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Quasimodular forms: an introduction

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Abstract

Quasimodular forms were the heroes of a Summer school held June 20 to 26, 2010 at Besse et Saint-Anastaise, France. We give a short introduction to quasimodular forms. More details on this topics may be found in [1].

Formes quasimodulaires : une introduction

Résumé

Les formes quasimodulaires furent les héroïnes d'une école d'été qui s'est tenue du 20 au 26 juin 2010 à Besse et Saint-Anastaise, France. On donne une courte introduction aux formes quasimodulaires. Une présentation plus large est donnée dans [1].

1. Besse summer school on quasimodular forms

The birth house of Bourbaki, at Besse et Saint-Anastaise, hosted from 20 to 26 June 2010 a summer school on quasimodular forms and applications. This summer school, organised with the financial support of the ANR project *Modunombres*, was attended by 29 participants. Beside short conferences by some of the participants, six experts of modular and quasimodular forms gave courses. The aim of this volume is to provide the lectures of four of the lecturers.

The algebra of modular forms is not stable by differentiation. This is one of the reasons why quasimodular forms are an interesting extension of modular forms. Rankin-Cohen brackets are combination of modular forms and their derivatives that are modular. Michaël Pevzner (Université de Reims) gave a lecture *Rankin–Cohen brackets and representations of conformal Lie groups*. He presented Rankin-Cohen brackets through the theory of unitary representations of conformal Lie groups. Min Ho Lee (University of Northern Iowa) is an expert on quasimodular forms. He presented a survey of his work in his course *Quasimodular forms and quasimodular polynomials*. In *Quasi-modular forms attached to elliptic curves,* *I*, Hossein Movasati (Instituto de Matematica Pura e Aplicada) developed the theory of quasimodular forms in the framework of Algebraic Geometry. He gave a geometric interpretation of quasimodular forms using moduli of elliptic curves with marked elements in their de Rham cohomologies. Marios Petropoulos (École polytechnique) gave a course on quasimodular forms in physics. He wrote notes with Pierre Vanhove (Institut des Hautes Études Scientifiques) *Gravity, strings, modular and quasimodular forms*. In thses notes, they exhibit the role played by modular and quasimodualar forms in gravity and string theory.

2. Modular forms

A reference for this part is [2]. Let

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : (a,b,c,d) \in \mathbb{Z}^4,\ ad-bc=1 \right\}.$$

be the modular group. It is well known that it is generated by

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The Poincaré upper half plane is the set

$$\mathcal{H} := \{z \in \mathbb{C} \colon\ Im\, z > 0\}.$$

We add to $\mathcal{H}$ the set $\mathbb{Q} \cup \{\infty\}$ to obtain $\overline{\mathcal{H}}$. The modular group acts on $\overline{\mathcal{H}}$: if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $z \in \overline{\mathcal{H}}$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z := \begin{cases} \frac{az+b}{cz+d} & \text{if } z \in \overline{\mathcal{H}} \setminus \{\infty, -\frac{d}{c}\} \\ \infty & \text{if } z = -\frac{d}{c} \\ \frac{a}{c} & \text{if } z = \infty. \end{cases}$$

Definition 2.1. Let k be an integer. A *modular form of weight k* over $\mathrm{SL}_2(\mathbb{Z})$ is a holomorphic function $f \colon \mathcal{H} \to \mathbb{C}$ such that :

(1) for any matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathbb{Z})$ and any $z \in \mathcal{H}$,

$$(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = f(z)\ ;$$

(2) the holomorphic function f is holomorphic at infinity.

The first condition implies that f is periodic of period 1. This implies that f admits a Fourier expansion. The second condition requires that this Fourier expansion has no coefficients with negative index:

$$f(z) = \sum_{n=0}^{+\infty} \widehat{f}(n) e^{2\pi i n z}.$$

We write $\mathcal{M}_k$ for the vector space of modular forms of weight k over $\mathrm{SL}_2(\mathbb{Z})$. It is finite dimensional.

Theorem 2.2. (1) *If $k < 0$ or if k is odd, then $\mathcal{M}_k = \{0\}$.*

(2) *if $k \geq 0$ is even, then*

$$\dim \mathcal{M}_k = \begin{cases} \left\lfloor \dfrac{k}{12} \right\rfloor & \text{if } k \equiv 2 \pmod{12} \\ \left\lfloor \dfrac{k}{12} \right\rfloor + 1 & \text{otherwise.} \end{cases}$$

We consider now that k is always nonnegative and even. A modular form is said to be parabolic if its Fourier expansion has no constant term:

$$f(z) = \sum_{n=1}^{+\infty} \widehat{f}(n) e^{2\pi i n z}.$$

We write $\mathcal{S}_k$ for the subspace of parabolic forms in $\mathcal{M}_k$. If $k \geq 4$, we have $\mathcal{M}_k = \mathcal{S}_k \oplus \mathbb{C}E_k$ where E_k is the weight k Eisenstein series defined by

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{+\infty} \sigma_{k-1}(n) e^{2i\pi n z}$$

where

$$\sigma_{k-1}(n) = \sum_{d \mid n} d^{k-1}$$

and

$$\sum_{n=0}^{+\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1}.$$

Moreover, if $k \geq 12$, then $\mathcal{S}_k = \Delta \mathcal{M}_{k-12}$ where

$$\Delta = \frac{1}{1728}(E_4^3 - E_6^2).$$

It can be shown that Δ vanishes only at ∞ and that this zero has order 1.

3. Definition of quasimodular forms

The notion of quasimodular forms is due to Kaneko & Zagier. Werner Nahm gave the definition we use. The theory has been developped by Don Zagier [3]. Define

$$\mathrm{D} := \frac{1}{2i\pi}\frac{\mathrm{d}}{\mathrm{d}z}.$$

The following proposition (proved recursively) implies that the derivatives of a modular form are not modular forms.

Proposition 3.1. *Let $f \in \mathcal{M}_k$ and $r \in \mathbb{N}$. Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Then*

$$\begin{aligned}(cz+d)^{-(k+2m)}\,\mathrm{D}^m f\left(\frac{az+b}{cz+d}\right)\\ = \sum_{j=0}^{m}\binom{m}{j}\frac{(k+m-1)!}{(k+m-j-1)!}\left(\frac{1}{2i\pi}\right)^j \mathrm{D}^{m-j}f(z)\left(\frac{c}{cz+d}\right)^j.\end{aligned}$$

This proposition is used to define quasimodular functions.

Definition 3.2. Let $f\colon \mathcal{H} \to \mathbb{C}$ be a holomorphic function, k and $s \geq 0$ be integers. The function f is a *quasimodular function* of weight k and depth s if there exist holomorphic functions $f_0, \ldots, f_s$ over $\mathcal{H}$ with f_s non identically zero, such that

$$(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = \sum_{j=0}^{s} f_j(z)\left(\frac{c}{cz+d}\right)^j \tag{3.1}$$

for any matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and any $z \in \mathcal{H}$. By convention, the zero function is quasimodular of depth 0 for any weight.

With the notation of definition 3.2, we write $Q_j(f) := f_j$. The application

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto f\Big|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where

$$f\Big|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) := (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) \tag{3.2}$$

is an action of $\mathrm{SL}_2(\mathbb{Z})$ on the holomorphic functions over $\mathcal{H}$. Finally, for any $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we define

$$\begin{array}{rcccc} \mathrm{X}(A) & : & \mathcal{H} & \to & \mathbb{C} \\ & & z & \mapsto & \dfrac{c}{cz+d}. \end{array}$$

With all these definitions (3.1) is rewritten

$$f\underset{k}{|}A = \sum_{j=0}^{s} Q_j(f)\,\mathrm{X}(A)^j. \tag{3.3}$$

For any quasimodular form f, we have $Q_0(f) = f$. Moreover, f is periodic of period 1. Indeed we prove in the next proposition that each $Q_j(f)$ is itself a quasimodular function.

Proposition 3.3. *Let f be a quasimodular form of weight k and depth s. For any $m \in \{0, \ldots, s\}$, we have*

$$Q_m(f)\underset{k-2m}{|}A = \sum_{v=0}^{s-m} \binom{m+v}{v} Q_{m+v}(f)\,\mathrm{X}(A)^v$$

for any $A \in \mathrm{SL}_2(\mathbb{Z})$.

Proof. Since $f\underset{k}{|}AB = \left(f\underset{k}{|}A\right)\underset{k}{|}B$, we have

$$f\underset{k}{|}(AB) = \sum_{n=0}^{s} \left(Q_n(f)\underset{k-2n}{|}B\right) \cdot \left(\mathrm{X}(A)\underset{2}{|}B\right)^n.$$

Since $\mathrm{X}(A)\underset{2}{|}B = \mathrm{X}(AB) - \mathrm{X}(B)$, it follows that

$$f\underset{k}{|}(AB) = \sum_{j=0}^{s} \left(\sum_{n=j}^{s} \binom{n}{j} (-\mathrm{X}(B))^{n-j} \left(Q_n(f)\underset{k-2n}{|}B\right)\right) \mathrm{X}(AB)^j.$$

Comparing with the expression of $f\underset{k}{|}AB$, we obtain

$$Q_j(f) = \sum_{n=j}^{s} \binom{n}{j} (-\mathrm{X}(B))^{n-j} \left(Q_n(f)\underset{k-2n}{|}B\right)$$

for any j. Let

$$M(x)=\left(\binom{\beta-1}{\alpha-1}x^{\beta-\alpha}\right)_{\substack{1\leq\alpha\leq s+1\\ \alpha\leq\beta\leq s+1}}.$$

The matrix $M(x)$ is invertible and upper triangular with inverse $M(x)^{-1}=M(-x)$. We rewrite these equations as

$$M\left(-\mathrm{X}(B)\right)\begin{pmatrix} Q_0(f)\underset{k}{|}B\\ \vdots\\ Q_s(f)\underset{k-2s}{|}B\end{pmatrix}=\begin{pmatrix}Q_0(f)\\ \vdots\\ Q_s(f)\end{pmatrix}$$

so that

$$\begin{pmatrix} Q_0(f)\underset{k}{|}B\\ \vdots\\ Q_s(f)\underset{k-2s}{|}B\end{pmatrix}=M\left(\mathrm{X}(B)\right)\begin{pmatrix}Q_0(f)\\ \vdots\\ Q_s(f)\end{pmatrix}.$$

Finally

$$Q_n(f)\underset{k-2n}{|}B=\sum_{n=j}^{s}\binom{n}{j}Q_n(f)\,\mathrm{X}(B)^{n-j}$$

for any n. This concludes the proof. □

It follows from proposition 3.3 that, if f is a quasimodular function of weight k and depth s then $Q_j(f)$ is a quasimodular function of weight $k-2j$ and depth $s-j$. We add a condition so that $Q_s(f)$ is indeed a modular form.

Definition 3.4. A quasimodular function f of weight k and depth s is a quasimodular form if the Fourier expansions of each $Q_j(f)$ have no terms of negative index :

$$Q_j(f)(z)=\sum_{n=0}^{+\infty}\widehat{Q_j(f)}(n)e^{2i\pi nz}$$

for any $j\in\{0,\dots,s\}$.

We denote by $\mathcal{M}_k^s$ the set of quasimodular forms of weight k and depth s and $\mathcal{M}_k^{\leq s}$ the $\mathbb{C}$-vector space of quasimodular forms of weight k and depth less than or equal to s. We define also

$$\mathcal{M}_k^\infty := \bigcup_{s\in\mathbb{N}} \mathcal{M}_k^{\leq s}.$$

If $f \in \mathcal{M}_k^s$ then $Q_s(f) \in \mathcal{M}_{k-2s}$. There are no modular forms of odd weight or negative weight hence k is even and $s \leq k/2$.

Theorem 3.5. *Let $f \in \mathcal{M}_k^s$ be non constant. Then $\mathrm{D}f \in \mathcal{M}_{k+2}^{s+1}$. More precisely,*

$$Q_0(\mathrm{D}f) = \mathrm{D}f,$$

$$Q_n(\mathrm{D}f) = \mathrm{D}(Q_n f) + \frac{k-n+1}{2i\pi} Q_{n-1}(f)$$

if $1 \leq n \leq s$ and

$$Q_{s+1}(\mathrm{D}f) = \frac{k-s}{2i\pi} Q_s(f).$$

Proof. Since

$$\mathrm{D}\,\mathrm{X}(A) = -\frac{1}{2i\pi}\,\mathrm{X}(A)^2$$

the derivation of (3.1) leads to

$$D\left(f \underset{k}{|} A\right) = \sum_{j=0}^{s} \Big(\mathrm{D}\left(Q_j(f)\right)\mathrm{X}(A)^j - \frac{j}{2i\pi} Q_j(f)\,\mathrm{X}(A)^{j+1}\Big). \tag{3.4}$$

We compare (3.4) with the equality

$$D\left(f \underset{k}{|} A\right) = -\frac{k}{2i\pi}(f \underset{k}{|} A)\,\mathrm{X}(A) + (\mathrm{D}f) \underset{k+2}{|} A$$

and use (3.3) to have

$$\begin{aligned}(\mathrm{D}f) \underset{k+2}{|} A = \sum_{j=0}^{s} \Big(\mathrm{D}\left(Q_j(f)\right)\mathrm{X}(A)^j &- \frac{j}{2i\pi} Q_j(f)\,\mathrm{X}(A)^{j+1} \\ &+ \frac{k}{2i\pi} Q_j(f)\,\mathrm{X}(A)^{j+1}\Big).\end{aligned} \tag{3.5}$$

□

Beside the derivatives of modular forms, we can build another quasimodular form. Let

$$E_2 = \frac{\mathrm{D}\Delta}{\Delta}.$$

Like Δ, this function has a Fourier expansion without any coefficient of negative index. Moreover, it follows from proposition 3.1 that

$$E_2 \underset{2}{|} A = E_2 + \frac{6}{i\pi} \mathrm{X}(A).$$

for any $A \in \mathrm{SL}_2(\mathbb{Z})$. Then E_2 is a quasimodular form of weight 2 and depth 1. It is less easy (but this justifies the notation) to prove that

$$E_2(z) = 1 - 24 \sum_{n=1}^{+\infty} \sigma_1(n) e^{2i\pi nz}.$$

4. **Structure theorems**

We give two structure theorems.

Theorem 4.1. *Let $f \in \mathcal{M}_k^s$. There exists modular forms $F_i \in \mathcal{M}_{k-2i}$ such that*

$$f = \sum_{i=0}^{s} F_i E_2^i.$$

In other words,

$$\mathcal{M}_k^{\leq s} = \bigoplus_{i=0}^{s} \mathcal{M}_{k-2i} E_2^i.$$

Proof. The proof is done recursively using the fact that if f has depth less than or equal to s then

$$f - \left(\frac{i\pi}{6}\right)^s Q_s(f) E_2^s$$

has depth less than or equal to $s - 1$. □

Theorem 4.2. *Let $f \in \mathcal{M}_k^s$. There exist a real number α and modular forms $F_i \in \mathcal{M}_{k-2i}$ such that*

$$f = \begin{cases} \displaystyle\sum_{i=0}^{s} \mathrm{D}^i F_i & si\ s < \dfrac{k}{2} \\ \displaystyle\sum_{i=0}^{k/2-2} \mathrm{D}^i F_i + \alpha\, \mathrm{D}^{k/2-1} E_2 & si\ s = \dfrac{k}{2}. \end{cases}$$

In other words,

$$\mathcal{M}_k^{\infty} = \mathcal{M}_k^{\leq k/2} = \bigoplus_{i=0}^{k/2-2} \mathrm{D}^i \mathcal{M}_{k-2i} \oplus \mathbb{C}\, \mathrm{D}^{k/2-1} E_2.$$

Proof. We proceed by descent on the depth s based to the following fact. We need a modular form $g \in \mathcal{M}_{k-2s}$ satisfying $Q_s(\mathrm{D}^s g) = Q_s(f)$. It would follow $f - D^s g \in \mathcal{M}_k^{\leq s-1}$. Consequently, let $g \in \mathcal{M}_{k-2s}$. Reiterating theorem 3.5, we get

$$Q_s(\mathrm{D}^s g) = \frac{s!}{(2i\pi)^s} \binom{k-s-1}{s} g. \tag{4.1}$$

If $\binom{k-s-1}{s} \neq 0$, wich is the case as soon as $s \neq \frac{k}{2}$, we choose

$$g \coloneqq \frac{(2i\pi)^s}{s!} \frac{1}{\binom{k-s-1}{s}} Q_s(f).$$

It belongs to $\mathcal{M}_{k-2s}$. If $s = \frac{k}{2}$, this procedure is not efficient since the binomial coefficient binomial is vanishing (this corresponds to the fact that $\mathcal{M}_0 = \mathbb{C}$). However, reiterating theorem 3.5, we have

$$Q_{k/2}\left(\mathrm{D}^{k/2-1} E_2\right) = \frac{(k/2-1)!}{(2i\pi)^{k/2-1}} Q_1(E_2) = \frac{(k/2-1)!}{(2i\pi)^{k/2-1}} \frac{6}{i\pi}.$$

Since $Q_{k/2}(f) \in \mathcal{M}_0 = \mathbb{C}$, we define

$$\alpha \coloneqq \frac{i\pi}{6} \frac{(2i\pi)^{k/2-1}}{(k/2-1)!} Q_{k/2}(f) \in \mathbb{C}$$

to obtain $f - \alpha\, \mathrm{D}^{k/2-1} E_2 \in \mathcal{M}_k^{\leq k/2-1}$. It follows that

$$\mathcal{M}_k^{\leq k/2} = \bigoplus_{i=0}^{k/2-1} \mathrm{D}^i \mathcal{M}_{k-2i} \oplus \mathbb{C}\, \mathrm{D}^{k/2-1} E_2.$$

We conclude using $\mathcal{M}_2 = \{0\}$. □

Remark 4.3. Do we really need E_2? We would like to have a modular form $f \in \mathcal{M}_\ell$ and an integer r such that $\mathrm{D}^r f \in \mathcal{M}_k^{k/2}$. This implies $k = \ell + 2r$ and $k/2 = \ell + r$ hence $\ell = 0$. However $\mathcal{M}_0 = \mathbb{C}$ and $\mathrm{D}\mathcal{M}_0 = \{0\}$. Hence, the function E_2 acts like "the nonzero derivative of a modular form of weight 0".

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