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# Finiteness of the image of the Reidemeister torsion of a splice 

Teruaki Kitano<br>Yuta Nozaki


#### Abstract

The set $R T(M)$ of values of the $\operatorname{SL}(2, \mathbb{C})$-Reidemeister torsion of a 3-manifold $M$ can be both finite and infinite. We prove that $R T(M)$ is a finite set if $M$ is the splice of two certain knots in the 3 -sphere. The proof is based on an observation on the character varieties and $A$-polynomials of knots.


## 1. Introduction

Let $K$ be the figure-eight knot and $E(K)$ the exterior of an open tubular neighborhood of $K$ in the 3 -sphere $S^{3}$. The first author [13] computed the $S L(2, \mathbb{C})$-Reidemeister torsion $\tau_{\rho}\left(E(K)\right.$ ) for any acyclic irreducible representation $\rho: \pi_{1}(E(K)) \rightarrow S L(2, \mathbb{C})$. As a consequence, for the double $M=E(K) \cup_{\mathrm{id}} E(K)$ of $E(K)$, the set $R T(M)$ of values of the $\operatorname{SL}(2, \mathbb{C})$-Reidemeister torsion $\tau_{\rho}(M)$ is the set of all complex numbers $\mathbb{C}$. In contrast, his computation also shows that $R T(\Sigma(K, K))$ is a finite set. Here, for knots $K_{1}$ and $K_{2}$ in $S^{3}$, let $\Sigma\left(K_{1}, K_{2}\right)$ denote the closed 3-manifold $E\left(K_{1}\right) \cup_{h} E\left(K_{2}\right)$, where $h$ is an orientation-reversing homeomorphism $\partial E\left(K_{1}\right) \rightarrow \partial E\left(K_{2}\right)$ interchanging meridians and preferred longitudes of the knots. We call $\Sigma\left(K_{1}, K_{2}\right)$ the splice of $E\left(K_{1}\right)$ and $E\left(K_{2}\right)$ (or simply the splice of $K_{1}$ and $K_{2}$ ). By definition, a splice is an integral homology 3-sphere. Recently, Zentner [20] showed that the fundamental group of any integral homology 3 -sphere $M$ admits an irreducible $S L(2, \mathbb{C})$-representation, and therefore, it is worth studying $R T(M)$.

The purpose of this paper is to generalize the above result on splices to a certain class of knots. We focus on the character variety $X\left(E(K)\right.$ ) and $A$-polynomial $A_{K}(L, M) \in \mathbb{Z}[L, M]$ of a knot $K$ and prove the following main theorem and its corollary.

Theorem 1.1. Suppose that knots $K_{1}$ and $K_{2}$ in $S^{3}$ satisfy the following conditions:

- for any irreducible component $C \subset X\left(E\left(K_{i}\right)\right)(i=1,2)$, either $\operatorname{dim} C=0$, or $\operatorname{dim} C=1$ and its image under the map $X\left(E\left(K_{i}\right)\right) \rightarrow X\left(\partial E\left(K_{i}\right)\right)$ is not a point.
- $\operatorname{gcd}\left(A_{K_{1}}(L, M), A_{K_{2}}(M, L)\right)=1$.

Then $\operatorname{RT}\left(\Sigma\left(K_{1}, K_{2}\right)\right)$ is a finite set.
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Corollary 1.2. For any 2-bridge knots $K_{1}$ and $K_{2}$, the set $R T\left(\Sigma\left(K_{1}, K_{2}\right)\right)$ is finite.
Curtis [5, 6] defined an $S L(2, \mathbb{C})$-Casson invariant $\lambda_{S L(2, C)}(M)$ for any homology 3-sphere $M$. Roughly speaking, this invariant counts the number of isolated points of $X(M)$. It is known that $\lambda_{S L(2, \mathrm{C})}\left(\Sigma\left(K_{1}, K_{2}\right)\right)$ is vanishing for any $K_{1}, K_{2}$ by Boden and Curtis [2]. By definition, this implies that there are no isolated points in $X\left(\Sigma\left(K_{1}, K_{2}\right)\right)$ and any connected component of $X\left(\Sigma\left(K_{1}, K_{2}\right)\right)$ has a positive dimension. However by the main theorem $R T\left(\Sigma\left(K_{1}, K_{2}\right)\right)$ is a finite set for any knots with the above conditions. In fact, we concretely describe $X(\Sigma(K, K))$ for the cases where $K$ is the trefoil knot or figure-eight knot in Section 4.

Recently Abouzaid and Manolescu defined an $\operatorname{SL}(2, \mathbb{C})$-Floer homology and also a full Casson invariant by taking its Euler characteristic in [1]. That is a problem to study a relation with our Reidemeister torsion for a splice.

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## 2. Character variety, $A$-polynomial and Reidemeister torsion

### 2.1. Representation variety and character variety

Let $\Gamma$ be a finitely generated group. We define the $S L(2, \mathbb{C})$-representation variety $R(\Gamma)$ of $\Gamma$ to be the affine algebraic set $\operatorname{Hom}(\Gamma, S L(2, \mathbb{C}))$ over $\mathbb{C}$. Considering the GIT quotient of $R(\Gamma)$ by the action of $S L(2, \mathbb{C})$ by conjugation, one obtains the $S L(2, \mathbb{C})$-character variety $X(\Gamma):=R(\Gamma) / / S L(2, \mathbb{C})$ of $\Gamma$ (see [9, Section 2] for instance). The character variety $X(\Gamma)$ is again an affine algebraic set and not necessarily irreducible. Let $R^{\mathrm{irr}}(\Gamma)$ denote the subset of irreducible representations and $X^{\mathrm{irr}}(\Gamma)$ the image of $R^{\mathrm{irr}}(\Gamma)$ under the projection $R(\Gamma) \rightarrow X(\Gamma)$. It is known that the induced map $R^{\mathrm{irr}}(\Gamma) / S L(2, \mathbb{C}) \rightarrow X^{\mathrm{irr}}(\Gamma)$ is bijective.

We focus on the case $\Gamma=\pi_{1}(M)$ for a connected compact manifold $M$ and call $R(M):=R\left(\pi_{1}(M)\right)\left(\right.$ resp. $\left.X(M):=X\left(\pi_{1}(M)\right)\right)$ the representation variety (resp. character
variety) of $M$. For instance, the character variety of a torus $T^{2}$ is described explicitly as follows: Let $\lambda, \mu$ be generators of $\pi_{1}\left(T^{2}\right)=\mathbb{Z}^{2}$ and $\rho \in R\left(T^{2}\right)$. Since $\lambda$ and $\mu$ commute, there exists a representation $\rho^{\prime}$ such that $\rho^{\prime}$ is conjugate to $\rho$ and both $\rho^{\prime}(\lambda)$ and $\rho^{\prime}(\mu)$ are upper triangular. Considering the (1,1)-entries of these matrices, one can define the map $\theta: R\left(T^{2}\right) \rightarrow\left(\mathbb{C}^{\times}\right)^{2} / \sim$ by $\theta(\rho)=\left(\rho^{\prime}(\lambda)_{11}, \rho^{\prime}(\mu)_{11}\right)$, where $(L, M) \sim\left(L^{\prime}, M^{\prime}\right)$ if $L=L^{\prime}, M=M^{\prime}$ or $L^{-1}=L^{\prime}, M^{-1}=M^{\prime}$.

It is easy to see that this map gives an identification $\theta: X\left(T^{2}\right) \rightarrow\left(\mathbb{C}^{\times}\right)^{2} / \sim$.
The character variety of the complement $E(K)$ of a knot $K$ is complicated in general. However, it is well known that if $K$ is a 2-bridge knot then $X(E(K))$ does not have an irreducible component of dimension larger than one. More generally, if a 3-manifold $M$ contains no irreducible closed surface and $\partial M \cong T^{2}$, then $\operatorname{dim} C=1$ for every irreducible component $C$ of $X(M)$ (see [3, Section 2.4]).

### 2.2. A-polynomial of knots

We briefly review the $A$-polynomial introduced by Cooper, Culler, Gillet, Long, and Shalen [3] (see also [4]) and a relation with the boundary slopes of knots. For an oriented knot $K$, let $r: X(E(K)) \rightarrow X(\partial E(K))$ denote the regular map between affine algebraic sets induced by the inclusion and let $\pi:\left(\mathbb{C}^{\times}\right)^{2} \rightarrow\left(\mathbb{C}^{\times}\right)^{2} / \sim$ be the natural projection. Here one takes $\lambda, \mu \in \pi_{1}(E(K))$ as a pair of a longitude $\lambda$ and a meridian $\mu$. We take $\lambda$ to be homologically trivial in $H_{1}(E(K) ; \mathbb{Z})$. By using these $\lambda$ and $\mu$ one can also identify $\pi_{1}(\partial E(K))$ with $\mathbb{Z}^{2}$.

For any $[\rho] \in X(E(K))$ one can take $\left[\rho^{\prime}\right]=[r(\rho)]$. To define the $A$-polynomial of a knot, we write $L$ for $\rho^{\prime}(\lambda)_{11}$ and $M$ for $\rho^{\prime}(\mu)_{11}$ as above.

Then, the Zariski closure of $\pi^{-1}(\theta \circ r(X(E(K)))) \subset \mathbb{C}^{2}$ is an affine algebraic set whose irreducible components are curves $C_{1}, \ldots, C_{n}$ and some points. Since codim $C_{j}=1$, the ideal $I\left(C_{j}\right)$ is known to be principal, namely $I\left(C_{j}\right)=\left(f_{j}\right)$ for some $f_{j} \in \mathbb{C}[L, M]$. It is known that there is $c \in \mathbb{C}$ such that $c f_{1}(L, M) \cdots f_{n}(L, M) \in \mathbb{Z}[L, M]$ and its coefficients have no common divisor. The $A$-polynomial $A_{K}(L, M)$ of $K$ is now defined by $A_{K}(L, M)=c f_{1}(L, M) \cdots f_{n}(L, M)$ up to sign, and it is independent of the choice of an orientation of $K$.

Remark 2.1. Since $A_{K}(L, M)$ has the factor $L-1$ coming from abelian representations of $\pi_{1}(E(K))$, the $A$-polynomial is sometimes defined to be $A_{K}(L, M) /(L-1)$. This is not essential in our main theorem due to Lemma 2.2.

Lemma 2.2. If $\theta \circ r(\rho)=(L, 1)$, then $L=1$. In particular, the $A$-polynomial $A_{K}(L, M)$ does not have the factor $M-1$.

Proof. It follows from $r(\rho)=(L, 1)$ that $\rho(\mu)$ is equal to the identity matrix $I_{2}$ or $\left(\begin{array}{ll}1 & 1 \\ 0 & 1\end{array}\right)$ up to conjugate. In the case $\rho(\mu)=I_{2}, \rho$ is trivial. In the latter case, $\rho(\lambda)$ is of the form $\left(\begin{array}{cc}1 & u \\ 0 & 1\end{array}\right)$ for some $u \in \mathbb{C}$, and hence $L=1$.

We next see a relation between the $A$-polynomial and boundary slopes of $K$. The rest of this subsection is devoted to proving Corollary 2.6 which is used in Corollary 1.2 , not in Theorem 1.1. Here, $p / q \in \mathbb{Q} \cup\{\infty\}$ is called a boundary slope of $K$ if there exists a properly embedded incompressible surface $S$ in $E(K)$ such that $\partial S$ is parallel copies of a simple closed curve of slope $p / q$, namely the homology class of each boundary component of $S$ equals $p \mu+q \lambda \in H_{1}(E(K))$ up to sign. We denote by $B S(K)$ the set of boundary slopes of $K$.

For a polynomial $f(L, M)=\sum_{i, j} a_{i j} L^{i} M^{j} \in \mathbb{Z}[L, M]$, the Newton polygon $N(f)$ of $f$ is defined by $N(f)=\operatorname{Conv}\left(\left\{(i, j) \in \mathbb{Z}^{2} \mid a_{i j} \neq 0\right\}\right)$, where $\operatorname{Conv}(T)$ denotes the convex hull of a subset $T$ in $\mathbb{R}^{2}$.

We write by $S S(P) \subset \mathbb{Q} \cup\{\infty\}$ the set of slopes of the sides of a polygon $P$. Note that $S S(N(f))=\emptyset$ if and only if $f$ is a monomial. The set $S S\left(N\left(A_{K}\right)\right)$ is closely related to $B S(K)$.

Theorem 2.3 ([3, Theorem 3.4]). The inclusion $S S\left(N\left(A_{K}\right)\right) \subset B S(K)$ holds for every knot $K$.

Let us review some facts about the Minkowski sum. For subsets $T$ and $U$ of $\mathbb{R}^{2}$, the Minkowski sum $T+U$ is defined by $T+U=\left\{t+u \in \mathbb{R}^{2} \mid t \in T, u \in U\right\}$. One can see that $\operatorname{Conv}(T+U)=\operatorname{Conv}(T)+\operatorname{Conv}(U)$, and hence $N(f g)=N(f)+N(g)$. The following proposition is well known and plays a key role in the next lemma.

Proposition 2.4 (see [7, Section 15.1] for example). Let $P$ and $Q$ be convex polygons. Then $S S(P+Q)=S S(P) \cup S S(Q)$.

For a subset $S$ of $\mathbb{Q} \cup\{\infty\}$, we denote by $S^{-1}$ the set $\left\{s^{-1} \in \mathbb{Q} \cup\{\infty\} \mid s \in S\right\}$, where we use the convention $0 \cdot \infty=1$. Also, for a polynomial $f \in \mathbb{Z}[L, M]$, we define $f^{T} \in \mathbb{Z}[L, M]$ by $f^{T}(L, M)=f(M, L)$.

Lemma 2.5. Let $f_{1}, f_{2} \in \mathbb{Z}[L, M]$. If $S S\left(N\left(f_{1}\right)\right) \cap S S\left(N\left(f_{2}\right)\right)^{-1}=\emptyset$, then $\operatorname{gcd}\left(f_{1}, f_{2}^{T}\right)$ is a monomial.

Proof. Let $g=\operatorname{gcd}\left(f_{1}, f_{2}^{T}\right)$. Then $g \mid f_{1}$ and $g^{T} \mid f_{2}$. By Proposition 2.4, we have $S S(N(g)) \subset S S\left(N\left(f_{1}\right)\right)$ and $S S\left(N\left(g^{T}\right)\right) \subset S S\left(N\left(f_{2}\right)\right)$. Since $S S(N(g))=S S\left(N\left(g^{T}\right)\right)^{-1}$, the assumption implies that $S S(N(g))=\emptyset$, namely $g$ is a monomial.

Corollary 2.6. If $K_{1}$ and $K_{2}$ be any 2-bridge knots, then it holds that $\operatorname{gcd}\left(A_{K_{1}}, A_{K_{2}}^{T}\right)=1$.

Proof. By [8, Theorem 1(b)], $B S\left(K_{i}\right) \subset 2 \mathbb{Z}$ holds. It follows from Theorem 2.3 that $S S\left(N\left(A_{K_{1}}\right)\right) \cap S S\left(N\left(A_{K_{2}}\right)\right)^{-1}=\emptyset$, and hence $\operatorname{gcd}\left(A_{K_{1}}, A_{K_{2}}^{T}\right)$ is a monomial by Lemma 2.5. Here, in general, the $A$-polynomial of a knot $K$ is divided by neither $L$ nor $M$ by definition. Therefore, the monomial must be 1 .

### 2.3. The $S L(2, \mathbb{C})$-Reidemeister torsion of $\mathbf{3}$-manifolds

For precise definitions of a Reidemeister torsion, please see Johnson [10], Kitano [12, 13] and Milnor [14, 15] as references.

Let $M$ be a 3-manifold and let $\rho \in R(M)$ be an acyclic representation. That is, $C_{*}\left(M ; \mathbb{C}_{\rho}^{2}\right)$ is an acyclic chain complex with twisted coefficients.

Then one gets a nonzero complex number $\tau_{\rho}(M) \in \mathbb{C}^{\times}$for an acyclic chain complex $C_{*}\left(M ; \mathbb{C}_{\rho}^{2}\right)$. We call it the $S L(2, \mathbb{C})$-Reidemeister torsion of $M$ for $\rho$.
Remark 2.7. Throughout this paper, we set $\tau_{\rho}(M)=0$ if $\rho$ is not acyclic. Then $\tau_{\rho}(M)$ can be regarded as a function on $R(M)$ and also on $X(M)$.

One can use the well-known multiplicativity of the Reidemeister torsion to compute it as below.

Proposition 2.8. Let $M$ be a 3-manifold decomposed into $M_{1}$ and $M_{2}$ by an embedded torus $T^{2}$. Let $\rho: \pi_{1}(M) \rightarrow S L(2, \mathbb{C})$ be a representation. Suppose that $\rho$ is acyclic on $\pi_{1}\left(T^{2}\right)$. Then it holds that $\rho$ is acyclic on $\pi_{1}(M)$ if and only if it is acyclic on both $\pi_{1}\left(M_{1}\right)$ and $\pi_{1}\left(M_{2}\right)$. Further in this case it holds that

$$
\tau_{\rho}(M)=\tau_{\rho}\left(M_{1}\right) \tau_{\rho}\left(M_{2}\right)
$$

One needs the acyclicity of representations to use the above. First we mention the following lemma.

Lemma 2.9. Let $\rho$ be a representation $\pi_{1}\left(T^{2}\right) \rightarrow S L(2, \mathbb{C})$. Then it holds that $\rho$ is acyclic if and only if $\rho$ is not parabolic. Here $\rho$ is said to be parabolic if $\operatorname{tr} \rho(x)=2$ for any $x \in \pi_{1}\left(T^{2}\right)$.

Proof. First note that for a basis $\{x, y\}$ of $\pi_{1}\left(T^{2}\right)$ the chain complex $C_{*}\left(T^{2} ; \mathbb{C}_{\rho}^{2}\right)$ is given by

$$
0 \rightarrow \mathbb{C}^{2} \xrightarrow{\partial_{2}} \mathbb{C}^{2} \oplus \mathbb{C}^{2} \xrightarrow{\partial_{1}} \mathbb{C}^{2} \rightarrow 0,
$$

where

$$
\partial_{2}=\left(\begin{array}{ll}
-\left(\rho(y)-I_{2}\right) & \rho(x)-I_{2}
\end{array}\right), \partial_{1}=\binom{\rho(x)-I_{2}}{\rho(y)-I_{2}} .
$$

We here show that $\rho$ is not parabolic if and only if $H_{0}\left(T^{2} ; \mathbb{C}_{\rho}^{2}\right)=0$. If $\rho$ is not parabolic, then there is a basis $\{x, y\}$ such that $\operatorname{det}\left(\rho(x)-I_{2}\right) \neq 0$, and thus $H_{0}\left(T^{2} ; \mathbb{C}_{\rho}^{2}\right)=0$.

Conversely, if $\rho$ is parabolic, then $\rho(x)$ and $\rho(y)$ are simultaneously of the form $\left(\begin{array}{ll}1 & * \\ 0 & 1\end{array}\right)$ by taking conjugate, and therefore $H_{0}\left(T^{2} ; \mathbb{C}_{\rho}^{2}\right) \neq 0$.

Next, if $H_{0}\left(T^{2} ; \mathbb{C}_{\rho}^{2}\right)=0$, then $\rho$ is acyclic. Indeed, by Kronecker duality (or the universal coefficient theorem) and Poincaré duality, $H_{2}\left(T^{2} ; \mathbb{C}_{\rho}^{2}\right) \cong H_{0}\left(T^{2} ; \mathbb{C}_{\check{\rho}}^{2}\right)$, where $\check{\rho}(\gamma):={ }^{t} \rho(\gamma)^{-1}$. When $\rho$ is parabolic, so is $\check{\rho}$. It follows from $\chi\left(T^{2}\right)=0$ that $H_{1}\left(T^{2} ; \mathbb{C}_{\rho}^{2}\right)=0$.

## 3. Proof of the main theorem

Recall that $\Sigma\left(K_{1}, K_{2}\right)$ denotes the splice. The following lemma is shown in [2, Proof of Corollary 3.3]. We give a proof to be self-contained.

Lemma 3.1. If $\rho$ is irreducible on $\pi_{1}\left(\Sigma\left(K_{1}, K_{2}\right)\right)$, then the restrictions of $\rho$ on $\pi_{1}\left(E\left(K_{1}\right)\right)$ and $\pi_{1}\left(E\left(K_{2}\right)\right)$ are also irreducible.

Proof. Assume that $\rho$ is reducible on $\pi_{1}\left(E\left(K_{1}\right)\right)$. Then we may take $\rho$ as an upper triangular representation on it. Since the longitude $\lambda_{1}$ of $K_{1}$ belongs to the commutator subgroup $\left[\pi_{1}\left(E\left(K_{1}\right)\right), \pi_{1}\left(E\left(K_{1}\right)\right)\right]$, then one can see that $L_{1}$ is an upper triangular parabolic matrix as $L_{1}=\rho\left(\lambda_{1}\right)=\left(\begin{array}{cc}1 & \alpha \\ 0 & 1\end{array}\right)$.

If $\alpha=0$, then $L_{1}$ is the identity and hence $X_{2}=L_{1}$ is also the identity matrix. This means that $\rho$ must be trivial on $\pi_{1}\left(E\left(K_{2}\right)\right)$ and this is a contradiction.

Therefore we may assume $\alpha \neq 0$. Since $X_{1}$ commutes with $L_{1}$, then $X_{1}$ is also an upper triangular matrix as $X_{1}=\left(\begin{array}{cc} \pm 1 & \beta \\ 0 & \pm 1\end{array}\right)(\beta \neq 0)$. Hence the image $\rho\left(\pi_{1}\left(E\left(K_{1}\right)\right)\right)$ is an upper triangular subgroup. Since this is an abelian subgroup in $\operatorname{SL}(2, \mathbb{C})$, then $L_{1}$ must be also the identity. This is a contradiction.

Remark 3.2. By the above arguments, it can be seen that there exists no reducible representation except the trivial representation.

Next we can see the following.
Proposition 3.3. If $\rho: \pi_{1}\left(\Sigma\left(K_{1}, K_{2}\right)\right) \rightarrow S L(2, \mathbb{C})$ be an acyclic representation, then its restriction $\left.\rho\right|_{\pi_{1}\left(T^{2}\right)}$ is also acyclic.

Proof. Assume that $\left.\rho\right|_{\pi_{1}\left(T^{2}\right)}$ is not acyclic. Consider the homology long exact sequence for

$$
0 \rightarrow C_{*}\left(T^{2} ; \mathbb{C}_{\rho}^{2}\right) \rightarrow C_{*}\left(E\left(K_{1}\right) ; \mathbb{C}_{\rho}^{2}\right) \oplus C_{*}\left(E\left(K_{2}\right) ; \mathbb{C}_{\rho}^{2}\right) \rightarrow C_{*}\left(\Sigma\left(K_{1}, K_{2}\right) ; \mathbb{C}_{\rho}^{2}\right) \rightarrow 0 .
$$



Since $C_{*}\left(\Sigma\left(K_{1}, K_{2}\right) ; \mathbb{C}_{\rho}^{2}\right)$ is acyclic, we have the exact sequences

$$
\begin{aligned}
& 0 \rightarrow H_{2}\left(T^{2} ; \mathbb{C}_{\rho}^{2}\right) \rightarrow H_{2}\left(E\left(K_{1}\right) ; \mathbb{C}_{\rho}^{2}\right) \oplus H_{2}\left(E\left(K_{2}\right) ; \mathbb{C}_{\rho}^{2}\right) \rightarrow 0, \\
& 0 \rightarrow H_{1}\left(T^{2} ; \mathbb{C}_{\rho}^{2}\right) \rightarrow H_{1}\left(E\left(K_{1}\right) ; \mathbb{C}_{\rho}^{2}\right) \oplus H_{1}\left(E\left(K_{2}\right) ; \mathbb{C}_{\rho}^{2}\right) \rightarrow 0, \\
& 0 \rightarrow H_{0}\left(T^{2} ; \mathbb{C}_{\rho}^{2}\right) \rightarrow H_{0}\left(E\left(K_{1}\right) ; \mathbb{C}_{\rho}^{2}\right) \oplus H_{0}\left(E\left(K_{2}\right) ; \mathbb{C}_{\rho}^{2}\right) \rightarrow 0 .
\end{aligned}
$$

Since $\rho$ is not acyclic on $\pi_{1}\left(T^{2}\right), \rho$ is parabolic on it by Lemma 2.9. If it is trivial on $\pi_{1}\left(T^{2}\right)$, it should be trivial on $\pi_{1}\left(\Sigma\left(K_{1}, K_{2}\right)\right)$. Then it is not acyclic on $\Sigma\left(K_{1}, K_{2}\right)$. For any non-trivial parabolic representation $\rho$ on $\pi_{1}\left(T^{2}\right)$, it is easy to see

$$
H_{2}\left(T^{2} ; \mathbb{C}_{\rho}^{2}\right) \cong H_{0}\left(T^{2} ; \mathbb{C}_{\rho}^{2}\right) \cong \mathbb{C}, H_{1}\left(T^{2} ; \mathbb{C}_{\rho}^{2}\right) \cong \mathbb{C}^{2}
$$

by the proof of Lemma 2.9. If $\rho$ is irreducible, then both $\left.\rho\right|_{\pi_{1}\left(E\left(K_{1}\right)\right)}$ and $\left.\rho\right|_{\pi_{1}\left(E\left(K_{2}\right)\right)}$ are irreducible by Lemma 3.1. Then it holds that $H_{0}\left(E\left(K_{1}\right) ; \mathbb{C}_{\rho}^{2}\right)$ and $H_{0}\left(E\left(K_{2}\right) ; \mathbb{C}_{\rho}^{2}\right)$ are vanishing. Therefore $H_{0}\left(T^{2} ; \mathbb{C}_{\rho}^{2}\right)$ is vanishing in this case by the above exact sequences. It is contradiction.

Next assume that $\rho$ is reducible. Now we may assume that the image of $\rho$ belongs to the upper triangular subgroup. It is easily seen that the images of the longitudes are trivial $I_{2}$ or $\left(\begin{array}{ll}1 & * \\ 0 & 1\end{array}\right)$ since the longitudes belong to the commutator subgroup. Therefore the image of each meridian is in the upper triangular parabolic subgroup by the definition of a splice, and thus $\rho$ is abelian. This contradicts the fact that the abelianization of $\pi_{1}\left(\Sigma\left(K_{1}, K_{2}\right)\right)$ is trivial.

Lemma 3.4. Let $f: X \rightarrow Y$ be a non-constant regular map between affine algebraic sets $X$ and $Y$. If $X$ is irreducible and $\operatorname{dim} X=1$, then $f^{-1}(\{y\})$ is a finite (possibly empty) set for any $y \in Y$.

Proof. The inverse image $f^{-1}(\{y\})$ is a closed subset of $X$, namely $f^{-1}(\{y\})$ is a finite union of irreducible algebraic sets. Since they are proper algebraic subsets of $X$, they are of dimension zero.

The next lemma follows from Lemma 3.4 or Bézout's theorem.
Lemma 3.5. Let $f, g \in \mathbb{C}[L, M]$. Then $\{f=g=0\} \subset \mathbb{C}^{2}$ is a finite set if and only if $\operatorname{gcd}(f, g)=1$.

Using the above lemmas and propositions, we prove the main theorem.
Proof of Theorem 1.1. First note that $\operatorname{gcd}_{\mathbb{Z}[L, M]}(f, g)=\operatorname{gcd}_{\mathbb{C}[L, M]}(f, g)$ holds for $f, g \in$ $\mathbb{Z}[L, M]$ up to multiplication by elements of $\mathbb{C}^{\times}$. By Lemma 3.5 , the intersection

$$
\left\{(L, M) \in \mathbb{C}^{2} \mid A_{K_{1}}(L, M)=A_{K_{2}}^{T}(L, M)=0\right\}
$$

of the algebraic curves defined by $A_{K_{1}}$ and $A_{K_{2}}^{T}$ is a finite set $A$. Let us prove that the image of $X\left(\Sigma\left(K_{1}, K_{2}\right)\right) \rightarrow X\left(E\left(K_{i}\right)\right)$ is a finite set $X_{i}$ for $i=1,2$. Then Propositions 2.8 and 3.3 complete the proof.

By the definition of the $A$-polynomial, $\theta \circ r_{i}\left(X_{i}\right) \subset A$. It follows from Lemma 3.4 and the second condition in Theorem 1.1 that $r_{i}^{-1}\left(\theta^{-1}(A)\right)$ is a finite set. Thus, $X_{i}$ is also a finite set.

We next prove Corollary 1.2. Let $K$ be a 2-bridge knot. Take and fix a presentation of $\pi_{1}(E(K))$ and write $\phi(s, t)$ to its Riley polynomial (see Section 4). Then the following lemma is a consequence of [19, Lemma 2].

Lemma 3.6. The coefficient of the leading term of $\phi(s, t) \in \mathbb{Z}\left[s^{ \pm 1}, t\right]$ with respect to $t$ is a monomial of $s$.

Proof of Corollary 1.2. It suffices to check that any pair of 2-bridge knots $K_{1}$ and $K_{2}$ satisfies the conditions in Theorem 1.1. First, Corollary 2.6 implies $\operatorname{gcd}\left(A_{K_{1}}(L, M)\right.$, $\left.A_{K_{2}}(M, L)\right)=1$. Let $C$ be an irreducible component of $X\left(E\left(K_{i}\right)\right)$.

If $C$ consists of reducible representations, then $\operatorname{dim} C=1$ and $r_{i}(C) \subset X\left(\partial E\left(K_{i}\right)\right)$ is not a point. Otherwise, $C$ is described by an irreducible factor of the Riley polynomial of $K_{i}$, and hence $\operatorname{dim} C=1$. Assume that $r_{i}(C)$ is a point $\theta^{-1}(L, M)$. Then the function $\operatorname{tr} \rho(\mu)$ is the constant $M+M^{-1}$ on $C$, and thus $s-M \mid \phi(s, t)$. Since $M \neq 0$, this contradicts Lemma 3.6.

We put the following problem.
Problem 3.7. When $R T\left(\Sigma\left(K_{1}, K_{2}\right)\right)$ is an infinite set? Or is it always a finite set?
Here we give an observation when $\operatorname{dim} C>1$ in Theorem 1.1.
Example 3.8. Let $K$ be the Montesinos knot $M(1 / 3,1 / 3,1 / 3,1 / 3,1 / 2)$ (see Figure 3.1). Then $\pi_{1}(E(K))$ has the presentation

$$
\left\langle\mu_{1}, \ldots, \mu_{5} \left\lvert\, \begin{array}{ll}
\mu_{i} \mu_{i+1}^{-1} \mu_{i}^{-1} \mu_{i+1} \mu_{i}^{-1}=\mu_{i+1} \mu_{i+2}^{-1} \mu_{i+1} \mu_{i+2}^{-1} \mu_{i+1}^{-1} \mu_{i+2} \mu_{i+1}^{-1} & (i=1,2,3) \\
\mu_{4} \mu_{5}^{-1} \mu_{4}^{-1} \mu_{5} \mu_{4}^{-1}=\mu_{5} \mu_{1}^{-1} \mu_{5} \mu_{1} \mu_{5}^{-1}
\end{array}\right.\right\rangle .
$$

Note that $\mu_{1}$ is conjugate to $\mu_{2}^{-1}, \mu_{3}, \mu_{4}^{-1}$ and $\mu_{5}$. It follows from [18, Theorem 1] that there is an irreducible component of $X(E(K))$ with $\operatorname{dim} \geq 2$. In fact, we construct a 2-parameter family $C$ of representations by a bending (see Section 4) along the sphere $S$ intersecting $K$ at 4 points illustrated in Figure 3.1.

For $s \in \mathbb{C}^{\times} \backslash\{1\}$, we first define the representation $\rho_{s}: \pi_{1}(E(K)) \rightarrow S L(2, \mathbb{C})$ by $\rho_{s}\left(\mu_{j}\right)=\left(\begin{array}{cc}s^{-1} & 0 \\ s^{2}-1+s^{-2} & s\end{array}\right)$ if $j=1,3,5$ and $\rho_{s}\left(\mu_{j}\right)=\left(\begin{array}{cc}s & 1 \\ 0 & s^{-1}\end{array}\right)$ if $j=2$, 4. Note that $\rho_{s}$ factors through $\pi_{1}\left(E\left(\overline{3}_{1}\right)\right)=\langle x, y \mid x y x=y x y\rangle$, where $\overline{3}_{1}$ denotes the right-handed trefoil knot.

That is, $\rho_{s}$ comes from $\bar{\rho}_{s}: \pi_{1}\left(E\left(\overline{3}_{1}\right)\right) \rightarrow S L(2, \mathbb{C})$ by $\bar{\rho}_{s}(x)=\rho_{s}\left(\mu_{j}\right)$ if $j=1,3,5$ and $\bar{\rho}_{s}(y)=\rho_{s}\left(\mu_{j}\right)^{-1}$ if $j=2,4$. Here we have $\rho_{s}\left(\mu_{1}^{\prime}\right)=\rho_{s}\left(\mu_{1}\right)$ since $\rho_{s}\left(\mu_{1}\right)=\rho_{s}\left(\mu_{5}\right)$. By finding the tangle enclosed by the dotted circle drawn in Figure 3.1 which is a part of $\overline{3}_{1}$, we can see that the restriction of $\rho_{s}$ to $\pi_{1}(S \backslash K)$ is invariant under conjugation by

$$
P_{u}=\left(\begin{array}{cc}
\left(\frac{s^{2}-1+s^{-2}}{u}\right)^{1 / 2} & \left.\frac{\left(s^{2}-1+s^{-2}\right.}{u}\right)^{1 / 2}-\left(\frac{s^{2}-1+s^{-2}}{u}\right)^{-1 / 2} \\
0 & \left(\frac{s^{2}-1+s^{-2}}{u}\right)^{-1 / 2}
\end{array}\right),
$$

where $u \in \mathbb{C}^{\times}$(see Lemma 4.1). Therefore, one obtains representations $\rho_{s, u}: \pi_{1}(E(K)) \rightarrow$ $S L(2, \mathbb{C})$ by

$$
\rho_{s, u}\left(\mu_{j}\right)= \begin{cases}\rho_{s}\left(\mu_{j}\right) & \text { if } j=3 \\ P_{u} \rho_{s}\left(\mu_{j}\right) P_{u}^{-1} & \text { if } j=1,2,4,5\end{cases}
$$

By the above bending construction, the set $C=\left\{\rho_{s, u}\right\}$ is still a 2-parameter family in $X(E(K))$. Now one can also check it directly

$$
\tau_{\rho_{s, u}}(E(K))=\frac{144 s^{-4}(s-1)^{8}}{-s^{-1}(s-1)^{2}}=-144\left(\operatorname{tr} \rho_{s, u}\left(\mu_{2}\right)-2\right)^{3}
$$

and hence $\tau_{\rho_{s, u}}(E(K))$ depends only on $\operatorname{tr} \rho_{s, u}\left(\mu_{2}\right)=s+s^{-1}$.
On the other hand, to be independent of $u$, it can be explained by the generalized multiplicativity of the Reidemeister torsion to the decomposition $E(K)=M_{1} \cup_{S_{0}} M_{2}$ along the surface $S_{0}=S \cap E(K)$ with 4 boundary components. Although $M_{1}, M_{2}$ and $S_{0}$ are not acyclic, after fixing suitable bases of $H_{1}\left(M_{1} ; \mathbb{C}_{\rho_{s, u}}^{2}\right), H_{1}\left(M_{2} ; \mathbb{C}_{\rho_{s, u}}^{2}\right)$ and $H_{1}\left(S_{0} ; \mathbb{C}_{\rho_{s, u}}^{2}\right)$, we obtain $\tau_{\rho_{s, u}}(E(K))=\tau_{\rho_{s, u}}\left(M_{1}\right) \tau_{\rho_{s, u}}\left(M_{2}\right) / \tau_{\rho_{s, u}}\left(S_{0}\right)$. By the construction of $\rho_{s, u}$, we see that the value of the right-hand side is independent of $u$.

Let $R T_{C}$ be the subset of $R T(\Sigma(K, K))$ consisting of $\tau_{\rho}(\Sigma(K, K))$ 's where the restriction of $\rho$ to each $E(K)$ belongs to $C$. Then $R T_{C}$ is a finite set even though $C \subset X(\Sigma(K, K))$ is 2-dimensional. Indeed, one can check that $\operatorname{tr} \rho_{s, u}(\lambda)=s^{24}+s^{-24}$, and thus there are finitely many solutions $\left(s_{1}, s_{2}\right)$ of $\operatorname{tr} \rho_{s_{1}, u_{1}}(\mu)=\operatorname{tr} \rho_{s_{2}, u_{2}}(\lambda)$ and $\operatorname{tr} \rho_{s_{1}, u_{1}}(\lambda)=\operatorname{tr} \rho_{s_{2}, u_{2}}(\mu)$. We conclude that there are finitely many possibilities of the value $\tau_{\rho}(\Sigma(K, K))=\tau_{\rho_{s_{1}, u_{1}}}(E(K)) \tau_{\rho_{s_{2}, u_{2}}}(E(K))$.

Problem 3.9. Can we relax the assumption "either $\operatorname{dim} C=0$, or $\operatorname{dim} C=1$ and its image under the map $X\left(E\left(K_{i}\right)\right) \rightarrow X\left(\partial E\left(K_{i}\right)\right)$ is not a point" in Theorem 1.1?


Figure 3.1. The Montesinos knot $K=M(1 / 3,1 / 3,1 / 3,1 / 3,1 / 2)$.

## 4. Computational observation

### 4.1. Concrete description of $X^{\mathrm{irr}}\left(\Sigma\left(K_{1}, K_{2}\right)\right)$

In this section we observe some examples of splices. We use Mathematica to compute matrices. Recall that $X^{\mathrm{irr}}(M)$ is identified with $R^{\mathrm{irr}}(M) / S L(2, \mathbb{C})$. The construction of deformations of a representation used in this section is called a bending construction or simply a bending. See $[11,17]$ as a reference.

Here we compute $X^{\text {irr }}(\Sigma(K, K))$ for the trefoil knot and the figure-eight knot $K$ by using a presentation of a twist knot. Let $J(2,2 q)$ be a twist knot where $q$ is a nonzero integer. Please see [16] as a reference for twist knots.

A presentation of $\pi_{1}(E(J(2,2 q)))$ is given as

$$
\pi_{1}(E(J(2,2 q)))=\left\langle x, y \mid z^{q} x=y z^{q}\right\rangle, z=\left[y, x^{-1}\right]
$$

We take a representation $\rho:\langle x, y\rangle \rightarrow S L(2, \mathbb{C})$ from the free group $\langle x, y\rangle$ in $S L(2, \mathbb{C})$ by the correspondence

$$
\rho(x)=\left(\begin{array}{cc}
s & 1 \\
0 & 1 / s
\end{array}\right), \rho(y)=\left(\begin{array}{cc}
s & 0 \\
-t & 1 / s
\end{array}\right) \quad\left(s, t \in \mathbb{C}^{\times}\right) .
$$

We use a small letter for a group element and its capital letter for the image of a small letter, like $X$ for $\rho(x)$. For $\rho\left(z^{q}\right)=Z^{q}$, we put the matrix $Z^{q}=\left(\begin{array}{cc}z_{11} & z_{12} \\ z_{21} & z_{22}\end{array}\right)$.

We define the Riley polynomial to be $\phi_{q}(s, t)=z_{11}+(1 / s-s) z_{12}$. It can be checked that the previous representation gives an irreducible representation of $\pi_{1}(E(J(2,2 q)))$ in $S L(2, \mathbb{C})$ if and only if $(s, t)$ satisfies $\phi_{q}(s, t)=0$.

It is seen that any $[\rho] \in X(E(J(2,2 q)))$ can be parametrized by

$$
\begin{gathered}
\xi=\operatorname{tr} \rho(x)=\operatorname{tr} \rho(y)=s+1 / s \\
\operatorname{tr} \rho(x y)=s^{2}+1 / s^{2}-t=(s+1 / s)^{2}-t-2=\xi^{2}-t-2
\end{gathered}
$$

and then by $\xi$ and $t$.

Here we take other words $\widetilde{z}=\left[x, y^{-1}\right]$ and $\lambda=\widetilde{z}^{q} z^{q}$. This $x$ gives a meridian of $J(2,2 q)$ and this $\lambda$ does the corresponding longitude for $x$. Here $\lambda$ is homologically trivial. Therefore $\langle x, \lambda\rangle$ is the free abelian group of rank 2 and $\rho(x)=X$ commutes with $\rho(\lambda)=L$. We can find another matrix which commutes with $X$ and $L$ by direct computations.

Lemma 4.1. Any matrix $A$ which commutes with $X=\left(\begin{array}{cc}s & c^{2} \\ 0 & 1 / s\end{array}\right)(s \neq \pm 1, c \neq 0)$ has a form of

$$
A=\left(\begin{array}{cc}
a & \frac{a-1 / a}{s-1 / s} c^{2} \\
0 & 1 / a
\end{array}\right)
$$

for some $a \in \mathbb{C}^{\times}$.
Now we consider two copies $K_{1}, K_{2}$ of $J(2,2 q)$ and

$$
\begin{aligned}
& \pi_{1}\left(E\left(K_{1}\right)\right)=\left\langle x_{1}, y_{1} \mid z_{1}^{q} x_{1}=y_{1} z_{1}^{q}\right\rangle, z_{1}=\left[y_{1}, x_{1}^{-1}\right], \\
& \pi_{1}\left(E\left(K_{2}\right)\right)=\left\langle x_{2}, y_{2} \mid z_{2}^{q} x_{2}=y_{2} z_{2}^{q}\right\rangle, z_{2}=\left[y_{2}, x_{2}^{-1}\right] .
\end{aligned}
$$

Further

$$
\begin{aligned}
\pi_{1}\left(\Sigma\left(K_{1}, K_{2}\right)\right) & =\pi_{1}\left(E\left(K_{1}\right)\right) *_{\pi_{1}\left(T^{2}\right)} \pi_{1}\left(E\left(K_{2}\right)\right) \\
& =\left\langle x_{1}, y_{1}, x_{2}, y_{2} \mid z_{1}^{q} x_{1}=y_{1} z_{1}^{q}, z_{2}^{q} x_{2}=y_{2} z_{2}^{q}, x_{1}=\lambda_{2}, \lambda_{1}=x_{2}\right\rangle .
\end{aligned}
$$

We consider an irreducible representation $\rho: \pi_{1}\left(\Sigma\left(K_{1}, K_{2}\right)\right) \rightarrow S L(2, \mathbb{C})$. Up to conjugate, we can set that

$$
X_{1}=\rho\left(x_{1}\right)=\left(\begin{array}{cc}
s_{1} & 1 \\
0 & 1 / s_{1}
\end{array}\right), Y_{1}=\rho\left(y_{1}\right)=\left(\begin{array}{cc}
s_{1} & 0 \\
-t_{1} & 1 / s_{1}
\end{array}\right) .
$$

First note that we treat cases of $s_{1} \neq \pm 1$. Further we may assume that $X_{2}$ is conjugate to $\left(\begin{array}{cc}s_{2} & 1 \\ 0 & 1 / s_{2}\end{array}\right)$ and $Y_{2}$ to $\left(\begin{array}{cc}s_{2} & 0 \\ -t_{2} & 1 / s_{2}\end{array}\right)$ simultaneously, as

$$
X_{2}=H\left(\begin{array}{cc}
s_{2} & 1 \\
0 & 1 / s_{2}
\end{array}\right) H^{-1}, Y_{2}=H\left(\begin{array}{cc}
s_{2} & 0 \\
-t_{2} & 1 / s_{2}
\end{array}\right) H^{-1}
$$

for some $H \in S L(2, \mathbb{C})$.
Here we require the conditions

$$
X_{1}=L_{2}, L_{1}=X_{2}
$$

to get a representation on $\pi_{1}\left(\Sigma\left(K_{1}, K_{2}\right)\right)$. It can be seen that $L_{1}$ is an upper triangular matrix and then $X_{2}$ is also an upper triangular matrix. By taking

$$
H=\left(\begin{array}{cc}
c & 0 \\
0 & 1 / c
\end{array}\right)(c \neq 0),
$$

one has

$$
\begin{aligned}
X_{2} & =H\left(\begin{array}{cc}
s_{2} & 1 \\
0 & 1 / s_{2}
\end{array}\right) H^{-1}=\left(\begin{array}{cc}
s_{2} & c^{2} \\
0 & 1 / s_{2}
\end{array}\right), \\
Y_{2} & =H\left(\begin{array}{cc}
s_{2} & 0 \\
-t_{2} & 1 / s_{2}
\end{array}\right) H^{-1}=\left(\begin{array}{cc}
s_{2} & 0 \\
-t_{2} / c^{2} & 1 / s_{2}
\end{array}\right) .
\end{aligned}
$$

Here $L_{2}$ is also an upper triangular matrix and $L_{2}=X_{1}$. Now any $[\rho]=\left[\rho_{1} * \rho_{2}\right] \in$ $X\left(\Sigma\left(K_{1}, K_{2}\right)\right)$ is corresponding to ( $\left.X_{1}, Y_{1}, X_{2}, Y_{2}\right)=\left(X_{1}, Y_{1}, L_{1}, Y_{2}\right)$ of the above forms. For $a \in \mathbb{C}^{\times}$we define $A_{a}$ by

$$
A_{a}=\left(\begin{array}{cc}
a & \frac{a-1 / a}{s_{1}-1 / s_{1}} \\
0 & 1 / a
\end{array}\right)
$$

and now consider deformations $\left[\rho_{a}\right]=\left[\left(A_{a} \rho_{1} A_{a}^{-1}\right) * \rho_{2}\right]$ of $[\rho]=\left[\rho_{1} * \rho_{2}\right]$ as

$$
\left(A_{a} X_{1} A_{a}^{-1}, A_{a} Y_{1} A_{a}^{-1}, X_{2}, Y_{2}\right)=\left(X_{1}, A_{a} Y_{1} A_{a}^{-1}, X_{2}, Y_{2}\right) .
$$

Lemma 4.2. It holds that $A_{a} L_{1} A_{a}^{-1}=L_{1}$.
Proof. We prove $A_{a} L_{1} A_{a}^{-1}=L_{1}$. We may assume $s \neq \pm 1$. Here we take eigenvectors $\mathbf{u}_{1}=\binom{1}{0}, \mathbf{u}_{2} \in \mathbb{C}^{2}$ for $X_{1}$ such that $X_{1} \mathbf{u}_{1}=s_{1} \mathbf{u}_{1}, X_{1} \mathbf{u}_{2}=s_{1}^{-1} \mathbf{u}_{2}$. Since $X_{1} L_{1}=L_{1} X_{1}$, one has

$$
\begin{aligned}
X_{1} L_{1} \mathbf{u}_{2} & =L_{1} X_{1} \mathbf{u}_{2} \\
& =L_{1} s_{1}^{-1} \mathbf{u}_{2} \\
& =s_{1}^{-1} L_{1} \mathbf{u}_{2} .
\end{aligned}
$$

Hence there exists a nonzero constant $\gamma$ such that $L_{1} \mathbf{u}_{2}=\gamma \mathbf{u}_{2}$. This means that $L_{1}$ has also $\mathbf{u}_{1}, \mathbf{u}_{2} \in \mathbb{C}^{2}$ as eigenvectors. By similar arguments for $A_{a} X_{1}=X_{1} A_{a}$, one sees $A_{a}$ has $\mathbf{u}_{1}, \mathbf{u}_{2} \in \mathbb{C}^{2}$ as eigenvectors. Therefore it is seen that $X_{1}, L_{1}, A_{a}$ are simultaneously diagonalizable and in particular $A_{a} L_{1} A_{a}^{-1}=L_{1}$.

By the above lemma, one can see $A_{a} \rho_{1} A_{a}^{-1}=\rho_{1}$ on the subgroup $\pi_{1}\left(T^{2}\right)$ generated by $\left\{x_{1}, l_{1}\right\}=\left\{x_{2}, l_{2}\right\}$ and then $\rho_{a}=\left(A_{a} \rho_{1} A_{a}^{-1}\right) * \rho_{2}$ gives an irreducible representation of $\pi_{1}\left(\Sigma\left(K_{1}, K_{2}\right)\right)$.

Further if $a \neq 1$, then $A_{a} Y_{1} A_{a}^{-1} \neq Y_{1}$. This implies $\rho_{a} \neq \rho \in R\left(\Sigma\left(K_{1}, K_{2}\right)\right)$. It can be seen by the following computations. First one sees that

$$
\begin{aligned}
\operatorname{tr}\left(\rho_{1} * \rho_{2}\left(y_{1} x_{2}\right)\right) & =\operatorname{tr}\left(Y_{1} X_{2}\right) \\
& =s_{1} s_{2}+\frac{1}{s_{1} s_{2}}-c^{2} t_{1} .
\end{aligned}
$$

On the other hand, one sees that

$$
\begin{aligned}
\left.\operatorname{tr}\left(\left(A_{a} \rho_{1} A_{a}^{-1}\right) * \rho_{2}\right)\left(y_{1} x_{2}\right)\right) & =\operatorname{tr}\left(A_{a} Y_{1} A_{a}^{-1} X_{2}\right) \\
& =s_{1} s_{2}+\frac{1}{s_{1} s_{2}}+\left\{\frac{\left(s_{2}-\frac{1}{s_{2}}\right)}{\left(s_{1}-\frac{1}{s_{1}}\right)}\left(\frac{1}{a^{2}}-1\right)-\frac{c^{2}}{a^{2}}\right\} t_{1} .
\end{aligned}
$$

Therefore we can find one character

$$
[\rho] \mapsto \operatorname{tr} \rho\left(y_{1} x_{2}\right)
$$

which is not constant on $X\left(\Sigma\left(K_{1}, K_{2}\right)\right)$ and we know $X\left(\Sigma\left(K_{1}, K_{2}\right)\right)$ has at least one dimension near $[\rho]$.

Proposition 4.3. $X\left(\Sigma\left(K_{1}, K_{2}\right)\right)$ has just one dimension near $[\rho]$.
Proof. Take and fix any $[\rho]=\left[\rho_{1} * \rho_{2}\right] \in X\left(\Sigma\left(K_{1}, K_{2}\right)\right)$. It is seen that the character variety $X\left(\Sigma\left(K_{1}, K_{2}\right)\right)$ has at least one dimension near $[\rho]$ by a bending construction.

Consider another one parameter family

$$
\left\{\left[\rho_{u}\right]\right\}_{u}=\left\{\left[\rho_{1, u} * \rho_{2, u}\right]\right\}_{u} \subset X\left(\Sigma\left(K_{1}, K_{2}\right)\right)
$$

such that $\left[\rho_{0}\right]=[\rho]$. Here recall there exist only finitely many quadruples $\left\{\left(s_{1}, t_{1}, s_{2}, t_{2}\right)\right.$ 's $\}$ for this fixed $[\rho]=\left[\rho_{1} * \rho_{2}\right] \in X\left(\Sigma\left(K_{1}, K_{2}\right)\right)$ by the proof of the main theorem. Then we may assume that $\left[\rho_{u}\right]=\left[\rho_{1, u} * \rho_{2}\right]$ and $\left[\rho_{1, u}\right]=\left[\rho_{1}\right] \in X\left(K_{1}\right)$ for any $u$. Hence one gets $\left[\rho_{u}\right]=\left[\rho_{1, u} * \rho_{2}\right]=\left[\left(B \rho_{1} B^{-1}\right) * \rho_{2}\right]$, where $B \in S L(2, \mathbb{C})$. Because $B$ must commute with $X_{1}$ and $L_{1}$, then $B$ has a similar form as $A_{a}$ in Lemma 4.1. Therefore this is a bending construction and the dimension of deformations is one.

## 4.2. $q=1$ Case

Here we put $q=1$. This $J(2,2)$ is the trefoil knot. We write again

$$
X=\left(\begin{array}{cc}
s & 1 \\
0 & 1 / s
\end{array}\right), Y=\left(\begin{array}{cc}
s & 0 \\
-t & 1 / s
\end{array}\right)
$$

and

$$
\begin{aligned}
Z=\left[Y, X^{-1}\right] & =\left(\begin{array}{cc}
1-s^{2} t & \frac{1}{s}-s(1+t) \\
-\frac{t}{s}+s t(1+t) & 1+\left(2-\frac{1}{s^{2}}\right) t+t^{2}
\end{array}\right), \\
\widetilde{Z}=\left[X, Y^{-1}\right] & =\left(\begin{array}{cc}
1-\left(-2+s^{2}\right) t+t^{2} & \frac{-1+s^{2}-t}{s} \\
\frac{t\left(1-s^{2}+t\right)}{s} & 1-\frac{t}{s^{2}}
\end{array}\right), \\
Z X-Y Z & =\left(\begin{array}{cc}
0 & -1+1 / s^{2}+s^{2}-t \\
s\left(-1+1 / s^{2}+s^{2}-t\right) & 0
\end{array}\right) .
\end{aligned}
$$

The condition that $(s, t)$ gives a representation is $-1+1 / s^{2}+s^{2}-t=0$. On the other hand,

$$
\begin{aligned}
\phi_{1}(s, t) & =w_{11}+(1 / s-s) w_{12} \\
& =1-s^{2} t+(1 / s-s)(1 / s-s(1+t)) \\
& =1-s^{2} t+1 / s^{2}-1-t-1+s^{2}+s^{2} t \\
& =-1+1 / s^{2}+s^{2}-t \\
& =\xi^{2}-3-t,
\end{aligned}
$$

where $m=s+1 / s$.
Hence in the case of the trefoil knot, one sees

$$
t=\xi^{2}-3
$$

and $X^{\mathrm{irr}}(E(J(2,2)))$ is given by

$$
\left\{(\xi, t) \in \mathbb{C}^{2} \mid t=\xi^{2}-3, t \neq 0\right\}
$$

If $t=0$, then the corresponding representation is not irreducible.
Remark 4.4. If $s=1$, that is, $\xi=2$, then the chain complex is not acyclic. In the other cases, $\tau_{\rho}(E(J(2,2)))=2$.

Compute

$$
\begin{aligned}
L & =\widetilde{Z} Z \\
& =\left(\begin{array}{cc}
1-t^{2}+s^{4} t^{2}-t^{3}+\frac{t(1+t)}{s^{2}}-s^{2} t\left(1+t+t^{2}\right) & \frac{\left(1+s^{2}\right) t\left(1+t+s^{4}(1+t)-s^{2}\left(3+3 t+t^{2}\right)\right)}{s^{3}} \\
\frac{t^{2}\left(1+s^{6}-s^{2} t-s^{4} t\right)}{s^{3}} & 1-t^{2}+\frac{t^{2}}{s^{4}}-t^{3}+s^{2} t(1+t)-\frac{t\left(1+t+t^{2}\right)}{s^{2}}
\end{array}\right) .
\end{aligned}
$$

By putting $t=1-\left(1 / s^{2}+s^{2}\right)$, one obtains

$$
L=\left(\begin{array}{cc}
-s^{6} & -\frac{\left(1+s^{2}+s^{4}\right)\left(1+s^{6}\right)}{s^{5}} \\
0 & -1 / s^{6}
\end{array}\right)
$$

and

$$
\operatorname{tr}(L)=-s^{6}-1 / s^{6}=-T_{6}(m)
$$

Here $T_{6}(x)=x^{6}-6 x^{4}+9 x^{2}-2$ is the normalized Chebyshev polynomial of degree 6 .
Remark that $T_{6}(x)$ has the property $T_{6}(2 \cos \theta)=2 \cos 6 \theta$.
By relations $x_{1}=\lambda_{2}, \lambda_{1}=x_{2}$, one has

$$
\operatorname{tr}\left(X_{1}\right)=\operatorname{tr}\left(L_{2}\right), \operatorname{tr}\left(L_{1}\right)=\operatorname{tr}\left(X_{2}\right) .
$$

By putting $\xi_{1}=s_{1}+1 / s_{1}, \xi_{2}=s_{2}+1 / s_{2}$, one obtains

$$
\xi_{1}=-T_{6}\left(\xi_{2}\right),-T_{6}\left(\xi_{1}\right)=\xi_{2}
$$

Hence we obtain only one equation

$$
\xi=-T_{6}\left(-T_{6}(\xi)\right)=-T_{6}\left(T_{6}(\xi)\right)
$$

This equation $\xi+T_{6}\left(T_{6}(\xi)\right)=0$ is a polynomial equation of degree 36 with distinct 36 roots as follows:

$$
-2=2 \cos \pi, 2 \cos \frac{k \pi}{35}(k=1,3, \ldots, 33), 2 \cos \frac{k \pi}{37}(k=1,3, \ldots, 35) .
$$

It is seen that they are the roots as

$$
\begin{aligned}
T_{6}\left(T_{6}(-2)\right) & =T_{6}\left(T_{6}(2 \cos \pi)\right) \\
& =2 \cos 36 \pi=2, \\
T_{6}\left(T_{6}\left(2 \cos \frac{k \pi}{35}\right)\right) & =2 \cos \frac{36 k \pi}{35}=-2 \cos \frac{k \pi}{35}, \\
T_{6}\left(T_{6}\left(2 \cos \frac{k \pi}{37}\right)\right) & =2 \cos \frac{36 k \pi}{37}=-2 \cos \frac{k \pi}{37} .
\end{aligned}
$$

Further one easily sees that $\xi=-2$ does not give a representation on the splice. The roots $2 \cos \frac{k \pi}{35}(k=1,3, \ldots, 33)$ are corresponding to the condition $s_{1}^{36}=s_{1}$ coming from matrix equations $L_{1}=X_{2}$ and $L_{2}=X_{1}$.

It can be seen that there exists a $k$ such that $\operatorname{tr}\left(\rho\left(x_{1}\right)\right)=2 \cos \frac{k \pi}{35}$ and $\operatorname{tr}\left(\rho\left(x_{2}\right)\right)=$ $-T_{6}\left(2 \cos \frac{k \pi}{35}\right)$ for any $[\rho] \in X^{\text {irr }}\left(\Sigma\left(K_{1}, K_{2}\right)\right)$. On the other hand, the roots $2 \cos \frac{k \pi}{37}(k=$ $1,3, \ldots, 35$ ) are corresponding to the condition $s_{1}^{36}=s_{1}^{-1}$ coming from equations $L_{1}=X_{2}^{-1}$ and $L_{2}=X_{1}$. They give representations of the splicing of $3_{1}$ and its mirror image, not $3_{1}$.

Take $[\rho]=\left[\rho_{1} * \rho_{2}\right] \in X^{\text {irr }}\left(\Sigma\left(K_{1}, K_{2}\right)\right)$ and identify it with $\left(X_{1}, Y_{1}, X_{2}, Y_{2}\right)$. Consider

$$
A_{a}=\left(\begin{array}{cc}
a & \frac{a-1 / a}{s_{1}-1 / s_{1}} \\
0 & 1 / a
\end{array}\right)
$$

where $a \in \mathbb{C}^{\times}, s_{1}, s_{2} \in \mathbb{C}^{\times}$are satisfying $s_{1}+1 / s_{1}=\xi_{1}, s_{2}+1 / s_{2}=\xi_{2}$ and $\xi_{1}=$ $-T_{6}\left(T_{6}\left(\xi_{1}\right)\right), \xi_{2}=-T_{6}\left(\xi_{1}\right)$. In this case, one gets

$$
\begin{aligned}
& \operatorname{tr}\left(\left(A_{a} \rho_{1} A_{a}^{-1}\right) *\left(\rho_{2}\right)\left(y_{1} x_{2}\right)\right) \\
& \quad=\operatorname{tr}\left(X_{2} A_{a} Y_{1} A_{a}^{-1}\right) \\
& \quad=s_{1} s_{2}+\frac{1}{s_{1} s_{2}}-\frac{c^{2}}{a^{2}}\left(s_{1}^{2}+1 / s_{1}^{2}-1\right)+\frac{\left(1-a^{2}\right)\left(s_{2}-1 / s_{2}\right)}{\left(s_{1}-1 / s_{1}\right)}\left(s_{1}^{2}+1 / s_{1}^{2}-1\right) .
\end{aligned}
$$

Here $c$ is determined by $X_{2}=L_{1}$, namely

$$
c^{2}=-\frac{\left(1+s_{1}^{2}+s_{1}^{4}\right)\left(1+s_{1}^{4}\right)}{s_{1}^{5}} .
$$

## 4.3. $q=-1$ Case

We put $q=-1$. This $J(2,-2)$ is the figure-eight knot. In this case the Riley polynomial $\phi_{-1}(s, t)$ is given by

$$
\begin{aligned}
\phi_{-1}(s, t) & =t^{2}-\left(s^{2}+1 / s^{2}-3\right) t-s^{2}-1 / s^{2}+3 \\
& =t^{2}-\left(\xi^{2}-5\right) t-\xi^{2}+5,
\end{aligned}
$$

where $\xi=s+1 / s$.
Then the irreducible representation part of $X^{\mathrm{irr}}(E(J(2,4)))$ is

$$
\left\{(\xi, t) \in \mathbb{C}^{2} \mid t^{2}-\left(\xi^{2}-5\right) t-\xi^{2}+5=0, t \neq 0\right\} .
$$

Under the same notations, one obtains

$$
\xi_{1}=\xi_{2}^{4}-5 \xi_{2}^{2}+2, \xi_{1}^{4}-5 \xi_{1}^{2}+2=\xi_{2} .
$$

Hence we obtain only one equation

$$
\xi=\xi^{16}-20 \xi^{14}+158 \xi^{12}-620 \xi^{10}+1244 \xi^{8}-1190 \xi^{6}+487 \xi^{4}-60 \xi^{2}-2
$$

and 16 roots $v_{0}=-2, v_{i} \neq \pm 1(i=1, \ldots, 15)$. For any $v_{i}(i \neq 0)$, we can take a bending construction to do deformations in $X^{\mathrm{irr}}(\Sigma(J(2,-2), J(2,-2)))$.

Remark 4.5. In this case, $t$ is a root of $t^{2}-\left(v_{i}^{2}-5\right) t-v_{i}^{2}+5=0$.

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