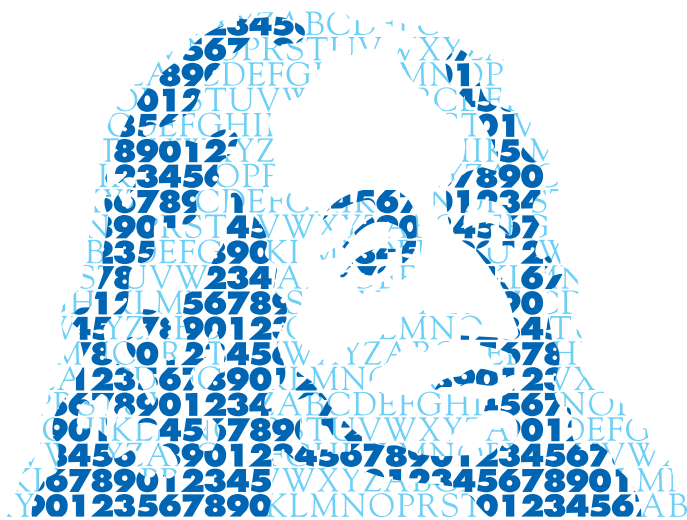


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Norm inequalities in some subspaces of Morrey space

JUSTIN FEUTO

Abstract

We give norm inequalities for some classical operators in amalgam spaces and in some subspaces of Morrey space.

*Inégalités en norme dans certains sous-espaces
d'espaces de Morrey*

Résumé

Nous établissons des inégalités en norme pour certains opérateurs classiques dans les amalgames et certains sous-espaces d'espaces de Morrey.

1. Introduction

For $1 \leq p, q \leq \infty$, the amalgam of L^q and L^p is the space (L^q, ℓ^p) of functions f on the d -dimensional euclidean space \mathbb{R}^d which are locally in L^q and such that the sequence $\left\{ \|f \chi_{Q_k}\|_q \right\}_{k \in \mathbb{Z}^d}$ belongs to $\ell^p(\mathbb{Z}^d)$, where $Q_k = \prod_{i=1}^d [k_i, k_i + 1)$, χ_{Q_k} denoting the characteristic function of Q_k and $\|\cdot\|_q$ the usual Lebesgue norm in L^q .

Amalgams arise naturally in harmonic analysis and were introduced by N. Wiener in 1926. But its systematic study goes back to the work of Holland [18]. We refer the reader to the survey paper of Fournier and Stewart [14] for more information about these spaces. We list here some of their basic properties.

Let $1 \leq p, q \leq \infty$.

- $(L^q, \ell^q) = L^q$

Keywords: Amalgams spaces, fractional maximal operator, Riesz potential, Hilbert transform.

Math. classification: 42B35, 42B20, 42B25.

- $L^q \cup L^p \subset (L^q, \ell^p)$ if $q \leq p$,
- $(L^q, \ell^p) \subset L^q \cap L^p$ if $p \leq q$,
- (L^q, ℓ^p) is a Banach space when equipped with the norm

$$\|f\|_{q,p} = \left\| \left\{ \|f\chi_{Q_k}\|_q \right\}_{k \in \mathbb{Z}^d} \right\|_{\ell^p}$$

if we identify functions that differ only on null subset of \mathbb{R}^d .

- For $1 \leq p, q < \infty$, the dual space of (L^q, ℓ^p) is $(L^{q'}, \ell^{p'})$.

In this definition of amalgam spaces, we can replace the cubes Q_k of side length 1 by cubes $Q_k^r = \prod_{i=1}^d [rk_i, r(k_i + 1))$ of side length r or by balls, and we can also consider continuous summation instead of discrete. More precisely, for $r > 0$, we put

$$r \|f\|_{q,p} := \left\| \left\{ \|f\chi_{Q_k^r}\|_q \right\}_{k \in \mathbb{Z}^d} \right\|_{\ell^p}$$

and

$$r \widetilde{\|f\|}_{q,p} = \begin{cases} \left(\int_{\mathbb{R}^d} \|f\chi_{B(y,r)}\|_q^p dy \right)^{\frac{1}{p}} & \text{if } p < \infty \\ \text{ess sup}_{y \in \mathbb{R}^d} \|f\chi_{B(y,r)}\|_q & \text{if } p = \infty \end{cases},$$

where $B(y, r)$ is the ball centered at y with radius r . It is easy to see that for any $r > 0$, and $f \in (L^q, \ell^p)$, there exists a constant $C_r > 0$ depending only on r such that

$$C_r^{-1} \|f\|_{q,p} \leq r \|f\|_{q,p} \leq C_r \|f\|_{q,p},$$

while

$$r \widetilde{\|f\|}_{q,p} \approx r^{\frac{d}{p}} \|f\|_{q,p}. \tag{1.1}$$

We can also consider on amalgam spaces, the continuous norm described in Dobler's master thesis [6] (see also [9] in the case of Wiener algebra).

Many classical results established in Fourier analysis on Lebesgue spaces have extensions in amalgams. For example, Hölder and Young inequalities are just a consequence of the analog in Lebesgue space [14, 3]. The Hardy-Littlewood-Sobolev inequality for fractional integrals has been generalized

¹Hereafter we propose the following abbreviation $\mathbf{A} \approx \mathbf{B}$ for the inequalities $C^{-1}\mathbf{A} \leq \mathbf{B} \leq C\mathbf{A}$, where C is a positive constant independent of the main parameters.

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to amalgam spaces by Cowling et al in [5]. In fact, they proved a more general result which can be formulated as follows: let $\mathbf{Q} = [-1, 1]^d$ and $K : \mathbb{R}^d \rightarrow \mathbb{C}$ be a measurable function such that

$$|K(x)| \leq D |x|^{\gamma-d} \chi_{\mathbf{Q}}(x) + D |x|^{\beta-d} \chi_{\mathbb{R}^d \setminus \mathbf{Q}}(x) \quad (1.2)$$

where $0 < \gamma, \beta < d$. Then the operator \mathcal{I}_γ^β defined by

$$\mathcal{I}_\gamma^\beta f(x) = \int_{\mathbb{R}^d} K(x-y) f(y) dy$$

is bounded from (L^q, ℓ^p) to (L^{q^*}, ℓ^{p^*}) , with $0 < \frac{1}{q^*} = \frac{1}{q} - \frac{\gamma}{d}$ and $0 < \frac{1}{p^*} = \frac{1}{p} - \frac{\beta}{d}$. An immediate consequence of the above result is the boundedness of the Riesz potential I_γ from (L^q, ℓ^p) to (L^{q^*}, ℓ^{p^*}) . We recall that $I_\gamma f$ is defined by

$$I_\gamma f(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-\gamma}} dy \quad (1.3)$$

when the integral exists.

There are classical properties of Lebesgue spaces which are not fulfilled in amalgam spaces. For example, when $q < p$ the translation operators $\tau_x : f \mapsto f(\cdot - x)$ for $x \in \mathbb{R}^d$ which are isometric in Lebesgue spaces are just uniformly bounded in amalgam spaces equipped with the norm $\|\cdot\|_{q,p}$ (it is isometric when one uses the continuous norm of Dobler). Dilation operators $\delta_r^q : f \mapsto r^{\frac{d}{q}} f(r \cdot)$ also behave differently in these spaces. In fact, there is no real number $\alpha > 0$ for which we have

$$\|\delta_r^q f\|_{q,p} \approx \|f\|_{q,p} \quad r > 0. \quad (1.4)$$

Fofana in [13] (see also [11, 12]) considered normed spaces denoted $(L^q, \ell^p)^\alpha$ which are subspaces of (L^q, ℓ^p) , satisfying property (1.4), and named these spaces integrable fractional mean function spaces. For $1 \leq q < \alpha$ fixed and p going from α to ∞ , these spaces form a chain of Banach spaces beginning with Lebesgue space L^α and ending by the classical Morrey's space $L^{q, d(1-\frac{q}{\alpha})} = (L^q, \ell^\infty)^\alpha$. We will see in the next paragraph that the spaces in the chain are distinct.

In this paper we give extensions of norm inequalities in Lebesgue or Morrey spaces to the setting of $(L^q, \ell^p)^\alpha$ with $1 \leq q \leq \alpha \leq p \leq \infty$. This is often done by using the relation (1.4) and known results in amalgam spaces.

The remaining of this paper is organized as follows:

In the second paragraph, we recall the definition of $(L^q, \ell^p)^\alpha$ spaces and some of their basic properties. Paragraph three is devoted to some norm inequalities in amalgams of Lebesgue and Lorentz spaces which we do not see in the literature, while in paragraph four we establish norm inequalities for some classical operators in the context of our spaces.

Throughout the paper, the letter C is used for non-negative constants that may change from one occurrence to another. Constants with subscript, such as C_0 , do not change in different occurrences. If E is a measurable subset of \mathbb{R}^d , then $|E|$ stands for its Lebesgue measure. The notation $\mathbf{A} \preccurlyeq \mathbf{B}$ will always mean that the ratio \mathbf{A}/\mathbf{B} is bounded away from zero by a constant independent of the relevant variables in \mathbf{A} and \mathbf{B} .

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2. Definition and basic properties of $(L^q, \ell^p)^\alpha$ spaces

For $1 \leq q, p, \alpha \leq \infty$, the space $(L^q, \ell^p)^\alpha := (L^q, \ell^p)^\alpha(\mathbb{R}^d)$ consists of those elements of (L^q, ℓ^p) such that

$$\|f\|_{q,p,\alpha} := \sup_{r>0} \|\delta_r^\alpha f\|_{q,p} < \infty,$$

with the usual convention that $\frac{1}{\infty} = 0$. As proved in [11, 13] the space $(L^q, \ell^p)^\alpha$ is non trivial if and only if $q \leq \alpha \leq p$; thus in the remaining of the paper we will always assume that this condition is fulfilled. We have the following properties.

Proposition 2.1 ([11, 12, 13]).

- (1) $((L^q, \ell^p)^\alpha, \|\cdot\|_{q,p,\alpha})$ is a complex Banach space.
- (2) If $1 \leq q_1 \leq q_2 \leq \alpha \leq p_1 \leq p_2 \leq \infty$, then we have

$$\|f\|_{q_1,p_2,\alpha} \leq \|f\|_{q_2,p_1,\alpha} \leq C \|f\|_\alpha.$$

Notice that

$$r \|f\|_{q,p,\alpha} := \|\delta_r^\alpha f\|_{q,p} = r^{d(\frac{1}{\alpha}-\frac{1}{q})} r \|f\|_{q,p}, \quad (2.1)$$

so that

$$\|f\|_{q,p,\alpha} = \sup_{r>0} r^{d(\frac{1}{\alpha}-\frac{1}{q})} r \|f\|_{q,p} \approx \sup_{r>0} r^{d(\frac{1}{\alpha}-\frac{1}{p}-\frac{1}{q})} r \widetilde{\|f\|}_{q,p}. \quad (2.2)$$

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Proposition 2.2. *Let $1 \leq q \leq \alpha \leq p \leq \infty$.*

(1) *For $f \in (L^q, \ell^p)^\alpha$, we have*

$$\|\delta_r^\alpha f\|_{q,p,\alpha} = \|f\|_{q,p,\alpha}.$$

(2) *The space $(L^q, \ell^p)^\alpha$ is the biggest norm space which is continuously included in (L^q, ℓ^p) and for which $\sup_{r>0} \|\delta_r^\alpha f\| < \infty$.*

Proof. The first assertion is an immediate consequence of the definition of the norm $\|\cdot\|_{q,p,\alpha}$. For the second, let $(E, \|\cdot\|)$ be a norm space for which there exists $C > 0$ such that we have

$$\|f\|_{q,p} \leq C \|f\| \quad \text{and} \quad \sup_{r>0} \|\delta_r^\alpha f\| < \infty, \quad (2.3)$$

for all $f \in E$. Then for $f \in E$, we have

$$\sup_{r>0} \|\delta_r^\alpha f\|_{q,p} \leq C \sup_{r>0} \|\delta_r^\alpha f\| < \infty,$$

so that $f \in (L^q, \ell^p)^\alpha$ by definition. □

As we say in the introduction, the family $\{(L^q, \ell^p)^\alpha\}_{\alpha \leq p \leq \infty}$ consists of distinct spaces. To see this on the real line, we let $q = 1$. A positive function f on \mathbb{R} belongs to $(L^1, \ell^p)^\alpha$ if and only if there exists a constant $C < \infty$ such that

$$r^{\frac{1}{\alpha}} \|E_r f\|_{\ell^p} \leq C, \quad r > 0 \quad (2.4)$$

where $E_r f$ is the sequence defined by

$$(E_r f)_k = \frac{1}{r} \int_{r(k+I)} f(y) dy,$$

where $I = [0, 1)$. Notice that it is enough to have condition (2.4) just for $r = 2^m$, $m \in \mathbb{Z}$. Let $1 \leq p_1 < p_2 < \infty$, and $\mathbf{a} = (\mathbf{a}_n)_{n \in \mathbb{N}}$ be a sequence of positive real numbers which belongs to ℓ^{p_2} without being in ℓ^{p_1} . Put for all $k \in \mathbb{Z}$

$$\lambda_k = \begin{cases} 0 & \text{if } k \neq 2^n \text{ for all } n \in \mathbb{N} \\ \mathbf{a}_n & \text{if } k = 2^n \end{cases},$$

and let us consider the function f defined by $f(x) = \lambda_k$ if $x \in k+I$, $k \in \mathbb{Z}$. We claim that $f \in (L^1, \ell^{p_2})^\alpha \setminus (L^1, \ell^{p_1})^\alpha$. The function does not belong to (L^1, ℓ^{p_1}) , since the sequence $\mathbf{a} \notin \ell^{p_1}$. Let us prove now that $f \in (L^1, \ell^{p_2})^\alpha$. Fix $r = 2^m$, with $m \in \mathbb{Z}$.

- If $m \leq 0$ then $(rk, r(k+1)) \cap \mathbb{Z} = \emptyset$, so that $f(x) = \lambda_{\lfloor rk \rfloor}$ for all $x \in I_k^r$. Thus

$$\frac{1}{r} \|f \chi_{I_k^r}\|_1 = \frac{1}{r} \int_{rk}^{r(k+1)} f(x) dx = \lambda_{\lfloor rk \rfloor},$$

and

$$\begin{aligned} r^{\frac{1}{\alpha}} \left\| \left\{ \frac{1}{r} \|f \chi_{I_k^r}\|_1 \right\}_{k \in \mathbb{Z}} \right\|_{\ell^{p_2}} &= r^{\frac{1}{\alpha}} \left\| \left\{ \lambda_{\lfloor rk \rfloor} \right\}_{k \in \mathbb{Z}} \right\|_{\ell^{p_2}} \\ &= r^{\frac{1}{\alpha}} \|\mathbf{a}\|_{\ell^{p_2}} \\ &\leq \|\mathbf{a}\|_{\ell^{p_2}}. \end{aligned}$$

- If $m > 0$ then $[rk, r(k+1)) = \cup_{j=rk}^{r(k+1)-1} [j, j+1)$, such that

$$\begin{aligned} r^{\frac{1}{\alpha}} \left\| \left\{ \frac{1}{r} \|f \chi_{I_k^r}\|_1 \right\}_{k \in \mathbb{Z}} \right\|_{\ell^{p_2}} &= r^{\frac{1}{\alpha}-1} \left\| \left\{ \sum_{j=rk}^{r(k+1)-1} \lambda_j \right\}_{k \in \mathbb{Z}} \right\|_{\ell^{p_2}} \\ &= r^{\frac{1}{\alpha}-1} \|\mathbf{a}\|_{\ell^{p_2}} \\ &\leq \|\mathbf{a}\|_{\ell^{p_2}}. \end{aligned}$$

The assumption follows.

For $q < \alpha < p$, the weak Lebesgue space $L^{\alpha, \infty}$ is a subset of $(L^q, \ell^p)^\alpha$. Moreover,

$$\|f\|_{q,p,\alpha} \leq C \|f\|_{\alpha, \infty}^*,$$

where the quasi-norm $\|\cdot\|_{p,q}^*$ is defined by

$$\|f\|_{p,q}^* = \begin{cases} \left[\frac{p}{q} \int_0^\infty \left(t^{\frac{1}{p}} f^*(t) \right)^q \frac{dt}{t} \right]^{\frac{1}{q}} & \text{if } 1 \leq p, q < \infty \\ \sup_{t>0} t^{\frac{1}{p}} f^*(t) & \text{if } 1 \leq p \leq \infty \text{ and } q = \infty \end{cases},$$

and f^* being the non increasing function rearrangement of f on \mathbb{R}^d , i.e.,

$$f^*(t) = \inf \left\{ \alpha > 0 : \left| \left\{ x \in \mathbb{R}^d : |f(x)| > \alpha \right\} \right| \leq t \right\}, \quad t > 0.$$

It is well known that $\sup_{t>0} t^{\frac{1}{p}} f^*(t) = \sup_{\alpha>0} \alpha \left| \left\{ x \in \mathbb{R}^d : |f(x)| > \alpha \right\} \right|^{\frac{1}{p}}$.

3. Some new results in amalgams

It is a classical result (see [3, 11, 14]) that if $f \in (L^{q_1}, \ell^{p_1})$ and $g \in (L^{q_2}, \ell^{p_2})$ with $\frac{1}{p_1} + \frac{1}{p_2} \geq 1$ and $\frac{1}{q_1} + \frac{1}{q_2} \geq 1$, then $f * g \in (L^q, \ell^p)$, where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} - 1$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} - 1$. Moreover,

$$\|f * g\|_{q,p} \leq C \|f\|_{q_1,p_1} \|g\|_{q_2,p_2}.$$

The proof of this result uses the Young inequality in Lebesgue spaces and in the space of real sequences. We can weaken the second member of the inequality if instead of the Young inequality we use the following result.

Theorem 3.1 (Theorem 2.10.1 [20]). *Let $1 \leq p_1, p_2, q_1, q_2 \leq \infty$ with $\frac{1}{p_1} + \frac{1}{p_2} > 1$. If $f \in L^{p_1, q_1}$ and $g \in L^{p_2, q_2}$ then $f * g \in L^{p, q}$, where*

$$\frac{1}{p_1} + \frac{1}{p_2} - 1 = \frac{1}{p}$$

and $q \geq 1$ such that

$$\frac{1}{q_1} + \frac{1}{q_2} \geq \frac{1}{q}.$$

Moreover, we have

$$\|f * g\|_{p,q}^* \leq C \|f\|_{p_1, q_1}^* \|g\|_{p_2, q_2}^*.$$

Let $1 \leq p, q, t, s \leq \infty$. The amalgam of the Lorentz space $L^{p,q}$ and its discrete version $\ell^{t,s}$ is the space of measurable functions f locally in $L^{p,q}$ and such that the sequence $(\|f \chi_{Q_k}\|_{p,q}^*)_{k \in \mathbb{Z}^d}$ belongs to $\ell^{t,s}$. We put $\|f\|_{(L^{p,q}, \ell^{t,s})} = \left\| (\|f \chi_{Q_k}\|_{p,q}^*)_{k \in \mathbb{Z}^d} \right\|_{t,s}^*$. The following result is established as the classical one in amalgams, just by replacing the Young inequality by that of Theorem 3.1.

Theorem 3.2. *Let*

$$\begin{cases} 1 \leq p_1, p_2, q_1, q_2 \leq \infty \\ 1 \leq r_1, r_2, s_1, s_2 \leq \infty \end{cases} \quad \text{with} \quad \begin{cases} \frac{1}{p_1} + \frac{1}{p_2} > 1 \\ \frac{1}{r_1} + \frac{1}{r_2} > 1 \end{cases},$$

$f \in (L^{p_1, q_1}, \ell^{r_1, s_1})$ and $g \in (L^{p_2, q_2}, \ell^{r_2, s_2})$. Then $f * g \in (L^{p, q}, \ell^{r, s})$, where

$$\begin{cases} \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} - 1 \\ \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} - 1 \end{cases} \quad \text{and} \quad \begin{cases} \frac{1}{s_1} + \frac{1}{s_2} \geq \frac{1}{s} \\ \frac{1}{s_1} + \frac{1}{s_2} \geq \frac{1}{s} \end{cases}.$$

Moreover, there exists a constant $C > 0$ such that

$$\|f * g\|_{(L^{p,q}, \ell^{r,s})} \leq C \|f\|_{(L^{p_1, q_1}, \ell^{r_1, s_1})} \|g\|_{(L^{p_2, q_2}, \ell^{r_2, s_2})}.$$

This result can be seen as one realization of the general result (convolution triples) stated in Theorem 3 of [10].

The next result follows immediately using the fact that $L^{p,p} = L^p$.

Corollary 3.3. *Let $1 \leq p_1, p_2 \leq \infty$ with $\frac{1}{p_1} + \frac{1}{p_2} > 1$ and $1 < q_1, q_2 < \infty$ with $\frac{1}{q_1} + \frac{1}{q_2} > 1$. If $f \in (L^{q_1}, \ell^{p_1})$ and $g \in (L^{q_2, \infty}, \ell^{p_2, \infty})$ then $f * g \in (L^s, \ell^r)$, where $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} - 1$ and $\frac{1}{s} = \frac{1}{q_1} + \frac{1}{q_2} - 1$. Moreover, there exists $C > 0$ such that*

$$\|f * g\|_{s,r} \leq C \|f\|_{q_1, p_1} \|g\|_{(L^{q_2, \infty}, \ell^{p_2, \infty})}.$$

Notice that a function K which satisfies (1.2) belongs to $(L^{\frac{d}{d-\gamma}, \infty}, \ell^{\frac{d}{d-\beta}, \infty})$ so that the boundedness of the operator I_γ^β from (L^q, ℓ^p) to (L^{q^*}, ℓ^{p^*}) is just a consequence of Corollary 3.3.

The next result which gives the continuity of the Hilbert transform in the amalgam spaces is well known. However, since we couldn't find its proof in the literature, we give one here. We recall that in the case $d = 1$, the Hilbert transform of a function f is the function Hf defined by

$$Hf(x) = \frac{1}{\pi} \text{v.p.} \int_{-\infty}^{+\infty} \frac{f(y)}{x-y} dy,$$

where v.p. denotes the Cauchy principal value.

Proposition 3.4. *Let $d = 1$ and $1 < q, p < \infty$. The Hilbert transform H is bounded on (L^q, ℓ^p) .*

Proof. To simplify the formulas, we adopt the abbreviation f_k for $f \chi_{Q_k}$.

Let $m \in \mathbb{Z}$. We have

$$(Hf)_m = \sum_{n-m \in \tilde{Q}} Hf_n \chi_{[m, m+1)} + \sum_{n-m \notin \tilde{Q}} Hf_n \chi_{[m, m+1)} = F_m + G_m$$

where $\tilde{Q} = (-2, 2)$. Since the Hilbert transform is bounded in L^q it follows that

$$\|F_m\|_q \leq C \sum_{n-m \in \tilde{Q}} \|f_n\|_q$$

so that

$$\left(\sum_m \|F_m\|_q^p \right)^{\frac{1}{p}} \leq C \left(\sum_n \|f_n\|_q^p \right)^{\frac{1}{p}}.$$

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For $x \in Q_m$ and $n - m \notin \tilde{Q}$, we have

$$Hf_n(x) = \frac{1}{\pi} \left(\frac{1}{m-n} \int_{\mathbb{R}} f_n(y) dy + \int_{\mathbb{R}} \left(\frac{1}{x-y} - \frac{1}{m-n} \right) f_n(y) dy \right),$$

with

$$\left| \frac{1}{x-y} - \frac{1}{m-n} \right| \leq \frac{2}{(|m-n|-1)|m-n|}, \text{ for } y \in Q_n.$$

So, if u is the sequence defined by $u_n = \int_n^{n+1} f(y) dy$, then we have

$$\|Hu\|_{\ell^p} \leq C \|u\|_{\ell^p} = C \|f\|_{q,p}.$$

Besides,

$$\begin{aligned} & \left[\sum_{m \in \mathbb{Z}} \left(\int_m^{m+1} \left| \sum_{n-m \notin \tilde{Q}} \int_{\mathbb{R}} \left(\frac{1}{x-y} - \frac{1}{m-n} \right) f_n(y) dy \right|^q dx \right)^{\frac{p}{q}} \right]^{\frac{1}{p}} \\ & \leq \left\{ \sum_{m \in \mathbb{Z}} \left(\sum_{n-m \notin \tilde{Q}} \frac{2}{(|m-n|-1)|m-n|} \int_{\mathbb{R}} |f_n(y)| dy \right)^p \right\}^{\frac{1}{p}}. \end{aligned} \quad (3.1)$$

Thus, if we consider the sequences v and w defined respectively by $v_0 = 0$ and $v_n = \frac{1}{n^2}$ for $n \neq 0$, and $w_n = \int_{\mathbb{R}} |f_n(y)| dy$, we have $v \in \ell^1$ and $w \in \ell^p$ with

$$\|w\|_{\ell^p} \leq \|f\|_{q,p}.$$

Since the second member of (3.1) is the ℓ^p norm of $v * w$, it is less than $\|v\|_{\ell^1} \|w\|_{\ell^p}$. It follows that

$$\left(\sum_{m \in \mathbb{Z}} \|G_m\|_q^p \right)^{\frac{1}{p}} \leq C \|f\|_{q,p},$$

which ends the proof. □

4. Norm inequalities involved in $(L^q, \ell^p)^\alpha$ spaces

The Riesz potential $I_\gamma f$ ($0 < \gamma < d$) of a function f as defined by (1.3) is closely related to the fractional maximal function $\mathcal{M}_\gamma f$ defined by

$$\mathcal{M}_\gamma f(x) = \sup_{r>0} |B(x,r)|^{\frac{\gamma}{d}-1} \int_{B(x,r)} |f(y)| dy.$$

It follows from the definitions that

$$\mathcal{M}_\gamma f \leq I_\gamma |f|. \quad (4.1)$$

Let $r > 0$ and $\alpha \geq 1$. We have

$$I_\gamma(\delta_r^\alpha f) = \delta_r^{\alpha^*}(I_\gamma f) \text{ and } \mathcal{M}_\gamma(\delta_r^\alpha f) = \delta_r^\alpha(\mathcal{M}_\gamma f) \quad (4.2)$$

with $\frac{1}{\alpha^*} = \frac{1}{\alpha} - \frac{\gamma}{d}$ and, on the real line,

$$H(\delta_r^\alpha f) = \delta_r^\alpha(Hf), \quad (4.3)$$

so that the next proposition follows from the definition of $(L^q, \ell^p)^\alpha$ spaces, the boundedness of I_γ from (L^q, ℓ^p) to (L^{q^*}, ℓ^{p^*}) and the boundedness of Hilbert transform on (L^q, ℓ^p) in the case $d = 1$.

Proposition 4.1. *Let $1 < q \leq \alpha \leq p < \infty$ and $\frac{\gamma}{d} \leq \frac{1}{p}$.*

- (1) *The Riesz potential I_γ and the associated fractional maximal operator \mathcal{M}_γ are bounded from $(L^q, \ell^p)^\alpha$ to $(L^{q^*}, \ell^{p^*})^{\alpha^*}$.*
- (2) *When $d = 1$, the Hilbert transform is bounded on $(L^q, \ell^p)^\alpha$.*

We will prove that we can have a result better than the above for Riesz potential. For this purpose, we need a control of the classical Hardy-Littlewood maximal operator \mathcal{M}_0 . We recall that for a locally integrable function f , the (centered) Hardy-Littlewood maximal function $\mathcal{M}_0 f$ is defined by

$$\mathcal{M}_0 f(x) = \sup_{r>0} |B(x, r)|^{-1} \int_{B(x, r)} |f(z)| dz.$$

Proposition 4.2. (1) *Let $1 < q \leq \alpha \leq p \leq \infty$. Then*

$$\|\mathcal{M}_0 f\|_{q,p,\alpha} \leq C \|f\|_{q,p,\alpha}. \quad (4.4)$$

- (2) *For $q = 1$, we have*

$$\|\mathcal{M}_0 f\|_{(L^{1,\infty}, L^p)^\alpha} \leq C \|f\|_{q,p,\alpha}, \quad (4.5)$$

where for $f \in L_{loc}^{1,\infty}$, and $p < \infty$,

$$\|f\|_{(L^{1,\infty}, L^p)^\alpha} := \sup_{r>0} r^{d(\frac{1}{\alpha} - \frac{1}{p} - 1)} \left[\int_{\mathbb{R}^d} (\|f \chi_{B(y,r)}\|_{1,\infty}^*)^p dy \right]^{\frac{1}{p}}.$$

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Proof. If $\alpha \in \{q, p\}$ then we recover the classical result in Lebesgue spaces. We just have to consider the cases where $1 < q < \alpha < p$; but, if $p = \infty$ this is nothing but Theorem 1 of [4]. We suppose now that $p < \infty$. Using the classical inequality

$$\int_{\mathbb{R}^d} (\mathcal{M}_0 f)^q(x) \phi(x) dx \leq C \int_{\mathbb{R}^d} |f(x)|^q (\mathcal{M}_0 \phi)(x) dx, \quad (4.6)$$

for all measurable functions f and $\phi > 0$ given by Theorem 2.12 of [15] with the characteristic function of a ball as ϕ , and proceeding as in the proof of Theorem 1 of [4], we obtain

$$r \|\widetilde{\mathcal{M}_0 f}\|_{q,p}^q \leq C \left\{ 2r \|\widetilde{f}\|_{q,p}^q + \sum_{k=1}^{\infty} \frac{1}{(2^{k-1})^d} 2^{k+1} r \|\widetilde{f}\|_{q,p}^q \right\},$$

for $f \in (L^q, \ell^p)^\alpha$. Let us multiply both sides by $r^{dq(\frac{1}{\alpha} - \frac{1}{q} - \frac{1}{p})}$, we obtain

$$\begin{aligned} (r^{d(\frac{1}{\alpha} - \frac{1}{q} - \frac{1}{p})} r \|\widetilde{\mathcal{M}_0 f}\|_{q,p})^q &\leq 2^{-dq(\frac{1}{\alpha} - \frac{1}{q} - \frac{1}{p})} \|f\|_{q,p,\alpha}^q \\ &\quad + \sum_{k=1}^{\infty} \frac{2^{(k+1)dq(\frac{1}{q} + \frac{1}{p} - \frac{1}{\alpha})}}{(2^{k-1})^d} \|f\|_{q,p,\alpha}^q \\ &\leq \|f\|_{q,p,\alpha}^q, \end{aligned}$$

since $dq(\frac{1}{q} + \frac{1}{p} - \frac{1}{\alpha}) - d = d(\frac{q}{p} - \frac{q}{\alpha}) < 0$. Thus, taking the supremum over $r > 0$, we obtain

$$\|\mathcal{M}_0 f\|_{q,p,\alpha} \leq C \|f\|_{q,p,\alpha},$$

using Relation (2.2).

As for the case $q = 1$, the proof is the same using the following inequality

$$\int_{\{x \in \mathbb{R}^d : \mathcal{M}_0 f(x) > t\}} \chi_{B(y,r)}(x) dx \leq \frac{C}{t} \int_{\mathbb{R}^d} |f(x)| (\mathcal{M}_0 \chi_{B(y,r)})(x) dx$$

instead of (4.6). □

We now use the Proposition 4.2 to give some estimates of the Riesz potential in $(L^q, \ell^p)^\alpha$ spaces.

Theorem 4.3. *Let $1 \leq q \leq \alpha \leq p \leq \infty$ and $0 < \frac{\gamma}{d} < \frac{1}{\alpha}$. Put $\frac{1}{\alpha^*} = \frac{1}{\alpha} - \frac{\gamma}{d}$.*

(1) *If $1 < q$ then we have*

$$\|I_\gamma f\|_{\tilde{q}, \tilde{p}, \alpha^*} \leq C \|f\|_{q,p,\alpha}^{1 - \frac{\alpha}{d}\gamma} \|f\|_{q,\infty,\alpha}^{\frac{\alpha}{d}\gamma} \quad (4.7)$$

with $\frac{1}{\tilde{q}} = \frac{1}{q} - \frac{\alpha}{q} \frac{\gamma}{d}$ and $\frac{1}{\tilde{p}} = \frac{1}{p} - \frac{\alpha}{p} \frac{\gamma}{d}$.

(2) If $q = 1$ then we have

$$\|I_\gamma f\|_{(L^{1,\infty}, L^{\bar{p}})^{\alpha*}} \leq C \|f\|_{1,p,\alpha}^{1-\frac{\alpha}{d}\gamma} \|f\|_{1,\infty,\alpha}^{\frac{\alpha}{d}\gamma}. \quad (4.8)$$

Proof. Let $1 \leq q \leq \alpha \leq p \leq \infty$, $0 < \gamma < d$ and $f \in (L^q, \ell^p)^\alpha$.

If $p = \infty$, then the theorem is exactly Theorem 3.1 of [1]. We consider the case where $1 \leq q \leq \alpha \leq p < \infty$. Since for $\alpha \in \{q, p\}$ the space $(L^q, \ell^p)^\alpha$ is equal to the Lebesgue space L^α , we just have to look at the case where $1 \leq q < \alpha < p < \infty$.

For all $\epsilon > 0$, we have

$$I_\gamma f(x) = \int_{|x-y| \leq \epsilon} \frac{f(y)}{|x-y|^{d-\gamma}} dy + \int_{|x-y| > \epsilon} \frac{f(y)}{|x-y|^{d-\gamma}} dy. \quad (4.9)$$

We recall that $(L^q, \ell^p)^\alpha$ is a subspace of the Morrey space $L^{q,d(1-\frac{d}{\alpha})}$, so that $\mathcal{M}_0 f$ is finite almost everywhere in \mathbb{R}^d . Following the proof of Theorem 2 in [4], we can say that

$$\left| \int_{|x-y| \leq \epsilon} \frac{f(y)}{|x-y|^{d-\gamma}} dy \right| \leq C \epsilon^\gamma \mathcal{M}_0 f(x),$$

and

$$\left| \int_{|x-y| > \epsilon} \frac{f(y)}{|x-y|^{d-\gamma}} dy \right| \leq C \epsilon^{\gamma-\frac{d}{\alpha}} \|f\|_{q,\infty,\alpha}.$$

It comes that

$$|I_\gamma f| \leq C \left(\epsilon^\gamma \mathcal{M}_0 f + \epsilon^{\gamma-\frac{d}{\alpha}} \|f\|_{q,\infty,\alpha} \right).$$

Taking $\epsilon = \left(\frac{\mathcal{M}_0 f}{\|f\|_{q,\infty,\alpha}} \right)^{-\frac{\alpha}{d}}$, we have

$$I_\gamma f \leq C (\mathcal{M}_0 f)^{-\frac{\alpha\gamma}{d}+1} \|f\|_{q,\infty,\alpha}^{\frac{\alpha}{d}\gamma}.$$

The inequalities (4.7) and (4.8) follow respectively from the inequalities (4.4) and (4.5). \square

The following corollary is a consequence of the fact that $\|f\|_{q,\infty,\alpha} \leq C \|f\|_{q,p,\alpha}$ for all $1 \leq q \leq \alpha \leq p \leq \infty$.

Corollary 4.4. *Let $1 \leq q \leq \alpha \leq p \leq \infty$ and $0 < \frac{\gamma}{d} < \frac{1}{\alpha}$. Put $\frac{1}{\alpha^*} = \frac{1}{\alpha} - \frac{\gamma}{d}$*

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(1) If $1 < q$ then we have

$$\|I_\gamma f\|_{\tilde{q}, \tilde{p}, \alpha^*} \leq C \|f\|_{q, p, \alpha} \quad (4.10)$$

with $\frac{1}{\tilde{q}} = \frac{1}{q} - \frac{\alpha \gamma}{q d}$ and $\frac{1}{\tilde{p}} = \frac{1}{p} - \frac{\alpha \gamma}{p d}$.

(2) If $q = 1$ then we have

$$\|I_\gamma f\|_{(L^{1, \infty}, L^{\tilde{p}})^{\alpha^*}} \leq C \|f\|_{1, p, \alpha}. \quad (4.11)$$

Using a different method, Dosso et al in [7] proved that if $0 < \frac{\gamma}{d} < \frac{1}{\alpha} - \frac{1}{p}$ then

$$\|I_\gamma f\|_{q^*, p, \alpha^*} \leq C \|f\|_{q, p, \alpha} \quad (4.12)$$

where $\frac{1}{q^*} = \frac{1}{q} - \frac{\gamma}{d}$. The next result is a generalization of Theorem 2.1 of [8], and its proof is just an adaptation of that given there.

Theorem 4.5. *Let $1 < q < \infty$ and $q \leq \alpha \leq p$. If T is a sublinear operator which is bounded on L^q and satisfy*

$$|Tf(x)| \leq C \int_{\mathbb{R}^d} \frac{|f(y)|}{|x-y|^d} dy \quad x \notin \text{supp} f, \quad (4.13)$$

for any $f \in L^1$ with compact support, then T is also bounded on $(L^q, \ell^p)^\alpha$.

Proof. We assume that $1 < q < \alpha < p < \infty$, since the case $p = \infty$ is Theorem 2.1 of [8] and when $\alpha \in \{q, p\}$ we have nothing to prove. Fix $y \in \mathbb{R}^d$ and $r > 0$ we have

$$f = f\chi_{B(y, 2r)} + \sum_{k=1}^{\infty} f\chi_{B(y, 2^{k+1}r) \setminus B(y, 2^k r)}$$

so that taking the L^q -norm on the ball $B(y, r)$, we obtain

$$\|Tf\chi_{B(y, r)}\|_q \leq C \left(\|f\chi_{B(y, 2r)}\|_q + \sum_{k=1}^{\infty} (2^k)^{-\frac{d}{p}} \|f\chi_{B(y, 2^{k+1}r)}\|_q \right),$$

using the L^q boundedness of T and relation (4.13). Taking the L^p -norm of both sides with respect to y , it comes that

$$r \|\widetilde{Tf}\|_{q, p} \leq C \left({}_{2r}\widetilde{\|f\|}_{q, p} + \sum_{k=1}^{\infty} (2^k)^{-\frac{d}{q}} {}_{2^{k+1}r}\widetilde{\|f\|}_{q, p} \right). \quad (4.14)$$

We multiply both sides of (4.14) by $r^{d(\frac{1}{\alpha}-\frac{1}{p}-\frac{1}{q})}$. It follows that

$$r^{d(\frac{1}{\alpha}-\frac{1}{p}-\frac{1}{q})} {}_r\|\widetilde{Tf}\|_{q,p} \leq C \left(1 + \sum_{k=1}^{\infty} (2^k)^{d(\frac{1}{p}-\frac{1}{\alpha})} \right) \|f\|_{q,p,\alpha}, \quad r > 0.$$

Taking the supremum over $r > 0$, Relation (2.2) yields the result. \square

As mentioned in [8], Condition (4.13) can be satisfied by many operators such as Bochner-Riesz operators at the critical index, Ricci-Stein's oscillatory singular integral, C. Fefferman's singular multiplier, and some Calderón-Zygmund operators.

We define the linear commutator $[\mathfrak{b}, T]$ by $[\mathfrak{b}, T]f(x) = T(\mathfrak{b}f)(x) - \mathfrak{b}(x)Tf(x)$, and recall that the space BMO consists of functions $\mathfrak{b} \in L^1_{loc}$ satisfying $\|\mathfrak{b}\|_{BMO} < \infty$, where

$$\|\mathfrak{b}\|_{BMO} := \sup_{r>0, x \in \mathbb{R}^d} \frac{1}{|B(x, r)|} \int_{B(x, r)} |\mathfrak{b}(y) - \mathfrak{b}_{B(x, r)}| dy, \quad (4.15)$$

with $\mathfrak{b}_{B(x, r)}$ denoting the average over $B(x, r)$ of \mathfrak{b} .

Theorem 4.6. *Let $1 < q < \infty$, $q \leq \alpha \leq p \leq \infty$ and $\mathfrak{b} \in BMO$. If a linear operator T satisfies (4.13) and $[\mathfrak{b}, T]$ is bounded on L^q , then $[\mathfrak{b}, T]$ is also bounded on $(L^q, \ell^p)^\alpha$.*

Proof. If $\alpha < p = \infty$ then the result is just Theorem 2.2 of [8], and when $\alpha = q$ or $\alpha = p$, there is nothing to prove. We suppose now that $1 < q < \alpha < p < \infty$. Proceeding as in the proof of Theorem 2.2 [8], we have that for all $y \in \mathbb{R}^d$ and $r > 0$,

$$\begin{aligned} \left\| [\mathfrak{b}, T] f \chi_{B(y, r)} \right\|_q &\leq \left\| f \chi_{B(y, 2r)} \right\|_q \\ &+ \sum_{k=1}^{\infty} \frac{1}{(2^k r)^d} \left[\int_{B(y, r)} \left(\int_{B(y, 2^{k+1}r)} |\mathfrak{b}(x) - \mathfrak{b}(z)| |f(x)| dx \right)^q dz \right]^{\frac{1}{q}}, \end{aligned}$$

so that the use of John-Nirenberg theorem on BMO functions (see Theorem 7.1.6 in [17]) and the properties of BMO lead to

$$\left\| [\mathfrak{b}, T] f \chi_{B(y, r)} \right\|_q \leq \left\| f \chi_{B(y, 2r)} \right\|_q + \|\mathfrak{b}\|_{BMO} \sum_{k=1}^{\infty} (2^k)^{-\frac{d}{q}} \left\| f \chi_{B(y, 2^{k+1}r)} \right\|_q.$$

We end the proof as above. \square

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The next result gives a norm equivalence of the Riesz potential and associated fractional maximal operator when we deal with non negative measurable functions.

Theorem 4.7. *Let $1 \leq q \leq \alpha \leq p \leq \infty$ and $0 < \gamma < d$. For every positive $f \in L^q_{loc}$, we have*

$$\|I_\gamma f\|_{q,p,\alpha} \approx \|\mathcal{M}_\gamma f\|_{q,p,\alpha}. \quad (4.16)$$

Proof. In view of inequality (4.1) it suffices to prove that $\|I_\gamma f\|_{q,p,\alpha} \lesssim \|\mathcal{M}_\gamma f\|_{q,p,\alpha}$. We can assume that $1 \leq q < \alpha < p < \infty$, since the case $\alpha \in \{q, p\}$ is solved in Theorem 1 of [19] while the case $p = \infty$ is Theorem 1.2 of [16] (see also Theorem 4.2 of [2]). For $y \in \mathbb{R}^d$ and $r > 0$, we have

$$\left\| I_\gamma f \chi_{B(y,r)} \right\|_q \approx \left\| \mathcal{M}_\gamma f \chi_{B(y,r)} \right\|_q + |B(y,r)|^{\frac{1}{q}} \int_{\mathbb{R}^d \setminus B(y,r)} \frac{f(x)}{|x-y|^{d-\gamma}} dx, \quad (4.17)$$

according to Lemma 4.2 of [16]. But

$$\begin{aligned} & |B(y,r)|^{\frac{1}{q}} \int_{\mathbb{R}^d \setminus B(y,r)} \frac{f(x)}{|x-y|^{d-\gamma}} dx \\ &= \sum_{k=0}^{\infty} |B(y,r)|^{\frac{1}{q}} \int_{2^k r \leq |x-y| < 2^{k+1} r} \frac{f(x)}{|x-y|^{d-\gamma}} dx \\ &\lesssim \sum_{k=0}^{\infty} (2^k)^{-\frac{d}{q}} |B(y,r)|^{\frac{\gamma}{d} - (1-\frac{1}{q})} \left\| f \chi_{B(y,2^{k+1}r)} \right\|_1 \\ &\lesssim \sum_{k=0}^{\infty} (2^k)^{-\frac{d}{q}} \left\| \mathcal{M}_\gamma f \chi_{B(y,2^{k+1}r)} \right\|_q, \end{aligned}$$

where the last inequality comes from Theorem 5.2 of [16]. Taking into account this inequality and the relation (4.17), the L^p -norm of both sides of the inequality leads to

$$r \widetilde{\|I_\gamma f\|_{q,p}} \lesssim r \|\mathcal{M}_\gamma f\|_{q,p} + \sum_{k=0}^{\infty} (2^k)^{-\frac{d}{q}} \widetilde{\| \mathcal{M}_\gamma f \|_{q,p}}_{2^{k+1}r}.$$

It follows from the above inequality and Relation (2.2) that

$$r^{d(\frac{1}{\alpha} - \frac{1}{p} - \frac{1}{q})} r \widetilde{\|I_\gamma f\|_{q,p}} \lesssim \|\mathcal{M}_\gamma f\|_{q,p,\alpha} + \sum_{k=0}^{\infty} (2^k)^{d(\frac{1}{p} - \frac{1}{\alpha})} \|\mathcal{M}_\gamma f\|_{q,p,\alpha}, \quad r > 0.$$

Thus the result follows by taking the supremum over $r > 0$. \square

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