# ALGEBRAIC COMBINATORICS

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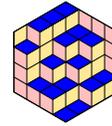
# Classification of Coxeter groups with finitely many elements of **a**-value 2

R. M. Green & Tianyuan Xu

ABSTRACT We consider Lusztig's **a**-function on Coxeter groups (in the equal parameter case) and classify all Coxeter groups with finitely many elements of **a**-value 2 in terms of Coxeter diagrams.

## 1. INTRODUCTION

This paper concerns Lusztig's **a**-function on Coxeter groups. The **a**-function was first defined for finite Weyl groups via their Hecke algebras by Lusztig in [20]; subsequently, the definition was extended to affine Weyl groups in [21] and to arbitrary Coxeter groups in [22]. The **a**-function is intimately related to the study of Kazhdan–Lusztig cells in Coxeter groups, the construction of Lusztig's asymptotic Hecke algebras, and the representation theory of Hecke algebras; see, for example, [20], [21], [22], [12] and [11].

For any Coxeter group $W$ and $w \in W$, $\mathbf{a}(w)$ is a non-negative integer obtained from the structure constants of the Kazhdan–Lusztig basis of the Hecke algebra of $W$. While **a**-values are often difficult to compute directly, it is known that $\mathbf{a}(w) = 0$ if and only if $w$ is the identity element and that $\mathbf{a}(w) = 1$ if and only if $w$ is a non-identity element with a unique reduced word (see Proposition 2.2). If we define $W$ to be $\mathbf{a}(n)$-*finite* for $n \in \mathbb{Z}_{\geqslant 0}$ if $W$ contains finitely many elements of **a**-value $n$ and $\mathbf{a}(n)$-*infinite* otherwise, then it is also known that $W$ is $\mathbf{a}(1)$-finite if and only if each connected component of the Coxeter diagram of $W$ is a tree and contains at most one edge of weight higher than 3 (see Proposition 2.3). The goal of this paper is to obtain a similar classification of $\mathbf{a}(2)$-finite Coxeter groups in terms of Coxeter diagrams.

Our interest in $\mathbf{a}(2)$-finite Coxeter groups comes from considerations about the asymptotic Hecke algebra $J$ of $W$. This is an associative algebra which may be viewed as a "limit" of the Hecke algebra of $W$, and each two-sided Kazhdan–Lusztig cell $E \subseteq W$ gives rise to a subalgebra $J_E$ of $J$ (see [22], Section 18). While $J$ has been interpreted geometrically for Weyl and affine Weyl groups by Bezrukavnikov et al. in [1], [2] and [3], it is not well understood for other Coxeter groups, and one approach to understand $J$ in these cases is to start with the subalgebras $J_E$ in the case $E$ is finite, whence $J_E$ is a *multi-fusion ring* in the sense of [9]. As the **a**-function is known to be constant on each cell, the presence of $\mathbf{a}(2)$-finite groups in our classification

that are not Weyl groups or affine Weyl groups potentially offers interesting examples of multi-fusion rings of the form $J_E$ where $E$ is a cell of $\mathbf{a}$-value 2. (For a study of algebras of the form $J_E$ where $E$ is a cell of $\mathbf{a}$-value 1, see [28].)

We now state our main results. For any (undirected) graph $G$, we define a *cycle* in $G$ to be a sequence $C = (v_1, v_2, \ldots, v_n, v_1)$ involving $n$ distinct vertices such that $n \geqslant 3$ and $\{v_1, v_2\}, \ldots, \{v_{n-1}, v_n\}$ and $\{v_n, v_1\}$ are all edges in $G$, and we say $G$ is *acyclic* if it contains no cycle. Our first main theorem is the following.

THEOREM 1.1. *Let $W$ be an irreducible Coxeter group with Coxeter diagram $G$.*

(1) *If $G$ contains a cycle, then $W$ is $\mathbf{a}(2)$-finite if and only if $G$ is a complete graph.*

(2) *If $G$ is acyclic, then $W$ is $\mathbf{a}(2)$-finite if and only if $G$ is one of the graphs in Figure 1, where $n$ denotes the number of vertices in a graph whenever it appears as a subscript in the label of the graph and there are $q$ and $r$ vertices strictly to the left and the right of the trivalent vertex in $E_{q,r}$.*
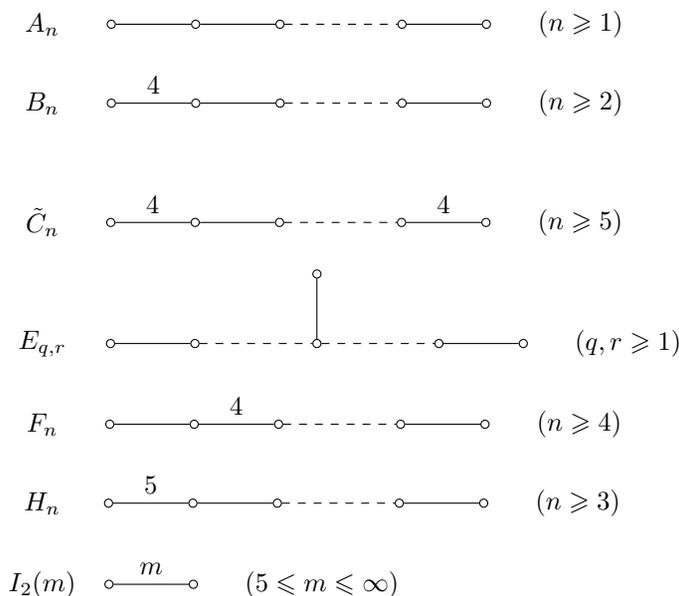


FIGURE 1. Irreducible $\mathbf{a}(2)$-finite Coxeter groups with acyclic diagrams.

REMARK 1.2. When $q = 1$, $E_{q,r}$ coincides with the Coxeter diagram for the Weyl group $D_{r+3}$. When $q = 2$ and $r = 2, 3, 4$, $E_{q,r}$ coincides with the Coxeter diagram of the Weyl group $E_6, E_7$ and $E_8$, respectively. More generally, for any larger value of $r$, $E_{2,r}$ coincides with $E_{r+4}$ in the notation of [24], which is considered an extension of the type $E$ Coxeter diagrams. In Section 4.2, we will recall a result from [24] which uses the notations $D_n$ and $E_n$.

Our second main theorem reduces the classification of reducible $\mathbf{a}(2)$-finite Coxeter groups to that of irreducible Coxeter groups in the following sense:

THEOREM 1.3. *Let $W$ be a reducible Coxeter group with Coxeter diagram $G$. Let $G_1, G_2, \ldots, G_n$ be the connected components of $G$, and let $W_1, W_2, \ldots, W_n$ be their corresponding Coxeter groups, respectively. Then the following are equivalent.*

(1) *$W$ is $\mathbf{a}(2)$-finite.*

(2) *The number $n$ is finite, i.e. $G$ has finitely many connected components, and $W_i$ is both **a**(1)-finite and **a**(2)-finite for each $1 \leqslant i \leqslant n$.*

(3) *The number $n$ is finite, and for each $1 \leqslant i \leqslant n$, $G_i$ is a graph of the form $A_n(n \geqslant 1), B_n(n \geqslant 2), E_{q,r}(q, r \geqslant 1), F_n(n \geqslant 4), H_n(n \geqslant 3)$ or $I_2(m)(5 \leqslant m \leqslant \infty)$, i.e. $G_i$ is a graph from Figure 1 other than $\tilde{C}_n(n \geqslant 5)$.*

Of the claims in the theorems, Part (2) of Theorem 1.1, i.e. the classification of **a**(2)-finite Coxeter groups with acyclic Coxeter diagrams, turns out to require the most amount of work. We describe our strategy for its proof below. A key fact we shall use is that each element of **a**-value 2 in a Coxeter group must be *fully commutative* in the sense of Stembridge (see Section 3.2). This implies, in particular, that we may associate to any element $w$ with $\mathbf{a}(w) = 2$ a poset called its *heap*, a notion well-defined for any fully commutative element. Heaps of fully commutative elements will be a fundamental tool for this paper.

In showing that $W$ is **a**(2)-finite if $G$ is a graph in Figure 1, the full commutativity of elements of **a**-value 2 will reduce our work to the cases $G = I_2(\infty), G = \tilde{C}_n$ or $G = E_{q,r}$ where $\min(q, r) \geqslant 3$. Indeed, thanks to a result of Stembridge's in [24], $W$ contains finitely many fully commutative elements if $G$ is any other graph from Figure 1, so $W$ must be **a**(2)-finite in these cases. It will be easy to show that $W$ is **a**(2)-finite when $G = I_2(\infty)$, and the case $G = \tilde{C}_n$ will also be easy thanks to a result of Ernst from [8] on the generalized Temperley–Lieb algebra of type $\tilde{C}_n$, therefore the only case requiring more work is $G = E_{q,r}$ where $\min(q, r) \geqslant 3$. We will prove $W$ is **a**(2)-finite in this case via a series of lemmas in Section 4.2, using arguments that involve heaps.

To show that $G$ must be a graph in Figure 1 if $W$ is **a**(2)-finite, we first prove that $W$ would be **a**(2)-infinite whenever $G$ contains certain subgraphs, then show that to avoid these subgraphs $G$ has to be in Figure 1. For each of these subgraphs, we will construct an infinite family of fully commutative elements that we call "witnesses" and verify that they have **a**-value 2. We will use three methods for these verifications:

(1) First, we recall a powerful result of Shi from [23] that says each fully commutative element $w$ in a Weyl or affine Weyl group satisfies $\mathbf{a}(w) = n(w)$, where $n$ is a statistic defined using heaps. We prove the same result for *star reducible* groups (in the sense of [14], see Proposition 3.15), and use these results to show our witnesses have **a**-value 2 by showing they have $n$-value 2.

(2) In our second method, we recall that the **a**-function is constant on each two-sided Kazhdan–Lusztig cells of $W$ and that each cell is closed under the so-called *generalized star operations* (see Section 3.1), then show our witnesses have **a**-value 2 by relating them to elements of **a**-value 2 by these operations.

(3) In our third and most technical method, we again show our witnesses have **a**-value 2 by showing they are in the same cell as some other element of **a**-value 2, but the proof will require more careful arguments involving certain leading coefficients, or "$\mu$-coefficients", from Kazhdan–Lusztig polynomials.

The rest of the paper is organized as follows. In Section 2, we briefly recall the background on Coxeter groups and Hecke algebras leading to the definition of the **a**-function, as well as the definition and some properties of Kazhdan–Lusztig cells. In Section 3, we introduce our main technical tools for computing and verifying **a**-values, namely, generalized star operations and heaps of fully commutative elements. Sections 4 and 5 prove the sufficiency and necessity of the diagram criterion of Theorem 1.1, respectively. The first three subsections of Section 5 contain a list of lemmas on the subgraphs that $G$ must avoid for $W$ to be **a**(2)-finite. The lemmas are grouped according to the method of verifying the witnesses' **a**-values, with 5.3

being the most technical part of the paper. We prove Theorem 1.3 in Section 6. Finally, in Section 7 we briefly discuss several open problems naturally arising from this paper, as well as their connections to other works in the literature. The problems include generalizing the aforementioned result of Shi to arbitrary Coxeter groups, enumerating elements of **a**-value 2 in **a**(2)-finite Coxeter groups, and the classification of **a**(3)-finite Coxeter groups.

## 2. Preliminaries

The **a**-function arises from the Kazhdan–Lusztig theory of Coxeter groups. We review the relevant basic notions and facts in this section. Besides defining **a**, we will recall the definitions of Kazhdan–Lusztig cells and the "$\mu$-coefficients" of Kazhdan–Lusztig polynomials. Both these notions will be key to the proofs of our main theorems.

2.1. COXETER GROUPS. Throughout the article, $W$ shall denote a Coxeter group with a finite generating set $S$ and Coxeter matrix $M = [m(s,t)]_{s,t \in S}$. Thus, $m(s,s) = 1$ for all $s \in S$, $m(s,t) = m(t,s) \in \mathbb{Z}_{\geqslant 2} \cup \{\infty\}$ for all distinct $s, t \in S$, and $W$ is generated by $S$ subject to the relations $(st)^{m(s,t)} = 1$ for all $s, t$ for which $m(s,t)$ is finite.

The defining data of each Coxeter group can be encoded via its *Coxeter diagram*. This is the weighted, undirected graph with vertex set $S$ and edge set $\{\{s,t\} : s,t \in S, m(s,t) \geqslant 3\}$ such that each edge $\{s,t\}$ has weight $m(s,t)$. Each edge is labelled by its weight except when the weight is 3. A Coxeter group is called *irreducible* if its Coxeter diagram is connected; otherwise the group is *reducible*. Note that any reducible Coxeter group $W$ with Coxeter diagram $G$ is isomorphic to the direct product of the Coxeter groups encoded by the connected components of $G$.

Let $S^*$ be the free monoid generated by $S$. For any $w \in W$, we define the *length* of $w$, written $l(w)$, to be the minimum length of all words in $S^*$ that express $w$. We call any such minimum-length word a *reduced word* of $w$. For any distinct $s, t \in S$, we call the relation

$$sts\cdots = tst\cdots$$

where both sides have $m(s,t)$ factors a *braid relation*. Since $s^2 = (ss)^{m(s,s)} = 1$ for all $s \in S$, the braid relation is equivalent to the relation $(st)^{m(s,t)} = 1$ from the definition of $W$. When $m(s,t) = 2$, we call the relation $st = ts$ a *commutation relation*, for $s$ and $t$ commute.

We can now recall the useful Matsumoto–Tits Theorem.

PROPOSITION 2.1 ([26]; [22, Theorem 1.9]). *Let $w \in W$. Then any pair of reduced words of $w$ can be obtained from each other by a finite sequence of braid relations.*

For more basic notions and facts about Coxeter groups such as the Bruhat order and its subword property, see [6].

2.2. THE **a**-FUNCTION. Let $W$ be an arbitrary Coxeter group. We recall Lusztig's definition of the function $\mathbf{a} : W \to \mathbb{Z}_{\geqslant 0}$ below.

Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. Following [22], we define the *Hecke algebra* of $W$ to be the unital $\mathcal{A}$-algebra $H$ generated by the set $\{T_s : s \in S\}$ subject to the relations

$$(1) \qquad\qquad (T_s - v)(T_s + v^{-1}) = 0$$

for all $s \in S$ and

$$T_s T_t T_s \cdots = T_t T_s T_t \cdots$$

for all $s, t \in S$, where both sides have $m(s,t)$ factors.

It is well-known that $H$ has a *standard basis* $\{T_w : w \in W\}$ where $T_w = T_{s_1} \cdots T_{s_q}$ for any reduced word $s_1 \cdots s_q$ $(s_1, \ldots, s_q \in S)$ of $w$, as well as a *Kazhdan–Lusztig*

*basis* $\{C_w : w \in W\}$ with remarkable properties (see [7]). Now let $h_{x,y,z}(x, y, z \in W)$ be the elements of $\mathcal{A}$ such that

$$C_x C_y = \sum_{z \in W} h_{x,y,z} C_z$$

for all $x, y$. By Lemma 13.5 of [22], for each $z \in W$, there exists a unique integer $\mathbf{a}(z) \geqslant 0$ that satisfies the conditions

   (1) $h_{x,y,z} \in v^{\mathbf{a}(z)} \mathbb{Z}[v^{-1}]$ for all $x, y \in W$,
   (2) $h_{x,y,z} \notin v^{\mathbf{a}(z)-1} \mathbb{Z}[v^{-1}]$ for some $x, y \in W$.

This defines the function $\mathbf{a} : W \to \mathbb{Z}_{\geqslant 0}$.

   Elements of $\mathbf{a}$-value 0 or 1 are well understood in the following sense.

PROPOSITION 2.2 ([22, Proposition 13.7], [28, Corollary 4.10]). *Let $W$ be an arbitrary Coxeter group, and let $1_W$ be the identity of $W$. For all $w \in W$, we have*

   (1) $\mathbf{a}(w) = 0$ *if and only if* $w = 1_W$.
   (2) $\mathbf{a}(w) = 1$ *if and only if* $w \neq 1_W$ *and $w$ has a unique reduced word.*

Here, the set of non-identity elements with a unique reduced word is known to be a *two-sided Kazhdan–Lusztig cell* (which we will define in the next subsection), and is sometimes called the *subregular cell* (see [19] and [28]).

   The following result classifies **a**(1)-finite Coxeter groups, i.e. Coxeter groups with finite subregular cells, in terms of Coxeter diagrams. We will use it in the proof of Theorem 1.3 in Section 6.

PROPOSITION 2.3 ([19, Proposition 3.8]). *Let $W$ be an irreducible Coxeter group with Coxeter diagram $G$. Then $W$ is $\mathbf{a}(1)$-finite if and only if $G$ is a tree and there is at most one edge of weight higher than 3 in $G$.*

   Besides the identity element and the elements in the subregular cell, it is also easy to compute the **a**-values of products of commuting generators in a Coxeter group, thanks to the following two results of Lusztig.

PROPOSITION 2.4 ([22, Section 14]). *Let $W$ be a Coxeter group with generating set $S$. Let $I \subseteq S$ and let $W_I$ be the subgroup of $W$ generated by $I$. If $w \in W_I$, then $\mathbf{a}(w)$ computed in terms of $W_I$ is equal to $\mathbf{a}(w)$ computed in terms of $W$.*

REMARK 2.5. The above statement appears as part of Conjecture 14.2 in [22]. However, it is known to hold in the setting of this paper, which is called the *equal parameter* or the *split* case (see [22], Section 15). The same remark applies to Proposition 2.8.

PROPOSITION 2.6 ([22, Proposition 13.8]). *Let $W$ be a finite Coxeter group, and let $w_0$ be the longest element of $W$. Then $\mathbf{a}(w_0) = l(w_0)$.*

COROLLARY 2.7. *Let $W$ be a Coxeter group with generating set $S$. Let $I = \{s_1, s_2, \ldots, s_k\}$ be a subset of $S$ such that $m(s_i, s_j) = 2$ for all $1 \leqslant i < j \leqslant k$, and let $w_0 = s_1 s_2 \cdots s_k$. Then $\mathbf{a}(w_0) = k$.*

*Proof.* The elements of $I$ commute with each other since $m(s_i, s_j) = 2$ for all distinct $i, j$, therefore the subgroup $W_I$ of $W$ generated by $I$ is isomorphic to the direct product of $k$ copies of the cyclic group of order 2. In particular, $W_I$ is finite. Furthermore, $w_0$ is clearly the longest element of $W_I$, therefore $\mathbf{a}(w_0) = l(w_0) = k$ by Propositions 2.4 and 2.6. $\square$

2.3. Kazhdan–Lusztig cells. We define the Kazhdan–Lusztig cells of a Coxeter group $W$ in this subsection. Let $H$ be the Hecke algebra of $W$, and let $\{C_w : w \in W\}$ be the Kazhdan–Lusztig basis of $H$. For each $x \in W$, let $D_x : H \to \mathcal{A}$ be the linear map such that
$$D_x(C_y) = \delta_{x,y}$$
for all $y \in W$, where $\delta$ is the Kronecker delta symbol. Furthermore, for $x, y \in W$,

(1) define $x \prec_L y$ if $D_x(C_s C_y) \neq 0$ for some $s \in S$;
(2) define $x \leqslant_L y$ if there is a sequence $x = z_1, z_2, \ldots, z_n = y$ in $W$ such that $z_i \prec_L z_{i+1}$ for all $1 \leqslant i \leqslant n - 1$;
(3) define $x \sim_L y$ if $x \leqslant_L y$ and $y \leqslant_L x$.

By the construction, $\sim_L$ defines an equivalence relation on $W$. We call the equivalence classes the *left Kazhdan–Lusztig cells*, or simply the *left cells*, of $W$, and we define the *right (Kazhdan–Lusztig) cells* and *two-sided (Kazhdan–Lusztig) cells* of $W$ similarly. Here, to define the two-sided cells, start by declaring $x \prec_{LR} y$ if either $D_x(C_s C_y) \neq 0$ for some $s \in S$ or $D_x(C_y C_s) \neq 0$ for some $s$ (i.e. if either $x \prec_L y$ or $x \prec_R y$). Note that each two-sided cell of $W$ must be a union of left cells as well as a union of right cells.

The Kazhdan–Lusztig cells of $W$ have the following key connection with the **a**-function on $W$.

PROPOSITION 2.8 ([22, Section 14]). *Let $x, y \in W$. If $x \leqslant_{LR} y$, then $\mathbf{a}(x) \geqslant \mathbf{a}(y)$. In particular, if $x \sim_{LR} y$, then $\mathbf{a}(x) = \mathbf{a}(y)$.*

By the proposition, one way to establish that an element $x \in W$ has a certain **a**-value is to find another element $y$ of that **a**-value and prove that $x$ and $y$ are in the same cell. We will repeatedly use this strategy in Sections 5.2 and 5.3.

As we may see from their construction, the key to understanding Kazhdan–Lusztig cells lies in understanding the products of the form $C_s C_y$. These products are controlled by the *Kazhdan–Lusztig polynomials*, which are defined to be the elements $p_{x,y} \in \mathcal{A}$ ($x, y \in W$) such that
$$C_y = \sum_{x \in W} p_{x,y} T_x$$
for all $y \in W$, where the elements $\{T_w : w \in W\}$ form the standard basis of $T$. More precisely, for each $x, y \in W$, let $\mu_{x,y}$ be the coefficient of the term $v^{-1}$ in $p_{x,y}$, then we have the following formulae.

PROPOSITION 2.9 ([22, Theorem 6.6]). *Let $y \in W$, $s \in S$, and let $\leqslant$ be the Bruhat order on $W$. Then in the Hecke algebra $H$ of $W$,*

$$C_s C_y = \begin{cases} (v + v^{-1}) C_y & \text{if } sy < y, \\ C_{sy} + \sum_{x : sx < x < y} \mu_{x,y} C_x & \text{if } sy > y, \end{cases}$$

$$C_y C_s = \begin{cases} (v + v^{-1}) C_y & \text{if } ys < y, \\ C_{ys} + \sum_{x : xs < x < y} \mu_{x^{-1}, y^{-1}} C_x & \text{if } ys > y. \end{cases}$$

REMARK 2.10. It is known that $\mu_{x,y} = \mu_{x^{-1}, y^{-1}}$ for any $x, y \in W$ (see Section 5.6 and Corollary 6.5 of [22]), therefore the last formula in the proposition also holds with $\mu_{x,y}$ in place of $\mu_{x^{-1}, y^{-1}}$.

REMARK 2.11. The paper [18] uses a normalization of the Hecke algebra that is different from ours, namely, it uses the relation $(T_s + 1)(T_s - q) = 0$ in place of our Equation (1). Consequently, the Kazhdan–Lusztig polynomials $P_{x,y}$ obtained in [18],

which are polynomials in $q$, do not exactly agree with our Kazhdan–Lusztig polynomial $p_{x,y}$. However, it is straightforward to check that we may convert $P_{x,y}$ to $p_{x,y}$ by first substituting $q$ by $v^2$ in $P_{x,y}$ and then multiplying the result by $v^{l(x)-l(y)}$. In particular, our definition of the numbers $\mu_{x,y}$ agrees with that in [18].

The $\mu$-coefficients are often called the "leading coefficients of Kazhdan–Lusztig polynomials" in the literature. Note that the elements $C_s(s \in S)$ generate $H$ by Proposition 2.9, so in a sense the $\mu$-coefficients control the multiplication of the Kazhdan–Lusztig basis elements in the Hecke algebra. As such, they also lead to an alternative characterization of the relations $\prec_L$ and $\prec_R$: for each $y \in W$, define the *left descent set* and *right descent set* of $y$ to be the sets

$$\mathcal{L}(y) = \{s \in S : sy < y\}, \qquad \mathcal{R}(y) = \{s \in S : ys < y\},$$

respectively. Then the following proposition holds.

PROPOSITION 2.12. *Let $x, y \in W$. Then*

(1) *$x \prec_L y$ if and only if one of the following conditions holds: (a) $x = y \neq 1_W$; (b) $x = sy$ for some $s \notin \mathcal{L}(y)$; (c) $x < y$, $\mathcal{L}(x) \nsubseteq \mathcal{L}(y)$, and $\mu_{x,y} \neq 0$.*
(2) *$x \prec_R y$ if and only if one of the following conditions holds: (a) $x = y \neq 1_W$; (b) $x = ys$ for some $s \notin \mathcal{R}(y)$; (c) $x < y$, $\mathcal{R}(x) \nsubseteq \mathcal{R}(y)$, and $\mu_{x,y} \neq 0$.*

*Proof.* By Proposition 2.9, we have $D_x(C_s C_y) \neq 0$ for some $s \in S$ if and only if one of the following occurs:

(a) $x = y \neq 1_W$, so that $\mathcal{L}(y) \neq \varnothing$ and $D_x(C_s C_y) \neq 0$ for each $s \in \mathcal{L}(y)$;
(b) $C_x$ appears in $C_s C_y$ for some $s \notin \mathcal{L}(y)$, with $x = sy$;
(c) $C_x$ appears in $C_s C_y$ for some $s \notin \mathcal{L}(y)$, and $x$ satisfies $x < y, sx < x$ and $\mu_{x,y} \neq 0$. Note that in this case we have $\mathcal{L}(x) \nsubseteq \mathcal{L}(y)$ as $s \in \mathcal{L}(x) \smallsetminus \mathcal{L}(y)$. Conversely, if $x < y$, $\mathcal{L}(x) \nsubseteq \mathcal{L}(y)$, and $\mu_{x,y} \neq 0$, then $D_x(D_s D_y) \neq 0$ for each $s \in \mathcal{L}(x) \smallsetminus \mathcal{L}(y)$ by Proposition 2.9.

Statement (1) now follows. The proof of (2) is similar. □

In Propositions 3.6 and 3.7, we will describe ways to compute certain $\mu$-coefficients combinatorially without referring to the Hecke algebra. This will allow us to avoid difficult computations of Kazhdan–Lusztig polynomials and understand Kazhdan–Lusztig cells by using only the combinatorics of Coxeter groups.

To end this section, we record several facts for future use.

COROLLARY 2.13. *Let $x, y \in W$, and let $s \in S$.*

(1) *We have $sy \leqslant_L y$ if $sy > y$, and $ys \leqslant_R y$ if $ys > y$;*
(2) *If there exist elements $u, v \in W$ such that $y = uxv$ and $l(y) = l(u)+l(x)+l(v)$ where $l$ is the length function on $W$ (see Section 2.1 for the definition of $l$), then we have $y \leqslant_{LR} x$ and $\mathbf{a}(y) \geqslant \mathbf{a}(x)$.*

*Proof.* This is a simple corollary of Propositions 2.9 and 2.8. Note that (2) follows from repeated application of (1) and Proposition 2.8, hence it suffices to prove (1). Suppose $sy > y$. Then $D_{sy}(C_s C_y) = 1$ by Proposition 2.9, therefore $sy \prec y$ and $sy \leqslant_L y$ by definition. Similarly, we have $ys \leqslant_R y$ if $ys > y$. □

PROPOSITION 2.14 ([22, Proposition 5.4]). *Let $x, y \in W$. If $x \leqslant y$, then $p_{x,y} = v^{-l(y)+l(x)} \mod v^{-l(y)+l(x)+1}\mathbb{Z}[v]$.*

COROLLARY 2.15. *Let $x, y \in W$. If $x \leqslant y$ and $l(x) = l(y) - 1$, then $p_{x,y} = v^{-1}$ and hence $\mu_{x,y} = 1$.*

*Proof.* This is immediate from the well-known fact that $p_{x,y} \in \mathbb{Z}[v^{-1}]$ (see [22, Section 5.3]) and Proposition 2.14. □

PROPOSITION 2.16 ([27, Fact 5]). *Let $x, y \in W$ be such that $l(x) < l(y) - 1$. If $\mathcal{L}(y) \nsubseteq \mathcal{L}(x)$ or $\mathcal{R}(y) \nsubseteq \mathcal{R}(x)$, then $\mu(x, y) = 0$.*

## 3. TOOLS FOR COMPUTATION OF **a**

We introduce our main tools for verification and computation of **a**-values in this section. The first tool is the so-called generalized star operations, which we will often use to show two elements are in a same Kazhdan–Lusztig cell and hence of the same **a**-value. The second tool involves heaps of fully commutative elements and will allow us to directly compute **a**-values in certain cases.

3.1. GENERALIZED STAR OPERATIONS. We review the notion of a *generalized star operation* in this subsection. We highlight a direct connection between the operation and Kazhdan–Lusztig cells, then describe a more subtle recurrence relation involving the operation and the $\mu$-coefficients from Proposition 2.9.

Let $W$ be an arbitrary Coxeter group, and let $s, t \in S$ be a pair of generators of $W$ with $3 \leqslant m(s, t) < \infty$. Set $I = \{s, t\}$, let $W_I = \langle s, t \rangle$, the subgroup of $W$ generated by $s$ and $t$, and set ${}^I W = \{w \in W : \mathcal{L}(w) \cap I = \varnothing\}$. It is known that every $w \in W$ admits a unique factorization $w = w_I \cdot {}^I w$, called a *coset decomposition*, where ${}^I w \in {}^I W$ and $w_I \in W_I$; moreover, we have $l(w) = l(w_I) + l({}^I w)$ in this case. (For proofs of these facts and an algorithm to compute the factors $w_I$ and ${}^I w$, see [6, Proposition 2.4.4].) Consider the following situations:

(1) $w_I = 1$;
(2) $w_I$ is the longest element $sts \cdots$ of length $m(s, t)$ in $W_I$;
(3) $w$ is one of the $(m - 1)$ elements $s \cdot {}^I w, ts \cdot {}^I w, sts \cdot {}^I w, tsts \cdot {}^I w, \ldots$;
(4) $w$ is one of the $(m - 1)$ elements $t \cdot {}^I w, st \cdot {}^I w, tst \cdot {}^I w, stst \cdot {}^I w, \ldots$.

We call the sequences appearing in (3) and (4) *left $\{s, t\}$-strings* or *left $I$-strings*, or simply *left strings* if the pair $\{s, t\}$ is clear from context. For any element $w$ in a left $\{s, t\}$-string other than the longest, we define ${}^* w$ to be the element to the right of $w$. Otherwise, we leave ${}^* w$ undefined. We call the map $w \mapsto {}^* w$ the *upper left star operation with respect to $I$*.

Similarly, we define the *lower left star operation* to be the operation $w \mapsto {}_* w$ where $w$ is an element in a left string other than the shortest and ${}_* w$ is the element to the left of $w$ in the same string. In addition, we say $w$ is *left star reducible to ${}_* w$ with respect to $I$* whenever the latter is defined. More generally, dropping the reference to a particular pair of generators, we say $y$ is *left star reducible* to $x$ for $x, y \in W$ if there is a sequence $x = z_1, z_2, \ldots, z_n = y$ in $W$ such that for each $1 \leqslant i \leqslant n - 1$, there is some pair $I_i = \{s_i, t_i\} \subseteq S$ with $3 \leqslant m(s_i, t_i) < \infty$ such that $z_{i+1}$ is left star reducible to $z_i$ with respect to $I_i$.

The concepts and notations above have obvious right-handed counterparts, where the coset decompositions to be considered are of the form $w = w^I \cdot w_I$ where $w_I \in W_I$ and the factor $w^I$ is from the set $W^I := \{w \in W : \mathcal{R}(w) \cap I = \varnothing\}$. We refer to the two types of left star operations and their right-handed counterparts collectively as *generalized star operations*. Finally, for $x, y \in W$, we say that $y$ is *star reducible* to $x$ if there is a sequence $x = z_1, z_2, \ldots, z_n = y$ in $W$ such that $z_{i+1}$ is either left reducible or right reducible to $z_i$ for each $1 \leqslant i \leqslant n - 1$.

REMARK 3.1. For each pair $I = \{s, t\} \subseteq S$ with $m(s, t) = 3$ and each member of a left $\{s, t\}$-string, only one of the lower and upper left star operations is defined for each member of a left $\{s, t\}$-string. The one that does is simply called the *left star operation* in the paper [18] where the operation was first introduced by Kazhdan and Lusztig. Similarly, it makes sense to simply speak of a *right star operation* with respect to $I$.

EXAMPLE 3.2. Let $W$ be a Coxeter group with generating set $S = \{a, b, c\}$. Suppose $m(a, b) = 3$, $m(b, c) = 4$, $m(a, c) = 2$, let $I = \{a, b\}$, $J = \{b, c\}$, and let $x = abcab$. Then with respect to $I$, the coset decompositions of $x$ are given by

$$x = x_I \cdot {}^I x = aba \cdot cb, \quad x = x^I \cdot x_I = abc \cdot ab.$$

It follows that $x$ is not in a left $I$-string but is in a right $I$-string. Moreover, as pointed out in Remark 3.1, only one right star operation with respect to $I$ is defined on $x$ since $m(a, b) = 3$: only the lower star operation is defined, and we have $x_* = abca$. With respect to $J$, the coset decompositions of $x$ are given by

$$x = x_J \cdot {}^J x = b \cdot abcb, \quad x = x^J \cdot x_J = ba \cdot bcb.$$

It follows that $x$ is both in a left $J$-string and in a right $J$-string. Moreover, we have ${}^* x = cbabcb$ and $x_* = babc$, but ${}_* x$ and $x^*$ are not defined.

Generalized star operations are intimately related to Kazhdan–Lusztig cells:

PROPOSITION 3.3. *Let $W$ be an arbitrary Coxeter group, and let $I = \{s, t\}$ be a pair of generators of $W$ for which $3 \leqslant m(s, t) < \infty$. Then the following hold, where all star operations are performed with respect to $I$.*

(1) *Let $y$ be an element of a left $\{s, t\}$-string such that ${}_* y$ makes sense, then $y \sim_L {}_* y$.*

(2) *Let $y$ be an element of a right $\{s, t\}$-string such that $y_*$ makes sense, then $y \sim_R y_*$.*

REMARK 3.4. The above facts are well-known to experts, but we have not found a reference stating it explicitly in this way, so we include a brief proof below.

*Proof.* We first prove (1). Without loss of generality, suppose $I \cap \mathcal{L}(y) = \{s\}$. Then the definition of left strings guarantees that $I \cap \mathcal{L}({}_* y) = \{t\}$. Since ${}_* y < y$, $t \in \mathcal{L}({}_* y) \setminus \mathcal{L}(y)$ and $\mu_{{}_* y, y} = 1$ by Corollary 2.15, we have ${}_* y \leqslant_L y$ by Proposition 2.12. On the other hand, $y \leqslant_L {}_* y$ by Corollary 2.13, therefore $y \sim_L {}_* y$. The proof of (2) is similar. $\square$

COROLLARY 3.5. *Let $x, y \in W$. If $y$ is star reducible to $x$, then $\mathbf{a}(x) = \mathbf{a}(y)$.*

*Proof.* Suppose $y$ is star reducible to $x$. Then $x \sim_{LR} y$ by repeated application of Proposition 3.3, therefore $\mathbf{a}(x) = \mathbf{a}(y)$ by Proposition 2.8. $\square$

Generalized star operations are also connected with $\mu$-coefficients:

PROPOSITION 3.6 ([18, Theorem 4.2]). *Let $W$ be an arbitrary Coxeter group, and let $I = \{s, t\}$ be a pair of generators of $W$ for which $m(s, t) = 3$. Then the following hold, where all star operations are performed with respect to $I$.*

(1) *Let $x, y \in W$ be elements of left $\{s, t\}$-strings such that $xy^{-1} \notin W_I$. Then $\mu(x, y) = \mu(*x, *y)$, where $*\alpha$ stands for the result of applying the left star operation on $\alpha$ for each string $\alpha$ (see Remark 3.1);*

(2) *Let $x, y \in W$ be elements of right $\{s, t\}$-strings such that $x^{-1} y \notin W_I$. Then $\mu(x, y) = \mu(x*, y*)$, where $\alpha*$ stands for the result of applying the right star operation on $\alpha$ for each string $\alpha$ (see Remark 3.1).*

PROPOSITION 3.7 ([20, Section 10.4]; [15, Proposition 5.9]). *Let $W$ be an arbitrary Coxeter group, and let $I = \{s, t\}$ be a pair of generators of $W$ for which $3 \leqslant m(s, t) < \infty$. Then the following hold, where all star operations are performed with respect to $I$.*

(1) *Let $x, y \in W$ be elements of left $\{s, t\}$-strings such that $\mathcal{L}(x) \cap I \neq \mathcal{L}(y) \cap I$. Then*

$$\mu({}_* x, y) + \mu({}^* x, y) = \mu(x, {}_* y) + \mu(x, {}^* y);$$

(2) *Let $x, y \in W$ be elements of right $\{s,t\}$-strings such that $\mathcal{R}(x) \cap I \neq \mathcal{R}(y) \cap I$. Then*

$$\mu(x_*, y) + \mu(x^*, y) = \mu(x, y_*) + \mu(x, y^*).$$

*Here, we define $\mu(\alpha, \beta) = 0$ if either $\alpha$ or $\beta$ is an undefined symbol.*

In Section 5.2, we will frequently use the two propositions above to compute certain $\mu$-coefficients $\mu_{x,y}$ recursively, then use Proposition 2.12 to conclude that $x \sim_L y$ or $x \sim_R y$. This provides a very useful connection, albeit a less direct one than Proposition 3.3, between generalized star operations and cells.

3.2. FULL COMMUTATIVITY AND HEAPS. Let $W$ be an arbitrary Coxeter group. In this subsection, we show that any element with **a**-value 2 must be *fully commutative* in the sense of [24]. We then recall a combinatorial characterization of the **a**-values of fully commutative elements in a Weyl or affine Weyl group in terms of *heaps*. This characterization will allow us to compute certain **a**-values without recourse to Kazhdan–Lusztig theory in Section 5.1.

An element $w \in W$ is said to be *fully commutative* if any pair of reduced words of $w$ can be obtained from each other by means of only commutation relations. It is well-known that $w$ is fully commutative if and only if no reduced word of $w$ contains a contiguous subword of $sts\cdots$ of length $m(s,t)$ where $s, t \in S$ and $m(s,t) \geqslant 3$ (see [24, Proposition 2.1]).

REMARK 3.8. Let $w$ be a fully commutative element with a reduced word $w = stw'$ where $l(w) = l(w') + 2$ and $m(s,t) \geqslant 3$. Consider the coset decomposition $w = w_I \cdot {}^I w$ with respect to the pair $I = \{s, t\}$. Since $w$ is fully commutative, $w_I$ cannot be the word $sts\cdots$ of length $m(s,t)$, therefore $w$ is an element of a left $\{s,t\}$-string, with ${}_*w = tw'$ with respect to $I$. That is, whenever a reduced word of a fully commutative element starts with a pair of letters $s, t \in S$ with $m(s,t) \geqslant 3$, the lower left star operation with respect to $\{s, t\}$ simply removes the leftmost letter of $w$. Similarly, whenever a reduced word of a fully commutative element ends with a pair of letters $s, t \in S$ with $m(s,t) \geqslant 3$, the lower right star operation with respect to $\{s, t\}$ simply removes the rightmost letter of $w$.

Problems related to fully commutative elements, such as the classification of Coxeter groups with finitely many fully commutative elements, the enumeration of fully commutative elements for those groups, and connections fully commutative elements have to Kazhdan–Lusztig cells and so-called *generalized Temperley–Lieb algebras*, have been studied extensively; see, for example, [4], [10], [16], [23], [24] and [25]. Fully commutative elements also provide a suitable framework for studying elements of **a**-value 2 because of the following fact.

PROPOSITION 3.9. *Let $w \in W$. If $\mathbf{a}(w) = 2$, then $w$ is fully commutative.*

*Proof.* We prove the contrapositive of the statement, i.e. that if $w$ is not fully commutative, then $\mathbf{a}(w) \neq 2$.

Suppose $w$ is not fully commutative. Then $w$ can be written in the form $w = uxv$ where $l(w) = l(u) + l(x) + l(v)$ and $x$ is of the form $x = sts\cdots$ with $s, t \in S, m(s,t) \geqslant 3$ and $l(x) = m(s,t)$. By Propositions 2.4 and 2.6, we have $\mathbf{a}(x) = m(s,t)$ in this case, therefore $\mathbf{a}(w) \geqslant \mathbf{a}(x) = m(s,t) \geqslant 3$ by Corollary 2.13. This completes the proof. $\square$

Next, we define the *heap* of an arbitrary word $s_1 s_2 \cdots s_q$ in the free monoid $S^*$: this is the poset $([q], \preccurlyeq)$ where $[q] = \{1, 2, \ldots, q\}$ and $\preccurlyeq$ is the partial order on $[q] = \{1, 2, \ldots, q\}$ obtained via the reflexive transitive closure of the relations

$$i \prec j \quad \text{if} \quad i < j \quad \text{and} \quad m(s_i, s_j) \neq 2.$$

In particular, $i \prec j$ if $i < j$ and $s_i = s_j$. We refer to the generator $s_i$ as the *label* of $i$ for each $i \in [q]$. It is well-known that the heaps of any two words in $S^*$ related by a commutation relation from $W$ are isomorphic as posets (see [24], Section 2.2), therefore for any fully commutative element $w \in W$, it makes sense to define the *heap of $w$* to be the heap of any reduced word of $w$. In this case, we denote the heap of $w$ by $H(w)$. On the other hand, given any word in $S^*$, there is also a criterion for determining if its heap is that of a fully commutative element:

PROPOSITION 3.10 ([24, Proposition 3.3]). *The heap $P$ of a word $s_1 s_2 \cdots s_q$ in $S^*$ is the heap of some fully commutative element in $W$ if and only if*

(1) *There is no covering relation $i \prec j$ such that $s_i = s_j$;*
(2) *There is no convex chain $i_1 < i_2 \cdots < i_m$ in $P$ such that $s_{i_1} = s_{i_3} = \cdots = s$ and $s_{i_2} = s_{i_4} = \cdots = \cdots = t$, where $s, t \in S$ and $m = m(s, t) \geqslant 3$.*

There is an intuitive way to visualize heaps of words in $S^*$. Consider the lattice $S \times \mathbb{N}$, with $S$ indexing the *columns* of the lattice and $\mathbb{N}$ indexing the *levels* or *heights*. We say two columns $s, t$ are *adjacent* if the corresponding vertices are adjacent in the Coxeter graph, i.e. if $m(s, t) \geqslant 3$.

For any word $s_1 s_2 \cdots s_q \in S^*$, we may embed its heap $P$ as a set of lattice points in $S \times \mathbb{N}$ as follows: read the word from left to right, and drop a point in the column representing $s_i$ as we read each letter. Here, we envision each point as being under the influence of "gravity" in the sense that the point must fall to the lowest possible row in its column subject to one condition, namely, it must fall higher than every point that was placed before it in the same column or in an adjacent column. We define the index of this lowest possible row to be the *level* of the point.

REMARK 3.11. A poset $P$ is said to be *ranked* if there exists a function $\rho : P \to \mathbb{Z}$, called a *rank function for $P$*, such that $\rho(b) = \rho(a) + 1$ whenever $a, b$ are elements in $P$ for which $a \prec b$ is a covering relation in $P$. It is worth noting that for a fully commutative element $w$ in $W$, while we have described how to assign each element in the heap $P$ of $w$ a well-defined level, the poset $P$ is not necessarily ranked, i.e. the level function may not be a rank function. For an example where $P$ is not ranked and for criteria for $P$ to be ranked, see [13].

As we embed $P$ in the lattice $S \times \mathbb{N}$, after a point $k \in P$ $(1 \leqslant k \leqslant q)$ falls into position, it is customary to label the point with $s_k$ rather than $k$. Furthermore, to indicate the covering relations of the heap, we connect the point with edges to the highest existing points in its column and its adjacent columns. The resulting graph therefore recovers the Hasse diagram of the poset $P$. For example, in Figure 2, the picture on the right shows the heap of the element *abcabd* in the Coxeter group whose Coxeter diagram is drawn on the left. Note that all reduced words of a fully commutative element result in an identical graph when we embed them in $S \times \mathbb{N}$, so we may identify the element with its embedding. For more on the lattice embeddings of heaps, see [5].

Note that the criterion from Proposition 3.10 can now be translated as follows.

PROPOSITION 3.12. *The heap $P$ of a word in $S^*$ is the heap of a fully commutative element in $W$ if and only if in the lattice embedding of $P$ in $S \times \mathbb{N}$,*

(1) *No column contains two points connected by an edge.*
(2) *For every pair $s, t \in S$ such that $m(s, t) \geqslant 3$, whenever there is a chain of edges connecting a sequence $s, t, s, \ldots$ of $m(s, t)$ points, there is another chain connecting two points in this sequence.*

For example, from Figure 2, we easily see that the element *abcabd* is fully commutative. In particular, although the heap contains the chains with labels $(a, b, a, b)$ and $(b, c, b)$
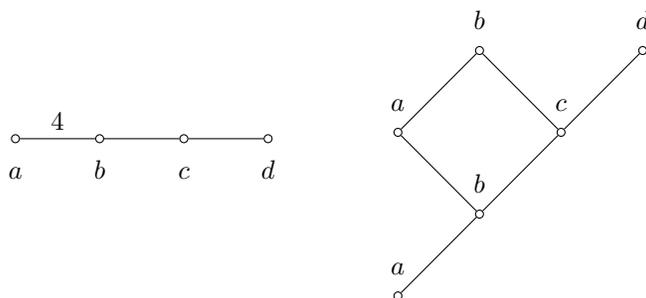
Figure 2. Lattice embedding of a heap.

with $m(a, b)$ and $m(b, c)$ letters, respectively, each of these chains contains two points connected by the other chain.

In addition to detecting fully commutative elements, heaps provide a convenient tool for visualizing generalized star operations on a fully commutative element $w$. More precisely, note that for any $u \in S$, $w$ admits a reduced word starting with $u$ if and only if $H(w)$, as a poset, contains a minimal element with label $u$. Thus, for each pair $I = \{s, t\} \subseteq S$ of generators with $m(s, t) \geqslant 3$, we may perform a lower left star operation with respect to $I$ on $w$ if and only if the following conditions hold:

  (1) the heap $H(w)$ contains a minimal element $i$ labeled by an element in $I$,
  (2) the element $i$ is connected to an element $j$ in $H(w)$ labeled by the other element in $I$,
  (3) upon removal of $i$ from $H(w)$, the element $j$ becomes a minimal element in the resulting poset.

When these conditions are met, performing the lower left star operation with respect to $I$ on $w$ corresponds to removing the vertex $i$ and all edges incident to $i$ in $H(w)$. Similarly, we can easily detect when we can perform lower right star opeartions on $w$ via $H(w)$, by examining the maximal elements of suitable heaps. We shall refer back to these visualizations frequently in Section 5.2. Upper star operations can also be described in terms of heaps, but we will not need them, so we omit the descriptions.

The final feature of heaps that we are interested in concerns the computation of **a**-values. Thanks to a powerful result of Shi in [23], heaps can sometimes be used to compute **a**-values of fully commutative elements in the following fashion.

PROPOSITION 3.13 ([23, Theorem 3.1]). *Let $W$ be a Weyl group or an affine Weyl group. Let $w$ be a fully commutative element of $W$, let $\mathcal{AC}$ be the collection of all antichains in the heap $H(w)$, and let $n(w) = \max(|A| : A \in \mathcal{AC})$, where $|A|$ denotes the cardinality of $A$ for each antichain $A \in \mathcal{AC}$. Then $\mathbf{a}(w) = n(w)$.*

REMARK 3.14. In [23], the author does not explicitly use heaps to describe the **a**-values of fully commutative elements. Rather, he associates a directed graph $G(w)$ to each fully commutative element $w$, defines a number $n(w)$ using $G(w)$, then shows $\mathbf{a}(w) = n(w)$. However, as the author points out at the end of Section 2.2, $G(w)$ can be reformulated in terms of heaps, and it is not difficult to see that his definition of $n(w)$ is identical with ours.

The equality $\mathbf{a}(w) = n(w)$ from Proposition 3.13 also holds in another situation: define a *star reducible* Coxeter group to be a Coxeter group where each fully commutative element is star reducible to a product of mutually commuting generators, then the following holds.

PROPOSITION 3.15. *Let $W$ be a star reducible Coxeter group, and let $w \in W$ be a fully commutative element. Then $\mathbf{a}(w) = n(w)$, where $n(w)$ is defined as in Proposition 3.13.*

*Proof.* Suppose $w$ can be reduced to a product $w' = s_1 \cdots s_k$ of $k$ mutually commuting generators of $W$ via a series of lower star operations. Then $\mathbf{a}(w) = \mathbf{a}(w') = k$ by Corollary 3.5 and Corollary 2.7, and $n(w') = k$ since no two elements in the heap of $w'$ are comparable. Thus, to show $\mathbf{a}(w) = n(w)$, it suffices to show that $n(w') = n(w)$. We do so below by showing that lower star operations preserve $n$-values of fully commutative elements.

Let $x \in W$ be fully commutative, and suppose $y = x_*$ with respect to some lower right star operation. Since any antichain in the heap $H(y)$ is also one in $H(x)$, $n(x) \geqslant n(y)$ by the definition of $n$. Meanwhile, by assumption, $x$ admits a reduced word $x = s_1 s_2 \cdots s_q$ such that $y = s_1 s_2 \cdots s_{q-1}$. Note that $H(x)$ must contain an element $p$ such that $q$ is the unique element in $H(x)$ larger than $p$, for otherwise the right star operation removing $s_q$ from $x$ would not be possible. Now, if an antichain $A$ in $H(x)$ contains $q$, then $p$ is not in $A$ since $A$ is an antichain. Furthermore, let $a \in A \smallsetminus \{q\}$, then $p \not\leqslant a$ since $q$ is the unique element larger than $p$ in $H(x)$, and $a \not\leqslant p$ since otherwise $a \leqslant q$ by transitivity, contradicting the fact that $A$ is an antichain. Thus, for any antichain $A$ of $H(x)$ that has length $n(x)$ and contains $q$, the set $(A \smallsetminus \{q\}) \cup \{p\}$ forms an antichain of the same length. This new antichain is also an antichain in $H(y)$, therefore we have $n(y) \geqslant n(x)$. We have thus proved $n(y) = n(x)$, i.e. that lower right star operations preserve $n$-values of fully commutative elements. A similar argument shows that the same is true for lower left star operations, so we are done. $\square$

For more on star reducible Coxeter groups, including the classification of all star reducible Coxeter groups, see [14].

By Propositions 3.9, 3.13 and 3.15, to show $\{w \in W : \mathbf{a}(w) = 2\}$ is infinite for a Weyl group, affine Weyl group or a star reducible Coxeter group, it suffices to produce infinitely many distinct fully commutative elements, examine their heaps, then use the antichain characterization to verify that the elements have $\mathbf{a}$-value 2. We will repeatedly use this strategy in Section 5.1.

## 4. PROOF OF THEOREM 1.1: SUFFICIENCY OF THE DIAGRAM CRITERIA

Let $W$ be an irreducible Coxeter group with Coxeter diagram $G$. We prove the "if" directions of the two parts of Theorem 1.1 in this section, i.e. we show that $W$ is $\mathbf{a}(2)$-finite if $G$ is as described in the theorem.

4.1. CASE 1. $G$ CONTAINS A CYCLE. We first prove the "if" direction of Theorem 1.1 (1). Since $W$ is certainly irreducible and $G$ certainly contains a cycle when $G$ is a complete graph with 3 or more vertices, it suffices to prove the following.

PROPOSITION 4.1. *If $G$ is a complete graph, then $W$ is $\mathbf{a}(2)$-finite.*

*Proof.* We claim that $W$ actually contains no element of $\mathbf{a}$-value 2 if $G$ is complete. To see this, suppose $a(w) = 2$ for some $w \in W$. Then $w$ is fully commutative by Proposition 3.9. But as $G$ is complete, no two elements of the generating set of $W$ commute, therefore an element in $W$ is fully commutative if and only if it has a unique reduced word. Proposition 2.2 then implies that $\mathbf{a}(w) \leqslant 1$, a contradiction. $\square$

4.2. CASE 2. $G$ IS ACYCLIC. We now prove the "if" part of Theorem 1.1 (2), which is restated below.

PROPOSITION 4.2. *Let $W$ be an irreducible Coxeter group with Coxeter diagram $G$. If $G$ is one of the graphs shown in Figure 1, then $W$ is $\mathbf{a}(2)$-finite.*

It turns out that when $G$ is any graph from Figure 1 other than $E_{q,r}$ where $\min(q,r) \geqslant 3$, we may use two key results not yet stated in the paper to prove that $W$ is $\mathbf{a}(2)$-finite. The first of these results is the following classification of Stembridge.

PROPOSITION 4.3 ([24, Theorem 5.1]). *An irreducible Coxeter group has finitely many fully commutative elements if and only if its Coxeter diagram is of the form $A_n (n \geqslant 1)$, $B_n (n \geqslant 2)$, $D_n (n \geqslant 4)$, $E_n (n \geqslant 6)$, $F_n (n \geqslant 4)$, $H_n (n \geqslant 3)$ or $I_2(m)(5 \leqslant m \leqslant \infty)$.*

Recall that the graphs of type $D_n$ and $E_n$ here are special cases of the graphs $E_{q,r}$ from Figure 1 (see Remark 1.2).

The second external result was established by D. Ernst in [8].

PROPOSITION 4.4 ([8, Corollary 5.16]). *Let $W$ be the affine Coxeter group of type $\tilde{C}_n$ for some $n \geqslant 5$, i.e. suppose its Coxeter diagram is of the form $\tilde{C}_n$ from Figure 1. Then $W$ is $\mathbf{a}(2)$-finite.*

We now deal with the case where $G$ is of the form $E_{q,r}$ where $\min(q,r) \geqslant 3$. We will prove that $W$ is $\mathbf{a}(2)$-finite in this case in Proposition 4.9, after we prove a series of lemmas. We will then combine the external results and Proposition 4.9 to finish the proof of Proposition 4.2.

Throughout the following four lemmas, let $w$ be a fully commutative element in $W$. For any $s \in S$ that labels at least two elements in $H(w)$, define an *open $s$-interval* in $H(w)$ to be an interval $(i,j) = \{k \in H(w) : i < k < j\}$ where $i$ and $j$ are consecutive elements labelled by $s$; similarly, define a *closed $s$-interval* to be an interval of the form $[i,j] = \{k \in H(w) : i \leqslant k \leqslant j\}$ where $i$ and $j$ are consecutive elements labelled by $s$.

LEMMA 4.5. *Suppose $G$ is of type $A$, and let $s \in S$. Every open $s$-interval in $H(w)$ contains exactly two elements whose labels are adjacent to $s$ in $G$, and the labels of these elements are distinct. In particular, if $s$ is an endpoint of $G$, then $w$ contains at most one occurrence of $s$.*

*Proof.* This is well-known; see, for example, [13, Remark 3.3.7]. $\square$

LEMMA 4.6. *Suppose $G$ is the Coxeter diagram shown in Figure 3, and suppose $\mathbf{a}(w) \leqslant 2$. Then any open $d$-interval in $H(w)$ contains exactly two elements with labels from the set $\{c, e, h\}$, and their labels are distinct.*
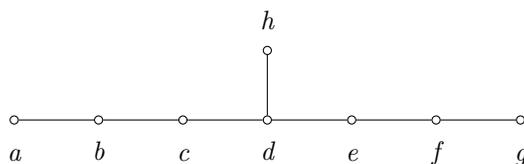


FIGURE 3.

*Proof.* Let $I$ be an open $d$-interval. Deleting $d$ from $G$ produces a union of three subgraphs of type $A$ in which $c, e$ and $h$ appear as endpoints, therefore Lemma 4.5 implies that each of $c, e$ and $h$ can appear at most once (as the label of an element) in $I$. Since $w$ is fully commutative, at least two of them must appear in $I$. Finally, since $G$ is the Coxeter diagram $\tilde{E}_7$, an affine Weyl group of type $E$, $c, e, h$ cannot all appear in $I$ because otherwise the corresponding elements would form an antichain of length 3 and we would have $\mathbf{a}(w) = n(w) \geqslant 3$ by Proposition 3.13. It follows that $I$ contains exactly two elements with distinct labels from $\{c, e, h\}$. $\square$

LEMMA 4.7. *Let $G$ and $w$ be as in Lemma 4.6. Then any open $h$-interval in $H(w)$ must contain precisely two occurrences of $d$.*

*Proof.* Let $I$ be an open $h$-interval. Since $w$ is fully commutative and $d$ is the only vertex adjacent to $h$ in $G$, $I$ must contain at least two occurrences of $d$, so it suffices to show that $I$ cannot contain three or more occurrences of $d$.

For a contradiction, suppose $I$ contains three elements $d_1, d_2, d_3$ with label $d$. By definition, all elements in $I$ are labelled by vertices from the subgraph of type $A$ induced by $a, b, \ldots, g$, therefore the interval $(d_1, d_2)$ contains exactly one element with label $c$ and exactly one element with label $e$ by Lemma 4.5; call them $c_1$ and $e_1$, respectively. Similarly, $(d_2, d_3)$ contains unique elements $c_2$ and $e_2$ with labels $c$ and $e$, respectively. Thus, the interval $(d_1, d_3)$ contains the sequence

$$d_1, c_1, e_1, d_2, c_2, e_2, d_3$$

in weakly increasing order.

Now consider the interval $J = (c_1, c_2)$. Since it contains exactly one occurrence of $d$ and $w$ is fully commutative, $J$ contains at least one occurrence of $b$. Moreover, since deletion of $c$ from $G$ leaves $b$ as an endpoint on a subgraph of type $A$, the appearance of $b$ in $J$ must be unique by Lemma 4.5. Similarly, by considering the interval $J' = (e_1, e_2)$, we may conclude that $J'$ contains a unique occurrence of $f$. But then the elements with labels $b, d, f$ in $(d_1, d_3)$ form an antichain of length 3 in $H(w)$, therefore $\mathbf{a}(w) = n(w) \geqslant 3$ since $G$ is of type $\tilde{E}$. This contradicts our assumption that $\mathbf{a}(w) \leqslant 2$, therefore $I$ contains precisely two occurrences of $d$. $\qquad\square$

LEMMA 4.8. *Let $G$ and $w$ be as in Lemma 4.6. Then $w$ cannot contain three or more occurrences of $h$.*

*Proof.* Suppose for a contradiction that $H(w)$ contains three consecutive elements $h_1, h_2$ and $h_3$ with label $h$. By Lemma 4.7, there are precisely two elements $d_1, d_2$ with label $d$ in $(h_1, h_2)$ and two elements $d_3, d_4$ with label $d$ in $(h_2, h_3)$. Moreover, by Lemma 4.6, $(d_1, d_2)$ and $(d_3, d_4)$ each contains exactly one occurrence each of $c$ and $e$, and $(d_2, d_3)$ contains one occurrence of $c$ or $e$. Without loss of generality, suppose $(d_2, d_3)$ contains an element labelled by $c$. Then the interval $[h_1, h_3]$ contains the sequence

$$h_1, d_1, c_1, e_1, d_2, c_2, h_2, d_3, c_3, e_2, d_4, h_3$$

in weakly increasing order. In this sequence, each $c_i$ is labelled by $c$, each $e_i$ is labelled by $e$, and all elements in $[h_1, h_3]$ with labels $c, d, e$ or $h$ have been listed.

Arguing as in the proof of Lemma 4.7, we see that $(c_1, c_2)$ must contain a unique element, say $b_1$, with label $b$, and $(c_2, c_3)$ must contain a unique element, say $b_2$, with label $b$. Moreover, any two occurrences of $b$ must be separated by an occurrence of $c$, therefore $b_1$ and $b_2$ are consecutive elements with label $b$. But then there must be an element, say $a_1$, with label $a$ in $(b_1, b_2)$. The elements $a_1, c_2, h_2$ now form an antichain of length 3 in $H(w)$, therefore $\mathbf{a}(w) \geqslant 3$, contradicting the assumption that $\mathbf{a}(w) \leqslant 2$. $\qquad\square$

PROPOSITION 4.9. *Let $G$ be of the form $E_{q,r}$ from Figure 1, and suppose $\min(q, r) \geqslant 3$. Then $W$ is $\mathbf{a}(2)$-finite.*

*Proof.* Denote the top vertex on the shortest branch of $G$ by $s$, and let $W'$ be the Coxeter group generated by $S \setminus \{s\}$. Any element $w \in W$ with $\mathbf{a}(w) = 2$ can be written in the form $w = w_1 s w_2 s \cdots s w_n$ for some $n \geqslant 0$ and $w_1, \ldots, w_n \in W'$. By Lemma 4.8, we must have $n \leqslant 3$ if $\mathbf{a}(w) = 2$. Since $W'$ is of type $A$ and thus finite, this implies that $W$ is $\mathbf{a}(2)$-finite. $\qquad\square$

We can now prove Proposition 4.2, i.e. the "if" direction of Theorem 1.1 (2).

*Proof of Proposition 4.2.* Recall that any element of $\mathbf{a}$-value 2 is necessarily fully commutative by Proposition 3.9. Thus, Proposition 4.3 implies that $W$ is $\mathbf{a}(2)$-finite if $G$ is of type $A, B, E_{1,r}, E_{2,r}, F, H$ or $I_2(m)$ where $5 \leqslant m < \infty$ from Figure 1. If $G = I_2(\infty)$, $W$ is $\mathbf{a}(2)$-finite by Proposition 4.1. If $G$ is of type $\tilde{C}$, $W$ is $\mathbf{a}(2)$-finite by Proposition 4.4. Finally, Proposition 4.9 says that $W$ is $\mathbf{a}(2)$-finite if $G$ is of the form $E_{q,r}$ where $q, r \geqslant 3$. This completes the proof. □

## 5. Proof of Theorem 1.1: Necessity of the diagram criteria

Let $W$ be an irreducible Coxeter group with Coxeter diagram $G$. We now prove the "only if" direction of Theorem 1.1. To do so, we first prove a series of lemmas that each says that $W$ is $\mathbf{a}(2)$-infinite if $G$ contains a certain subgraph. We then argue in the last subsection that in order for $G$ not to contain these subgraphs, it has to be a graph from Figure 1.

We shall call the elements of $\mathbf{a}$-value 2 in our lemmas *witnesses*. Based on the method we use to prove that the witnesses have $\mathbf{a}$-value 2, we will group our lemmas into three subsections.

By Proposition 2.4, to show that $W$ is $\mathbf{a}(2)$-infinite when $G$ contains a certain subgraph $G'$, it suffices to find infinitely many witnesses of $\mathbf{a}$-value 2 in the Coxeter group with $G'$ as its Coxeter diagram. We will use this fact without comment throughout the rest of the paper.

5.1. Lemmas with heap arguments. For our first set of lemmas, the proofs that the witnesses have $\mathbf{a}$-value 2 will rely only on Propositions 3.13 and 3.15 from Section 3.2. In particular, no star operations will be involved in the arguments.

Lemma 5.1. *Suppose $G$ contains a subgraph of the form shown in Figure 4, where $n \geqslant 1$ and all edges other than $\{v_{n-1}, v_n\}$ have weight 3. Then $W$ is $\mathbf{a}(2)$-infinite.*
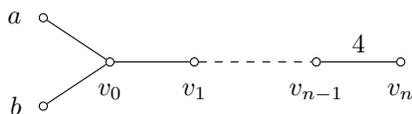


Figure 4.

*Proof.* Let
$$w_k = (abv_0v_1 \cdots v_{n-1}v_nv_{n-1} \cdots v_1v_0)^k.$$
for $k \in \mathbb{Z}_{\geqslant 1}$. The heap of $w_k$ is shown in Figure 5, where the dashed rectangles correspond to the parenthesized expression in $w_k$ and are repeated $k$ times.

For each $k \geqslant 1$, it is clear from the figure that $w_k$ is reduced and fully commutative by Proposition 3.12. Furthermore, observe that any two elements from consecutive levels of $H(w_k)$ are comparable, hence any antichain of maximal length in $H(w_k)$ must contain exactly the two elements labelled by $a$ and $b$ on a same level, therefore $n(w_k) = 2$. Since the subgraph in question is the Coxeter diagram of an affine Weyl group of type $B$, it follows from Proposition 3.13 that $\mathbf{a}(w_k) = 2$ for all $k \geqslant 1$, therefore $W$ is $\mathbf{a}(2)$-infinite. □
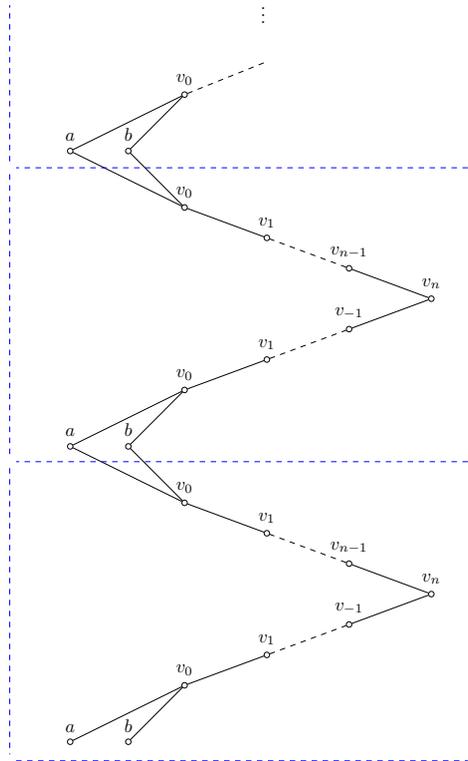
FIGURE 5.

LEMMA 5.2. *Suppose $G$ contains a subgraph of the form shown in Figure 6. Then $W$ is **a**(2)-infinite.*
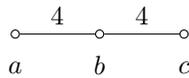


FIGURE 6.

*Proof.* Let $w_k = (acb)^k$ for $k \in \mathbb{Z}_{\geqslant 1}$. The heap of $w_k$ is shown in Figure 7.
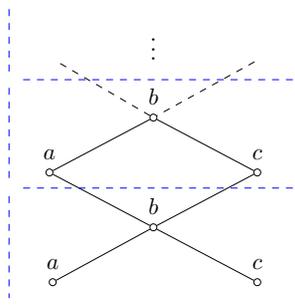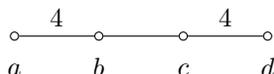


FIGURE 7.

As in the previous lemma, it is clear that $w_k$ is reduced and fully commutative by Proposition 3.12 and that $n(w_k) = 2$ for each $k$. Since the subgraph in question is the Coxeter diagram of $\tilde{C}_3$, an affine Weyl group of type $C$, Proposition 3.13 implies that $\mathbf{a}(w_k) = 2$ for all $k \geqslant 1$, therefore $W$ is $\mathbf{a}(2)$-infinite. $\qquad\square$

LEMMA 5.3. *Suppose $G$ contains a subgraph of the form shown in Figure 8. Then $W$ is $\mathbf{a}(2)$-infinite.*

$$\underset{a}{\circ}\overset{4}{\rule{2cm}{0.4pt}}\underset{b}{\circ}\rule{2cm}{0.4pt}\underset{c}{\circ}\overset{4}{\rule{2cm}{0.4pt}}\underset{d}{\circ}$$

FIGURE 8.

*Proof.* The proof is similar to that of Lemma 5.2: the graph in question is still of type $\tilde{C}_n$, and the elements $w_k = (acbd)^k$ where $k \in \mathbb{Z}_{\geqslant 1}$ now suffice as our witnesses. The fact that $\mathbf{a}(w_k) = n(w_k) = 2$ for each $k \geqslant 1$ is evident from the heap of $w_k$, which is shown in Figure 9. $\qquad\square$
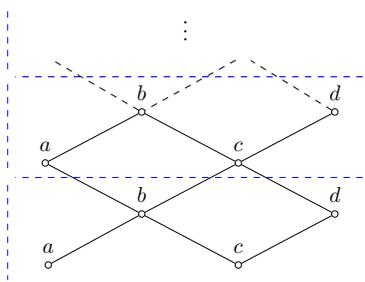


FIGURE 9.

LEMMA 5.4. *Suppose $G$ contains a subgraph of the form shown in Figure 10, where $n \in \mathbb{Z}_{\geqslant 1}$ and all edges have weight 3. Then $W$ is $\mathbf{a}(2)$-infinite.*
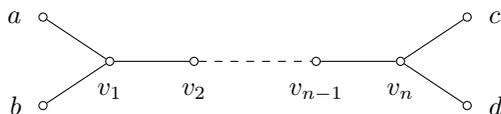


FIGURE 10.

*Proof.* Consider the elements

$$w_k = ab(v_1 v_2 \cdots v_n cd v_n \cdots v_2 v_1 ab)^k$$

for $k \in \mathbb{Z}_{\geqslant 0}$. The heap of $w_k$ is shown in Figure 11.

From the figure, it is clear that $w_k$ is reduced and fully commutative for each $k \geqslant 0$ by Proposition 3.12. As in Lemma 5.2, it is also clear that $n(w_k) = 2$ for all $k \geqslant 0$. Since the subgraph in question is the Coxeter diagram of an affine Weyl group of type $D$, Proposition 3.13 implies that $\mathbf{a}(w_k) = 2$ for all $k \geqslant 2$, therefore $W$ is $\mathbf{a}(2)$-infinite. $\qquad\square$
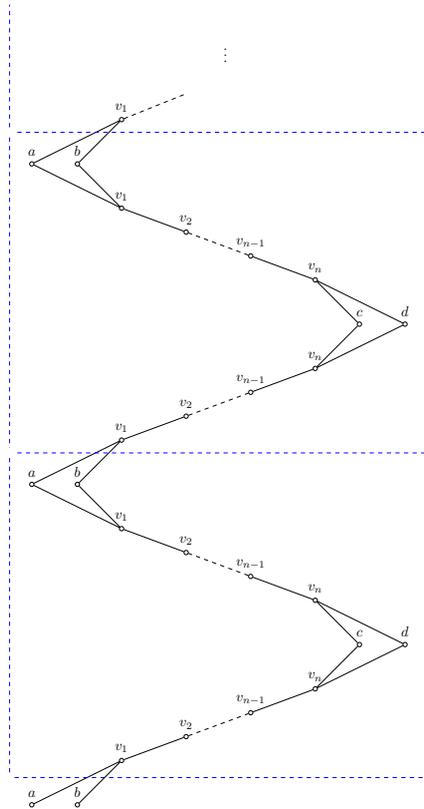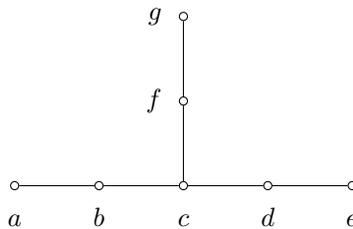
FIGURE 11.



FIGURE 12.

LEMMA 5.5. *Suppose $G$ contains a subgraph of the form shown in Figure* 12. *Then $W$ is $\mathbf{a}(2)$-infinite.*

*Proof.* Consider the elements

$$w_k = (acbfcgdfecdb)^k$$

for $k \in \mathbb{Z}_{\geqslant 1}$. The heap of $w_k$ is shown in Figure 13.

As in the previous lemmas, we may observe from the above figure that $w_k$ is reduced and fully commutative, and that $n(w_k) = 2$, for each $k \geqslant 1$. Since the subgraph in question is the Coxeter diagram of $\tilde{E}_6$, an affine Weyl group of type $E$, Proposition 3.13 implies that $\mathbf{a}(w_k) = 2$ for all $k \geqslant 1$, therefore $W$ is $\mathbf{a}(2)$-infinite. $\qquad\square$

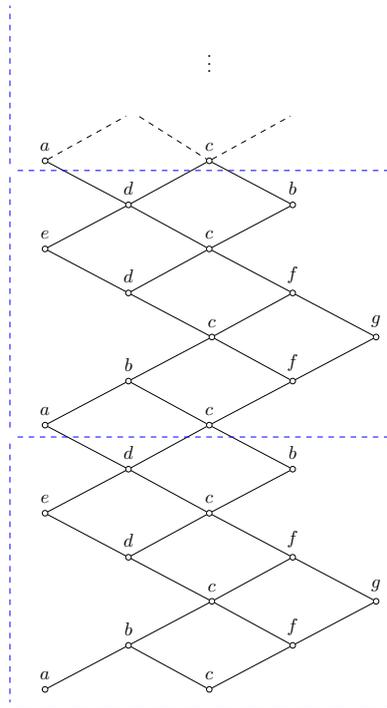Our next lemma will rely on the following non-trivial result from [14].

FIGURE 13.

PROPOSITION 5.6 ([14, Lemma 5.5]). *The Coxeter group whose Coxeter diagram is shown in Figure* 14 *is star reducible.*
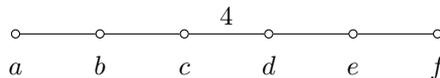


FIGURE 14.

LEMMA 5.7. *Suppose $G$ contains a subgraph of the form shown in Figure* 14. *Then $W$ is $\mathbf{a}(2)$-infinite.*

*Proof.* Consider the elements

$$w_k = (bdacbdcedfce)^k$$

for $k \in \mathbb{Z}_{\geqslant 1}$. The heap of $w_k$ is shown in Figure 15.

As in the previous lemmas, it is clear from the figure that $w_k$ is reduced and fully commutative and that $n(w_k) = 2$ for each $k \geqslant 1$. Since the subgraph in question corresponds to a star reducible Coxeter group by Proposition 5.6, it follows from Proposition 3.15 that $\mathbf{a}(w_k) = 2$ for all $k \geqslant 1$, therefore $W$ is $\mathbf{a}(2)$-infinite. $\qquad\square$

5.2. LEMMAS WITH STAR OPERATION ARGUMENTS. For our second set of lemmas, the proofs that our witnesses have $\mathbf{a}$-value 2 will involve the star operations introduced in Section 3.1. Our main tools will be Corollary 3.5 and Remark 3.8.

The first lemma in this set deals with the case where $G$ contains a cycle. It will be used to prove the "only if" direction of Theorem 1.1 (1).
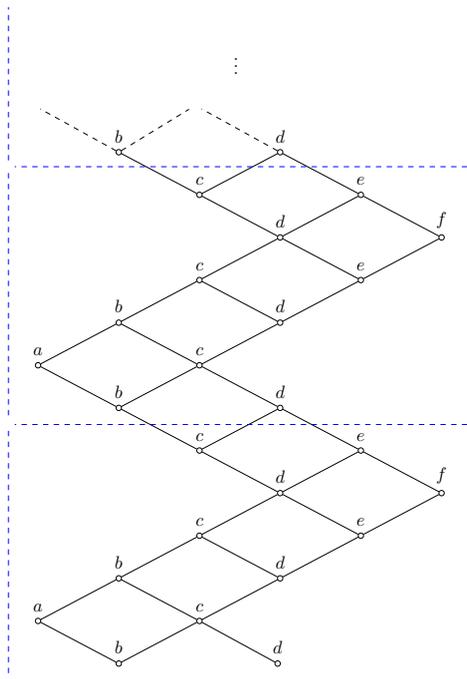
FIGURE 15.

LEMMA 5.8. *Suppose $G$ contains a cycle $C = (v_1, v_2, \ldots, v_n, v_1)$ for some $n \geqslant 3$.*

> (1) *If $G$ contains a vertex $v$ that does not not appear in $C$ and is not adjacent to all vertices in $C$, then $W$ is $\mathbf{a}(2)$-infinite.*
>
> (2) *If $C$ contains two vertices that are not adjacent, then $W$ is $\mathbf{a}(2)$-infinite.*

*Proof.* (1) Suppose $v$ is not adjacent to $v_j$ for some $1 \leqslant j \leqslant n$. Consider the elements

$$x_k = v v_j (v_{j+1} v_{j+2} \cdots v_n v_1 \cdots v_{j-1} v_j)^k$$

for $k \in \mathbb{Z}_{\geqslant 0}$. For each $k \geqslant 1$, note that $x_k$ is reduced (and is actually a reduced word of a fully commutative element) by Proposition 3.10. Moreover, by Remark 3.8, we may reduce $x_k$ to $x_{k-1}$ via $n$ lower right star operations, successively with respect to the pairs

$$\{v_j, v_{j-1}\}, \{v_{j-1}, v_{j-2}\}, \ldots, \{v_n, v_{n-1}\}, \ldots, \{v_{j+1}, v_j\}.$$

It follows that $\mathbf{a}(x_k) = \mathbf{a}(x_0)$ for all $k \geqslant 0$. Since $\mathbf{a}(x_0) = \mathbf{a}(v v_j) = 2$ by Corollary 2.7, it further follows that $\mathbf{a}(x_k) = 2$ for all $k \geqslant 0$, therefore $W$ is $\mathbf{a}(2)$-infinite.

(2) Suppose $v_i, v_j$ are not adjacent for some $1 \leqslant i, j \leqslant n$. Let

$$y_k = v_i v_j (v_{j+1} v_{j+2} \cdots v_n v_1 \cdots v_{j-1} v_j)^k$$

for $k \in \mathbb{Z}_{\geqslant 0}$. Then by an argument similar to the one in (1), $y_k$ is right star reducible to $y_0$ and $\mathbf{a}(y_k) = \mathbf{a}(y_0) = 2$ for all $k \geqslant 0$, therefore $W$ is $\mathbf{a}(2)$-infinite. $\qquad\square$

LEMMA 5.9. *Suppose $G$ contains a subgraph of the form shown in Figure 16, where $n \geqslant 1$, $m_1 \geqslant 5, m_2 \geqslant 4$ and all the middle edges have weight 3. Then $W$ is $\mathbf{a}(2)$-infinite.*

*Proof.* Consider the elements

$$w_k = v_0 (v_{n+1} v_n v_{n-1} \cdots v_1 v_0 v_1 \cdots v_n)^k$$
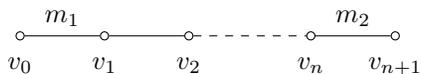
FIGURE 16.

for $k \in \mathbb{Z}_{\geqslant 1}$. The heap of $w_k$ is shown in Figure 17. Note that by Proposition 3.12, it is clear from the figure that $w_k$ is reduced for each $k \geqslant 1$.
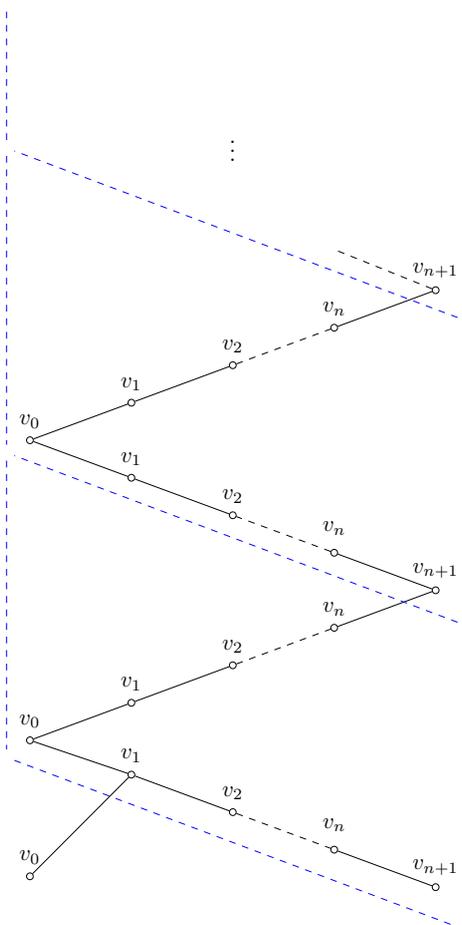


FIGURE 17.

We may commute $v_0$ past the first occurrences of $v_{n+1}, v_n, \ldots, v_2$ to write

$$w_k = v_{n+1}v_n \cdots v_3 v_2 \cdot v_0 \cdot v_1 v_0 v_1 v_2 \cdots v_n \cdot (v_{n+1}v_n \cdots v_1 v_0 v_1 \cdots v_n)^{k-1}.$$

By Remark 3.8, we may then use suitable lower star operations to remove letters from the left and right of $w_k$ to obtain $v_2 v_0$. (Note that by the discussion following Proposition 3.12, these operations can be visualized as the successive removal of the vertices $v_{n+1}, v_n, \ldots, v_3$ in the bottom right of the heap $H(w_k)$ and of the vertices $v_n, v_{n-1}, \ldots, v_1, v_0, v_1, \ldots, v_1, \ldots$ along the top "zig-zag" part of $H(w_k)$.) This implies that $\mathbf{a}(w_k) = \mathbf{a}(v_2 v_0) = 2$ for all $k \geqslant 1$ by Corollaries 3.5 and 2.7, therefore $W$ is $\mathbf{a}(2)$-infinite. $\qquad \square$

LEMMA 5.10. *Suppose $G$ contains a subgraph of the form shown in Figure* 18, *where $n \geqslant 1$ and all edges between $v_1$ and $v_n$ have weight 3. Then $W$ is **a**(2)-infinite.*
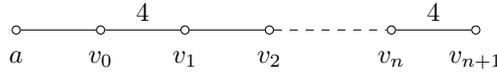


FIGURE 18.

*Proof.* Consider the elements

$$w_k = av_1(v_0 v_1 \cdots v_n v_{n+1} v_n \cdots v_2 v_1)^k$$

for $k \in \mathbb{Z}_{\geqslant 0}$. The heap of $w_k$ is shown in Figure 19. Observe that $w_k$ is reduced by Proposition 3.12.
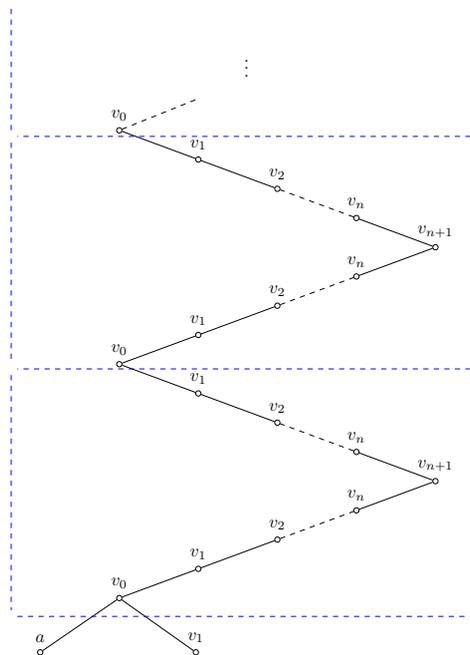


FIGURE 19.

By Remark 3.8, we may easily obtain $w_k$ from $w_{k+1}$ via suitable lower right star operations for any $k \geqslant 0$. (As in the proof of the previous lemma, these operations can be easily visualized in terms of the heap $H(w_{k+1})$, this time as the successive removal of the suitable vertices from the top, wedge-shaped part of the heap.) Corollaries 3.5 and 2.7 then imply that $\mathbf{a}(w_k) = \mathbf{a}(w_0) = \mathbf{a}(av_1) = 2$ for all $k \geqslant 0$, therefore $W$ is $\mathbf{a}(2)$-infinite. $\square$

5.3. LEMMAS WITH $\mu$-COEFFICIENT COMPUTATIONS. For our third set of lemmas, the proofs will all involve showing $x \prec_R y$ for some elements $x, y$ by using Proposition 2.12. The proofs will be more technical than those for the previous lemmas, as we will frequently need to use Propositions 3.6 and 3.7 to deduce $\mu$-values.

LEMMA 5.11. *Suppose $G$ contains a subgraph of the form shown in Figure* 20, *where $m \geqslant 6$. Then $W$ is **a**(2)-infinite.*
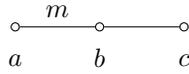
FIGURE 20.

*Proof.* Let $w_k = (cabab)^k$ for each $k \in \mathbb{Z}_{\geqslant 0}$. The heap of $w_k$ is shown in Figure 21. Observe that for each $k \geqslant 0$, $w_k$, $w_k a$ and $w_k ca$ are all reduced by Proposition 3.12.
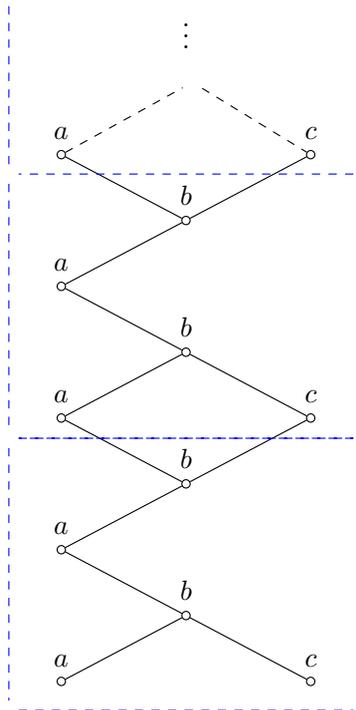


FIGURE 21.

We shall prove that

(2) $$w_k ca \leqslant_R w_{k+2} a \leqslant_R w_{k+2} \leqslant_R w_{k+1} \leqslant_R w_k ca$$

for all $k \geqslant 0$. This implies that $w_k \sim_R ca$ and hence $\mathbf{a}(w_k) = \mathbf{a}(ca) = 2$ for all $k \geqslant 1$ by Proposition 2.8 and Corollary 2.7. It then follows that $W$ is $\mathbf{a}(2)$-infinite.

Now let $k \geqslant 0$ be fixed. To prove (2), first note that $w_{k+2} a \leqslant_R w_{k+2} \leqslant_R w_{k+1} \leqslant_R w_k ca$ by Corollary 2.13, therefore it suffices to show that $w_k ca \leqslant_R w_{k+2} a$. Let $x = w_k ca$ and $y = w_{k+2} a$. We will show that in fact $x \prec_R y$. Since $x < y$ and $c \in \mathcal{R}(x) \smallsetminus \mathcal{R}(y)$, it further suffices to show that $\mu(x, y) \neq 0$ by Proposition 2.12. We do so below.

Consider the coset decompositions of $x$ and $y$ with respect to $I = \{a, b\}$, where

$$x = \cdots cababca = x^I \cdot x_I \quad \text{with} \quad x^I = w_k c, \quad x_I = a,$$
$$y = \cdots cababcababa = y^I \cdot y_I \quad \text{with} \quad y^I = w_{k+1} c, y_I = ababa.$$

For any integer $0 \leqslant i \leqslant m$, let $\alpha_i$ be the word $ab\cdots$ that alternates in $a$ and $b$, starts in $a$, and has length $i$, then let $x_i = x^I \cdot \alpha_i$ and $y_i = y^I \cdot \alpha_i$. Set

$$[i, j] = \mu(x_i, y_j)$$

for all $0 \leqslant i, j \leqslant m$. Then

    (1) $[3, 1] = 0$ by Proposition 2.16, since $l(x_3) < l(y_1) - 1$ and $c \in \mathcal{R}(y_1) \smallsetminus \mathcal{R}(x_3)$;

    (2) $[2, 2] = [1, 1]$ by Proposition 3.6, since $x_1 = x_2*, y_1 = y_2*$ with respect to the pair $I = \{b, c\}$ and $x_2^{-1} y_2 = abcab \notin W_I$;

    (3) $[2, 2] = [1, 3] + [1, 1]$ by Proposition 3.7, hence $[1, 3] = 0$ by (2);

    (4) $[1, 3] + [3, 3] = [2, 4] + [2, 2]$, hence $[3, 3] = [2, 4] + [2, 2]$ by (3);

    (5) $[4, 2] + [2, 2] = [3, 3] + [3, 1]$, hence $[3, 3] = [4, 2] + [2, 2]$ by (1);

    (6) $[2, 4] = [4, 2]$ by (4) and (5);

    (7) $[2, 4] = [1, 5] + [1, 3]$ by Proposition 3.7, hence $[1, 5] = [2, 4]$ by (3);

    (8) $[4, 2] = \mu(w_{k+1}c, w_{k+1}ca) = 1$, where the second equation follows from Corollary 2.15 and the first equation holds by Proposition 3.6 because with respect to $I = \{b, c\}$, $(w_{k+1}c)* = x_4$, $(w_{k+1}ca)* = y_2$ and $(w_{k+1}c)^{-1}(w_{k+1}ca) = a \notin W_I$;

    (9) $\mu(x, y) = [1, 5] = [2, 4] = [4, 2] = 1$ by (7), (6), and (8).

We have now shown that $\mu(x, y) \neq 0$, and our proof is complete. $\qquad\square$

LEMMA 5.12. *Suppose $G$ contains a subgraph of the form shown in Figure 22. Then $W$ is $\mathbf{a}(2)$-infinite.*
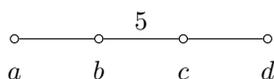


FIGURE 22.

*Proof.* Let $X = acbc$ and $Y = bdcb$. For each $k \in \mathbb{Z}_{\geqslant 0}$, let $w_k = XY \cdots$ be the string that starts in $X$ and contains $k$ alternating occurrences of $X$ and $Y$. The heap of $w_k$ is shown in Figure 23, where the dashed rectangles alternately correspond to the expression $X$ and $Y$ and appear a total of $k$ times. Observe that $w_k$ is reduced for each $k$ by Proposition 3.12.

We shall prove that

$$(3) \qquad\qquad w_k ac \leqslant_R w_{k+2}c \leqslant_R w_{k+2} \leqslant_R w_{k+1} \leqslant_R w_k ac$$

for all even integers $k \geqslant 0$ and that

$$(4) \qquad\qquad w_k bd \leqslant_R w_{k+2}b \leqslant_R w_{k+2} \leqslant_R w_{k+1} \leqslant_R w_k bd$$

for all odd integers $k \geqslant 1$. It then follows that $w_k \sim_R ac$ and hence $\mathbf{a}(w_k) = \mathbf{a}(ac) = 2$ for all $k \geqslant 1$, therefore $W$ is $\mathbf{a}(2)$-infinite.

To prove (3), let $k \geqslant 0$ be an even integer. Note that $w_{k+2}c \leqslant_R w_{k+2} \leqslant_R w_{k+1} \leqslant_R w_k ac$ follows from Corollary 2.13, therefore it suffices to show that $w_k ac \leqslant_R w_{k+2}c$. Let $x = w_k ac$ and $y = w_{k+2}c$. We will show in fact $x \prec_R y$. Since $x < y$ and $a \in \mathcal{R}(x) \smallsetminus \mathcal{R}(y)$, it further suffices to show that $\mu(x, y) \neq 0$ by Proposition 2.12.

To compute $\mu(x, y)$, we consider the coset decompositions of $x$ and $y$ with respect to $I = \{b, c\}$, where

$$x = \cdots bdcbac = x^I \cdot x_I \quad \text{with} \quad x^I = w_k a, \quad x_I = c,$$
$$y = \cdots acbcdbcbc = y^I \cdot y_I \quad \text{with} \quad y^I = w_{k+1}d, \quad {}^I y = bcbc.$$

For any integers $0 \leqslant i, j \leqslant 4$, let $p_i$ be the word $cb \cdots$ that alternates in $b$ and $c$, starts in $c$, and has length $i$, and similarly let $q_j$ be the alternating word $bc \cdots$ of length $j$. Let $x_i = x^I \cdot p_i$ and $y_j = y^I \cdot q_j$, and set
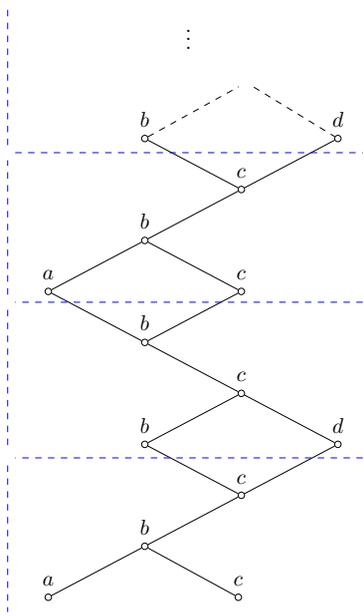
$$[i, j] = \mu(x_i, y_j)$$

Figure 23.

for all $0 \leqslant i, j \leqslant 4$. We have

$$
\begin{aligned}
[1,4] &= -[3,4] + [2,3] \\
&= -[3,4] + (-[4,3] + [3,2] + [3,4]) \\
&= -[4,3] + [3,2] \\
&= -[4,3] + ([4,1] + [4,3]) \\
&= [4,1],
\end{aligned}
$$

where the first, second and fourth equalities follow from applications of Part (1) of Proposition 3.7 with $(x_2, y_4), (x_3, y_3)$ and $(x_4, y_2)$ in place of the pair $(x, y)$, respectively. Now,

$$
[4,1] = \mu(x_4, y_1) = \mu(w_k acbcb, w_k acbcdb) = 1,
$$

where the last equality follows from Corollary 2.15, therefore $\mu(x, y) = [1,4] = [4,1] \neq 0$, and we have proved (3).

The proof of (4) is similar to that of (3), thanks to the symmetry $a \leftrightarrow d, b \leftrightarrow c$ in the subgraph in question. We have now completed our proof. □

Lemma 5.13. *Suppose $G$ contains a subgraph of the form shown in Figure* 24. *Then $W$ is $\mathbf{a}(2)$-infinite.*
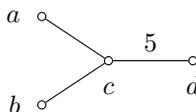


Figure 24.

*Proof.* We present a proof similar to that of Lemma 5.12. Let $X = abcdc$ and $Y = bdcdc$. For each $k \in \mathbb{Z}_{\geqslant 0}$, let $w_k = XY \cdots$ be the alternating string starting with $X$ and containing $k$ total occurrences of $X$ and $Y$. The heap of $w_k$ is shown in Figure 25, from which we observe that $w_k$ is reduced for each $k$.
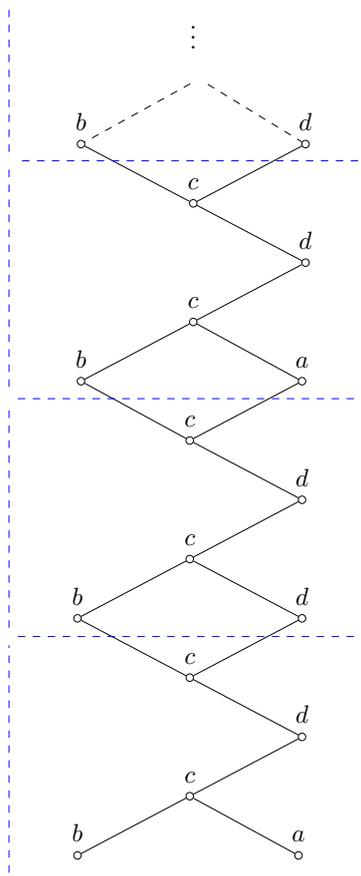


FIGURE 25.

We shall prove that

$$(5) \qquad w_k ab \leqslant_R w_{k+2} a \leqslant_R w_{k+2} \leqslant_R w_{k+1} \leqslant_R w_k ab$$

for all even integers $k \geqslant 0$ and that

$$(6) \qquad w_k bd \leqslant_R w_{k+2} d \leqslant_R w_{k+2} \leqslant_R w_{k+1} \leqslant_R w_k bd$$

for all odd integers $k \geqslant 1$. It then follows that $w_k \sim_R ab$ and hence $\mathbf{a}(w_k) = \mathbf{a}(ab) = 2$ for all $k \geqslant 1$, therefore $W$ is $\mathbf{a}(2)$-infinite.

We first prove (5). As in Lemma 5.12, this is easily reduced to showing $w_k ab \leqslant_R w_{k+2} a$ and then to showing $\mu(w_k ab, w_{k+2} a) \neq 0$. By Proposition 3.6, we have $\mu(w_k ab, w_{k+2} a) = \mu(w_k abc, w_{k+2})$ by considerations with respect to the pair $\{a, c\}$, therefore it further suffices to show that $\mu(x, y) \neq 0$ for $x = w_k abc$ and $y = w_{k+2}$. As before, we do so by considering the coset decomposition of $x$ and $y$

with respect to $I = \{c, d\}$, where

$$x = \cdots abc = x^I \cdot x_I \quad \text{with} \quad x^I = w_k ab, \; x_I = c,$$
$$y = \cdots bdcdc = y^I \cdot y_I \quad \text{with} \quad y^I = w_{k+1}b, \; {}^I y = dcdc.$$

For any integers $0 \leqslant i, j \leqslant 4$, let $p_i$ be the word $cd \cdots$ that alternates in $c$ and $d$, starts in $c$, and has length $i$, and similarly let $q_j$ be the alternating word $dc \cdots$ of length $j$. Let $x_i = x^I \cdot p_i$ and $y_j = y^I \cdot q_j$, and set

$$[i, j] = \mu(x_i, y_j)$$

for all $0 \leqslant i, j \leqslant 4$. By the same calculations as in Lemma 5.12, we have

$$\mu(x, y) = [1, 4] = [4, 1] = \mu(w_k abcdcd, w_k abcdcbd) = 1,$$

by Corollary 2.15, which proves (5).

We now prove (6). As usual, it suffices to show that $\mu(w_k bd, w_{k+2}d) \neq 0$. By setting $x = w_k bd, y = w_{k+2}d$, considering their coset decompositions with respect to $I = \{c, d\}$, and defining $[i, j]$ for $0 \leqslant i, j \leqslant 4$ in the usual way, we may again conclude that

$$\mu(x, y) = [1, 4] = [4, 1] = \mu(w_k bdcdc, w_{k+1}abc) = \mu(w_{k+1}, w_{k+1}abc).$$

Proposition 3.6, applied with respect to the pair $\{a, c\}$, then implies that

$$\mu(x, y) = \mu(w_{k+1}, w_{k+1}abc) = \mu(w_{k+1}a, w_{k+1}ab) = 1 \neq 0,$$

where the last equality follows from Corollary 2.15. Our proof is now complete. $\quad\square$

**Lemma 5.14.** *Suppose $G$ contains a subgraph of the form shown in Figure 26, where $n \geqslant 2$ and all edges other than $\{v_{n-1}, v_n\}$ have weight 3. Then $W$ is $\mathbf{a}(2)$-infinite.*
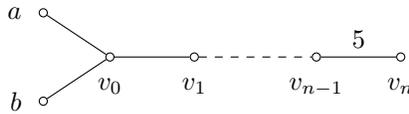


FIGURE 26.

*Proof.* We present a proof similar to that of Lemma 5.13. Let

$$X = abv_0 v_1 \cdots v_{n-1} v_n v_{n-1}, \quad Y = v_{n-2} v_n v_{n-1} v_n v_{n-1} \cdots v_1 v_0,$$

and define $w_k = XYX \cdots$ for each $k \in \mathbb{Z}_{\geqslant 0}$ as in the previous lemma. The heap of $w_k$ is in Figure 27, from which we observe that $w_k$ is reduced for each $k$.

We shall prove that

$$(7) \qquad w_k ab \leqslant_R w_{k+2}a \leqslant_R w_{k+2} \leqslant_R w_{k+1} \leqslant_R w_k ab$$

for all even integers $k \geqslant 0$ and that

$$(8) \qquad w_k v_{n-2} v_n \leqslant_R w_{k+2} v_n \leqslant_R w_{k+2} \leqslant_R w_{k+1} \leqslant_R w_k v_{n-2} v_n$$

for all odd integers $k \geqslant 1$. It then follows that $w_k \sim_R ab$ and hence $\mathbf{a}(w_k) = \mathbf{a}(ab) = 2$ for all $k \geqslant 1$, therefore $W$ is $\mathbf{a}(2)$-infinite.

We first prove (7). As in the previous lemma, we may reduce this to showing $\mu(w_k ab, w_{k+2}a) \neq 0$. Applying Proposition 3.6 with respect to the pairs $\{a, v_0\}, \{v_0, v_1\}, \ldots, \{v_{n-2}, v_{n-1}\}$ successively, we get $\mu(w_k ab, w_{k+2}a) = \mu(x, y)$ where

$$x = w_k abv_0 v_1 \cdots v_{n-2} v_{n-1}, y = w_{k+1} v_{n-2} v_n v_{n-1} v_n v_{n-1}.$$
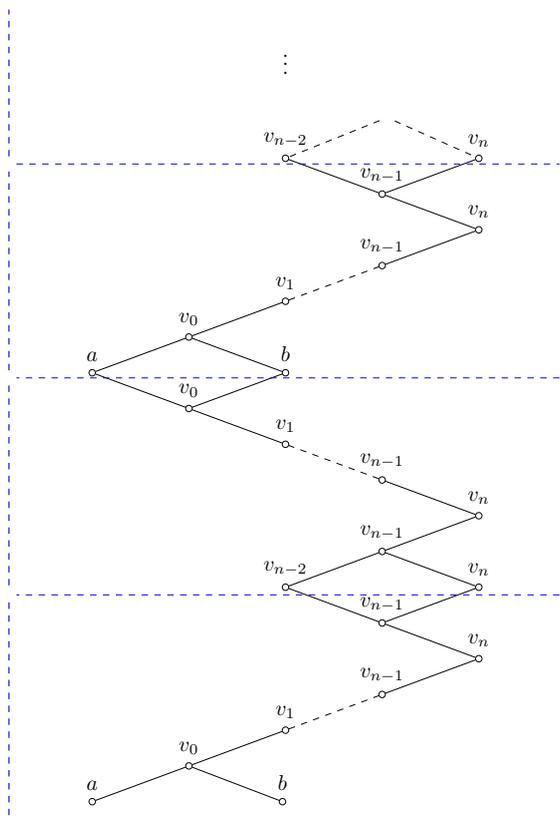
FIGURE 27.

Furthermore, as before, but this time using the coset decomposition of $x$ and $y$ with respect to $I = \{v_{n-1}, v_n\}$, we get

$$\mu(x, y) = \mu(w_k abv_0 v_1 \cdots v_{n-2} v_{n-1} v_n v_{n-1} v_n, w_{k+1} v_{n-2} v_n) = \mu(w_{k+1} v_n, w_{k+1} v_{n-2} v_n).$$

Corollary 2.15 then implies that $\mu(x, y) = 1 \neq 0$, which proves (7).

We now prove (8). We may reduce this to showing that $\mu(w_k v_{n-2} v_n, w_{k+2} v_n) \neq 0$. By setting $x = w_k v_{n-2} v_n, y = w_{k+2} v_n$, considering their coset decompositions with respect to $I = \{v_{n-1}, v_n\}$, and defining $[i, j]$ for $0 \leqslant i, j \leqslant 4$ in the usual way, we again have

$$\mu(x, y) = [1, 4] = [4, 1] = \mu(w_k v_{n-2} v_n v_{n-1} v_n v_{n-1}, w_{k+1} abv_0 v_1 \cdots v_{n-2} v_{n-1}).$$

Finally, applying Proposition 3.6 repeatedly with respect to the suitable pairs of vertices, we have

$$\mu(w_k v_{n-2} v_n v_{n-1} v_n v_{n-1}, w_{k+1} abv_0 v_1 \cdots v_{n-2} v_{n-1}) = \mu(w_{k+1} a, w_{k+1} ab) = 1$$

where the last equality follows from Corollary 2.15. This implies that $\mu(x, y) \neq 0$, and our proof is now complete. □

5.4. FINISHING THE PROOF. We may now combine the lemmas to finish the proof of the "only if" directions of Theorem 1.1 (2). We first deal with the case where $G$ contains a cycle:

PROPOSITION 5.15. *Let $W$ be a Coxeter group with Coxeter diagram $G$. If $G$ contains a cycle and $W$ is **a**(2)-finite, then $W$ is a complete graph.*

*Proof.* Consider a cycle $C = (v_1, v_2, \ldots, v_n, v_1)$ of maximal length in $G$. We claim that $v_1, v_2, \ldots, v_n$ must be all the vertices of $G$. To see this, suppose otherwise and let $v$ be any other vertex of $G$ not in $C$. Then $v$ must be adjacent to all vertices in $C$ by Part (1) of Lemma 5.8. However, in this case $C' = (v, v_1, v_2, \ldots, v_n, v)$ would form a longer cycle in $G$ than $C$, contradicting our maximality assumption.

To prove $G$ is complete, it now suffices to show that $v_i$ and $v_j$ are adjacent for all $1 \leqslant i, j \leqslant n$. This follows from Part (2) of Lemma 5.8, which says that otherwise $W$ would be $\mathbf{a}(2)$-infinite. $\qquad\square$

REMARK 5.16. Note that the above proposition is slightly stronger than the "only if" condition of Theorem 1.1 (1) since we do not need to assume $W$ is irreducible in its statement or proof. This is because Lemma 5.8 implies that the diagram of any $\mathbf{a}(2)$-finite Coxeter group must be connected if it contains a cycle.

Next, we deal with Part (2) of the theorem. For convenience, we define a *path graph* to be a weighted graph such that the underlying unweighted graph looks like a "straight line", i.e. a graph of type $A_n$ from Figure 1.

PROPOSITION 5.17. *Let $W$ be an irreducible Coxeter group with Coxeter diagram $G$. If $G$ is acyclic and $W$ is $\mathbf{a}(2)$-finite, then $G$ is one of the graphs from Figure 1.*

*Proof.* Suppose $G$ is acyclic and $W$ is $\mathbf{a}(2)$-finite. Since $W$ is irreducible, $G$ is connected and hence a tree.

Let $h$ be the largest weight of an edge in $G$. This is well-defined because $G$ contains finitely many vertices and hence edges. If $h \geqslant 6$, all other edges in $G$ must have weight 3, for otherwise we must be able to find a subgraph of the form shown in Lemma 5.9 so that $W$ would be $\mathbf{a}(2)$-infinite. Lemma 5.11 then further implies that $G$ must be exactly of rank 2, therefore $G$ is of the form $I_2(h)$ from Figure 1.

Next, suppose $h = 5$. Then again, in light of Lemma 5.9, $G$ must have only one edge of weight 5, and all other edges of $G$ must have weight 3. Moreover, no vertices of $G$ can have degree 3 or higher by Lemmas 5.13 and 5.14, therefore $G$ must be a path graph. By Lemma 5.12, the unique edge of weight 5 cannot have edges both to its left and to its right in the path graph, therefore $G$ is either $I_2(5)$ or $H_n$ for some $n \geqslant 3$.

Now suppose $h = 4$. We claim that $G$ must be a path graph. Otherwise, let $v$ be a vertex of degree at least 3, then either $v$ is incident to at least two edges of weight 4 and $W$ is $\mathbf{a}(2)$-infinite by Lemma 5.2, or, if $v$ is incident to one or no edge of weight 4, then $W$ is $\mathbf{a}(2)$-infinite by Lemma 5.1 as $G$ must contain a subgraph of the form shown in the lemma. Given that $G$ is a path graph, we also claim that $G$ can contain at most two edges of weight 4. Otherwise, since $G$ cannot contain a subgraph of the form shown in Lemma 5.2, there must be at least one edge of weight 3 between each pair of edges of weight 4. This would force $G$ to contain a subgraph of the form shown in Lemma 5.10, so $W$ would be $\mathbf{a}(2)$-infinite, a contradiction. Moreover, Lemmas 5.2, 5.3 and 5.10 also imply that in the case where $G$ contains two edges of weight 4, they must appear on the two ends of the path graph and $G$ must of the form $\tilde{C}_n$ for some $n \geqslant 5$. Finally, if $G$ contains exactly one edge of weight 4, then Lemma 5.7 implies that $G$ must be of the form $B_n$ for some $n \geqslant 2$ or $F_n$ for some $n \geqslant 4$.

Finally, we consider the case $h = 3$. If $G$ contains no vertex of degree 3 or higher, then $G$ is a path graph and hence of the form $A_n$ from Figure 1 for some $n \geqslant 2$. On the other hand, by Lemma 5.4, $G$ cannot contain any vertex of degree 4, nor can it contain two vertices of degree at least 3, therefore if $G$ has a vertex of degree at least 3 at all, $G$ must be of the form shown in Figure 28, where removal of the trivalent vertex results in three path graphs containing $p, q, r$ vertices for some $1 \leqslant p \leqslant q \leqslant r$. Note

that $p \geqslant 2$ would imply that $G$ contains a subgraph of the form shown in Lemma 5.5, therefore $p = 1$ by the lemma. But then $G$ is of the form $E_{q,r}$ from Figure 1. This completes our proof. $\qquad\square$
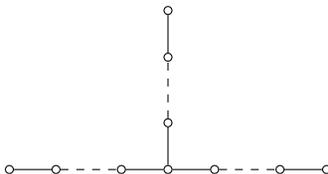


FIGURE 28.

By proving Propositions 4.1, 4.2, 5.15 and 5.17, we have now completed the proof of Theorem 1.1.

## 6. Reducible **a**(2)-finite Coxeter groups

We now prove Theorem 1.3, which is restated below for convenience.

THEOREM. *Let $W$ be a reducible Coxeter group with Coxeter diagram $G$. Let $G_1, G_2, \ldots, G_n$ be the connected components of $G$, and let $W_1, W_2, \ldots, W_n$ be their corresponding Coxeter groups, respectively. Then the following are equivalent.*

  (1) *$W$ is **a**(2)-finite.*
  (2) *The number $n$ is finite, i.e. $G$ has finitely many connected components, and $W_i$ is both **a**(1)-finite and **a**(2)-finite for each $1 \leqslant i \leqslant n$.*
  (3) *The number $n$ is finite, and for each $1 \leqslant i \leqslant n$, $G_i$ is a graph of the form $A_n(n \geqslant 1), B_n(n \geqslant 2), E_{q,r}(q, r \geqslant 1), F_n(n \geqslant 4), H_n(n \geqslant 3)$ or $I_2(m)(5 \leqslant m \leqslant \infty)$, i.e. $G_i$ is a graph from Figure 1 other than $\tilde{C}_n(n \geqslant 5)$.*

*Proof.* The equivalence of (2) and (3) is immediate from Proposition 2.3 and Theorem 1.1, so we just need to prove the equivalence of (1) and (2).

We first prove that (1) implies (2). Suppose $W$ is **a**(2)-finite. Then clearly $W_i$ is **a**(2)-finite for each $i$. Also note that if we pick an element $t_i$ from the generating set of $W_i$ for each $i$, then $\mathbf{a}(t_i t_j) = 2$ for all distinct $1 \leqslant i, j \leqslant n$ by Corollary 2.7, therefore $n$ must be finite. Finally, let $1 \leqslant i \leqslant n$, and consider words of the form $tw_1$ where $w_1 = s_1 s_2 \cdots s_q$ is the reduced word of an element of **a**-value 1 in $W_i$ and $t$ is a vertex in $G_j$ for some $j \neq i$. Clearly, the word $tw_1$ is still reduced. Furthermore, since $w_1$ must have a unique reduced word by Proposition 2.2, no two adjacent letters in $w_1$ can commute, i.e. $m(s_k, s_{k+1}) \geqslant 3$, for all $1 \leqslant k \leqslant q - 1$. This means $tw_1$ can be reduced to $ts_1$ via suitable lower star operations by Remark 3.8, therefore $\mathbf{a}(tw_1) = \mathbf{a}(ts_1) = 2$ by Corollaries 3.5 and 2.7. It follows that $W_i$ is **a**(1)-finite, for otherwise we can find infinitely many distinct elements of the form $tw_1$ in $W$.

It remains to prove that (2) implies (1). Suppose $W_i$ is both **a**(1)-finite and **a**(2)-finite for each $1 \leqslant i \leqslant n$, and let $w \in W$ be an element of **a**-value 2. Since every generator of $W_i$ commutes with every generator of $W_j$ for any distinct $i, j$, $w$ admits a reduced word $w = w_1 \cdot w_2 \cdot \cdots \cdot w_n$ where each $w_i$ is a (possibly empty) reduced word for an element in $W_i$. Note that at most two of of $w_1, \ldots, w_n$ can be nonempty, for otherwise if we have reduced words $w_i = r_1 r_2 \cdots, w_j = s_1 s_2 \cdots, w_k = t_1 t_2 \cdots$ for some $i < j < k$, then we may commute $s_1$ and $t_1$ past letters to their left to form the reduced word of the form $w = w_1 \cdots w_{i-1}(r_1 s_1 t_1) r_2 \cdots$, therefore $\mathbf{a}(w) \geqslant \mathbf{a}(r_1 s_1 t_1) = 3$ by Corollaries 2.13 and 2.7. It follows $w$ must be of the form $w = w_i \cdot w_j$ for some $1 \leqslant i < j \leqslant n$. By Corollary 2.13, this forces $\mathbf{a}(w_i) \leqslant 2$ and $\mathbf{a}(w_j) \leqslant 2$ now that

$\mathbf{a}(w) = 2$. Since $W_i$ is both $\mathbf{a}(1)$-finite and $\mathbf{a}(2)$-finite for all $1 \leqslant i \leqslant n$, this implies that there are only finitely many possibilities for $w$, therefore $W$ is $\mathbf{a}(2)$-finite. This completes the proof. □

## 7. Concluding remarks

We discuss below some open problems naturally arising from this paper.

7.1. Generalizing Shi's result. The "$\mathbf{a} = n$" result of Shi from Proposition 3.13, that is, the result that $\mathbf{a}(w) = n(w)$ for every fully commutative element $w$ in a Weyl or affine Weyl group, is a very powerful tool used in this paper. One can verify that all witnesses of $\mathbf{a}$-value 2 given in Section 5 have $n$-value 2, therefore we actually have $\mathbf{a}(w) = n(w)$ for every witness $w$ we used, regardless of whether we showed $\mathbf{a}(w) = 2$ by using the result $\mathbf{a}(w) = n(w)$. In fact, the $\mathbf{a} = n$ result is exactly how we found the witnesses heuristically: we constructed heaps of fully commutative elements with $n$-value 2, then tried to verify that the elements have $\mathbf{a}$-value 2 by using various methods.

We extended Shi's result to star reducible Coxeter groups in Proposition 3.15, and we believe it would be very interesting to know whether the result can be further generalized to arbitrary Coxeter groups. As direct computations of $\mathbf{a}$-values require understanding products of Kazhdan–Lusztig basis elements in Hecke algebras, which is usually difficult, the generalization of the $\mathbf{a} = n$ result would provide a powerful shortcut to computing $\mathbf{a}$-values for fully commutative elements. Moreover, in Section 6.6 of the paper [4], Biagioli, Jouhet and Nadeau mention that it would be interesting to explore statistics on fully commutative elements which can be studied naturally on heaps; the generalization would suggest that the $n$-value is a such a statistic with interesting connections to representation theory.

On a more technical level, a generalization of the $\mathbf{a} = n$ result would also be remarkable in the following sense. If a word $w$ represents a fully commutative element in a Coxeter group $W$ with Coxeter diagram $G$, then it also represents a fully commutative element in any Coxeter group $W'$ whose Coxeter diagram $G'$ is obtained from $G$ by increasing the weights of some edges. The increase in edge weights does not affect the heap of $w$, hence these two elements have the same $n$-value, yet it is not obvious why the increase should not affect the $\mathbf{a}$-value.

7.2. Enumerating elements of $\mathbf{a}$-value 2. Given our classification of $\mathbf{a}(2)$-finite Coxeter groups, it is natural to wonder how many elements of $\mathbf{a}$-value 2 there are in each group $W$ from the classification. This can be related to the following questions. First, how is the set of the elements of $\mathbf{a}$-value 2 in $W$ partitioned into two-sided Kazhdan–Lusztig cells? Second, for each two-sided Kazhdan–Lusztig cell $E$ of $W$ with $\mathbf{a}$-value 2, how can we describe the structure of the associated subalgebra $J_E$ of the asymptotic Hecke algebra $J$ of $W$? By definition, the algebra $J_E$ is a free abelian group with a basis indexed by the elements of $E$, therefore such a description would allow us to recover the cardinality of $E$ as the rank of $J_E$ and understanding all subalgebras of the form $J_E$ where $E$ is a cell of $\mathbf{a}$-value 2 would allow us to recover the total number of elements of $\mathbf{a}$-value 2.

We intend to investigate the above questions elsewhere. We should mention that the analogous questions for elements of $\mathbf{a}$-value 1 have been solved in the following sense. Let $W$ be an arbitrary Coxeter group, let $C = \{w \in W : \mathbf{a}(w) = 1\}$, and let $U = \{w \in W : w \text{ has a unique reduced word}\}$. Then we have $U = C \sqcup \{1_W\}$ (see Proposition 2.2), therefore $W$ is $\mathbf{a}(1)$-finite if and only if $U$ is finite, and the problem of counting $C$ for $\mathbf{a}(1)$-finite Coxeter groups is equivalent to that of counting $U$ for those Coxeter groups where $U$ is finite. The latter problem is solved by Hart in [17].

As for the partition of $C$ into cells, it is well-known that $C$ always forms a single two-sided cell. In addition, the structure of the algebra $J_C$ for general Coxeter groups is studied in [28], and the structure of $J_C$ for $\mathbf{a}(1)$-finite Coxeter groups can be easily deduced from the results of [28].

7.3. CLASSIFICATION OF $\mathbf{a}(3)$-FINITE COXETER GROUPS. Yet another natural way to extend the results of this paper is to classify $\mathbf{a}(3)$-finite Coxeter groups. Note that unlike elements of $\mathbf{a}$-value 2, elements of $\mathbf{a}$-value 3 in a Coxeter group are not necessarily fully commutative (cf. Proposition 3.9); for example, in the Coxeter group of type $A_2$, the longest element has $\mathbf{a}$-value 3 by Proposition 2.6 but is not fully commutative. This means that in general an element of $\mathbf{a}$-value 3 does not have an intrinsically associated heap. As heaps of fully commutative elements are fundamental to our arguments in this paper, a classification of $\mathbf{a}(3)$-finite Coxeter groups will likely require a different set of ideas from those used in this paper.

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