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Polynomiality of shifted Plancherel averages and content evaluations

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Abstract

The shifted Plancherel measure is a natural probability measure on strict partitions. We prove a polynomiality property for the averages of the shifted Plancherel measure. As an application, we give alternative proofs of some content evaluation formulas, obtained by Han and Xiong very recently. Our main tool is factorial Schur Q-functions.

Résumé

La mesure de Plancherel décalée est une mesure de probabilité naturelle sur les partitions strictes. Nous démontrons une propriété de polynomialité pour les moyennes de mesures de Plancherel décalées. Comme application, nous donnons une nouvelle preuve de certaines formules d'évaluation des contenus obtenues par Han et Xiong très récemment. Nous utilisons, comme outil principal, les Q-fonctions de Schur factorielles.

1. Introduction

1.1. Partitions

Following to Macdonald's book [16], let us recall the basic knowledge on strict partitions. A *partition* is a finite weakly-decreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ of positive integers. The integer $\ell(\lambda) = l$ is the *length* of λ and $|\lambda| = \sum_{i=1}^{l} \lambda_i$ is the *size* of λ. If $|\lambda| = n$, we say that λ is a partition of n.

A partition λ is said to be *strict* if all λ_i are pairwise distinct, and λ is said to be *odd* if all λ_i are odd integers. Let $\mathcal{SP}_n$ be the set of all strict partitions of n and $\mathcal{OP}_n$ the set of all odd partitions of n. The fact that their cardinalities coincide is well known: $|\mathcal{SP}_n| = |\mathcal{OP}_n|$. Set $\mathcal{SP} = \bigcup_{n=0}^{\infty} \mathcal{SP}_n$ and $\mathcal{OP} = \bigcup_{n=0}^{\infty} \mathcal{OP}_n$. For convenience, we deal with

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the empty partition $\emptyset$, which is the unique partition in $\mathcal{SP}_0 = \mathcal{OP}_0$ with length 0.

For each $\lambda \in \mathcal{SP}$, we consider the set

$$S(\lambda) = \{(i,j) \in \mathbb{Z}^2 \mid 1 \le i \le \ell(\lambda),\ i \le j \le \lambda_i + i - 1\}.$$

The set $S(\lambda)$ is usually drawn in a graphical way, and called the *shifted Young diagram* of λ. Each element $\square = (i,j) \in S(\lambda)$ is often called a *box* of λ.

Let λ, μ be strict partitions such that $S(\lambda) \supset S(\mu)$. Put $k = |\lambda| - |\mu|$. A *standard tableau* of shape $S(\lambda/\mu)$ is a sequence of strict partitions $(\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(k)})$ such that $\lambda^{(0)} = \mu$; $\lambda^{(k)} = \lambda$; and that for each $i = 1, 2, \dots, k$, the diagram $S(\lambda^{(i)})$ is obtained from $S(\lambda^{(i-1)})$ by adding exactly one box. We denote by $g^{\lambda/\mu}$ the number of standard tableaux of shape $S(\lambda/\mu)$. We set $g^{\lambda/\mu} = 0$ unless $S(\lambda) \supset S(\mu)$. Define $g^\lambda = g^{\lambda/\emptyset}$.

1.2. Shifted Plancherel measure

In this paper, we consider the following probability measure on $\mathcal{SP}_n$, studied in many papers, e.g., in [1, 10, 17].

Definition 1.1. The *shifted Plancherel measure* $\mathbb{P}_n$ on $\mathcal{SP}_n$ is defined by

$$\mathbb{P}_n(\lambda) = \frac{2^{n-\ell(\lambda)}(g^\lambda)^2}{n!}.$$

This indeed defines a *probability* since the identity [9, Corollary 10.8]

$$\sum_{\lambda \in \mathcal{SP}_n} 2^{n-\ell(\lambda)}(g^\lambda)^2 = n!$$

holds. For example, if $n = 5$ then $\mathbb{P}_5((5)) = \frac{16}{5!}$, $\mathbb{P}_5((4,1)) = \frac{72}{5!}$, and $\mathbb{P}_5((3,2)) = \frac{32}{5!}$.

For a function φ on $\mathcal{SP}$, we call

$$\mathbb{E}_n[\varphi] = \sum_{\lambda \in \mathcal{SP}_n} \mathbb{P}_n(\lambda)\varphi(\lambda) = \sum_{\lambda \in \mathcal{SP}_n} \frac{2^{n-\ell(\lambda)}(g^\lambda)^2}{n!}\varphi(\lambda)$$

the *shifted Plancherel average* of φ.

Let $\{x_1, x_2, \dots\}$ be formal variables, then for each positive integer r, the r-th power-sum symmetric function is defined by

$$p_r(x_1, x_2, \dots) = x_1^r + x_2^r + \cdots.$$

It is well known that the algebra of symmetric functions is generated by the family $\{p_r\}$. We denote by Γ the subalgebra generated by $\{p_{2m+1}\}_{m=0,1,2,\ldots}$. Elements in Γ are sometimes called *supersymmetric functions*, adapted to strict partitions. The following theorem claims a polynomiality of $\mathbb{E}_n[f]$ for any supersymmetric function f.

Theorem 1.2. *Suppose that f is a supersymmetric function. Then*

$$\mathbb{E}_n[f] = \sum_{\lambda \in \mathcal{SP}_n} \frac{2^{n-\ell(\lambda)}(g^\lambda)^2}{n!} f\Big(\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)}\Big)$$

is a polynomial function in n.

We introduce a notation $x^{\downarrow k}$ as

$$x^{\downarrow k} = x(x-1)(x-2)\cdots(x-k+1)$$

for a variable x and a positive integer k. If n is an integer with $0 \le n < k$ then $n^{\downarrow k} = 0$. We also set $x^{\downarrow 0} = 1$.

In some supersymmetric functions f, we give the explicit expressions of $\mathbb{E}_n[f]$ as linear combinations of descending powers $n^{\downarrow j}$. In fact, we will show

$$\mathbb{E}_n[p_3] = \sum_{\lambda \in \mathcal{SP}_n} \frac{2^{n-\ell(\lambda)}(g^\lambda)^2}{n!} \Big(\lambda_1^3 + \lambda_2^3 + \cdots + \lambda_{\ell(\lambda)}^3\Big) = 3n^{\downarrow 2} + n,$$

$$\mathbb{E}_n[p_5] = \sum_{\lambda \in \mathcal{SP}_n} \frac{2^{n-\ell(\lambda)}(g^\lambda)^2}{n!} \Big(\lambda_1^5 + \lambda_2^5 + \cdots + \lambda_{\ell(\lambda)}^5\Big) = \frac{40}{3}n^{\downarrow 3} + 15n^{\downarrow 2} + n,$$

$$\begin{aligned}\mathbb{E}_n[p_3^2] = \sum_{\lambda \in \mathcal{SP}_n} \frac{2^{n-\ell(\lambda)}(g^\lambda)^2}{n!} &\Big(\lambda_1^3 + \lambda_2^3 + \cdots + \lambda_{\ell(\lambda)}^3\Big)^2 \\ &= 9n^{\downarrow 4} + 54n^{\downarrow 3} + 31n^{\downarrow 2} + n.\end{aligned}$$

1.3. A deformation

Fix a strict partition μ of m. We define the measure $\mathbb{P}_{\mu,n}$ on $\mathcal{SP}_{n+m}$ by

$$\mathbb{P}_{\mu,n}(\lambda) = \frac{m!}{(n+m)!} 2^{n-\ell(\lambda)+\ell(\mu)} \frac{g^\lambda}{g^\mu} g^{\lambda/\mu} \qquad (\lambda \in \mathcal{SP}_{n+m}). \tag{1.1}$$

Note that $\mathbb{P}_{\mu,n}(\lambda) = 0$ unless $S(\lambda) \supset S(\mu)$. We will prove that $\mathbb{P}_{\mu,n}$ is a *probability*, i.e., $\sum_{\lambda \in \mathcal{SP}_{n+m}} \mathbb{P}_{\mu,n}(\lambda) = 1$. For a function φ on $\mathcal{SP}$, define

$$\begin{aligned}\mathbb{E}_{\mu,n}[\varphi] = &\sum_{\lambda \in \mathcal{SP}_{n+m}} \mathbb{P}_{\mu,n}(\lambda)\varphi(\lambda) \\ &= \sum_{\lambda \in \mathcal{SP}_{n+m}} \frac{m!}{(n+m)!} 2^{n-\ell(\lambda)+\ell(\mu)} \frac{g^\lambda}{g^\mu} g^{\lambda/\mu} \varphi(\lambda).\end{aligned}$$

The summation $\mathbb{E}_{\mu,n}[\varphi]$ is considered in [8]. Note that $\mathbb{E}_{\emptyset,n}[\varphi]$ is nothing but $\mathbb{E}_n[\varphi]$. The following theorem is a slight extension of Theorem 1.2.

Theorem 1.3. *Let μ be a strict partition. Suppose that f is a supersymmetric function. Then $\mathbb{E}_{\mu,n}[f]$ is a polynomial function in n.*

1.4. Content evaluations

For each box $\square = (i,j)$ in $S(\lambda)$, we define $c_\square = j - i$ and call it the *content* of $\square$. We deal with symmetric functions evaluated by quantities $\widehat{c}_\square$, where

$$\widehat{c}_\square = \frac{1}{2} c_\square (c_\square + 1).$$

Theorem 1.4. *For any symmetric function F, there exists a unique supersymmetric function $\widehat{F}$ in Γ such that*

$$\widehat{F}(\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)}) = F\left(\widehat{c}_\square : \square \in S(\lambda)\right)$$

for any $\lambda \in \mathcal{SP}$. Here $F\left(\widehat{c}_\square : \square \in S(\lambda)\right)$ is the specialization of the symmetric function $F(x_1, x_2, \dots)$ such that the first $|\lambda|$ variables are substituted by $\widehat{c}_\square$ for boxes $\square$ in $S(\lambda)$, and all other variables by 0.

From Theorems 1.3 and 1.4, we obtain the following corollary immediately. This result was first obtained in [8] very recently.

Corollary 1.5. *Let μ be a strict partition of m. For any symmetric function F,*

$$\sum_{\lambda \in \mathcal{SP}_{n+m}} \frac{m!}{(n+m)!} 2^{n-\ell(\lambda)+\ell(\mu)} \frac{g^\lambda}{g^\mu} g^{\lambda/\mu} F\left(\widehat{c}_\square : \square \in S(\lambda)\right)$$

is a polynomial function in n.

In [8], the reason why they consider the quantity $F(\widehat{c}_\square : \square \in S(\lambda))$ is not presented. We note that the multi-set $\{\widehat{c}_\square \mid \square \in S(\lambda)\}$ forms the collection of squares of eigenvalues with respect to projective analogs of Jucys–Murphy elements, see [20, 23, 26, 27] and also [1, Theorem 3.2].

We give the explicit expressions of some content evaluations. In fact, we will show

$$\mathbb{E}_n[\widehat{p_2}] = \sum_{\lambda \in \mathcal{SP}_n} \frac{2^{n-\ell(\lambda)}(g^\lambda)^2}{n!} \sum_{\square \in S(\lambda)} (\widehat{c}_\square)^2 = \frac{2}{3} n^{\downarrow 3} + \frac{1}{2} n^{\downarrow 2},$$

$$\mathbb{E}_n[\widehat{p}_{(1^2)}] = \sum_{\lambda \in \mathcal{SP}_n} \frac{2^{n-\ell(\lambda)}(g^\lambda)^2}{n!} \left\{ \sum_{\square \in S(\lambda)} \widehat{c}_\square \right\}^2 = \frac{1}{12}(n^{\downarrow 4} + 4n^{\downarrow 3} - 8n^{\downarrow 2} - 2n)\,.$$

Furthermore, we will give a new algebraic proof of the identity

$$\mathbb{E}_{\mu,n}[\widehat{p}_1 - \widehat{p}_1(\mu)] = \frac{1}{2} n^{\downarrow 2} + |\mu| n\,,$$

which is given in [8, Theorem 1.3].

1.5. **Related research and the aim**

Let $\mathcal{P}_n$ be the set of all (not necessary strict) partitions of n. The (traditional) *Plancherel probability measure* $\mathbb{P}_n^{\mathrm{Plan}}$ on $\mathcal{P}_n$ is defined by

$$\mathbb{P}_n^{\mathrm{Plan}}(\lambda) = \frac{(f^\lambda)^2}{n!}\,,$$

where f^λ is the number of standard tableaux of shape $Y(\lambda)$. Here $Y(\lambda)$ is the ordinary Young diagram of λ: $Y(\lambda) = \{(i,j) \in \mathbb{Z}^2 \mid 1 \le i \le \ell(\lambda),\ 1 \le j \le \lambda_i\}$. Let F be a symmetric function. In [25] (see also [7]), Stanley proves that the summation

$$\sum_{\lambda \in \mathcal{P}_n} \mathbb{P}_n^{\mathrm{Plan}}(\lambda) F(h_\square^2 : \square \in Y(\lambda))$$

is a polynomial in n. Here $h_\square$ denotes the hook length of the square $\square$ in the Young diagram $Y(\lambda)$. Panova [22] shows an explicit identity for the symmetric function $F = F_r(x_1, x_2, \dots) = \sum_{j \ge 1} \prod_{i=1}^r (x_j - i^2)$.

Moreover, Stanley [25] proves that the content evaluation

$$\sum_{\lambda \in \mathcal{P}_n} \mathbb{P}_n^{\mathrm{Plan}}(\lambda) F(c_\square : \square \in Y(\lambda)) \tag{1.2}$$

is also a polynomial in n. Olshanski [21] finds that the functions $\lambda \mapsto F(c_\square : \square \in Y(\lambda))$ are seen as shifted-symmetric functions in variables

$\lambda_1, \lambda_2, \dots$ and obtains an alternative algebraic proof for the polynomiality of (1.2). Some explicit formulas for particular F are obtained in [5, 6, 14, 15, 18, 19]. Just as an example, in [6] the identity

$$\sum_{\lambda \in \mathcal{P}_n} \mathbb{P}_n^{\mathrm{Plan}}(\lambda) \sum_{\Box \in Y(\lambda)} \prod_{i=0}^{k-1} \left(c_\Box^2 - i^2 \right) = \frac{(2k)!}{((k+1)!)^2} n^{\downarrow(k+1)}$$

is obtained.

We emphasize the fact that the content evaluations are related to matrix integrals ([18, 19]). For example, let us consider the unitary group $U(N)$ with the normalized Haar measure $\mathrm{d}U$ and suppose $n \leq N$. Then Weingarten calculus gives the following identity

$$\int_{U(N)} |u_{11} u_{22} \cdots u_{nn}|^2 \, \mathrm{d}U = \sum_{k=0}^{\infty} (-1)^k N^{-(n+k)} \sum_{\lambda \in \mathcal{P}_n} \mathbb{P}_n^{\mathrm{Plan}}(\lambda) h_k(c_\Box : \Box \in Y(\lambda)).$$

Here h_k are complete symmetric functions. More general identities (for other classical groups) can be seen in [18, 19].

The quantity $F(\widehat{c}_\Box : \Box \in S(\lambda))$ in Subsection 1.4 is a natural projective analog of $F(c_\Box : \Box \in Y(\lambda))$, because the $c_\Box$ are eigenvalues of Jucys–Murphy elements of the symmetric groups, while the $\widehat{c}_\Box$ come from their projective version. Unfortunately, it is not known any direct connection between matrix integrals and the projective content evaluation $F(\widehat{c}_\Box : \Box \in S(\lambda))$.

Our results in this paper are seen as the counterparts of the content evaluation [21] in the theory of the shifted Plancherel measure. As Olshanski does in [21], we employ factorial versions of symmetric functions. Specifically, we introduce a new family of supersymmetric functions $(\mathfrak{p}_\rho)_{\rho \in \mathcal{OP}}$. The function $\mathfrak{p}_\rho$ is also regarded as projective (or spin) irreducible character values of the symmetric groups. For ordinary partitions, the counterpart is the normalized linear character, which has been studied in e.g. [2, 3, 4, 12, 24], and written as $p_\rho^\#$, $\widehat{\chi}_\rho$, Ch_ρ, $\dots$ in their articles. We will provide explicit values of shifted Plancherel averages $\mathbb{E}_{\mu,n}[\mathfrak{p}_\rho]$ for all strict partitions μ and odd partitions ρ.

As mentioned above, Corollary 1.5 is obtained by Han and Xiong [8]. Our purpose in this paper is to provide more insight for their result, based on the theory of factorial Schur Q-functions. As a result, we can obtain some new identities given in Subsections 1.2 and 1.4 in a simple way.

1.6. **Outline of the paper**

The paper is organized as follows. Section 2 gives definitions and basis properties of Schur P- and factorial Schur P-functions. A more detailed description can be seen in [10, 11, 16]. In Section 3 we introduce new supersymmetric functions $\mathfrak{p}_\rho$ and provide some necessary properties. In Section 4 we give the proofs of Theorem 1.2 and Theorem 1.3. New identities presented in Subsection 1.2 are also proved. In Section 5 we give a proof of Theorem 1.4 and present some examples of content evaluations. In Section 6 we deal with some family of functions on $\mathcal{SP}$ introduced in [8] and show that they are supersymmetric functions. We comment on some remaining questions in Section 7.

2. **Supersymmetric functions**

2.1. **The algebra of supersymmetric functions**

A *symmetric function* is a collection of polynomials $F = (F_N)_{N=1,2,\dots}$ with rational coefficients such that

- each F_N is symmetric in N commutative variables $x_1, x_2, \dots, x_N$;
- the stability relation $F_{N+1}(x_1, \dots, x_N, 0) = F_N(x_1, \dots, x_N)$ holds for all $N \geq 1$.

We often write F as $F(x_1, x_2, \dots)$ in infinitely-many variables $x_1, x_2, \dots$.

For each $r = 1, 2, \dots,$ the r-th power-sum symmetric function p_r is given by

$$p_r(x_1, x_2, \dots) = x_1^r + x_2^r + \cdots .$$

It is well known that the p_r generate the algebra of all symmetric functions and are algebraically independent over $\mathbb{Q}$.

Definition 2.1. Let Γ denote the subalgebra of symmetric functions generated by p_{2m+1}, $m = 0, 1, 2, \dots$. We say elements in Γ to be *supersymmetric functions.*

Let us review supersymmetric functions along [16, Chapter III.8]. Define

$$p_\rho = p_{\rho_1} p_{\rho_2} \cdots p_{\rho_l}$$

for $\rho = (\rho_1, \rho_2, \dots, \rho_l) \in \mathcal{OP}$. The p_ρ form a linear basis of Γ by definition. The scalar product on Γ is defined by

$$\langle p_\rho, p_\sigma \rangle = 2^{-\ell(\rho)} z_\rho \delta_{\rho\sigma} \tag{2.1}$$

for $\rho, \sigma \in \mathcal{OP}$, where

$$z_\rho = \prod_{r \geq 1} r^{m_r(\rho)} m_r(\rho)!\,,$$

and $m_r(\rho)$ is the multiplicity of r in ρ: $m_r(\rho) = |\{i \in \{1, 2, \dots, \ell(\rho)\} \mid \rho_i = r\}|$.

Given an element f in Γ and a strict partition λ, we denote by $f(\lambda)$ the value

$$f(\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)}, 0, 0, \dots).$$

For example, $p_3(\lambda) = \lambda_1^3 + \lambda_2^3 + \cdots + \lambda_{\ell(\lambda)}^3$. Elements in Γ are uniquely defined by their values on $\mathcal{SP}$, i.e. two elements f, g in Γ coincide with each other if and only if it holds that $f(\lambda) = g(\lambda)$ for every strict partition λ.

2.2. Schur P-functions

Let us review the Schur P-function, which is the particular $t = -1$ case of the Hall–Littlewood function with parameter t. We use the definition in [10]. See also [16, Chapter III.8] and [9] for details.

Definition 2.2. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l) \in \mathcal{SP}$. Suppose that $N \geq l = \ell(\lambda)$. We define a polynomial $P_{\lambda|N}$ by

$$P_{\lambda|N}(x_1, \dots, x_N) = \frac{1}{(N-l)!} \sum_{\omega \in S_N} \omega \left(x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_l^{\lambda_l} \prod_{\substack{i: 1 \leq i \leq l, \\ j: i < j \leq N}} \frac{x_i + x_j}{x_i - x_j} \right),$$

where the symmetric group S_N acts by permuting the variables $x_1, \dots, x_N$. If $\ell(\lambda) > N$, then we set $P_{\lambda|N}(x_1, \dots, x_N) = 0$. Then the collection $(P_{\lambda|N})_{N=1,2,\dots}$ defines an element P_λ in Γ. Define Q_λ by $Q_\lambda = 2^{\ell(\lambda)} P_\lambda$. We call P_λ and Q_λ a *Schur P-function* and *Schur Q-function*, respectively.

Proposition 2.3.

(1) *The family* $(P_\lambda)_{\lambda \in \mathcal{SP}}$ *forms a linear basis of* Γ.

(2) $\langle P_\lambda, Q_\mu \rangle = \delta_{\lambda\mu}$ *for* $\lambda, \mu \in \mathcal{SP}$.

(3) $\sum_{\lambda \in \mathcal{SP}_n} 2^{-\ell(\lambda)} \langle f, Q_\lambda \rangle \langle g, Q_\lambda \rangle = \langle f, g \rangle$ *for* $f, g \in \Gamma$.

(4) *Suppose* $|\lambda| \geq |\mu|$. *Then* $g^{\lambda/\mu} = \langle p_1^{|\lambda|-|\mu|} P_\mu, Q_\lambda \rangle$ *for* $\lambda, \mu \in \mathcal{SP}$, *where* $g^{\lambda/\mu}$ *is the number of standard tableaux of shape* $S(\lambda/\mu)$. *In particular,* $g^\lambda = g^{\lambda/\emptyset} = \langle p_1^{|\lambda|}, Q_\lambda \rangle$.

Proof. (1), (2): See [16, Chapter III, (8.9) and (8.12)].

(3): By linearity, it is enough to show the identity for $f = P_\mu$ and $g = Q_\nu$ with $|\mu| = |\nu| = n$. Then both sides are equal to $\delta_{\mu\nu}$ by (2).

(4): We recall a special case of the Pieri-type formula for Schur P-functions ([16, Chapter III, (8.15)]): for $\mu \in \mathcal{SP}$,

$$p_1 P_\mu = \sum_{\mu^+ : \mu^+ \searrow \mu} P_{\mu^+},$$

where the sum runs over strict partitions μ^+ obtained from μ by adding one box. Put $k = |\lambda| - |\mu|$. The integer $g^{\lambda/\mu}$ is the number of sequences $(\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(k)})$ of strict partitions such that $\lambda^{(0)} = \mu$, $\lambda^{(k)} = \lambda$, and such that $\lambda^{(i)} \searrow \lambda^{(i-1)}$ for each $i = 1, 2, \dots, k$. Therefore we find the formula

$$p_1^k P_\mu = \sum_{\lambda \in \mathcal{SP}_{|\mu|+k}} g^{\lambda/\mu} P_\lambda.$$

(4) now follows from (2). $\square$

For a strict partition λ and odd partition ρ of sizes k, we define

$$X_\rho^\lambda = \langle p_\rho, Q_\lambda \rangle. \tag{2.2}$$

Equivalently, the quantities X_ρ^λ are determined as transition matrices via

$$p_\rho = \sum_{\lambda \in \mathcal{SP}_k} X_\rho^\lambda P_\lambda \qquad \text{or} \qquad Q_\lambda = \sum_{\rho \in \mathcal{OP}_k} 2^{\ell(\rho)} z_\rho^{-1} X_\rho^\lambda p_\rho. \tag{2.3}$$

The quantity X^λ_ρ is a character value for a projective representation of symmetric groups, see [9, Chapter 8]. One can compute values X^λ_ρ recursively if we use a Murnaghan–Nakayama rule ([16, Chapter III.8, Example 11]). Note that $X^\lambda_{(1^k)} = g^\lambda$ and $X^{(k)}_\rho = 1$. Equivalently,

$$p_1^k = \sum_{\lambda \in \mathcal{SP}_k} g^\lambda P_\lambda \qquad \text{and} \qquad Q_{(k)} = \sum_{\rho \in \mathcal{OP}_k} 2^{\ell(\rho)} z_\rho^{-1} p_\rho .$$

Proposition 2.4. *For $\rho, \sigma \in \mathcal{OP}_k$,*

$$\sum_{\lambda \in \mathcal{SP}_k} 2^{-\ell(\lambda)} X^\lambda_\rho X^\lambda_\sigma = \delta_{\rho,\sigma} 2^{-\ell(\rho)} z_\rho .$$

Proof. It follows from (2.3) and Proposition 2.3 (2) that

$$\langle p_\rho, p_\sigma \rangle = \Big\langle \sum_\lambda X^\lambda_\rho P_\lambda, \sum_\mu X^\mu_\sigma P_\mu \Big\rangle = \sum_{\lambda,\mu} X^\lambda_\rho X^\mu_\sigma \langle P_\lambda, P_\mu \rangle = \sum_\lambda X^\lambda_\rho X^\lambda_\sigma 2^{-\ell(\lambda)},$$

the left hand side of which equals $2^{-\ell(\rho)} z_\rho \delta_{\rho\sigma}$ by (2.1). □

2.3. Factorial Schur P-functions

The next definition is due to A. Okounkov and given in [10].

Definition 2.5. Let $\lambda = (\lambda_1, \dots, \lambda_l) \in \mathcal{SP}$. Suppose that $N \geq l = \ell(\lambda)$. We introduce a polynomial $P^*_{\lambda|N}$ by

$$P^*_{\lambda|N}(x_1, \dots, x_N) = \frac{1}{(N-l)!} \sum_{\omega \in S_N} \omega \left(x_1^{\downarrow \lambda_1} x_2^{\downarrow \lambda_2} \cdots x_l^{\downarrow \lambda_l} \prod_{\substack{i:1 \leq i \leq l, \\ j: i<j \leq N}} \frac{x_i + x_j}{x_i - x_j} \right).$$

If $\ell(\lambda) > N$, then we set $P^*_{\lambda|N}(x_1, \dots, x_N) = 0$. The collection $(P^*_{\lambda|N})_{N=1,2,\dots}$ defines an element P^*_λ in Γ. Define Q^*_λ by $Q^*_\lambda = 2^{\ell(\lambda)} P^*_\lambda$. We call P^*_λ and Q^*_λ the *factorial Schur P-function* and *Q-function*, respectively.

Remark that P_λ is homogeneous, whereas P^*_λ is not. Let us review some properties for factorial Schur P-functions. See [10, 11] for detail. We note that (1)–(3) in the next proposition are immediately comfirmed from definitions, while the proof of (4) requires a more careful work.

Proposition 2.6.

(1) $P^*_\lambda = P_\lambda + g$, *where* g *is a supersymmetric function of degree less than* $|\lambda|$.

(2) *The family* $(P^*_\lambda)_{\lambda \in \mathcal{SP}}$ *forms a linear basis of* Γ.

(3) $P^*_\mu(\lambda) = 0$ *unless* $S(\mu) \subset S(\lambda)$.

(4) *It holds that*

$$P^*_\mu(\lambda) = |\lambda|^{\downarrow|\mu|} \frac{g^{\lambda/\mu}}{g^\lambda} = |\lambda|^{\downarrow|\mu|} \frac{\langle p_1^{|\lambda|-|\mu|} P_\mu, Q_\lambda \rangle}{g^\lambda} . \tag{2.4}$$

On the right hand side of (2.4), we can think $|\lambda| - |\mu|$ being nonnegative, because if $|\lambda| < |\mu|$ then $|\lambda|^{\downarrow|\mu|} = 0$.

Next we give a formula for an expansion of P_λ in terms of P^*_μ. Recall the Stirling numbers $T(k, j)$ of the second kind defined by

$$x^k = \sum_{j=1}^{k} T(k,j) x^{\downarrow j} \qquad (k = 1, 2, \dots) . \tag{2.5}$$

Proposition 2.7. *Let* λ *be a strict partition of length* l. *Then*

$$P_\lambda = \sum_{j_1=1}^{\lambda_1} \cdots \sum_{j_l=1}^{\lambda_l} T(\lambda_1, j_1) \cdots T(\lambda_l, j_l) P^*_{(j_1,\dots,j_l)} .$$

Here we set $P^*_{(j_1,\dots,j_l)} = 0$ *if* $j_1, \dots, j_l$ *are not pairwise distinct, and*

$$P^*_{(j_1,\dots,j_l)} = (\operatorname{sgn} \pi) P^*_\mu$$

if $(j_1, \dots, j_l) = (\mu_{\pi(1)}, \dots, \mu_{\pi(l)})$ *for some strict partition* $\mu = (\mu_1, \dots, \mu_l)$ *of length* l *with a permutation* π.

Proof. The definition of $P^*_{\lambda|N}(x_1, \dots, x_N)$ in Definition 2.5 makes sense even if $(\lambda_1, \dots, \lambda_l)$ is replaced with any sequence of positive integers $(j_1, \dots, j_l)$. From the alternating property

$$\pi \left(\prod_{\substack{i:1 \le i \le l,\\ j: i < j \le N}} \frac{x_i + x_j}{x_i - x_j} \right) = (\operatorname{sgn} \pi) \prod_{\substack{i:1 \le i \le l,\\ j: i < j \le N}} \frac{x_i + x_j}{x_i - x_j}$$

for a permutation $\pi \in S_l$ acting on variables $x_1, \dots, x_l$, we have

$$P^*_{(j_{\pi(1)},\dots,j_{\pi(l)})|N}(x_1,\dots,x_N) = (\operatorname{sgn}\pi) P^*_{(j_1,\dots,j_l)|N}(x_1,\dots,x_N).$$

In particular, $P^*_{(j_1,\dots,j_l)} = 0$ if $j_s = j_t$ for some $s \neq t$. Our proposition follows from the definitions of P_λ, P^*_μ, and Stirling numbers. □

3. New supersymmetric functions

Recall the fact that two families $(P_\lambda)_{\lambda\in\mathcal{SP}}$ and $(P^*_\lambda)_{\lambda\in\mathcal{SP}}$ are liner bases of Γ. Define a linear isomorphism $\Psi : \Gamma \to \Gamma$ by

$$\Psi(P_\lambda) = P^*_\lambda \qquad (\lambda \in \mathcal{SP}).$$

Note that $\Psi^{-1}(f)$ coincides with the top-degree term of f by Proposition 2.6 (1).

Definition 3.1. For each $\rho \in \mathcal{OP}_k$, we define the supersymmetric function $\mathfrak{p}_\rho$ by $\mathfrak{p}_\rho = \Psi(p_\rho)$. From (2.3), we have

$$\mathfrak{p}_\rho = \sum_{\lambda\in\mathcal{SP}_k} X^\lambda_\rho P^*_\lambda.$$

For an odd partition ρ, we denote by $\tilde{\rho}$ the odd partition obtained from ρ by erasing parts equal to 1. For example, if $\rho = (5,5,3,1,1)$, then $\tilde{\rho} = (5,5,3)$. Note that $|\rho| = |\tilde{\rho}| + m_1(\rho)$ and $\ell(\rho) = \ell(\tilde{\rho}) + m_1(\rho)$.

Proposition 3.2.

(1) *$\mathfrak{p}_\rho = p_\rho + g$, where g is a supersymmetric function of degree less than $|\rho|$.*

(2) *The family $(\mathfrak{p}_\rho)_{\rho\in\mathcal{OP}}$ forms a linear basis of Γ.*

(3) *For $\rho \in \mathcal{OP}$ and $\lambda \in \mathcal{SP}$,*

$$\mathfrak{p}_\rho(\lambda) = |\lambda|^{\downarrow|\rho|} \frac{\langle p_1^{|\lambda|-|\tilde{\rho}|} p_{\tilde{\rho}}, Q_\lambda\rangle}{g^\lambda} = |\lambda|^{\downarrow|\rho|} \frac{X^\lambda_{\tilde{\rho}\cup(1^{|\lambda|-|\tilde{\rho}|})}}{g^\lambda}.$$

Here $\tilde{\rho}\cup(1^k)$ denotes the odd partition $(\tilde{\rho}, \underbrace{1,1,\dots,1}_{k})$.

(4) *For $\rho \in \mathcal{OP}$ and $\lambda \in \mathcal{SP}$,*

$$\mathfrak{p}_\rho(\lambda) = (|\lambda| - |\tilde{\rho}|)^{\downarrow m_1(\rho)} \mathfrak{p}_{\tilde{\rho}}(\lambda).$$

(5) *If* $|\tilde{\rho}| > |\lambda|$, *then* $\mathfrak{p}_\rho(\lambda) = 0$.

Proof. (1): It follows immediately from (2.3) and Proposition 2.6 (1).

(2): It follows immediately from (1) and the fact that $(p_\rho)_{\rho \in \mathcal{OP}}$ form a linear basis of Γ.

(3): Set $k = |\rho|$. From Proposition 2.6 (iv), (2.3), and (2.2), we have

$$\begin{aligned}
\mathfrak{p}_\rho(\lambda) &= \sum_{\mu \in \mathcal{SP}_k} X^\mu_\rho P^*_\mu(\lambda) = \sum_{\mu \in \mathcal{SP}_k} X^\mu_\rho |\lambda|^{\downarrow k} \frac{\langle p_1^{|\lambda|-k} P_\mu, Q_\lambda \rangle}{g^\lambda} \\
&= |\lambda|^{\downarrow k} \frac{\langle p_1^{|\lambda|-k} p_\rho, Q_\lambda \rangle}{g^\lambda} = |\lambda|^{\downarrow k} \frac{X^\lambda_{\rho \cup (1^{|\lambda|-k})}}{g^\lambda} .
\end{aligned}$$

Note that $\rho \cup (1^{|\lambda|-k}) = \tilde{\rho} \cup (1^{|\lambda|-|\tilde{\rho}|})$.

(4): Since $|\lambda|^{\downarrow|\rho|} = |\lambda|^{\downarrow|\tilde{\rho}|} \cdot (|\lambda| - |\tilde{\rho}|)^{\downarrow m_1(\rho)}$, (3) implies that

$$\mathfrak{p}_\rho(\lambda) = |\lambda|^{\downarrow|\tilde{\rho}|} \cdot (|\lambda| - |\tilde{\rho}|)^{\downarrow m_1(\rho)} \frac{X^\lambda_{\tilde{\rho} \cup (1^{|\lambda|-|\tilde{\rho}|})}}{g^\lambda} = (|\lambda| - |\tilde{\rho}|)^{\downarrow m_1(\rho)} \mathfrak{p}_{\tilde{\rho}}(\lambda) .$$

(5): If $|\lambda| < |\tilde{\rho}|$, then $|\lambda|^{\downarrow|\tilde{\rho}|} = |\lambda|(|\lambda|-1)\cdots(|\lambda|-|\tilde{\rho}|+1) = 0$, and therefore we obtain $\mathfrak{p}_\rho(\lambda) = 0$ from (3). $\square$

Substituting $\rho = (1^k)$ in Proposition 3.2 (iv), we obtain $\mathfrak{p}_{(1^k)}(\lambda) = |\lambda|^{\downarrow k}$. Using Stirling numbers defined in (2.5), we find

$$p_{(1^k)} = \sum_{j=1}^{k} T(k,j) \mathfrak{p}_{(1^j)} .$$

More generally, a power-sum function p_ρ can be expanded as a linear combination of $\mathfrak{p}_\sigma$ in the following way. First, we expand p_ρ in terms of P_λ by using (2.3). Second, each P_λ is expanded in terms of factorial Schur P-functions P^*_μ by Proposition 2.7. Finally, each P^*_μ is expanded in terms of $\mathfrak{p}_\sigma$ by the formula

$$P^*_\mu = \sum_{\sigma \in \mathcal{OP}_{|\mu|}} 2^{-\ell(\mu)+\ell(\sigma)} z_\sigma^{-1} X^\mu_\sigma \mathfrak{p}_\sigma ,$$

which is the image of the second equation on (2.3) under Ψ.

Example 3.3.

$$
\begin{aligned}
p_{(1)} &= \mathfrak{p}_{(1)}\,,\\
p_{(1^2)} &= \mathfrak{p}_{(1^2)} + \mathfrak{p}_{(1)}\,,\\
p_{(3)} &= \mathfrak{p}_{(3)} + 3\mathfrak{p}_{(1^2)} + \mathfrak{p}_{(1)}\,,\\
p_{(1^3)} &= \mathfrak{p}_{(1^3)} + 3\mathfrak{p}_{(1^2)} + \mathfrak{p}_{(1)}\\
p_{(3,1)} &= \mathfrak{p}_{(3,1)} + 3\mathfrak{p}_{(3)} + 3\mathfrak{p}_{(1^3)} + 7\mathfrak{p}_{(1^2)} + \mathfrak{p}_{(1)}\,,\\
p_{(1^4)} &= \mathfrak{p}_{(1^4)} + 6\mathfrak{p}_{(1^3)} + 7\mathfrak{p}_{(1^2)} + \mathfrak{p}_{(1)}\,,\\
p_{(5)} &= \mathfrak{p}_{(5)} + 10\mathfrak{p}_{(3,1)} + \frac{35}{3}\mathfrak{p}_{(3)} + \frac{40}{3}\mathfrak{p}_{(1^3)} + 15\mathfrak{p}_{(1^2)} + \mathfrak{p}_{(1)}\,,\\
p_{(3,1,1)} &= \mathfrak{p}_{(3,1,1)} + 7\mathfrak{p}_{(3,1)} + 3\mathfrak{p}_{(1^4)} + 9\mathfrak{p}_{(3)} + 16\mathfrak{p}_{(1^3)} + 15\mathfrak{p}_{(1^2)} + \mathfrak{p}_{(1)}\,,\\
p_{(1^5)} &= \mathfrak{p}_{(1^5)} + 10\mathfrak{p}_{(1^4)} + 25\mathfrak{p}_{(1^3)} + 15\mathfrak{p}_{(1^2)} + \mathfrak{p}_{(1)}\,.
\end{aligned}
$$

Remark 3.4. For two ordinary partitions λ, μ, we consider

$$
\mathrm{Ch}_\mu(\lambda) = \begin{cases} |\lambda|^{\downarrow|\mu|}\frac{\chi^\lambda_{\mu\cup(1^{|\lambda|-|\mu|})}}{f^\lambda} & \text{if } |\lambda| \geq |\mu|,\\ 0 & \text{if } |\lambda| < |\mu|, \end{cases}
$$

where $\chi^\lambda_{\mu\cup(1^{|\lambda|-|\mu|})}$ is the value of the irreducible character χ^λ of the symmetric group $S_{|\lambda|}$ at conjugacy class associated with $\mu \cup (1^{|\lambda|-|\mu|})$. The functions Ch_μ on the set of all partitions are called the normalized characters of symmetric groups, and have rich properties and applications. See [4, 13, 24]. Note that the function is written as $p^\#_\mu$ in [13]. Our function $\mathfrak{p}_\rho$ is a *projective* analog of Ch_μ since X^λ_ρ is a character value for a projective representation of symmetric groups.

4. **Shifted Plancherel averages**

4.1. **Proof of Polynomiality**

In the present section we give a proof of Theorems 1.2 and 1.3. Let m be a nonnegative integer. Fix $\mu \in \mathcal{SP}_m$. For each $\rho \in \mathcal{OP}$, we consider the summation

$$
\mathbb{E}_{\mu,n}[\mathfrak{p}_\rho] = \sum_{\lambda\in\mathcal{SP}_{n+m}} \frac{m!}{(n+m)!}\, 2^{n-\ell(\lambda)+\ell(\mu)}\,\frac{g^\lambda}{g^\mu}\, g^{\lambda/\mu}\mathfrak{p}_\rho(\lambda)\,.
$$

Since Proposition 3.2 (iv) implies that

$$\mathbb{E}_{\mu,n}[\mathfrak{p}_\rho] = (n+m-|\tilde{\rho}|)^{\downarrow m_1(\rho)}\mathbb{E}_{\mu,n}[\mathfrak{p}_{\tilde{\rho}}], \tag{4.1}$$

it is sufficient to compute $\mathbb{E}_{\mu,n}[\mathfrak{p}_\rho]$ for odd partitions ρ with no part equal to 1.

The following lemma is seen in [16, Chapter III.8, Example 11].

Lemma 4.1. *For $f, g \in \Gamma$,*

$$\langle p_1 f, g\rangle = \frac{1}{2}\left\langle f, \frac{\partial}{\partial p_1} g\right\rangle.$$

Here the differential operator $\frac{\partial}{\partial p_1}$ acts on functions in Γ expressed as polynomials in $p_1, p_3, p_5, \ldots$.

Theorem 4.2. *Let ρ be an odd partition such that $m_1(\rho) = 0$. Then*

$$\mathbb{E}_{\mu,n}[\mathfrak{p}_\rho] = \mathfrak{p}_\rho(\mu). \tag{4.2}$$

Proof. First of all, if $n+m < |\rho|$, then we have $\mathfrak{p}_\rho(\lambda) = 0$ for all $\lambda \in \mathcal{SP}_{n+m}$ and $\mathfrak{p}_\rho(\mu) = 0$ by virtue of Proposition 3.2 (5), so that $\mathbb{E}_{\mu,n}[\mathfrak{p}_\rho] = 0 = \mathfrak{p}_\rho(\mu)$. Consequently, we may assume $|\rho| \le n+m$.

Using Proposition 2.3 (4) and Proposition 3.2 (3), we see that

$$\begin{aligned}
\mathbb{E}_{\mu,n}[\mathfrak{p}_\rho] &= \sum_{\lambda \in \mathcal{SP}_{n+m}} \frac{m!}{(n+m)!} 2^{n-\ell(\lambda)+\ell(\mu)} \frac{g^\lambda}{g^\mu} \langle p_1^n P_\mu, Q_\lambda\rangle \\
&\qquad\qquad \times (n+m)^{\downarrow|\rho|} \frac{\langle p_1^{n+m-|\rho|} p_\rho, Q_\lambda\rangle}{g^\lambda} \\
&= \frac{m!}{g^\mu} \frac{(n+m)^{\downarrow|\rho|}}{(n+m)!} \sum_{\lambda \in \mathcal{SP}_{n+m}} 2^{n-\ell(\lambda)} \langle p_1^n Q_\mu, Q_\lambda\rangle \langle p_1^{n+m-|\rho|} p_\rho, Q_\lambda\rangle \\
&= \frac{m!}{g^\mu} \frac{(n+m)^{\downarrow|\rho|}}{(n+m)!} 2^n \langle p_1^n Q_\mu, p_1^{n+m-|\rho|} p_\rho\rangle,
\end{aligned} \tag{4.3}$$

where we have used Proposition 2.3 (3) for the last equality.

We next compute the scalar product $\langle p_1^n Q_\mu, p_1^{n+m-|\rho|} p_\rho\rangle$. Assume that $|\rho| > m$. Expanding Q_μ in p_σ (see (2.3)), we have

$$\langle p_1^n Q_\mu, p_1^{n+m-|\rho|} p_\rho\rangle = \sum_{\sigma \in \mathcal{OP}_m} 2^{\ell(\sigma)} z_\sigma^{-1} X_\sigma^\mu \langle p_1^{n+m_1(\sigma)} p_{\tilde{\sigma}}, p_1^{n-(|\rho|-m)} p_\rho\rangle.$$

Here all the scalar products of the right hand side vanish by (2.1) because $m_1(\rho) = 0$ and $n + m_1(\sigma) \geq n > n - (|\rho| - m)$. Therefore it follows from (4.3) that

$$\mathbb{E}_{\mu,n}[\mathfrak{p}_\rho] = 0 \qquad \text{if } |\rho| > m\,.$$

On the other hand, $\mathfrak{p}_\rho(\mu) = 0$ if $|\rho| > m$ by Proposition 3.2 (5), hence $\mathbb{E}_{\mu,n}[\mathfrak{p}_\rho] = 0 = \mathfrak{p}_\rho(\mu)$ in that case.

We finally assume that $|\rho| \leq m$. Using Lemma 4.1 we have

$$\begin{aligned}\langle p_1^n Q_\mu, p_1^{n+m-|\rho|} p_\rho\rangle &= \frac{1}{2^n}\left\langle Q_\mu, \left(\frac{\partial}{\partial p_1}\right)^n p_1^{n+m-|\rho|} p_\rho\right\rangle \\ &= \frac{(n+m-|\rho|)^{\downarrow n}}{2^n}\langle Q_\mu, p_1^{m-|\rho|} p_\rho\rangle \\ &= \frac{(n+m-|\rho|)^{\downarrow n}}{2^n}\frac{g^\mu}{m^{\downarrow|\rho|}}\,\mathfrak{p}_\rho(\mu)\,,\end{aligned}$$

where in the last equality Proposition 3.2 (3) is applied. Combining this with (4.3) gives

$$\mathbb{E}_{\mu,n}[\mathfrak{p}_\rho] = \frac{(n+m)^{\downarrow|\rho|}\,(n+m-|\rho|)^{\downarrow n} m!}{(n+m)!\, m^{\downarrow|\rho|}}\,\mathfrak{p}_\rho(\mu)\,.$$

A straightforward computation gives the desired expression. □

Substituting $\rho = \emptyset$ in Theorem 4.2, we obtain

$$\mathbb{E}_{\mu,n}[1] = 1\,,$$

which shows that $\mathbb{P}_{\mu,n}$ defined in (1.1) is a probability measure on $\mathcal{SP}_{n+m}$ and that $\mathbb{E}_{\mu,n}$ is the average with respect to $\mathbb{P}_{\mu,n}$.

Corollary 4.3. *For $\rho \in \mathcal{OP}_k$,*

$$\mathbb{E}_n[\mathfrak{p}_\rho] = \delta_{\rho,(1^k)} n^{\downarrow k}.$$

Proof. Notice that $\mathbb{E}_n[\mathfrak{p}_\rho] = n^{\downarrow m_1(\rho)}\mathbb{E}_n[\mathfrak{p}_{\tilde{\rho}}]$ by (4.1). Substituting $\mu = \emptyset$ in Theorem 4.2 implies that $\mathbb{E}_n[\mathfrak{p}_{\tilde{\rho}}] = \delta_{\tilde{\rho},\emptyset}$. Therefore $\mathbb{E}_n[\mathfrak{p}_\rho]$ survives only if $\rho = (1^k)$, and $\mathbb{E}_n[\mathfrak{p}_{(1^k)}] = n^{\downarrow k}$. □

Now the polynomiality of $\mathbb{E}_{\mu,n}[f]$ becomes trivial.

Proof of Theorems 1.2 and 1.3. Since the $\mathfrak{p}_\rho$, $\rho \in \mathcal{OP}$, form a linear basis of the algebra Γ of supersymmetric functions, we obtain Theorem 1.3 from Theorem 4.2. Theorem 1.2 is a special case of Theorem 1.3 with $\mu = \emptyset$. □

4.2. **Orthogonality for** $\mathfrak{p}_\rho$

We can also easily compute the $\mathbb{P}_n$-average of products $\mathfrak{p}_\rho\mathfrak{p}_\sigma$.

Theorem 4.4. *Let* ρ *and* σ *be odd partitions such that* $m_1(\rho) = m_1(\sigma) = 0$*. Then*

$$\mathbb{E}_n[\mathfrak{p}_\rho\mathfrak{p}_\sigma] = \delta_{\rho,\sigma} 2^{|\rho|-\ell(\rho)} z_\rho n^{\downarrow|\rho|}.$$

Proof. We may suppose $n \geq \max\{|\rho|, |\sigma|\}$ by virtue of Proposition 3.2 (5). It follows from Proposition 3.2 (3) that

$$\begin{aligned}\mathbb{E}_n[\mathfrak{p}_\rho\mathfrak{p}_\sigma] &= \sum_{\lambda\in\mathcal{SP}_n} \frac{2^{n-\ell(\lambda)}(g^\lambda)^2}{n!} \cdot \frac{n^{\downarrow|\rho|}}{g^\lambda} X^\lambda_{\rho\cup(1^{n-|\rho|})} \cdot \frac{n^{\downarrow|\sigma|}}{g^\lambda} X^\lambda_{\sigma\cup(1^{n-|\sigma|})} \\ &= \frac{n^{\downarrow|\rho|}n^{\downarrow|\sigma|}}{n!} \sum_{\lambda\in\mathcal{SP}_n} 2^{n-\ell(\lambda)} X^\lambda_{\rho\cup(1^{n-|\rho|})} X^\lambda_{\sigma\cup(1^{n-|\sigma|})}.\end{aligned}$$

Using Proposition 2.4 and the fact that $m_1(\rho) = m_1(\sigma) = 0$, it equals

$$= \frac{n^{\downarrow|\rho|}n^{\downarrow|\sigma|}}{n!} 2^{n-(\ell(\rho)+n-|\rho|)} z_{\rho\cup(1^{n-|\rho|})}\delta_{\rho,\sigma} = n^{\downarrow|\rho|}2^{|\rho|-\ell(\rho)} z_\rho \delta_{\rho,\sigma}. \qquad \square$$

Substituting $\sigma = \emptyset$ in Theorem 4.4 recovers Corollary 4.3.

4.3. **Bound for degrees**

The following proposition is a direct consequence of Corollary 4.3.

Proposition 4.5. *Let* f *be a supersymmetric function. If we expand* f *as a linear combination of* $\mathfrak{p}_\rho$*:*

$$f = \sum_{\substack{\rho\in\mathcal{OP} \\ \text{(finite sum)}}} a_\rho(f)\mathfrak{p}_\rho,$$

then

$$\mathbb{E}_n[f] = \sum_{r\geq 0} a_{(1^r)}(f) n^{\downarrow r}.$$

In particular, if $a_{(1^r)}(f)$ *vanish for all* $r > k$*, then* $\mathbb{E}_n[f]$ *is of degree at most* k*.*

Inspired by [12], we define a degree filtration of the vector space Γ by

$$\deg_1(\mathfrak{p}_\rho) = |\rho| + m_1(\rho).$$

More generally, for $f = \sum_\rho a_\rho(f)\mathfrak{p}_\rho \in \Gamma$, define

$$\deg_1(f) = \max_{\rho : a_\rho(f)\neq 0} (|\rho| + m_1(\rho)).$$

Proposition 4.6. *The degree of $\mathbb{E}_n[f]$ as a polynomial in n is at most $\frac{1}{2}\deg_1(f)$.*

Proof. It is sufficient to check this for $f = \mathfrak{p}_\rho$. By virtue of Corollary 4.3, we have: if $\rho \neq (1^k)$ then $\mathbb{E}_n[\mathfrak{p}_\rho] = 0$; if $\rho = (1^k)$ then $\mathbb{E}_n[\mathfrak{p}_\rho] = n^{\downarrow k}$ and $\deg_1(\mathfrak{p}_\rho) = 2k$. $\square$

4.4. **Examples**

We show some explicit expressions of $\mathbb{E}_n[p_\rho]$, which are presented in Subsection 1.2. In Example 3.3, we give expansions of some p_ρ in $\mathfrak{p}_\sigma$. By Corollary 4.3, we obtain the following identities immediately.

$$\begin{aligned}
\mathbb{E}_n[p_3] &= \mathbb{E}_n\left[\mathfrak{p}_{(3)} + 3\mathfrak{p}_{(1^2)} + \mathfrak{p}_{(1)}\right] = 3n^{\downarrow 2} + n,\\
\mathbb{E}_n[p_5] &= \mathbb{E}_n\left[\mathfrak{p}_{(5)} + 10\mathfrak{p}_{(3,1)} + \frac{35}{3}\mathfrak{p}_{(3)} + \frac{40}{3}\mathfrak{p}_{(1^3)} + 15\mathfrak{p}_{(1^2)} + \mathfrak{p}_{(1)}\right]\\
&= \frac{40}{3}n^{\downarrow 3} + 15n^{\downarrow 2} + n.
\end{aligned}$$

Moreover, Theorem 4.4 gives

$$\begin{aligned}
\mathbb{E}_n\left[p_3^2\right] &= \mathbb{E}_n\left[(\mathfrak{p}_{(3)} + 3\mathfrak{p}_{(1^2)} + \mathfrak{p}_{(1)})^2\right]\\
&= \mathbb{E}_n\Big[\mathfrak{p}_{(3)}\mathfrak{p}_{(3)} + 6\mathfrak{p}_{(3)}\mathfrak{p}_{(1^2)} + 2\mathfrak{p}_{(3)}\mathfrak{p}_{(1)} + 9\mathfrak{p}_{(1^2)}\mathfrak{p}_{(1^2)}\\
&\qquad\qquad + 6\mathfrak{p}_{(1^2)}\mathfrak{p}_{(1)} + \mathfrak{p}_{(1)}\mathfrak{p}_{(1)}\Big]\\
&= 12n^{\downarrow 3} + 0 + 0 + 9n^{\downarrow 2}\cdot n^{\downarrow 2} + 6n^{\downarrow 2}\cdot n + n^2\\
&= 9n^{\downarrow 4} + 54n^{\downarrow 3} + 31n^{\downarrow 2} + n.
\end{aligned}$$

5. **Content evaluations**

5.1. **Supersymmetry**

In this section, we give a proof of Theorem 1.4. Let λ be a strict partition and recall the shifted Young diagram

$$S(\lambda) = \left\{(i,j) \in \mathbb{Z}^2 \;\middle|\; 1 \le i \le \ell(\lambda),\ i \le j \le \lambda_i + i - 1\right\}.$$

For each $\square = (i,j) \in S(\lambda)$, its content $c_\square$ is defined by $c_\square = j - i$. We find

$$\begin{aligned}\{c_\square \in \mathbb{Z} \mid \square \in S(\lambda)\} &= \{j - i \mid 1 \le i \le \ell(\lambda),\ i \le j \le \lambda_i + i - 1\} \\ &= \{j - 1 \mid 1 \le i \le \ell(\lambda),\ 1 \le j \le \lambda_i\}\end{aligned}$$

as multi-sets.

Lemma 5.1. *For each $m = 0, 1, 2, \dots,$*

$$p_{2m+1}(\lambda) = \sum_{\square \in S(\lambda)} \left\{ (c_\square + 1)^{2m+1} - c_\square^{2m+1} \right\}$$

for any strict partition λ.

Proof. Consider the function

$$\Phi(u;\lambda) = \prod_{i=1}^{\ell(\lambda)} \frac{1+\lambda_i u}{1-\lambda_i u}.$$

The Taylor expansion of $\log \Phi(u;\lambda)$ at $u = 0$ is

$$\begin{aligned}\log \Phi(u;\lambda) &= \sum_{i=1}^{\ell(\lambda)} \{\log(1+\lambda_i u) - \log(1-\lambda_i u)\} \\ &= \sum_{i=1}^{\ell(\lambda)} \sum_{r=1}^{\infty} \frac{u^r}{r} \{1 + (-1)^{r-1}\} \lambda_i^r = 2 \sum_{m=0}^{\infty} \frac{u^{2m+1}}{2m+1} p_{2m+1}(\lambda).\end{aligned}$$

On the other hand, since

$$\frac{1+\lambda_i u}{1-\lambda_i u} = \prod_{j=1}^{\lambda_i} \frac{(1+ju)(1-(j-1)u)}{(1-ju)(1+(j-1)u)},$$

we see that

$$\Phi(u;\lambda) = \prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} \frac{(1+ju)(1-(j-1)u)}{(1-ju)(1+(j-1)u)} = \prod_{\square \in S(\lambda)} \frac{(1+(c_\square+1)u)(1-c_\square u)}{(1-(c_\square+1)u)(1+c_\square u)}.$$

In this expression, the Taylor expansion of $\log \Phi(u;\lambda)$ at $u = 0$ is

$$\log \Phi(u;\lambda) = 2 \sum_{m=0}^{\infty} \frac{u^{2m+1}}{2m+1} \sum_{\square \in S(\lambda)} \left\{ (c_\square + 1)^{2m+1} - c_\square^{2m+1} \right\}.$$

Comparing coefficients in two expressions of the Taylor expansion, we obtain the desired formula. $\square$

Lemma 5.2. *Let $R(X)$ be a polynomial in a variable X. Put $Y=X(X+1)$. Then there exists a polynomial $\tilde{R}$ in Y such that $\tilde{R}(Y) = R(X)$ if and only if R satisfies the functional equation $R(X) = R(-X-1)$. Moreover, if the top-degree term of R is aX^{2m} then the top-degree term of $\tilde{R}$ is aY^m.*

Proof. First we suppose that $R(X)$ can be expressed as $R(X) = \tilde{R}(Y)$. Since $Y = X(X+1)$ is invariant under the change of variable $X \mapsto \bar{X} = -X-1$, we obtain the functional equation $R(X) = R(-1-X)$.

Next suppose that $R(X)$ satisfies the functional equation $R(X) = R(-1-X)$. In general, a polynomial function $y = r(x)$ is symmetric with respect to the y-axis in the xy-plane if and only if r is of the form

$$r(x) = \sum_{j=0}^{m} a_j x^{2j}$$

with certain coefficients a_j. We may suppose $a_m \neq 0$. Put $R(X) = r(X+\frac{1}{2})$. We can observe that the symmetry $r(-x) = r(x)$ is equivalent to the functional equation $R(-X-1) = R(X)$. Moreover, $R(X)$ is of the form

$$\begin{aligned} R(X) = \sum_{j=0}^{m} a_j \left(X + \tfrac{1}{2}\right)^{2j} &= \sum_{j=0}^{m} a_j \left(X(X+1) + \tfrac{1}{4}\right)^{j} \\ &= \sum_{j=0}^{m} a_j \left(Y + \tfrac{1}{4}\right)^{j} =: \tilde{R}(Y). \end{aligned}$$

The top-degree term of $R(X)$ is $a_m X^{2m}$, whereas that of $\tilde{R}(Y)$ is $a_m Y^m$. $\square$

Define

$$\widehat{c}_\square = \frac{1}{2} c_\square (c_\square + 1)$$

for each $\square \in S(\lambda)$.

Proposition 5.3. *Let $k = 1, 2, 3, \ldots$. The function $\widehat{p}_k$ on $\mathcal{SP}$ defined by*

$$\widehat{p}_k(\lambda) = p_k\left(\widehat{c}_\square : \square \in S(\lambda)\right) = 2^{-k} \sum_{\square \in S(\lambda)} \left\{ c_\square (c_\square + 1) \right\}^k$$

is supersymmetric. Moreover, if we set $\widehat{p}_0(\lambda) = |\lambda|$, then $\widehat{p}_0$ is also supersymmetric.

Proof. First, we see that $\widehat{p}_0(\lambda) = |\lambda| = \lambda_1 + \lambda_2 + \cdots = p_1(\lambda)$, and hence $\widehat{p}_0$ is supersymmetric.

Let m be a nonnegative integer and λ a strict partition. Lemma 5.1 says that

$$p_{2m+1}(\lambda) = \sum_{\square \in S(\lambda)} \left\{ (c_\square + 1)^{2m+1} - c_\square^{2m+1} \right\}.$$

Since the polynomial function $R(X) := (X+1)^{2m+1} - X^{2m+1} = (2m+1)X^{2m} + \cdots$ clearly satisfies the functional equation $R(X) = R(-X-1)$, it can be expressed as a polynomial in $Y = X(X+1)$ of degree m by Lemma 5.2. Hence there exist universal coefficients a_{mr} ($r = 0, 1, 2, \ldots, m-1$) such that

$$\begin{aligned} p_{2m+1}(\lambda) &= \sum_{\square \in S(\lambda)} \left[(2m+1)\{c_\square(c_\square+1)\}^m + \sum_{r=0}^{m-1} a_{mr}\{c_\square(c_\square+1)\}^r \right] \\ &= 2^m(2m+1)\widehat{p}_m(\lambda) + \sum_{r=0}^{m-1} 2^r a_{mr} \widehat{p}_r(\lambda). \end{aligned}$$

This relation implies that for each $k = 0, 1, 2, \ldots,$

$$\widehat{p}_k = \frac{1}{2^k(2k+1)} p_{2k+1} + \sum_{r=0}^{k-1} b_{kr} p_{2r+1} \tag{5.1}$$

with some rational coefficients b_{kr}. Therefore $\widehat{p}_k$ belongs to Γ. $\square$

Proof of Theorem 1.4. Let F be any symmetric function. It is well known that F can be uniquely expressed as a polynomial in variables $p_1, p_2, \ldots$. Hence the function $\widehat{F}$ on $\mathcal{SP}$ defined by

$$\widehat{F}(\lambda) = F(\widehat{c}_\square \ : \ \square \in S(\lambda))$$

is a polynomial in $\widehat{p}_1, \widehat{p}_2, \ldots$. Theorem 1.4 follows from Proposition 5.3. $\square$

The family $(\widehat{p}_k)_{k=0,1,2,\ldots}$ is an algebraic basis of Γ by (5.1). This shows that Theorem 1.3 and Corollary 1.5 are *equivalent*. Furthermore, we have obtained the following proposition.

Proposition 5.4. *The algebra Γ coincides with the algebra generated by the function $\lambda \mapsto |\lambda|$ and the functions $\widehat{F}$, where F are (ordinary) symmetric functions.*

5.2. Examples

We give some examples of $\mathbb{E}_{\mu,n}[\widehat{F}]$, where F is a symmetric function. It is easy to see that

$$\widehat{p}_1 = \frac{1}{6}p_3 - \frac{1}{6}p_1\,, \qquad \widehat{p}_2 = \frac{1}{20}p_5 - \frac{1}{12}p_3 + \frac{1}{30}p_1\,.$$

Indeed, for example, $\widehat{p}_1$ is computed as follows:

$$\begin{aligned}\widehat{p}_1(\lambda) = \sum_{\square \in S(\lambda)} \frac{1}{2}c_\square(c_\square + 1) &= \frac{1}{2}\sum_{i=1}^{\ell(\lambda)}\sum_{j=1}^{\lambda_i}\left(j^2 - j\right) \\ &= \sum_{i=1}^{\ell(\lambda)} \frac{\lambda_i^3 - \lambda_i}{6} = \frac{1}{6}(p_3(\lambda) - p_1(\lambda))\,.\end{aligned}$$

Using Example 3.3 and Corollary 4.3, we have

$$\mathbb{E}_n[\widehat{p}_1] = \mathbb{E}_n\left[\frac{1}{6}\mathfrak{p}_{(3)} + \frac{1}{2}\mathfrak{p}_{(1^2)}\right] = \frac{1}{2}n^{\downarrow 2}\,.$$

Similarly, we have

$$\mathbb{E}_n[\widehat{p}_2] = \mathbb{E}_n\left[\frac{1}{20}\mathfrak{p}_{(5)} + \frac{1}{2}\mathfrak{p}_{(3,1)} + \frac{1}{2}\mathfrak{p}_{(3)} + \frac{2}{3}\mathfrak{p}_{(1^3)} + \frac{1}{2}\mathfrak{p}_{(1^2)}\right] = \frac{2}{3}n^{\downarrow 3} + \frac{1}{2}n^{\downarrow 2}.$$

Moreover, Theorem 4.4 gives

$$\begin{aligned}\mathbb{E}_n\left[(\widehat{p}_1)^2\right] &= \mathbb{E}_n\left[\left(\tfrac{1}{6}\mathfrak{p}_{(3)} + \tfrac{1}{2}\mathfrak{p}_{(1^2)}\right)^2\right] \\ &= \frac{1}{36}\mathbb{E}_n\left[\mathfrak{p}_{(3)}\mathfrak{p}_{(3)}\right] + \frac{1}{6}\mathbb{E}_n\left[\mathfrak{p}_{(3)}\mathfrak{p}_{(1^2)}\right] + \frac{1}{4}\mathbb{E}_n\left[\mathfrak{p}_{(1^2)}\mathfrak{p}_{(1^2)}\right] \\ &= \frac{1}{36}\cdot 12n^{\downarrow 3} + \frac{1}{4}n^2(n-1)^2 \\ &= 6\binom{n}{4} + 8\binom{n}{3} + \binom{n}{2}.\end{aligned}$$

The following example is seen in [8, Theorem 1.3]:

$$\mathbb{E}_{\mu,n}[\widehat{p}_1 - \widehat{p}_1(\mu)] = \frac{1}{2}n(n-1) + n|\mu|\,.$$

We can give its simple proof as follows. Since $\widehat{p}_1 = \frac{1}{6}\mathfrak{p}_{(3)} + \frac{1}{2}\mathfrak{p}_{(1^2)}$, we have $\widehat{p}_1(\mu) = \frac{1}{6}\mathfrak{p}_{(3)}(\mu) + \frac{1}{2}|\mu|(|\mu|-1)$, and $\mathbb{E}_{\mu,n}[\widehat{p}_1] = \frac{1}{6}\mathfrak{p}_{(3)}(\mu) + \frac{1}{2}(n+|\mu|)(n+$

$|\mu|-1)$ by virtue of Theorem 4.2 and (4.1). Therefore we have

$$\mathbb{E}_{\mu,n}[\widehat{p}_1-\widehat{p}_1(\mu)] = \frac{1}{2}(n+|\mu|)(n+|\mu|-1)-\frac{1}{2}|\mu|(|\mu|-1) = \frac{1}{2}n(n-1)+n|\mu|.$$

6. Remarks on functions introduced by Han and Xiong

We identify a strict partition λ with its shifted Young diagram $S(\lambda)$ as usual. A box $\square=(i,j)$ in $S(\lambda)$ is said to be an *outer corner* of λ if we obtain a new strict partition by removing the box $\square$ from $S(\lambda)$. A box $\square \in \mathbb{Z}^2$ is said to be an *inner corner* of λ if we obtain a new strict partition by adding the box $\square$ to $S(\lambda)$. Denote by $\mathbb{O}_\lambda$ and by $\mathbb{I}_\lambda$ the set of all outer and inner corners of λ, respectively. For example, if $\lambda=(5,4,2)$, then we have $\mathbb{O}_\lambda=\{(2,5),(3,4)\}$ and $\mathbb{I}_\lambda=\{(1,6),(3,5),(4,4)\}$.

For each integer $k\ge 1$, we define a function ψ_k on $\mathcal{SP}$ by

$$\psi_k(\lambda)=\sum_{\square\in\mathbb{I}_\lambda}\{c_\square(c_\square+1)\}^k-\sum_{\square\in\mathbb{O}_\lambda}\{c_\square(c_\square+1)\}^k.$$

In their paper [8], Han and Xiong first introduced those functions. Remark that these are denoted by q_k (or by Φ^k) in their articles with slight change $\psi_k=2^kq_k$. Our purpose in this short section is to give an alternative simple expression of ψ_k and to show that they are supersymmetric.

Proposition 6.1. *For each $\lambda\in\mathcal{SP}$, we have*

$$\prod_{i=1}^{\ell(\lambda)}\frac{1-\lambda_i(\lambda_i-1)u}{1-\lambda_i(\lambda_i+1)u}=\exp\left(\sum_{k=1}^{\infty}\frac{u^k}{k}\psi_k(\lambda)\right). \tag{6.1}$$

Proof. We prove the formula by induction on $|\lambda|$. If $\lambda=\emptyset$, then (6.1) holds true because $\psi_k(\emptyset)=0$ for all $k\ge 1$. Let λ be a strict partition and put

$$\widetilde{\Phi}(u;\lambda)=\prod_{i=1}^{\ell(\lambda)}\frac{1-\lambda_i(\lambda_i-1)u}{1-\lambda_i(\lambda_i+1)u}\qquad\text{and}\qquad\widetilde{\boldsymbol{\Phi}}(u;\lambda)=\exp\left(\sum_{k=1}^{\infty}\frac{u^k}{k}\psi_k(\lambda)\right).$$

Consider a strict partition λ^+ obtained by adding a box $\square$ to λ. The added box $\square$ is an inner corner of λ and of the form $\square=(r,\lambda_r+r)$, where r is an integer in $\{1,2,\dots,\ell(\lambda)+1\}$ and we set $\lambda_{\ell(\lambda)+1}=0$. Notice that $c_\square=\lambda_r$. It is easy to see that

$$\frac{\widetilde{\Phi}(u;\lambda^+)}{\widetilde{\Phi}(u;\lambda)}=\frac{(1-\lambda_r(\lambda_r+1)u)^2}{(1-(\lambda_r+1)(\lambda_r+2)u)(1-(\lambda_r-1)\lambda_r u)}.$$

If we show the same identity for $\widetilde{\mathbf{\Phi}}$ then the induction step is completed.

A careful observation of the difference between $\mathbb{I}_\lambda \sqcup \mathbb{O}_\lambda$ and $\mathbb{I}_{\lambda^+} \sqcup \mathbb{O}_{\lambda^+}$ implies that

$$\begin{aligned}\psi_k(\lambda^+) - \psi_k(\lambda) &= \{(c_\square + 1)(c_\square + 2)\}^k + \{(c_\square - 1)c_\square\}^k - 2\{c_\square(c_\square + 1)\}^k \\ &= \{(\lambda_r + 1)(\lambda_r + 2)\}^k + \{(\lambda_r - 1)\lambda_r\}^k - 2\{\lambda_r(\lambda_r + 1)\}^k\end{aligned}$$

see also [8, (3.10)]. Therefore we have

$$\begin{aligned}\frac{\widetilde{\mathbf{\Phi}}(u;\lambda^+)}{\widetilde{\mathbf{\Phi}}(u;\lambda)} &= \exp\left(\sum_{k=1}^{\infty} \frac{u^k}{k}(\psi_k(\lambda^+) - \psi_k(\lambda))\right) \\ &= \frac{(1 - \lambda_r(\lambda_r + 1)u)^2}{(1 - (\lambda_r + 1)(\lambda_r + 2)u)(1 - (\lambda_r - 1)\lambda_r u)},\end{aligned}$$

as desired. $\square$

The function ψ_k is simply given as a polynomial in variables $\lambda_1, \lambda_2, \dots$.

Proposition 6.2. *For each $k \geq 1$ and strict partition λ, we have*

$$\psi_k(\lambda) = \sum_{i=1}^{\ell(\lambda)} \lambda_i^k\{(\lambda_i + 1)^k - (\lambda_i - 1)^k\} = 2 \sum_{\substack{1 \leq s \leq k \\ \text{odd}}} \binom{k}{s} p_{2k-s}(\lambda).$$

In particular, ψ_k is a supersymmetric function.

Proof. Taking the logarithm of (6.1), we find

$$\sum_{k=1}^{\infty} \frac{u^k}{k}\psi_k(\lambda) = \sum_{i=1}^{\ell(\lambda)} \log \frac{1 - \lambda_i(\lambda_i - 1)u}{1 - \lambda_i(\lambda_i + 1)u}.$$

Expanding the logarithm functions and comparing the coefficient of u^k on both sides, we obtain the first equality in the theorem. The remaining equality is obtained by applying the binomial theorem for the first equality. $\square$

For example:

$$\psi_1 = 2p_1, \qquad \psi_2 = 4p_3, \qquad \psi_3 = 6p_5 + 2p_3, \qquad \psi_4 = 8p_7 + 8p_5.$$

7. Open problems

7.1. Degree functions on Γ

In Subsection 4.3, we introduced the filtration $\deg_1$ on the vector space Γ. We remain a conjecture:

Conjecture 7.1. *The filtration* $\deg_1$ *is compatible with the multiplication of* Γ *in the following sense. Define the structure constants* $f^{\rho}_{\sigma\tau}$ *by*

$$\mathfrak{p}_\sigma \mathfrak{p}_\tau = \sum_\rho f^{\rho}_{\sigma\tau} \mathfrak{p}_\rho .$$

For $f^{\rho}_{\sigma\tau} \neq 0$ *then,*

$$|\rho| + m_1(\rho) \leq (|\sigma| + m_1(\sigma)) + (|\tau| + m_1(\tau)) .$$

Hence $\deg_1$ *defines an algebra filtration of* Γ*.*

Remark that the corresponding result in the algebra of shifted-symmetric functions is obtained in [12], based on the theory of the partial permutation algebra. Can we find a spin-analog of the partial permutation algebra?

We showed that, if f is a supersymmetric function, $\mathbb{E}_n[f]$ is a polynomial in n of degree at most $\frac{1}{2}\deg_1(f)$. To consider various degree filtrations is of help for estimations of the degree of $\mathbb{E}_n[f]$, see [3, 13].

7.2. Polynomiality for non-supersymmetric functions

Assume that F is a symmetric function but not supersymmetric. It is natural to ask whether $\mathbb{E}_n[F]$ is a polynomial in n. As a trial, let us consider the second power-sum symmetric function $p_2(x_1, x_2, \dots) = \sum_{i \geq 1} x_i^2$. Some values are directly computed as follows:

$$\mathbb{E}_1[p_2] = 1, \qquad \mathbb{E}_2[p_2] = 4, \qquad \mathbb{E}_3[p_2] = \frac{23}{3},$$
$$\mathbb{E}_4[p_2] = 12, \qquad \mathbb{E}_5[p_2] = 17, \qquad \mathbb{E}_6[p_2] = \frac{1016}{45}.$$

Recall $\mathbb{E}_n[p_3] = 3n^2 - 2n$ and a trivial identity $\mathbb{E}_n[p_1] = n$. If $\mathbb{E}_n[p_2]$ is a polynomial in n, one may expect that it is of degree at most 2. However, there is no polynomial $\Phi_{p_2}(x)$ of degree 2 such that $\mathbb{E}_n[p_2] = \Phi_{p_2}(n)$ for all $1 \leq n \leq 6$.

7.3. Hook evaluations

Recall the ordinary Plancherel measure $\mathbb{P}_n^{\mathrm{Plan}}$ on partitions. As we mention in Subsection 1.5, Stanley [25] (see also [7]) proves that the summation

$$\sum_{\lambda \in \mathcal{P}_n} \mathbb{P}_n^{\mathrm{Plan}}(\lambda) F\left(h_\square^2 : \square \in Y(\lambda)\right)$$

is a polynomial in n for any symmetric function F, where $h_\square$ denotes the hook length of the square $\square$ in the Young diagram $Y(\lambda)$. What is the analog of this result for the shifted Plancherel measure on strict partitions?

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