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Quantum Isometry Group of Deformation: A Counterexample

Debashish Goswami
Arnab Mandal

Abstract

We give a counterexample to show that the quantum isometry group of a deformed finite dimensional spectral triple may not be isomorphic with a deformation of the quantum isometry group of the undeformed spectral triple.

1. Introduction

Quantum groups play an important role in several areas of mathematics and physics, often as some kind of generalised symmetry objects. Beginning from the pioneering work by Drinfeld, Jimbo, Manin, Woronowicz and others nearly three decades ago ([9, 12, 19, 21] and references therein) there is now a vast literature on quantum groups both from algebraic and analytic (operator algebraic) viewpoints. Generalizing the concept of group actions on spaces, notions of (co)actions of quantum groups on possibly noncommutative spaces have been formulated and studied by many mathematicians in recent years. In 1998, S. Wang [20] initiated this programme by defining quantum automorphism groups of certain mathematical structures (typically finite sets, matrix algebras etc.). After that, a number of mathematicians including Banica, Bichon and others ([1, 2, 8] and references therein) formulated the notion of quantum symmetries of finite metric spaces and finite graphs. With the motivation of connecting Wang's quantum automorphism groups with a more geometric framework, one of the authors of the present article [14] defined and proved existence of an analogue of the group of isometries of a Riemannian manifold, in the framework of the so-called compact quantum groups à la Woronowicz. In fact, he considered the more general setting of noncommutative manifold, given by spectral triples defined by Connes [10] and under some mild regularity conditions, he proved the existence of a universal compact quantum group (termed as the quantum isometry group) acting on the C^*-algebra underlying the noncommutative manifold such that the action also commutes with a natural analogue of Laplacian of the spectral triple. We refer the reader to the original article [14] and the recently published book [7] for the details of the theory of quantum isometry groups. In this context, a very natural and important question is whether the quantum isometry group of deformation of some noncommutative

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manifold is isomorphic with certain deformed version of the quantum isometry group of the original undeformed noncommutative manifold. That is, whether the functor "QISO" which assigns to a noncommutative manifold its quantum isometry group "commutes" with the functor of deformation. This question has been answered in the affirmative for Rieffel type cocycle deformation by Goswami, Bhowmick, Joardar [5, 15] and for the more general "monoidal deformation" by Sadeleer [11]. The computation of the quantum isometry groups of Podles spheres [6] indicates that there may be an affirmative answer for a bigger class of deformations. However, it is not true in general as the isometry group of a classical Riemannian manifold may drastically change by a slight perturbation of the Riemannian metric. The aim of the present article is to give an example of flat deformation of finite dimensional spectral triples on the C^*-algebras of finite groups for which the corresponding quantum isometry groups are not flat deformation. Note that such examples can't be produced for classical Riemannian geometry since there exists only one Riemannian metric, so that there is no room for deformations.

2. **Background Materials**

We very briefly discuss the basic definitions and recall some standard facts about noncommutative geometry and quantum isometry groups. We refer [10, 17, 21] for more details. Let us fix some notational convention. We denote the algebraic tensor product and spatial (minimal) C^*-tensor product by $\otimes$ and $\check{\otimes}$ respectively. We'll use the leg-numbering notation. Let $\mathcal{H}$ be a complex Hilbert space, $\mathcal{K}(\mathcal{H})$ the C^*-algebra of compact operators on it, and Q a unital C^*-algebra. The multiplier algebra $\mathcal{M}(\mathcal{K}(\mathcal{H}) \check{\otimes} Q)$ has two natural embeddings into $\mathcal{M}(\mathcal{K}(\mathcal{H}) \check{\otimes} Q \check{\otimes} Q)$, one obtained by extending the map $x \mapsto x \otimes 1$ and the second one is obtained by composing this map with the flip on the last two factors. We will write ω^{12} and ω^{13} for the images of an element $\omega \in \mathcal{M}(\mathcal{K}(\mathcal{H}) \check{\otimes} Q)$ under these two maps respectively. We'll denote by $\mathcal{H} \bar{\otimes} Q$ the Hilbert C^*-module obtained by completing $\mathcal{H} \otimes Q$ with respect to the norm induced by the Q valued inner product $\langle\langle \xi \otimes q, \xi' \otimes q' \rangle\rangle := \langle \xi, \xi' \rangle q^* q'$, where $\xi, \xi' \in \mathcal{H}$ and $q, q' \in Q$.

2.1. **Spectral triple and compact quantum groups**

Definition 2.1. A spectral triple is a triple $(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})$ where $\mathcal{H}$ is a separable Hilbert space, $\mathcal{A}^\infty$ is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$, (not necessarily norm closed) and $\mathcal{D}$ is a self adjoint (typically unbounded) operator such that for all $a \in \mathcal{A}^\infty$, the operator $[\mathcal{D}, a]$ has a bounded extension. Such spectral triple is also called an odd spectral triple. If in addition, we have $\gamma \in \mathcal{B}(\mathcal{H})$ satisfying $\gamma = \gamma^* = \gamma^{-1}$, $\mathcal{D}\gamma = -\gamma\mathcal{D}$ and $[a, \gamma] = 0$ for all $a \in \mathcal{A}^\infty$, then we say the quadruplet $(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D}, \gamma)$ is an even spectral triple. The operator $\mathcal{D}$ is called the Dirac operator corresponding to the spectral triple.

We say that the spectral triple is of compact type if $\mathcal{A}^\infty$ is unital and $\mathcal{D}$ has compact resolvent. In this article, we will consider only odd spectral triple.

Definition 2.2. A compact quantum group (CQG in short) is a pair (Q, Δ), where Q is a unital C^*- algebra and $\Delta : Q \to Q \check{\otimes} Q$ is a unital C^*-homomorphism (called the comultiplication), such that

(1) $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$ as homomorphism $Q \to Q \check{\otimes} Q \check{\otimes} Q$ (coassociativity).

(2) The spaces $\Delta(Q)(1 \otimes Q) = \mathrm{Span}\{\Delta(b)(1 \otimes a) \,|\, a, b \in Q\}$ and $\Delta(Q)(Q \otimes 1) = \mathrm{Span}\{\Delta(b)(a \otimes 1) \,|\, a, b \in Q\}$ are dense in $Q \check{\otimes} Q$.

A CQG morphism from (Q_1, Δ_1) to another (Q_2, Δ_2) is a unital C^*-homomorphism $\pi : Q_1 \mapsto Q_2$ such that $(\pi \otimes \pi)\Delta_1 = \Delta_2 \pi$.

Definition 2.3. We say that a CQG (Q, Δ) acts on a unital C^*-algebra $\mathcal{B}$ if there is a unital C^*-homomorphism (called action) $\alpha : \mathcal{B} \to \mathcal{B} \check{\otimes} Q$ satisfying the following:

(1) $(\alpha \otimes \mathrm{id})\alpha = (\mathrm{id} \otimes \Delta)\alpha$.

(2) The linear span of $\alpha(\mathcal{B})(1 \otimes Q)$ is norm-dense in $\mathcal{B} \check{\otimes} Q$.

Definition 2.4. Let (Q, Δ) be a CQG. A unitary representation of Q on a Hilbert space $\mathcal{H}$ is a $\mathbb{C}$-linear map U from $\mathcal{H}$ to the Hilbert module $\mathcal{H} \bar{\otimes} Q$ such that

(1) $\langle\langle U(\xi), U(\eta)\rangle\rangle = \langle \xi, \eta \rangle \, 1_Q$, where $\xi, \eta \in \mathcal{H}$.

(2) $(U \otimes \mathrm{id})U = (\mathrm{id} \otimes \Delta)U$.

(3) $\mathrm{Span}\{U(\mathcal{H})Q\}$ is dense in $\mathcal{H} \bar{\otimes} Q$.

Given such a unitary representation we have a unitary element $\widetilde{U}$ belonging to $\mathcal{M}(\mathcal{K}(\mathcal{H}) \check{\otimes} Q)$ given by $\widetilde{U}(\xi \otimes b) = U(\xi)b, (\xi \in \mathcal{H},\ b \in Q)$ satisfying $(\mathrm{id} \otimes \Delta)(\widetilde{U}) = \widetilde{U}^{12}\widetilde{U}^{13}$.

2.2. **Quantum isometry groups**

In [14] the first author introduced the notion of quantum isometry group of a spectral triple satisfying certain regularity conditions. We refer to [3, 4, 14] for the original formulation of quantum isometry groups and its various avatars including the quantum isometry group for orthogonal filtrations.

Definition 2.5. Let $(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})$ be a spectral triple of compact type (à la Connes). Consider the category $Q(\mathcal{D}) \equiv Q(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})$ whose objects are (Q, U), where (Q, Δ) is a CQG having a unitary representation U on the Hilbert space $\mathcal{H}$ satisfying the following:

(1) $\widetilde{U}$ commutes with $(\mathcal{D} \otimes 1_Q)$.

(2) $(\mathrm{id} \otimes \phi) \circ ad_{\widetilde{U}}(a) \in (\mathcal{A}^\infty)''$ for all $a \in \mathcal{A}^\infty$ and ϕ is any state on Q, where $ad_{\widetilde{U}}(x) := \widetilde{U}(x \otimes 1)\widetilde{U}^*$ for $x \in \mathcal{B}(\mathcal{H})$. Note that $ad_{\widetilde{U}}$ is faithful on $\mathcal{A}^\infty$.

A morphism between two such objects (Q, U) and (Q', U') is a CQG morphism $\psi : Q \to Q'$ such that $U' = (\mathrm{id} \otimes \psi)U$. If a universal object exists in $Q(\mathcal{D})$ then we denote it by $\widetilde{QISO^+(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})}$ and the corresponding largest Woronowicz subalgebra for which $ad_{\widetilde{U_0}}$ is faithful, where U_0 is the unitary representation of $\widetilde{QISO^+(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})}$, is called the quantum group of orientation preserving isometries and denoted by $QISO^+(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})$.

Let us state Theorem 2.23 of [4] which gives a sufficient condition for the existence of $QISO^+(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})$.

Theorem 2.6. *Let $(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})$ be a spectral triple of compact type. Assume that $\mathcal{D}$ has one dimensional kernel spanned by a vector $\xi \in \mathcal{H}$ which is cyclic and separating for $\mathcal{A}^\infty$ and each eigenvector of $\mathcal{D}$ belongs to $\mathcal{A}^\infty\xi$. Then $QISO^+(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})$ exists.*

Here we briefly discuss a specific case of interest for us. For more details see [16, Section 2.2].

Let Γ be a finitely generated discrete group with a symmetric generating set S not containing the identity of Γ (symmetric means $g \in S$ if and only if $g^{-1} \in S$) and let l be the corresponding word length function. We define an operator D_Γ by $D_\Gamma(\delta_g) = l(g)\delta_g$, where δ_g denotes the vector in $l^2(\Gamma)$ which takes value 1 at the point g and 0 at all other points. Note that $\delta_g, g \in \Gamma$ forms an orthonormal basis of $l^2(\Gamma)$. Let τ be the faithful positive functional on the reduced group C^*-algebra $C_r^*(\Gamma)$ given by $\tau(\sum_g c_g\lambda_g) = c_e$, where e is the identity element of Γ. Then $\mathrm{QISO}^+(\mathbb{C}\Gamma, l^2(\Gamma), D_\Gamma)$ exists by Theorem 2.6, taking δ_e as the cyclic separating vector for $\mathbb{C}\Gamma$.

We also refer to [3] for the related notion of orthogonal filtration and note that the above quantum isometry group is the quantum symmetry group of the orthogonal filtration on $\mathbb{C}\Gamma$ with respect to the tracial state τ and the filtration $\mathcal{F}_l = (V_{1,k})_{k\geq 0}$, where $V_{1,k} = \text{Span}\{\lambda_g : l(g) = k\}$. Moreover, we make the following useful observation:

Theorem 2.7. *Let P_0 and P_1 be the orthogonal projections onto the subspaces $V_{1,0}, V_{1,1}$ respectively and let $P_2 = 1 - (P_0 + P_1)$. Define $D = P_1 + 2P_2$. Then for a finite group Γ, $QISO^+(\mathbb{C}\Gamma, l^2(\Gamma), D)$ exists and is isomorphic with $QISO^+(\mathbb{C}\Gamma, l^2(\Gamma), D_\Gamma)$.*

Proof. Let $Q_1 = QISO^+(\mathbb{C}\Gamma, l^2(\Gamma), D_\Gamma)$ and $Q_2 = QISO^+(\mathbb{C}\Gamma, l^2(\Gamma), D)$. It follows from Theorem 2.10 and Theorem 3.2 of [3] that Q_2 is the quantum symmetry group of the orthogonal filtration $\mathcal{F}_D = (V_{2,k})_{k=0,1,2}$ and with respect to the trace τ, where $V_{2,k} = V_{1,k}$ for $k = 0, 1$ and $V_{2,2} = \cup_{k\geq 2} V_{1,k} = \text{Span}\{\lambda_g \mid l(g) \geq 2\}$. As $\mathcal{F}_l$ is a refinement of $\mathcal{F}_D$, it is clear that Q_1 is a subobject of Q_2 (in the category of $\mathcal{F}_D$ preserving quantum groups). Hence it suffices to show that Q_2 is also a subobject of Q_1 in the category of $\mathcal{F}_l$ preserving quantum groups. But by definition, the coaction of Q_2 leaves $V_{1,1}$ invariant and preserves the trace τ, hence it is an object in the category $\mathbf{C}_\tau$ introduced in [16]. By Lemma 2.14 of [16] we conclude that Q_2 is a subobject of Q_1. $\square$

Let Γ_1 and Γ_2 be two finite groups with identity elements e and e' respectively. Assume that $\mathcal{F}_1 = (V_i)_{i=0,1,\ldots,k}$ and $\mathcal{F}_2 = (\widetilde{V_j})_{j=0,1,\ldots,l}$ are two orthogonal filtrations of $C^*(\Gamma_1)$ and $C^*(\Gamma_2)$ with respect to the canonical traces τ_1 and τ_2 respectively. Moreover, let $Q_{\mathcal{F}_i}$ be the quantum isometry groups of $C^*(\Gamma_i)$ for $i = 1, 2$. For each $g \in \Gamma_1, g' \in \Gamma_2$ consider the subspaces $V_{(i,g')} = \text{Span}\{b \otimes \lambda_{g'} \mid b \in V_i\}$ and $\widetilde{V}_{(g,j)} = \text{Span}\{\lambda_g \otimes a \mid a \in \widetilde{V_j}\}$ forall $i = 0, 1, \ldots, k$ and $j = 0, 1, \ldots, l$ inside the vector space $C^*(\Gamma_1) \otimes C^*(\Gamma_2)$. Clearly, $\mathcal{F} = (V_{(i,g')})_{i=0,1,\ldots,k, g'\in\Gamma_2}$ and $\mathcal{F}' = (\widetilde{V}_{(g,j)})_{j=0,1,\ldots,l, g\in\Gamma_1}$ are two orthogonal filtrations for $C^*(\Gamma_1) \otimes C^*(\Gamma_2)$, i.e. $C^*(\Gamma_1) \otimes C^*(\Gamma_2) = \bigoplus_{i,g'} V_{(i,g')}$ and $C^*(\Gamma_1) \otimes C^*(\Gamma_2) = \bigoplus_{i,g} \widetilde{V}_{(g,i)}$ with respect to the state $\tau_1 \otimes \tau_2$. Let $Q_{\mathcal{F}}$ and $Q_{\mathcal{F}'}$ be the quantum isometry groups of $C^*(\Gamma_1) \otimes C^*(\Gamma_2)$ corresponding to the filtrations $\mathcal{F}$ and $\mathcal{F}'$ respectively. Let us assume that $\{s_1, \ldots, s_p\}$ is a generating set of the group Γ_1 and $V_1 = \text{Span}\{\lambda_{s_1}, \ldots, \lambda_{s_p}\}$. Furthermore, assume that the action of $Q_{\mathcal{F}_1}$ on $C^*(\Gamma_1)$ is defined by $\alpha_1(\lambda_{s_i}) = \sum_{j=1}^p \lambda_{s_j} \otimes q_{ji}$, where the underlying C^*-algebra of $Q_{\mathcal{F}_1}$ is generated by q_{ij}'s for $i, j = 1, \ldots, p$.

Theorem 2.8. $Q_{\mathcal{F}} \cong Q_{\mathcal{F}_1} \otimes C^*(\Gamma_2)$.

Proof. First of all, note that any action α on $C^*(\Gamma_1) \otimes C^*(\Gamma_2)$ is determined by $\alpha(\lambda_{s_i} \otimes \lambda_{e'})$ and $\alpha(\lambda_e \otimes \lambda_{g'})$ forall $i = 1, \ldots, p$ and $g' \in \Gamma_2$. Let $\alpha_{\mathcal{F}}$ be the action of $Q_{\mathcal{F}}$ on $C^*(\Gamma_1) \otimes C^*(\Gamma_2)$. As $V_{(0,g')}$ and $V_{(1,e')}$ are members of the filtration $\mathcal{F}$, $\alpha_{\mathcal{F}}$ must preserve each of them and hence $\alpha_{\mathcal{F}}$ must satisfy $\alpha_{\mathcal{F}}(\lambda_e \otimes \lambda_{g'}) = \lambda_e \otimes \lambda_{g'} \otimes q_{g'} \quad \forall\ g' \in \Gamma_2$ and $\alpha_{\mathcal{F}}(\lambda_{s_i} \otimes \lambda_{e'}) = \sum_{j=1}^p \lambda_{s_j} \otimes \lambda_{e'} \otimes q'_{ji}$ forall $i = 1, \ldots, p$ where $\{q_{g'}, q'_{ji}, g' \in \Gamma_2, i, j = 1, \ldots, p\} \subseteq Q_{\mathcal{F}}$. In fact, as $\{\lambda_e \otimes \lambda_{g'}, \lambda_{s_i} \otimes \lambda_{e'}, g' \in \Gamma_2, i = 1, \ldots, p\}$ is a set of generators for the C^*-algebra $C^*(\Gamma_1) \otimes C^*(\Gamma_2)$ and $\alpha_{\mathcal{F}}$ is faithful, the set

$\{q_{g'}, q'_{ji}, g' \in \Gamma_2, i,j = 1,\dots,p\}$ is a set of generators of the C^*-algebra $\mathcal{Q}_{\mathcal{F}}$. Now define $\beta : C^*(\Gamma_1) \otimes C^*(\Gamma_2) \mapsto C^*(\Gamma_1) \otimes C^*(\Gamma_2) \otimes \mathcal{Q}_{\mathcal{F}_1} \otimes C^*(\Gamma_2)$ by $\beta = \sigma_{23} \circ (\alpha_1 \otimes \Delta_{\Gamma_2})$, where σ_{23} denotes the isomorphism between $C^*(\Gamma_1) \otimes \mathcal{Q}_{\mathcal{F}_1} \otimes C^*(\Gamma_2) \otimes C^*(\Gamma_2)$ and $C^*(\Gamma_1) \otimes C^*(\Gamma_2) \otimes \mathcal{Q}_{\mathcal{F}_1} \otimes C^*(\Gamma_2)$ which interchanges the second and third tensor copies and Δ_{Γ_2} is the usual coproduct on $C^*(\Gamma_2)$. Clearly, β is a C^*-action of $\mathcal{Q}_{\mathcal{F}_1} \otimes C^*(\Gamma_2)$ on $C^*(\Gamma_1) \otimes C^*(\Gamma_2)$ and it satisfies $\beta(\lambda_e \otimes \lambda_{g'}) = \lambda_e \otimes \lambda_{g'} \otimes 1_{\mathcal{Q}_{\mathcal{F}_1}} \otimes \lambda_{g'} \quad \forall\ g' \in \Gamma_2$ and $\beta(\lambda_{s_i} \otimes \lambda_{e'}) = \sum_{j=1}^{p} \lambda_{s_j} \otimes \lambda_{e'} \otimes q_{ji} \otimes \lambda_{e'}$ forall $i = 1,\dots,p$, where $1_{\mathcal{Q}_{\mathcal{F}_1}}$ is the unit of $\mathcal{Q}_{\mathcal{F}_1}$. Thus, β preserves the filtration $\mathcal{F}$ which implies the existence of a well defined surjective C^*-homomorphism from $\mathcal{Q}_{\mathcal{F}}$ to $\mathcal{Q}_{\mathcal{F}_1} \otimes C^*(\Gamma_2)$ sending $q_{g'}$ to $\lambda_{g'}$ and q'_{ji} to $q_{ji}, g' \in \Gamma_2, i,j = 1,\dots,p$. We claim that there is an inverse of this morphism, namely a C^*-homomorphism from $\mathcal{Q}_{\mathcal{F}_1} \otimes C^*(\Gamma_2)$ to $\mathcal{Q}_{\mathcal{F}}$ which sends $\lambda_{g'}$ to $q_{g'}$ and q_{ji} to q'_{ji}. To this end, observe that as $\alpha_{\mathcal{F}}$ is a C^*-homomorphism and $\alpha_{\mathcal{F}}(\lambda_e \otimes \lambda_{g'}) = \lambda_e \otimes \lambda_{g'} \otimes q_{g'}$, we must have $(\lambda_{g'_1} \otimes q_{g'_1})(\lambda_{g'_2} \otimes q_{g'_2}) = \lambda_{g'_1 g'_2} \otimes q_{g'_1 g'_2}$ forall $g'_1, g'_2 \in \Gamma_2$, i.e. $q_{g'_1} \cdot q_{g'_2} = q_{g'_1 g'_2}$. Similarly, $q_{e'} = 1$ and $q_{g'^{-1}} = (q_{g'})^{-1} = q^*_{g'}$. Thus, $g' \mapsto q_{g'}$ is a group homomorphism from Γ_2 to the group of unitaries of $\mathcal{Q}_{\mathcal{F}}$, hence by the universality of $C^*(\Gamma_2)$ we get a C^*-homomorphism (say ρ) from $C^*(\Gamma_2)$ to $\mathcal{Q}_{\mathcal{F}}$ such that $\rho(\lambda_{g'}) = q_{g'} \quad \forall\ g' \in \Gamma_2$. Next, note that $\alpha_{\mathcal{F}}$ preserves the C^*-subalgebra $C^*(\Gamma_1) \otimes \lambda_{e'} (\cong C^*(\Gamma_1)) \subseteq C^*(\Gamma_1) \otimes C^*(\Gamma_2)$, and clearly the restriction of $\alpha_{\mathcal{F}}$ to this subalgebra gives a $\mathcal{F}_1$- preserving action on $C^*(\Gamma_1)$. Hence we get a C^*-homomorphism, say θ, from $\mathcal{Q}_{\mathcal{F}_1}$ to $\mathcal{Q}_{\mathcal{F}}$ such that $\theta(q_{ji}) = q'_{ji} \quad \forall$ $i,j = 1,\dots,p$. Moreover, as $C^*(\Gamma_1) \otimes \lambda_{e'}$ and $\lambda_e \otimes C^*(\Gamma_2)$ are commutative subalgebras of $C^*(\Gamma_1) \otimes C^*(\Gamma_2)$, their images under $\alpha_{\mathcal{F}}$ must commute too. From this, it easily follows that $q_{g'} \cdot q'_{ji} = q'_{ji} \cdot q_{g'} \quad \forall\ g' \in \Gamma_2, i,j = 1,\dots,p$. Thus, the images of θ and ρ commute implying the existence of a C^*-homomorphism $\theta \otimes \rho : \mathcal{Q}_{\mathcal{F}_1} \otimes C^*(\Gamma_2) \mapsto \mathcal{Q}_{\mathcal{F}}$ which sends $q_{ji} \otimes \lambda_{g'}$ to $q_{g'} \cdot q'_{ji}$. □

Similarly, it can be shown that $\mathcal{Q}_{\mathcal{F}'} \cong \mathcal{Q}_{\mathcal{F}_2} \otimes C^*(\Gamma_1)$.

2.3. **Deformation**

There are different notions of deformation of spaces, algebras and operators found in the literature. Let us specify the notion which we are concerned with. For a more general, abstract setting of deformation theory we refer to the seminal work of Gerstenhaber and others (see, e.g. [13] and references therein).

We will consider families of vector spaces W_h, $h \in I$, where I is an open interval in $\mathbb{R}$.

Definition 2.9. Let $\{W_h\}_{h \in I}$ be a family of vector subspaces of some finite dimensional vector space W. $\{W_h\}$ is called a continuous deformation of W_{h_0} if there are W-valued continuous functions $w_{i,h}, i = 1,\dots,n$ (where $\dim(W) = n$) such that $W_h = \mathrm{Span}\{w_1(h),\dots,w_n(h)\} \quad \forall\ h \in I$.

From now on, we just call it deformation by dropping the word continuous. Moreover, $\{W_h\}_{h\in I}$ is called a flat deformation of W_{h_0} if $\dim(W_h)$ does not depend on h.

Definition 2.10. Let $\{V_h\}_{h\in I}$ be a family of filtered vector spaces, i.e. $V_h = \cup_{i\geq 0}F_i(V_h)$ with the conditon $F_0(V_h) \subseteq \ldots F_i(V_h) \subseteq \cdots \subseteq V_h$ of finite dimensional subspaces of V_h. $\{V_h\}$ is called a deformation of V_{h_0} if for each i, the family of subspaces $\{F_i(V_h)\}$ is a deformation of $F_i(V_{h_0})$ as in Definition 2.9.

Such deformation is called a flat deformation if $\dim(F_i(V_h))$ does not depend on h for any fixed i.

Definition 2.11. Let $\{A_h\}_{h\in I}$ be a family of filtered unital algebras, i.e. $A_h = \cup_{i\geq 0}F_i(A_h)$ with the conditons $F_0(A_h) \subseteq \ldots F_i(A_h) \subseteq \cdots \subseteq A_h$ of finite dimensional subspaces of A_h and $F_i(A_h).F_j(A_h) \subseteq F_{i+j}(A_h)\ \forall\ i, j$. Moreover, we assume that $F_0(A_h) = \mathbb{C}.1$ for all h. Then A_h is called a deformation of A_{h_0} if for each i, there is a finite dimensional vector space V_i and V_i valued continuous functions $w_{i,j}(h), j = 1,\ldots,n_i$ (where $\dim(V_i) = n_i$) such that $F_i(V_h) = \mathrm{Span}\{w_{i,1}(h),\ldots,w_{i,n_i}(h)\}$ and the map $h \mapsto w_{i,k}(h).w_{j,l}(h)$ is continuous forall i, j, k, l with $1 \leq k \leq n_i,\ 1 \leq l \leq n_j$.

Remark 2.12. Any finitely generated algebra can be considered as a filtered algebra. Given such an algebra A with a unit 1 and a finite set of generators $\{a_1,\ldots,a_k\}$, we consider the filtration given by $A_0 = \mathbb{C}1$, $A_1 = \mathrm{Span}\{1, a_1,\ldots,a_k\}$, $A_n = \mathrm{Span}\{1, a_{i_1}\ldots a_{i_m}, 1 \leq m \leq n, i_l \in \{1,\ldots,k\}\ \ \forall\ l\}$. Hence the definition of deformation of filtered algebras applies to any arbitrary finitely generated algebra.

Definition 2.13. The family of filtered algebras $\{A_h\}_{h\in I}$ is called a flat deformation of A_{h_0} if for any fixed i, the dimension of $F_i(A_h)$ does not depend on h.

Remark 2.14. It is clear that a flat deformation $\{A_h\}$ of A_{h_0} is uniquely determined by a family $\bullet_h$ of associative algebra multiplication from $A_{h_0} \otimes A_{h_0}$ to A_{h_0} such that $h \mapsto \omega(a \bullet_h b)$ is continuous for all $\omega \in A'_{h_0}$ (dual vector space), $a, b \in A_{h_0}$. In fact, one can take this as a definition of flat deformation, for not necessarily filtered algebra A_{h_0}.

Definition 2.15. A family $(\mathcal{A}_h, \mathcal{H}_h, D_h)$ of finite dimensional spectral triples of compact type will be called a flat deformation if

(1) There exists continuous functions $\lambda_i(h),\ i = 1,\ldots,N$ such that $\lambda_i(h) \neq \lambda_j(h)\ \forall\ i \neq j$, $\{\lambda_i(h)\}$ is the set of eigenvalues of D_h.

(2) If $V_i(h)$ denotes the eigenspaces of D_h corresponding to $\lambda_i(h)$, then $\{V_i(h), h \in I\}$ is a flat deformation of finite dimensional vector spaces $\forall\ i$.

(3) $\{A_h\}_{h\in I}$ is a flat deformation of filtered unital *-algebras too.

Finally, we need an appropriate notion of flat deformation of Hopf algebras, which are finitely generated as algebras.

Definition 2.16. Let $\{(Q_h, \Delta_h)\}_{h \in I}$ be a family of Hopf algebras with Δ_h denoting the coproduct. Suppose that each $\{Q_h\}_{h \in I}$ is finitely generated as an algebra and $\{Q_h\}_{h \in I}$ is a flat deformation of Q_{h_0} with respect to the filtered algebra structure corresponding to a finite generating set of Q_{h_0}. Thus by Remark 2.14 we can identify each Q_h with Q_{h_0} for any fixed $h_0 \in I$ so that the multiplication map $\bullet_h$ is continuous in the sense discussed in that remark. If furthermore, Δ_h viewed as a map from Q_{h_0} to $Q_{h_0} \otimes Q_{h_0}$ is continuous in a similar sense, i.e. $h \mapsto (\beta \otimes \mathrm{id})(\Delta_h(q))$ is continuous $\forall\ q \in Q_{h_0}$ and for all linear functional β on $Q_{h_0} \otimes Q_{h_0}$, we say that $(Q_h, \Delta_h)\}_{h \in I}$ is a flat deformation of finitely generated Hopf algebras.

Remark 2.17. By construction of [3] the underlying Hopf algebra of the quantum isometry group for an orthogonal filtration is always finitely generated, hence the above definition applies to such Hopf algebras. A similar remark can be made about the underlying Hopf algebra of $QISO(\mathcal{A}, \mathcal{H}, D)$ of a finite dimensional spectral triple.

3. The Counterexample

It is now natural to ask the following question: does the quantum isometry groups of a flat deformation of spectral triples again form a flat deformation of compact quantum groups? The answer to this question is negative even if the spectral triples considered are finite dimensional, as shown by the counterexample given by the anonymous referee, which is briefly described below.

Consider $\mathcal{A}^\infty = \mathbb{C}^2$ acting on the 2 dimensional Hilbert space $\mathcal{H} = \mathbb{C}^2$ and two Dirac operators

$$D_0 = 1/2 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad D_1 = 1/5 \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix},$$

which are rank 1 projections. We take a homotopy D_t connecting D_0 and D_1 via rank 1 projections, then $(\mathcal{A}^\infty, \mathcal{H}, D_t)$ is a flat deformation of spectral triples, satisfying the assumption of Theorem 2.6. Notice that the only quantum automorphism group of $\mathcal{A}^\infty$ is $\mathbb{Z}_2$, which preserves D_0 but not D_1. Hence $QISO^+(\mathcal{A}^\infty, \mathcal{H}, D_0)$ is $\mathbb{Z}_2$ but $QISO^+(\mathcal{A}^\infty, \mathcal{H}, D_1)$ is trivial.

However, we would like to give another counterexample, where the initial and final spectral triples of the deformed family arise from a finite group with two different generating sets. From now onwards the two groups $\mathbb{Z}_9 \times \mathbb{Z}_3$ and $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$ are denoted by Γ_1 and Γ_2 respectively. Moreover, we denote the identity elements of Γ_1 and Γ_2 by e and

e' respectively. Note that

$$\Gamma_1 = \{a, b \mid ab = ba, a^9 = b^3 = e\}, \quad \Gamma_2 = \{x, y \mid xyx^{-1} = y^4, x^3 = y^9 = e'\}.$$

Consider $V_0 = \mathbb{C}\lambda_e, V_1 = \mathrm{Span}\{\lambda_a, \lambda_{a^{-1}}, \lambda_b, \lambda_{b^{-1}}\}$ and $V_2 = \mathrm{Span}\{\lambda_t \mid t \in \Gamma_1 - \{e, a, a^{-1}, b, b^{-1}\}\}$. Observe that $\mathcal{F}_1 = (V_i)_{i=0,1,2}$ gives an orthogonal filtration for $C^*(\Gamma_1)$. Similarly, consider $\widetilde{V}_0 = \mathbb{C}\lambda_{e'}, \widetilde{V}_1 = \mathrm{Span}\{\lambda_x, \lambda_{x^{-1}}, \lambda_y, \lambda_{y^{-1}}\}, \widetilde{V}_2 = \mathrm{Span}\{\lambda_s \mid s \in \Gamma_2 - \{e', x, x^{-1}, y, y^{-1}\}\}$ and $\mathcal{F}_2 = (\widetilde{V}_i)_{i=0,1,2}$ is an orthogonal filtration for $C^*(\Gamma_2)$. Moreover, for each $g \in \Gamma_1, g' \in \Gamma_2$ and $i = 0, 1, 2$ we can consider the subspaces $\widetilde{V}_{(g,i)} = \mathrm{Span}\{\lambda_g \otimes a \mid a \in \widetilde{V}_i\}$ and $V_{(i,g')} = \mathrm{Span}\{b \otimes \lambda_{g'} \mid b \in V_i\}$ inside $C^*(\Gamma_1) \otimes C^*(\Gamma_2)$. Clearly, $\mathcal{F} = (V_{(i,g')})_{i=0,1,2,g' \in \Gamma_2}$ and $\mathcal{F}' = (\widetilde{V}_{(g,i)})_{i=0,1,2,g \in \Gamma_1}$ are two orthogonal filtrations for $C^*(\Gamma_1) \otimes C^*(\Gamma_2)$, i.e. $C^*(\Gamma_1) \otimes C^*(\Gamma_2) = \bigoplus_{(i,g')} V_{(i,g')}$ and $C^*(\Gamma_1) \otimes C^*(\Gamma_2) = \bigoplus_{(g,i)} \widetilde{V}_{(g,i)}$. Assume that $g' \mapsto g$ is a bijective map from Γ_2 to Γ_1. We can consider a unitary operator U on $l^2(\Gamma_1) \otimes l^2(\Gamma_2)$ such that $U(V_{(i,g')}) = \widetilde{V}_{(g,i)}$. Observe that $\sigma(U)$ is a finite subset of the unit circle as $l^2(\Gamma_1) \otimes l^2(\Gamma_2)$ is a finite dimensional vector space. Using an appropriate branch of logarithm not intersecting the set $\sigma(U)$ we get a self adjoint matrix S such that $U = e^{iS}$. Then we can define a family of unitary operators $U^h = e^{ihS}\ \forall\ h \in [0, 1]$. Consider the spaces $V_{(i,g'),h} = U^h(V_{(i,g')})$ forall h and define the operators $D_h = \sum_{i=0,1,2,g' \in \Gamma_2} n(i, g') P_{(i,g'),h}$ on $l^2(\Gamma_1) \otimes l^2(\Gamma_2)$, where $n(i, g') \in \mathbb{N} \cup \{0\}, n(i_1, g'_1) \neq n(i_2, g'_2)$ if $(i_1, g'_1) \neq (i_2, g'_2)$ and $P_{(i,g'),h}$ is the orthogonal projection onto $V_{(i,g'),h}$. Observe that $V_{(i,g'),0} = V_{(i,g')}$ and $V_{(i,g'),1} = \widetilde{V}_{(g,i)}$ for each $g' \in \Gamma_2$ and $i = 0, 1, 2$. Consider the family of spectral triples $(\mathcal{A}_h, \mathcal{H}_h, D_h)$, where $\mathcal{A}_h = C^*(\Gamma_1) \otimes C^*(\Gamma_2), \mathcal{H}_h = l^2(\Gamma_1) \otimes l^2(\Gamma_2)$ forall h and D_h is defined as above. Note that this family is clearly a flat deformation in the sense of Definition 2.15 and $QISO^+(C^*(\Gamma_1) \otimes C^*(\Gamma_2), l^2(\Gamma_1) \otimes l^2(\Gamma_2), D_h)$ exists by Theorem 2.6, taking $\xi = \delta_e \otimes \delta_{e'}$ as a cyclic, separating vector for $C^*(\Gamma_1) \otimes C^*(\Gamma_2)$. Moreover, using Theorem 2.7, we have

$$Q_{\mathcal{F}_1} \cong QISO^+(C^*(\Gamma_1), l^2(\mathbb{Z}_9) \otimes l^2(\mathbb{Z}_3), D_{\Gamma_1}),$$
$$Q_{\mathcal{F}_2} \cong QISO^+(C^*(\Gamma_2), l^2(\mathbb{Z}_9) \otimes l^2(\mathbb{Z}_3), D_{\Gamma_2}).$$

However, $QISO^+(C^*(\Gamma_1), l^2(\mathbb{Z}_9) \otimes l^2(\mathbb{Z}_3), D_{\Gamma_1})$ and $QISO^+(C^*(\Gamma_2), l^2(\mathbb{Z}_9) \otimes l^2(\mathbb{Z}_3), D_{\Gamma_2})$ have been already computed in [16, 18] respectively and one has the following:

$$QISO^+(C^*(\Gamma_1), l^2(\mathbb{Z}_9) \otimes l^2(\mathbb{Z}_3), D_{\Gamma_1}) \cong [C^*(\mathbb{Z}_9) \oplus C^*(\mathbb{Z}_9)] \otimes [C^*(\mathbb{Z}_3) \oplus C^*(\mathbb{Z}_3)],$$
$$QISO^+(C^*(\Gamma_2), l^2(\mathbb{Z}_9) \otimes l^2(\mathbb{Z}_3), D_{\Gamma_2}) \cong C^*(\mathbb{Z}_9 \rtimes \mathbb{Z}_3) \oplus C^*(\mathbb{Z}_9 \rtimes \mathbb{Z}_3)$$

with the coproduct structures discussed in [16] and [18] respectively. From this, we can easily conclude the following:

Theorem 3.1. *$QISO^+(C^*(\Gamma_1) \otimes C^*(\Gamma_2), l^2(\Gamma_1) \otimes l^2(\Gamma_2), D_h)$ is not a flat deformation.*

Proof. Note that the dimension of the underlying vector space of $QISO^+(C^*(\Gamma_1), l^2(\mathbb{Z}_9) \otimes l^2(\mathbb{Z}_3), D_{\Gamma_1})$ is 108, whereas the dimension of the underlying vector space of $QISO^+(C^*(\Gamma_2), l^2(\mathbb{Z}_9) \otimes l^2(\mathbb{Z}_3), D_{\Gamma_2})$ is 54. Thus, by Theorem 2.8, the dimensions of the underlying vector spaces of $QISO^+(C^*(\Gamma_1) \otimes C^*(\Gamma_2), l^2(\Gamma_1) \otimes l^2(\Gamma_2), D_0)$ and $QISO^+(C^*(\Gamma_1) \otimes C^*(\Gamma_2), l^2(\Gamma_1) \otimes l^2(\Gamma_2), D_1)$ are 2916 and 1458 respectively. Hence the family $QISO^+(C^*(\Gamma_1) \otimes C^*(\Gamma_2), l^2(\Gamma_1) \otimes l^2(\Gamma_2), D_h)$ can't be a flat deformation. □

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