy $x_1 <_{\mathrm{lex}} t'$ and $y_1 <_{\mathrm{lex}} t'$, by Lemma 7.30 (ii). Hence it follows by the inductive hypothesis that the corresponding arc weights in the two formulae above are the same. Thus $\mu(c_u, c_t) = \mu^{(\lambda)}(u,t)$, and $\mu(c_{u'}, c_{t'}) = \mu^{(\lambda)}(u',t')$, as desired.

*Case 2.*
Suppose that $i = j - 1$, so that $m(s_i, s_j) = 3$. By Lemma 9.3, $\mathrm{col}_t(j-1) \neq \mathrm{col}_t(j+1)$, and so either $\mathrm{col}_t(j-1) < \mathrm{col}_t(j+1)$ or $\mathrm{col}_t(j-1) > \mathrm{col}_t(j+1)$.

If $\mathrm{col}_t(j-1) < \mathrm{col}_t(j+1)$, then the result follows by the same argument as used in Case 1, with $j-1$ replacing $i$ and using Lemma 7.29 (ii) in place of Lemma 7.29 (i).

Suppose that $\mathrm{col}_t(j-1) > \mathrm{col}_t(j+1)$. Since $j-1 \in \mathrm{SD}(v)$ by Lemma 7.29 (iii), we have $s_{j-1}v \in \mathrm{Std}(\lambda)$ and $s_{j-1}v < v$. Let $w = s_{j-1}v$. It follows by Lemma 7.29 (iii) that $j-1, j \notin \mathrm{D}(w)$, whereas $j-1, j \in \mathrm{D}(u)$ (as we have already seen).

We consider directed paths from $c_w$ to $c_u$ that have length three and are alternating of type $(j-1, j)$ or type $(j, j-1)$. We have $j \in \mathrm{D}(t)$ and $j-1 \notin \mathrm{D}(t)$ (as we have already seen), while $j-1 \in \mathrm{D}(v)$ and $j \notin \mathrm{D}(v)$ by Lemma 7.29 (iii).

If $(c_w, c_{x_1}, c_{x_2}, c_u)$ is any alternating directed path of type $(j-1, j)$, then, since $\Gamma$ is ordered, it follows that either $x_1 = s_{j-1}w = v > w$, or else $x_1 < w$. Moreover, since $\Gamma$ satisfies the Simply-Laced Bonding Rule, the fact that $j-1 \in \mathrm{D}(x_1)$ and $j \notin \mathrm{D}(x_1)$ shows that $c_{x_2}$ is the unique vertex adjacent to $c_{x_1}$ satisfying $j-1 \notin \mathrm{D}(x_2)$ and $j \in \mathrm{D}(x_2)$. That is, $x_2$ is the $(j-1)$-neighbour of $x_1$. Thus it follows that either $x_1 = v$ and $x_2 = s_j v = t$, or else $x_1 < w$ and either $x_2 = s_j x_1 > x_1$ or $x_2 = s_{j-1}x_1 < x_1$.

Similarly, if $(c_w, c_{y_1}, c_{y_2}, c_u)$ is any alternating directed path of type $(j, j-1)$, then it follows that either $y_1 = s_j w > w$ or $y_1 < w$, and $y_2$ is the $(j-1)$-neighbour of $y_1$. Note that if $y_1 = s_j w > w$, then since $y_1 \in \mathrm{Std}(\lambda)$, it follows that $j \in \mathrm{SA}(w)$. Thus, if $y_1 = s_j w > w$ then Lemma 7.29 (iii) tells us that $y_2 = s_{j-1}y_1 = s_{j-1}s_j w > s_j w = y_1$, and also that $j \in \mathrm{D}(s_j w)$ and $j-1 \notin \mathrm{D}(s_j w)$, and that $j-1 \in \mathrm{D}(s_{j-1}s_j w)$ and $j \notin \mathrm{D}(s_{j-1}s_j w)$. On the other hand if $y_1 < w$ then either $y_2 = s_{j-1}y_1 > y_1$ or $y_2 = s_j y_1 < y_1$.

Now since $\Gamma$ satisfies the Polygon Rule, we have $N^3_{j-1,j}(c_w, c_u) = N^3_{j,j-1}(c_w, c_u)$, and it follows that

$$\begin{aligned}(23)\quad \mu(c_u, c_t)\mu(c_t, c_v)\mu(c_v, c_w) &+ \sum_{\substack{x_1 \in X,\ x_1 < w\\ x_2 = (j-1)\text{-neb}(x_1)}} \mu(c_u, c_{x_2})\mu(c_{x_2}, c_{x_1})\mu(c_{x_1}, c_w)\\ &= \mu(c_u, c_{s_{j-1}s_j w})\mu(c_{s_{j-1}s_j w}, c_{s_j w})\mu(c_{s_j w}, c_w)\\ &\qquad + \sum_{\substack{y_1 \in Y,\ y_1 < w\\ y_2 = (j-1)\text{-neb}(y_1)}} \mu(c_u, c_{y_2})\mu(c_{y_2}, c_{y_1})\mu(c_{y_1}, c_w),\end{aligned}$$

where the term $\mu(c_u, c_{s_{j-1}s_j w})\mu(c_{s_{j-1}s_j w}, c_{s_j w})\mu(c_{s_j w}, c_w)$ on the right hand side of Eq. (23) should be omitted if $j \notin \mathrm{SA}(w)$. Note that if $j \in \mathrm{SA}(w)$ then $(c_w, c_{s_j w}, c_{s_{j-1}s_j w}, c_u)$ is not necessarily a directed path, since there need not be an arc from $c_{s_{j-1}s_j w}$ to $c_u$, but in this case $\mu(c_u, c_{s_{j-1}s_j w})\mu(c_{s_{j-1}s_j w}, c_{s_j w})\mu(c_{s_j w}, c_w) = 0$ since $\mu(c_u, c_{s_{j-1}s_j w}) = 0$. Similarly, $(c_w, c_v, c_t, c_u)$ is not necessarily a directed path, since there need not be an arc from $c_t$ to $c_u$, but $\mu(c_u, c_t)\mu(c_t, c_v)\mu(c_v, c_w) = 0$ in this case. So Eq. (23) still holds in these cases.

Since $\mu(c_v, c_w) = \mu(c_{s_j w}, c_w) = 1$ and $\mu(c_t, c_v) = \mu(c_{s_{j-1}s_j w}, c_{s_j w}) = 1$, by Corollary 7.23, and since $\mu(c_{x_2}, c_{x_1}) = \mu(c_{y_2}, c_{y_1}) = 1$, since $\{c_{x_1}, c_{x_2}\}$ and $\{c_{y_1}, c_{y_2}\}$ are simple edges, Eq. (23) yields the following formula for $\mu(c_u, c_t)$:

$$\begin{aligned}\mu(c_u, c_t) = \mu(c_u, c_{s_{j-1}s_j w}) &+ \sum_{\substack{y_1 \in Y,\ y_1 < w\\ y_2 = (j-1)\text{-neb}(y_1)}} \mu(c_u, c_{y_2})\mu(c_{y_1}, c_w)\\ &\qquad - \sum_{\substack{x_1 \in X,\ x_1 < w\\ x_2 = (j-1)\text{-neb}(x_1)}} \mu(c_u, c_{x_2})\mu(c_{x_1}, c_w),\end{aligned}$$

where $\mu(c_u, c_{s_{j-1}s_j w})$ should be interpreted as 0 if $s_j w \notin \mathrm{Std}(\lambda)$.

Working similarly on $\Gamma_\lambda$ yields the following formula for $\mu^{(\lambda)}(u,t)$:

$$\mu^{(\lambda)}(u,t) = \mu^{(\lambda)}(u, s_{j-1}s_j w) + \sum_{\substack{y_1\in Y,\ y_1<w \\ y_2=(j-1)\text{-neb}(y_1)}} \mu^{(\lambda)}(u,y_2)\mu^{(\lambda)}(y_1,w) - \sum_{\substack{x_1\in X,\ x_1<w \\ x_2=(j-1)\text{-neb}(x_1)}} \mu^{(\lambda)}(u,x_2)\mu^{(\lambda)}(x_1,w).$$

Now $w <_{\text{lex}} t'$ and $s_{j-1}s_j w <_{\text{lex}} t'$ (if $j \in \mathrm{SA}(w)$) by Lemma 7.30 (iii). Moreover, $x_2$, $y_2 <_{\text{lex}} t'$ by Lemma 7.30 (iv). Hence it follows by the inductive hypothesis that the corresponding arc weights in the two formulae above are the same. Thus $\mu(c_{u'}, c_{t'}) = \mu^{(\lambda)}(u',t')$, as desired. ☐

Proposition 9.6. *Let $\Gamma = \Gamma(C,\mu,\tau)$ be an admissible $W_n$-graph whose molecules are all of type $\lambda$, for some $\lambda \in P(n)$. Then there are no arcs between distinct molecules, and each of the molecules is isomorphic to $\Gamma_\lambda$.*

*Proof.* Slightly modifying the notation used in Theorem 8.1, let $\mathcal{I}$ be a set that indexes the molecules of $\Gamma$, and for each $\alpha \in \mathcal{I}$ let $C_\alpha$ be the vertex set of the corresponding molecule. We also write $C_\alpha = \{\, c_{\alpha,t} \mid t \in \mathrm{Std}(\lambda)\,\}$, and let $\Gamma_\alpha = \Gamma(C_\alpha)$ denote the molecule spanned by $C_\alpha$. If $\Gamma = \bigoplus_{\alpha\in\mathcal{I}} \Gamma_\alpha$, the direct sum of the $\Gamma_\alpha$, then each $\Gamma_\alpha$ is a monomolecular admissible $W_n$-cell of type $\lambda$, and hence isomorphic to $\Gamma_\lambda$, by Proposition 9.5. Hence it will suffice to show that $\Gamma = \bigoplus_{\alpha\in\mathcal{I}} \Gamma_\alpha$.

Suppose otherwise. Then there exists $\alpha \in \mathcal{I}$ such that $\mathrm{Ini}_\Gamma(\alpha) \neq \varnothing$, where

$$\mathrm{Ini}_\Gamma(\alpha) = \{\, t \in \mathrm{Std}(\lambda) \mid \mu(c_{\beta,u}, c_{\alpha,t}) \neq 0 \text{ for some } (\beta,u) \in (\mathcal{I} \smallsetminus \{\alpha\}) \times \mathrm{Std}(\lambda)\}.$$

Fix such an $\alpha$, and let $t'$ be the element of $\mathrm{Ini}_\alpha(\Gamma)$ that is minimal in the lexicographic order on $\mathrm{Std}(\lambda)$. Choose $(\beta,u') \in (\mathcal{I} \smallsetminus \{\alpha\}) \times \mathrm{Std}(\lambda)$ with $\mu(c_{\beta,u'}, c_{\alpha,t'}) \neq 0$. Since $\Gamma$ satisfies the Simplicity Rule (by Theorem 5.8), the assumption that $\alpha \neq \beta$ and $\mu(c_{\beta,u'}, c_{\alpha,t'}) \neq 0$ implies that $\mathrm{D}(t') \subsetneqq \mathrm{D}(u')$. Moreover, since $\Gamma$ is ordered (by Theorem 8.2), $\alpha \neq \beta$ implies that $u' < t'$. Hence $(u',t')$ is a probable pair.

Let $i$ be the restriction number of $(u',t')$ and $j = \max(\mathrm{SD}(t'))$. Let $(u,t) \in F(u',t')$, so that $(u,t)$ is $i$-restricted and favourable, and Lemma 9.4 shows that $(u,t)$ is a probable pair, $\max(\mathrm{SD}(t)) = \max(\mathrm{SD}(t')) = j$, and $\mu(c_{\beta,u}, c_{\alpha,t}) = \mu(c_{\beta,u'}, c_{\alpha,t'}) \neq 0$. Since $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$ (since $(u,t)$ is probable) and $i \in \mathrm{D}(u) \oplus \mathrm{D}(t)$ (since $(u,t)$ is favourable), $i \in \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$. Thus, since $j \in \mathrm{D}(t) \subsetneqq \mathrm{D}(u)$, it follows that $j \in \mathrm{D}(t)$ and $i \notin \mathrm{D}(t)$, and $i$, $j \in \mathrm{D}(u)$. Let $v = s_j t$, and note that $v \in \mathrm{Std}(\lambda)$ and $v < t$, by Remark 6.19. Lemma 9.2 yields that $i < j$.

Suppose first that $i < j-1$. Then Lemma 7.29 (i) yields that $i$, $j \notin \mathrm{D}(v)$. Moreover, since $\mu(c_{\alpha,t}, c_{\alpha,v}) = 1$ by Corollary 7.23, and since $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$, it follows that $(c_{\alpha,v}, c_{\alpha,t}, c_{\beta,u})$ is an alternating directed path of type $(j,i)$. Thus since $\Gamma$ is admissible it follows that $N^2_{j,i}(c_{\alpha,v}, c_{\beta,u}) > 0$, and so $N^2_{i,j}(c_{\alpha,v}, c_{\beta,u}) > 0$, since $\Gamma$ satisfies the Polygon Rule. Thus there exists at least one $(\delta,x_1) \in \mathcal{I} \times \mathrm{Std}(\lambda)$ such that $(c_{\alpha,v}, c_{\delta,x_1}, c_{\beta,u})$ is an alternating directed path of type $(i,j)$. If $\delta \neq \alpha$ then $v \in \mathrm{Ini}_\Gamma(\alpha)$, and since $(t')_{>i} = t_{>i}$ we have $v <_{\text{lex}} t'$, by Lemma 7.30, contradicting the definition of $t'$. So we must have $\delta = \alpha$, whence $x_1 \in \mathrm{Ini}_\Gamma(\alpha)$. Now Theorem 8.2 shows that either $x_1 = s_i v$ and $i \in \mathrm{SA}(v)$, or else $x_1 < v$. But $x_1 <_{\text{lex}} t'$ by Lemma 7.30 (i) in the former case, and $x_1 <_{\text{lex}} t'$ by Lemma 7.30 (ii) in the latter case. Both alternatives contradict the definition of $t'$, and we conclude that $i < j-1$ is impossible.

Thus $i = j-1$, and it now follows from Lemma 9.3 that $\mathrm{col}_t(j-1) \neq \mathrm{col}_t(j+1)$. But if $\mathrm{col}_t(j-1) < \mathrm{col}_t(j+1)$ then we obtain a contradiction by the same reasoning as in the $i < j-1$ case, using Lemma 7.29 (ii) in place of Lemma 7.29 (i). So $\mathrm{col}_t(j-1) > \mathrm{col}_t(j+1)$.

Let $w = s_{j-1}v$. Lemma 7.29 (iii) gives $j-1 \in \mathrm{SD}(v)$, whence $w \in \mathrm{Std}(\lambda)$ and $w < v$. Moreover, Lemma 7.29 (iii) also gives $j-1, j \notin \mathrm{D}(w)$, as well as $j-1 \in \mathrm{D}(v)$ and $j \notin \mathrm{D}(v)$. Next, since $\mu(c_{\alpha,v}, c_{\alpha,w}) = \mu(c_{\alpha,t}, c_{\alpha,v}) = 1$ by Corollary 7.23 and since $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$, it follows that $(c_{\alpha,w}, c_{\alpha,v}, c_{\alpha,t}, c_{\beta,u})$ is an alternating directed path of type $(j-1, j)$. Thus $N^3_{j-1,j}(c_{\alpha,w}, c_{\beta,u}) > 0$, and so $N^3_{j,j-1}(c_{\alpha,w}, c_{\beta,u}) > 0$, by the Polygon Rule.

It follows that there exists at least one $(\delta, x_1) \in \mathcal{I} \times \mathrm{Std}(\lambda)$ and one $(\gamma, x_2) \in \mathcal{I} \times \mathrm{Std}(\lambda)$ such that $(c_{\alpha,w}, c_{\delta,x_1}, c_{\gamma,x_2}, c_{\beta,u})$ is an alternating directed path of type $(j, j-1)$. If $\delta \neq \alpha$ then $w \in \mathrm{Ini}_\alpha(\Gamma)$, and since $t_{>(j-1)} = t'_{>(j-1)}$, we have $w <_{\mathrm{lex}} t'$ by Lemma 7.30 (iii). But this contradicts the definition of $t'$, and so $\delta = \alpha$.

Since $\mathrm{D}(x_1) \cap \{j-1, j\} = \{j\}$ and $\mathrm{D}(x_2) \cap \{j-1, j\} = \{j-1\}$, and $\mu(c_{\gamma,x_2}, c_{\delta,x_1}) \neq 0$, it follows from the Simplicity Rule that $\{c_{\delta,x_1}, c_{\gamma,x_2}\}$ is a simple edge. Thus $\gamma = \delta$, and $x_1$ and $x_2$ are related by a DKM. Thus $x_2$ is the $(j-1)$-neighbour of $x_1$, and $x_2 \in \mathrm{Ini}_\alpha(\Gamma)$. It will suffice to show that $x_2 <_{\mathrm{lex}} t'$, contradicting the definition of $t'$.

By Theorem 8.2 either $x_1 = s_j w > w$ or $x_1 < w$. If $x_1 < w$ then since $t_{>(j-1)} = (t')_{>(j-1)}$, the conclusion $x_2 <_{\mathrm{lex}} t'$ follows from Lemma 7.30 (iv). We are left with the case $x_1 = s_j w > w$. This gives $j \in \mathrm{SA}(w)$, and we see that the conditions of Lemma 7.29 (iii) are satisfied: we have $v = s_j t$ with $j \in \mathrm{SD}(t)$ and $\mathrm{col}_t(j+1) < \mathrm{col}_t(j-1)$, and $w = s_{j-1}v$. Since $j \in \mathrm{SA}(w)$ it follows that $j-1 \in \mathrm{SA}(x_1)$, and $s_{j-1}x_1$ is the $(j-1)$-neighbour of $x_1$. Thus $x_2 = s_{j-1}x_1 = s_{j-1}s_j w$, and since $t_{>(j-1)} = (t')_{>(j-1)}$, we have $x_2 <_{\mathrm{lex}} t'$ by Lemma 7.30 (iii). □

Remark 9.7. In the above proof, after noting that $\mu(c_{\beta,u'}, c_{\alpha,t'}) \neq 0$ and $\alpha \neq \beta$, Remark 8.10 can be used to deduce that $A(u', t') \neq \varnothing$, and then, choosing $(u, t) \in A(u', t')$, a proof similar to that of Proposition 8.13 (with $t'$ replacing $t_\lambda$) shows that $t_{\geqslant i}$ is $i$-critical. But then Lemma 9.2 and Definition 6.22 combined show that $i+1 = \max(\mathrm{SD}(t))$ and $\mathrm{col}_t(i+2) = \mathrm{col}_t(i)$, and this contradicts Lemma 9.3.

We are now in a position to state and prove the main result of the paper.

Theorem 9.8. *Admissible cells of type $A_{n-1}$ are Kazhdan–Lusztig.*

*Proof.* Let $\Gamma = \Gamma(C, \mu, \tau)$ be an admissible $W_n$-cell, and let $\Lambda$ be the set of molecule types for $\Gamma$. By Lemma 8.18, $\Lambda = \{\lambda\}$ for some $\lambda \in P(n)$. Let $\mathcal{I}$ index the molecules of $\Gamma$, and for each $\gamma \in \mathcal{I}$ let $C_\gamma = \{c_{\gamma,w} \mid w \in \mathrm{Std}(\lambda)\}$ be the vertex set of the corresponding molecule. Then Proposition 9.6 says that $\Gamma = \bigoplus_{\gamma \in \mathcal{I}} \Gamma(C_\gamma)$, with each $\Gamma(C_\gamma)$ isomorphic to $\Gamma_\lambda$. But $\Gamma$ must be connected, since it is a cell, and so $\mathcal{I}$ has only one element. Thus $\Gamma$ is isomorphic to $\Gamma_\lambda$. Since $\Gamma_\lambda$ is isomorphic to $\Gamma(C(\tau_\lambda))$, it follows from Corollary 6.37 that $\Gamma$ is isomorphic to a Kazhdan–Lusztig left cell. □

Remark 9.9. Let $\lambda \in P(n)$ and let $D(\lambda) = \bigsqcup_{t \in \mathrm{Std}(\lambda)} C(t)$, the Kazhdan–Lusztig two-sided cell corresponding to $\lambda$. By Remark 8.21, the singleton set $\{\lambda\}$ is the set of molecule types of the admissible $W_n$-graph $\Gamma(D(\lambda))$. It follows from Proposition 9.6 that $\Gamma(D(\lambda))$ is a disjoint union of the Kazhdan–Lusztig left cells $\Gamma(C(t))$. This implies the following well known result (see, for example, [8, Theorem 5.3]).

Theorem 9.10. *Let $\lambda \in P(n)$ and $y, w \in D(\lambda)$. If $y \preceq_{\mathsf{L}} w$ then $y, w \in C(t)$ for some $t \in \mathrm{Std}(\lambda)$.*

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Van Minh Nguyen, School of Mathematics and Statistics, University of Sydney, NSW 2006, Australia
*E-mail :* `Van.Nguyen@sydney.edu.au`