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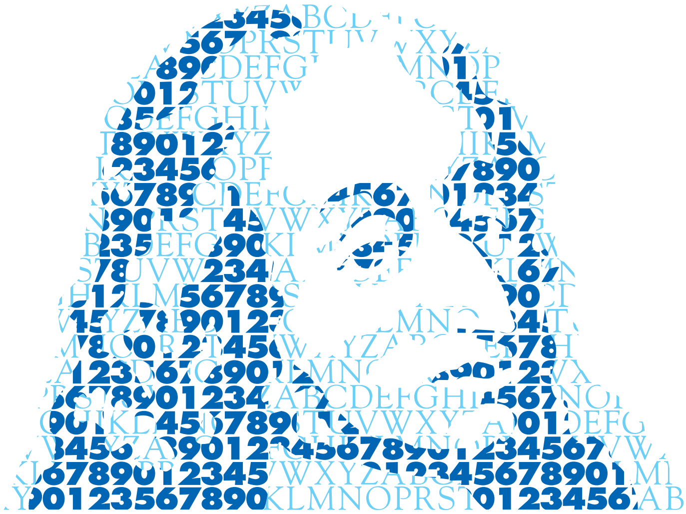

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Monotone Hurwitz Numbers and the HCIZ Integral

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Abstract

In this article, we prove that the complex convergence of the HCIZ free energy is equivalent to the non-vanishing of the HCIZ integral in a neighbourhood of $z = 0$. Our approach is based on a combinatorial model for the Maclaurin coefficients of the HCIZ integral together with classical complex-analytic techniques.

Les nombres de Hurwitz monotones et l'intégrale HCIZ

Résumé

Nous démontrons que la convergence de l'énergie libre de l'intégrale HCIZ dans le plan complexe est équivalente à la non-nullité de l'intégrale HCIZ autour de $z = 0$. Notre approche est basée sur un modèle combinatoire pour les coefficients de Maclaurin de l'intégrale HCIZ et sur des méthodes classiques d'analyse complexe.

1. Introduction

The Harish-Chandra-Itzykson-Zuber integral,

$$I_N(z) = \int_{U(N)} e^{zN\,\mathrm{Tr}(A_N U B_N U^{-1})} \mathrm{d}U, \tag{1.1}$$

is a ubiquitous special function which plays a key role in random matrix theory and related areas. It enters into both the fine-scale spectral analysis of a single random matrix and the macroscopic analysis of several

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interacting random matrices. For further information on these and other aspects of the HCIZ integral, we refer the reader to the surveys [5, 11, 22].

In (1.1), the integration is over the group of $N \times N$ unitary matrices against the normalized Haar measure, z is a complex parameter, and A_N, B_N are any two $N \times N$ complex diagonal matrices. Since $U(N)$ is compact, $I_N(z)$ is an entire function of the complex variable z. Let Ω_N be a simply connected open set in $\mathbb{C}$ which contains the origin and avoids the zeros of $I_N(z)$. The existence of such a domain is guaranteed by the discreteness of zeros of analytic functions together with the evaluation $I_N(0) = 1$. Let $\log I_N(z)$ denote the principal branch of the logarithm of $I_N(z)$ on Ω_N. The holomorphic function

$$F_N(z) = \frac{1}{N^2} \log I_N(z), \quad z \in \Omega_N,$$

is known in the random matrix literature as the *free energy* of the HCIZ integral.

A longstanding conjecture asserts that, if $(A_N)_{N=1}^{\infty}$ and $(B_N)_{N=1}^{\infty}$ are two sequences of complex diagonal matrices which grow in a suitably regular fashion, then $F_N(z)$ converges uniformly on compact subsets of a complex neighbourhood of $z = 0$. This conjecture is implicit in the classic paper [13], and explicitly stated in [1, 2]. Clearly, a necessary condition for this conjecture to hold is the existence of a domain Ω containing the origin such that $I_N(z)$ is non-vanishing on Ω for all but finitely many N. In this paper, we will prove that this condition is also sufficient.

2. **The Leading Derivatives Theorem**

Let $S(d)$ denote the symmetric group acting on $\{1, \ldots, d\}$, and identify $S(d)$ with its (right) Cayley graph as generated by the full conjugacy class of transpositions. Define an edge labelling of the Cayley graph as follows: each edge corresponding to the transposition $\tau = (s\ t)$ is marked by t, the larger of the two numbers interchanged. Thus, emanating from each vertex of the Cayley graph, there is one 2-edge, two 3-edges, three 4-edges, etc.

Definition 2.1. A walk on the Cayley graph of $S(d)$ is said to be *monotone* if the labels of the edges it traverses form a weakly increasing sequence.

Given two permutations $\rho, \sigma \in S(d)$, we denote by $\vec{w}^r(\rho, \sigma)$ the number of r-step monotone walks from ρ to σ. Given two Young diagrams $\alpha, \beta \vdash d$, we denote by

$$\vec{W}^r(\alpha, \beta) = \sum_{\rho \in C_\alpha} \sum_{\sigma \in C_\beta} \vec{w}^r(\rho, \sigma) \tag{2.1}$$

the number of monotone walks beginning in the conjugacy class C_α and ending in the conjugacy class C_β.

Theorem 2.2 (Leading Derivatives Theorem)**.** *For any $N \geq 1$ and any $1 \leq d \leq N$, we have*

$$I_N^{(d)}(0) = \sum_{r=0}^{\infty} \left(-\frac{1}{N}\right)^r \sum_{\alpha, \beta \vdash d} p_\alpha(A_N) p_\beta(B_N) \vec{W}^r(\alpha, \beta),$$

and this series is absolutely convergent. Equivalently, we have

$$I_N(z) = 1 + \sum_{d=1}^{N} \frac{z^d}{d!} \sum_{r=0}^{\infty} \left(-\frac{1}{N}\right)^r \sum_{\alpha, \beta \vdash d} p_\alpha(A_N) p_\beta(B_N) \vec{W}^r(\alpha, \beta) + O(z^{N+1}),$$

where the O-term is uniform on compact subsets of $\mathbb{C}$. Here

$$p_\alpha(A_N) = \prod_{i=1}^{\ell(\alpha)} \operatorname{Tr}(A_N^{\alpha_i}), \quad p_\beta(B_N) = \prod_{j=1}^{\ell(\beta)} \operatorname{Tr}(B_N^{\beta_j}),$$

are the power sum symmetric functions p_α, p_β evaluated on the eigenvalues of A_N, B_N.

In the remainder of this section, we give the proof of Theorem 2.2.

2.1. Differentiation under the integral sign

The derivatives of the entire function (1.1) may be computed by differentiation under the integral sign. In particular, the Maclaurin coefficients of $I_N(z)$ are given by

$$\begin{aligned} I_N^{(d)}(0) &= N^d \int\limits_{U(N)} (\operatorname{Tr} A_N U B_N U^{-1})^d \mathrm{d}U \\ &= N^d \sum_{i,j} a_{i(1)} \dots a_{i(d)} b_{j(1)} \dots b_{j(d)} \int\limits_{U(N)} |u_{i(1)j(1)} \dots u_{i(d)j(d)}|^2 \mathrm{d}U, \end{aligned}$$

where the summation is over all N^{2d} pairs of functions

$$i, j : \{1, \dots, d\} \to \{1, \dots, N\},$$

and $A_N = \operatorname{diag}(a_1, \dots, a_N)$, $B_N = \operatorname{diag}(b_1, \dots, b_N)$.

2.2. The Weingarten function

The integration of monomial functions of matrix elements over $U(N)$ can be addressed using the *Weingarten convolution formula* of Collins and Śniady [3]:

$$\int\limits_{U(N)} u_{i(1)j(1)} \dots u_{i(d)j(d)} \overline{u_{i'(1)j'(1)} \dots u_{i'(d)j'(d)}} \mathrm{d}U = \sum_{\rho,\sigma \in S(d)} \delta_{i,i'\rho} \delta_{j,j'\sigma} \operatorname{Wg}_N(\rho^{-1}\sigma),$$

where $\operatorname{Wg}_N : S(d) \to \mathbb{Q}$ is the *Weingarten function*, which is given by

$$\operatorname{Wg}_N(\pi) = \int\limits_{U(N)} u_{11} \dots u_{dd} \overline{u_{1\pi(1)} \dots u_{d\pi(d)}} \mathrm{d}U$$

for $N \geq d$.

Given a function $i : \{1, \dots, d\} \to \{1, \dots, N\}$, we denote by $\operatorname{Stab}(i)$ the set of permutations $\pi \in S(d)$ such that $i\pi = i$, and given a permutation $\pi \in S(d)$ we denote by $\operatorname{Fix}(\pi)$ the set of functions $i : \{1, \dots, d\} \to \{1, \dots, N\}$ such that $i\pi = i$. Applying the Weingarten convolution formula to $I_N^{(d)}(0)$ with $1 \leq d \leq N$, we obtain

$$\begin{aligned} I_N^{(d)}(0) &= N^d \sum_i \sum_j a_{i(1)} \dots a_{i(d)} b_{j(1)} \dots b_{j(d)} \sum_{\rho \in \mathrm{Stab}(i)} \sum_{\sigma \in \mathrm{Stab}(j)} \mathrm{Wg}_N(\rho^{-1}\sigma) \\ &= N^d \sum_\rho \sum_\sigma \mathrm{Wg}_N(\rho^{-1}\sigma) \sum_{i \in \mathrm{Fix}(\rho)} \sum_{j \in \mathrm{Fix}(\sigma)} a_{i(1)} \dots a_{i(d)} b_{j(1)} \dots b_{j(d)} \\ &= N^d \sum_{\rho,\sigma} \mathrm{Wg}_N(\rho^{-1}\sigma) p_{t(\rho)}(A_N) p_{t(\sigma)}(B_N). \end{aligned} \tag{2.2}$$

2.3. $1/N$-expansion of the Weingarten function

In [16] (see also [15]), it was shown that, for any $N \geq d$ and any $\pi \in S(d)$, the Weingarten function admits the following absolutely convergent expansion in powers of $1/N$:

$$\mathrm{Wg}_N(\pi) = \frac{1}{N^d} \sum_{r=0}^{\infty} (-1)^r \frac{\vec{w}^r(\mathrm{id}, \pi)}{N^r}.$$

Thus

$$\mathrm{Wg}_N(\rho^{-1}\sigma) = \frac{1}{N^d} \sum_{r=0}^{\infty} (-1)^r \frac{\vec{w}^r(\mathrm{id}, \rho^{-1}\sigma)}{N^r} = \frac{1}{N^d} \sum_{r=0}^{\infty} (-1)^r \frac{\vec{w}^r(\rho, \sigma)}{N^r}.$$

Plugging this expansion into (2.2) and changing order of summation, we arrive at

$$\begin{aligned} I_N^{(d)}(0) &= \sum_{\rho,\sigma \in S(d)} p_{t(\rho)}(A_N) p_{t(\sigma)}(B_N) \sum_{r=0}^{\infty} (-1)^r \frac{\vec{w}^r(\rho, \sigma)}{N^r} \\ &= \sum_{r=0}^{\infty} \left(-\frac{1}{N} \right)^r \sum_{\rho,\sigma \in S(d)} p_{t(\rho)}(A_N) p_{t(\sigma)}(B_N) \vec{w}^r(\rho, \sigma), \end{aligned}$$

where $t(\rho) \vdash d$ is the cycle type of ρ, and likewise for $t(\sigma)$. The internal sum may be written

$$\begin{aligned}\sum_{\rho,\sigma\in S(d)} p_{t(\sigma)}(A_N)p_{t(\rho)}(B_N)\vec{w}^r(\rho,\sigma) &= \sum_{\alpha\vdash d}\sum_{\beta\vdash d} p_\alpha(A_N)p_\beta(B_N)\sum_{\rho\in C_\alpha}\sum_{\sigma\in C_\beta}\vec{w}^r(\rho,\sigma)\\ &= \sum_{\alpha\vdash d}\sum_{\beta\vdash d} p_\alpha(A_N)p_\beta(B_N)\vec{W}^r(\alpha,\beta),\end{aligned}$$

and this completes the proof of Theorem 2.2.

3. (Monotone) Hurwitz theory and the HCIZ free energy

3.1. Classical Hurwitz numbers

As shown by Hurwitz in the 19th century [12], the enumeration of unrestricted walks on the symmetric group with given boundary conditions is equivalent to the enumeration of branched covers of the Riemann sphere with given singular data.

To state this precisely, consider the generating function

$$\mathbf{W} = 1 + \sum_{d=1}^{\infty}\frac{z^d}{d!}\sum_{r=0}^{\infty}\frac{t^r}{r!}\sum_{\alpha,\beta\vdash d} p_\alpha(A)p_\beta(B)W^r(\alpha,\beta),$$

where

$$A = \begin{bmatrix} a_1 & & \\ & a_2 & \\ & & \ddots \end{bmatrix}, \quad B = \begin{bmatrix} b_1 & & \\ & b_2 & \\ & & \ddots \end{bmatrix}$$

are a pair of formal infinite diagonal matrices and $W^r(\alpha,\beta)$ is the total number of r-step walks on $S(d)$ which begin in C_α and end in C_β. Thus $\mathbf{W}$ is an element of the formal power series algebra $\mathbb{Q}[[z,t,a_1,a_2,\dots,b_1,b_2,\dots]]$. Set

$$H^r(\alpha,\beta) = \left[\frac{z^d}{d!}\frac{t^r}{r!}p_\alpha(A)p_\beta(B)\right]\mathbf{H},$$

where $\mathbf{H} = \log\mathbf{W}$ and $[X]Y$ denotes the coefficient of term X in a series Y. By the exponential formula [9], the coefficient $H^r(\alpha,\beta)$ is the number of r-step walks beginning in C_α and ending in C_β whose endpoints and steps together generate a transitive subgroup of $S(d)$.

Hurwitz showed that $H^r(\alpha,\beta)/d!$ may be interpreted as a weighted count of degree d branched covers of the Riemann sphere by a compact, connected Riemann surface such that the covering map has profile α over ∞, β over 0, and simple branching over each of the r^{th} roots of unity. According to the Riemann-Hurwitz formula, such a cover exists if and only if

$$g = \frac{r+2-\ell(\alpha)-\ell(\beta)}{2}$$

is a nonnegative integer, in which case it is the topological genus of the covering surface. We will use the notation $H^r(\alpha,\beta) = H_g(\alpha,\beta)$, with the understanding that r and g determine one another via the Riemann-Hurwitz formula.

The numbers $H_g(\alpha,\beta)$ were first considered from a modern perspective by Okounkov [17], who called them the *double Hurwitz numbers*. Proving a conjecture of Pandharipande [18] in Gromov-Witten theory, Okounkov showed that $\mathbf{H}$ is a solution of the 2D Toda lattice hierarchy of Ueno and Takasaki. It was subsequently shown by Kazarian and Lando [14] that, when combined with the ELSV formula [4], Okounkov's result implies the celebrated Kontsevich-Witten Theorem relating intersection theory in moduli spaces of Riemann surfaces to the KdV hierarchy.

3.2. Monotone Hurwitz numbers

Mimicking the above classical construction, consider the generating function

$$\vec{\mathbf{W}} = 1 + \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{r=0}^{\infty} t^r \sum_{\alpha,\beta \vdash d} p_\alpha(A_N) p_\beta(B_N) \vec{W}^r(\alpha,\beta)$$

enumerating *monotone* walks of all possible lengths and boundary conditions on all of the finite symmetric groups. Note that, due to the monotonicity constraint, the variable t is now an ordinary rather than exponential marker for the walk length statistic. Define the *monotone double Hurwitz numbers* by

$$\vec{H}^r(\alpha,\beta) = \left[\frac{z^d}{d!} t^r p_\alpha(A) p_\beta(B)\right] \vec{\mathbf{H}},$$

where $\vec{\mathbf{H}} = \log \vec{\mathbf{W}}$. Then $\vec{H}^r(\alpha,\beta)$ is the number of r-step monotone walks beginning in C_α and ending in C_β whose endpoints and steps together generate a transitive subgroup of $S(d)$. Alternatively, $\vec{H}^r(\alpha,\beta)/d!$ counts a combinatorially restricted subset of the branched covers counted by $H^r(\alpha,\beta)/d!$. In [7, 8], an extensive combinatorial theory of monotone Hurwitz numbers was developed, and monotone analogues of most combinatorial results in classical Hurwitz theory were obtained. Here we will use those results in tandem with Theorem 2.2.

3.3. The HCIZ free energy

The monotone double Hurwitz numbers describe the Maclaurin coefficients of the HCIZ free energy.

As in the Introduction, let Ω_N be a simply connected open set in $\mathbb{C}$ which contains the origin and avoids the zeros of $I_N(z)$, and let $\log I_N(z)$ denote the principal branch of the logarithm of $I_N(z)$ on Ω_N. From Theorem 2.2 and the exponential formula, it follows that the Maclaurin series of the logarithm is

$$\log I_N(z) = \sum_{d=1}^{N} \frac{z^d}{d!} \sum_{r=0}^{\infty} \left(-\frac{1}{N}\right)^r \sum_{\alpha,\beta\vdash d} p_\alpha(A_N)p_\beta(B_N)\vec{H}^r(\alpha,\beta) + O(z^{N+1}),$$

where the O-term is uniform on compact subsets of Ω_N.

The Maclaurin series of $\log I_N(z)$ can be simplified using the Riemann-Hurwitz formula. We have:

$$\begin{aligned}
\log I_N(z) &= \sum_{d=1}^{N} \frac{z^d}{d!} \sum_{\alpha,\beta\vdash d} p_\alpha(A_N)p_\beta(B_N) \sum_{r=0}^{\infty} \left(-\frac{1}{N}\right)^r \vec{H}^r(\alpha,\beta) + O(z^{N+1}) \\
&= \sum_{d=1}^{N} \frac{z^d}{d!} \sum_{\alpha,\beta\vdash d} p_\alpha(A_N)p_\beta(B_N) \sum_{g=0}^{\infty} \left(-\frac{1}{N}\right)^{2g-2+\ell(\alpha)+\ell(\beta)} \vec{H}_g(\alpha,\beta) + O(z^{N+1}) \\
&= N^2 \sum_{d=1}^{N} \frac{z^d}{d!} \sum_{g=0}^{\infty} \frac{1}{N^{2g}} \sum_{\alpha,\beta\vdash d} (-1)^{\ell(\alpha)+\ell(\beta)} \frac{p_\alpha(A_N)}{N^{\ell(\alpha)}} \frac{p_\beta(B_N)}{N^{\ell(\beta)}} \vec{H}_g(\alpha,\beta) + O(z^{N+1}),
\end{aligned}$$

with the O-term uniform on compact subsets of Ω_N. Thus, parameterizing the monotone double Hurwitz numbers by g instead of r, we arrive at a description of $F_N(z) = N^{-2} \log I_N(z)$ which is well-poised for an $N \to \infty$ asymptotic analysis.

3.4. Asymptotic expansion of Maclaurin coefficients

We will now consider the asymptotic behaviour of the Maclaurin coefficients of $F_N(z)$ as $N \to \infty$, when A_N, B_N vary regularly with N.

Suppose there exists a nonnegative number M, a nonnegative integer h, and two complex sequences ϕ_k, ψ_k such that:

(1) $\|A_N\|, \|B_N\| \leq M$ for all $N \geq 1$;

(2) For each $k \geq 1$,

$$\frac{1}{N}\operatorname{Tr}(A_N^k) = \phi_k + o\left(\frac{1}{N^{2h}}\right), \quad \frac{1}{N}\operatorname{Tr}(B_N^k) = \psi_k + o\left(\frac{1}{N^{2h}}\right)$$

as $N \to \infty$.

We will summarize these conditions by saying that A_N, B_N are (M,h)-regular with limit moments ϕ_k, ψ_k.

Theorem 3.1. *Suppose that A_N, B_N are (M,h)-regular with limit moments ϕ_k, ψ_k. Then, for each $d \geq 1$, we have*

$$F_N^{(d)}(0) = \sum_{g=0}^{h} \frac{C_{g,d}}{N^{2g}} + o\left(\frac{1}{N^{2h}}\right)$$

as $N \to \infty$, where

$$C_{g,d} = \sum_{\alpha,\beta \vdash d} (-1)^{\ell(\alpha)+\ell(\beta)} \phi_\alpha \psi_\beta \vec{H}_g(\alpha,\beta),$$

and

$$\phi_\alpha = \prod_{i=1}^{\ell(\alpha)} \phi_{\alpha_i}, \quad \psi_\beta = \prod_{j=1}^{\ell(\beta)} \phi_{\beta_j}.$$

Proof. For $1 \leq d \leq N$, we have

$$F_N^{(d)}(0) = \sum_{g=0}^{\infty} \frac{C_{g,d,N}}{N^{2g}},$$

where

$$C_{g,d,N} = \sum_{\alpha,\beta \vdash d} (-1)^{\ell(\alpha)+\ell(\beta)} \frac{p_\alpha(A_N)}{N^{\ell(\alpha)}} \frac{p_\beta(B_N)}{N^{\ell(\beta)}} \vec{H}_g(\alpha,\beta).$$

Since $\|A_N\|, \|B_N\| \leq M$ for all $N \geq 1$, we have

$$|C_{g,d,N}| \leq M^{2d} \sum_{\alpha,\beta \vdash d} \vec{H}_g(\alpha,\beta)$$

for all $N \geq 1$. Since $\vec{H}_g(\alpha,\beta)$ counts certain solutions of the equation

$$\sigma = \rho\tau_1 \dots \tau_r$$

in $S(d)$, with $r = 2g-2+\ell(\alpha)+\ell(\beta)$, we have

$$\vec{H}_g(\alpha,\beta) \leq (d!)^{2g+2d}$$

for all $\alpha, \beta \vdash d$. Consequently,

$$|C_{g,d,N}| \leq (d!p(d)M)^{2d}(d!)^{2g},$$

where $p(d)$ is the number of Young diagrams with d cells. Moreover, since A_N, B_N are h-regular with limit moments ϕ_k, ψ_k, we have

$$C_{g,d,N} = C_{g,d} + o\left(\frac{1}{N^{2h}}\right)$$

as $N \to \infty$. We thus have

$$\begin{aligned} F_N^{(d)}(0) &= \sum_{g=0}^{h} \frac{C_{g,d,N}}{N^{2g}} + \sum_{g=h+1}^{\infty} \frac{C_{g,d,N}}{N^{2g}} \\ &= \sum_{g=0}^{h} \frac{C_{g,d}}{N^{2g}} + o\left(\frac{1}{N^{2h}}\right) + O\left(\frac{1}{N^{2h+2}}\right) \end{aligned}$$

as $N \to \infty$, which proves the claim. □

Remark 3.2. The convergence of $F_N^{(d)}(0)$ under the above hypotheses was first stated by Itzykson and Zuber [13], and proved by Collins [1]. Collins obtained the limit of $F_N^{(d)}(0)$ as a double sum over $S(d)$, whereas we present the same limit as a double sum over partitions of d.

Remark 3.3. To the best of our knowledge, Theorem 3.1 is the first result which addresses the sub-leading asymptotics of $F_N^{(d)}(0)$, clearly showing the emergence of a "topological expansion" in this context.

4. Absolute summability of genus-specific generating functions

Let us introduce a sequence of formal power series $C_g(z) \in \mathbb{C}[[z]]$ defined by

$$C_g(z) = \sum_{d=1}^{\infty} \frac{z^d}{d!} C_{g,d}, \quad g \geq 0, \tag{4.1}$$

where $C_{g,d}$ are the expansion coefficients from Theorem 3.1. The goal of this section is to prove that these formal power series are absolutely summable, and to obtain bounds on their radii of convergence. By the uniform boundedness of $\|A_N\|$ and $\|B_N\|$, this reduces to establishing the absolute summability of the series

$$\vec{\mathbf{H}}_g(z) = \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{\alpha,\beta \vdash d} \vec{H}_g(\alpha,\beta), \quad g \geq 0. \tag{4.2}$$

4.1. Monotone simple Hurwitz numbers

Consider the genus-specific generating functions

$$\vec{\mathbf{S}}_g = \sum_{d=1}^{\infty} \frac{z^d}{d!} \vec{H}_{g,d}$$

for the monotone *simple* Hurwitz numbers

$$\vec{H}_{g,d} = \vec{H}_g(1^d, 1^d).$$

The monotone simple Hurwitz number $\vec{H}_{g,d}$ counts monotone loops of length $r = 2g - 2 + 2d$, based at any given point of $S(d)$, whose steps generate a transitive subgroup of $S(d)$.

According to [8, Theorem 1.4], for any $g \geq 2$ we have

$$\vec{\mathbf{S}}_g = \frac{\zeta(1-2g)}{2-2g} + \frac{1}{(1-6s)^{2g-2}} \sum_{r=0}^{3g-3} \sum_{\mu \vdash r} \frac{c_{g,\mu}(6s)^{\ell(\mu)}}{(1-6s)^{\ell(\mu)}},$$

where ζ is the Riemann zeta function, the $c_{g,\mu}$'s are rational numbers, and s is the unique solution of the functional equation

$$s = z(1-2s)^{-2}$$

in the formal power series algebra $\mathbb{Q}[[z]]$. This equation may be solved by Lagrange inversion, yielding the solution

$$s = \sum_{n=1}^{\infty} \frac{2^{n-1}}{n} \binom{3n-2}{n-1} z^n.$$

Since

$$s' = \sum_{n=0}^{\infty} \binom{3n+1}{n} (2z)^n = {}_2F_1\left(\frac{2}{3}, \frac{4}{3}, \frac{3}{2}; \frac{27}{2} z\right),$$

where ${}_2F_1(a, b, c; z)$ is the Gauss hypergeometric function, $\vec{\mathbf{S}}_g$ extends to a holomorphic function of z on the domain $\mathbb{C} \backslash [z_c, \infty)$, where

$$z_c = \frac{2}{27}.$$

In the case $g = 0$, [7, Theorem 1.1] yields the exact formula

$$\frac{\vec{H}_{0,d}}{d!} = \frac{2^{d-1}}{d^2(d-1)} \binom{3d-3}{d-1},$$

so that, using Stirling's formula, we obtain $\frac{2}{27}$ as the radius of convergence of $\vec{\mathbf{S}}_0$. Moreover, since $\vec{\mathbf{S}}_0$ has positive coefficients, Pringsheim's Theorem (see [20, §7.21]) guarantees that $z_c = \frac{2}{27}$ is a singularity of $\vec{\mathbf{S}}_0$.

We have thus shown that:

Theorem 4.1. *The series $\vec{\mathbf{S}}_g$, $g \geq 0$, have a common dominant singularity at the critical point $z_c = \frac{2}{27}$.*

Remark 4.2. The $g = 0$ case of Theorem 4.1 was obtained by Zinn-Justin in [21] using the Toda lattice equations. For a combinatorial solution of the Toda equations encompassing those of Okounkov and Zinn-Justin, see [6].

Remark 4.3. P. Di Francesco has pointed out to us that the critical value $z_c = \frac{2}{27}$ also appears in the enumeration of finite groups, see [19].

4.2. **Monotone double Hurwitz numbers**

Consider the full genus-specific generating functions $\vec{\mathbf{H}}_g(z)$ defined in equation (4.2). Obviously,

$$\sum_{\alpha,\beta\vdash d} \vec{H}_g(\alpha,\beta) \geq \vec{H}_{g,d},$$

so the radius of convergence of $\vec{\mathbf{H}}_g$ is at most the radius of convergence of $\vec{\mathbf{S}}_g$.

We now explain how a lower bound on the radius of convergence of $\vec{\mathbf{H}}_g$ follows from the study of a refinement of the monotone double Hurwitz number $\vec{H}_g(\alpha,\beta)$. Given a positive integer c, let $\vec{H}_g(\alpha,\beta;c)$ denote the number of walks counted by $\vec{H}_g(\alpha,\beta)$ whose steps have c distinct labels. The following inequality is obtained in [10]:

$$\sum_{c=2}^{d} 3^c \vec{H}_g(1^d,1^d;c) \leq \sum_{\alpha,\beta\vdash d} \vec{H}_g(\alpha,\beta) \leq \sum_{c=2}^{d} 4^c \vec{H}_g(1^d,1^d;c).$$

The proof is combinatorial, and makes use of an action of the symmetric group $S(r)$ on the set of walks counted by the classical double Hurwitz number $H_g(\alpha,\beta)$. From this inequality and the definition of $\vec{H}_g(\alpha,\beta;c)$, we obtain

$$\sum_{\alpha,\beta\vdash d} \vec{H}_g(\alpha,\beta) \leq 4^{d-1} \sum_{c=2}^{d} \vec{H}_g(1^d,1^d;c) = 4^{d-1}\vec{H}_{g,d},$$

which implies that the radius of convergence of $\vec{\mathbf{H}}_g$ is at least one quarter the radius of convergence of $\vec{\mathbf{S}}_g$. Combining this with Theorem 4.1, we have:

Theorem 4.4. *For each $g \geq 0$, the series $\vec{\mathbf{H}}_g$ is absolutely summable, and has radius of convergence at least $\frac{1}{54}$ and at most $\frac{2}{27}$.*

Remark 4.5. The results of [2] also imply the absolute summability of $\vec{\mathbf{H}}_0$, but without effective bounds on the radius of convergence.

5. Convergence of the HCIZ free energy

5.1. Convergence under a non-vanishing hypothesis

Let A_N, B_N be $(M,0)$-regular with limit moments ϕ_k, ψ_k, respectively. Then, by Theorem 4.4, the series $C_0(z)$ is absolutely summable, with radius of convergence at least $\frac{1}{54M^2}$.

Theorem 5.1. *Suppose that there exists a positive number R such that $I_N(z)$ is non-vanishing on the open disc $D(0,R)$ for all but finitely many N. Let*

$$r = \min\left\{R, \frac{1}{54M^2}\right\}.$$

Then $F_N(z) \to C_0(z)$ uniformly on compact subsets of $D(0,r)$.

Proof. First, we show that the claim holds pointwise. The proof is a two circles argument. Let $z_0 \in D(0,r)$ and $\varepsilon > 0$ be arbitrary. Choose r_1, r_2 such that

$$|z_0| < r_1 < r_2 < r.$$

For N sufficiently large, we have

$$|F_N(z_0) - C_0(z_0)| \leq \sum_{d=1}^{\infty} |F_N^{(d)}(0) - C_{0,d}| \frac{|z_0|^d}{d!}.$$

By Cauchy's inequality,

$$\frac{1}{d!}|F_N^{(d)}(0) - C_{0,d}| \leq \frac{\|F_N - C_0\|_{r_1}}{r_1^d},$$

where $\|\cdot\|_{r_1}$ denotes sup-norm over the circle of radius r_1. We thus have

$$|F_N(z_0)-C_0(z_0)| \leq \sum_{d=1}^{E} |F_N^{(d)}(0)-C_{0,d}|\frac{|z_0|^d}{d!} + \frac{\|F_N\|_{r_1}+\|C_0\|_{r_1}}{1-\frac{|z_0|}{r_1}}\left(\frac{|z_0|}{r_1}\right)^{E+1}$$

for any positive integer E.

We will now bound the error term independently of N. Since

$$\begin{aligned}|I_N(z)| &\leq \int\limits_{U(N)} e^{|z|N|\operatorname{Tr}(A_NUB_NU^{-1})|}\mathrm{d}U \\ &\leq \int\limits_{U(N)} e^{|z|M^2N\sum_{i,j=1}^N |u_{ij}|^2}\mathrm{d}U \\ &\leq e^{M^2N^2|z|}\end{aligned}$$

for all $z \in \mathbb{C}$, where the last line follows from the fact that $(|u_{ij}|^2)$ is a doubly stochastic matrix, the inequality

$$\Re F_N(z) \leq M^2|z|$$

holds for all $z \in \Omega_N$, the domain of holomorphy of $F_N(z)$. Combining this with the Borel-Carathéodory inequality, which bounds the sup-norm of an analytic function on a circle in terms of the maximum of its real part on a circle of larger radius (see e.g. [20, §5.5]), we have

$$\|F_N\|_{r_1} \leq \frac{2r_1}{r_2-r_1}\sup_{|z|=r_2}\Re F_N(z) \leq \frac{2M^2r_1r_2}{r_2-r_1}.$$

Returning to our estimate on $|F_N(z_0)-C_0(z_0)|$, we now have the inequality

$$|F_N(z_0)-C_0(z_0)| \leq \sum_{d=1}^{E} |F_N^{(d)}(0)-C_{0,d}|\frac{|z_0|^d}{d!} + \frac{\frac{2M^2r_1r_2}{r_2-r_1}+\|C_0\|_{r_1}}{1-\frac{|z_0|}{r_1}}\left(\frac{|z_0|}{r_1}\right)^{E+1}$$

for all N sufficiently large and all $E \geq 1$. Choosing E_0 large enough so that

$$\frac{\frac{2M^2r_1r_2}{r_2-r_1}+\|C_0\|_{r_1}}{1-\frac{|z_0|}{r_1}}\left(\frac{|z_0|}{r_1}\right)^{E+1} < \frac{\varepsilon}{2},$$

and subsequently choosing N_0 large enough so that

$$\sum_{d=1}^{E_0} |F_{N_0}^{(d)}(0) - C_{0,d}| \frac{|z_0|^d}{d!} < \frac{\varepsilon}{2},$$

we obtain that

$$|F_N(z_0) - C_0(z_0)| < \varepsilon$$

for all $N \geq N_0$. This completes the proof.

We now explain how the mode of convergence can be boosted from pointwise to uniform on compact subsets of $D(0,r)$. It is easy to check that

$$\sup_N \|F_N\|_K < \infty$$

for each compact set $K \subset D(0,r)$; for example, one could prove this holds for closed discs by using the Borel-Carathéodory inequality again. Thus $\{F_N\}$ is a locally uniformly bounded family on $D(0,r)$, and Vitali's theorem [20, §5.21] implies pointwise and uniform-on-compact convergence are equivalent. □

5.2. Remarks on the non-vanishing hypothesis

Theorem 5.1 reduces the complex convergence of $F_N(z)$ near $z = 0$ to checking that the zeros of $I_N(z)$ do not encroach on $z = 0$ as $N \to \infty$. Therefore it is of significant interest to determine sufficient hypotheses on the matrix sequences A_N and B_N which ensure that this condition holds.

Determining the exact locations of the zeros of $I_N(z)$ seems to be a difficult problem in general. Let us give one example where it can be solved explicitly. Suppose that A_N and B_N are real diagonal with distinct eigenvalues

$$a_1 < \dots < a_N, \quad b_1 < \dots < b_N,$$

and suppose further that the eigenvalues of A_N form an arithmetic progression with constant difference $\hbar > 0$. In this case, all determinants in the HCIZ formula (see [13, Equation 3.28]) are Vandermonde determinants, and one computes that the zeros of $I_N(z)$ are the pure imaginary points

$$z = \frac{2k\pi}{N\hbar(b_j - b_i)}\mathbf{i}, \quad 1 \leq i < j \leq N, \quad k \in \mathbb{Z}.$$

Thus one has a sort of Lee-Yang theorem in this special case. If the eigenvalues of A_N and B_N are confined to a fixed interval $[-M, M]$ for all N, then $\hbar$ is of order $1/N$ and the above formula shows that the zeros of $I_N(z)$ remain at bounded distance from the origin as N increases.

Concerning the verification of the non-vanishing hypothesis more generally, we do not know much at present. It might be that the Toda equations could be of help here, but we have not pursued this seriously.

5.3. **A conjecture**

In this article, we have produced explicit holomorphic candidates $C_g(z)$ for the asymptotics of $F_N(z)$ near $z = 0$. Indeed, if $F_N(z)$ admits a uniform-on-compact asymptotic expansion on the asymptotic scale N^{-2} in some neighbourhood Ω of $z = 0$, then the coefficients in this expansion *must* be the functions $C_g(z)$. We therefore believe that the following conjecture is reasonable: If A_N, B_N are (M, h)-regular with limit moments ϕ_k, ψ_k, then there exists a neighbourhood Ω of $z = 0$ such that

$$F_N(z) = \sum_{g=0}^{h} \frac{C_g(z)}{N^{2g}} + o\left(\frac{1}{N^{2h}}\right)$$

uniformly on compact subsets of Ω.

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