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Finiteness of the image of the Reidemeister torsion of a splice

Teruaki Kitano Yuta Nozaki

Abstract

The set RT(M) of values of the SL(2, \mathbb{C})-Reidemeister torsion of a 3-manifold M can be both finite and infinite. We prove that RT(M) is a finite set if M is the splice of two certain knots in the 3-sphere. The proof is based on an observation on the character varieties and A-polynomials of knots.

1. Introduction

Let *K* be the figure-eight knot and E(K) the exterior of an open tubular neighborhood of *K* in the 3-sphere S^3 . The first author [13] computed the $SL(2, \mathbb{C})$ -Reidemeister torsion $\tau_{\rho}(E(K))$ for any acyclic irreducible representation $\rho: \pi_1(E(K)) \to SL(2, \mathbb{C})$. As a consequence, for the double $M = E(K) \cup_{id} E(K)$ of E(K), the set RT(M) of values of the $SL(2, \mathbb{C})$ -Reidemeister torsion $\tau_{\rho}(M)$ is the set of all complex numbers \mathbb{C} . In contrast, his computation also shows that $RT(\Sigma(K, K))$ is a finite set. Here, for knots K_1 and K_2 in S^3 , let $\Sigma(K_1, K_2)$ denote the closed 3-manifold $E(K_1) \cup_h E(K_2)$, where *h* is an orientation-reversing homeomorphism $\partial E(K_1) \to \partial E(K_2)$ interchanging meridians and preferred longitudes of the knots. We call $\Sigma(K_1, K_2)$ the *splice* of $E(K_1)$ and $E(K_2)$ (or simply the *splice* of K_1 and K_2). By definition, a splice is an integral homology 3-sphere. Recently, Zentner [20] showed that the fundamental group of any integral homology 3-sphere *M* admits an irreducible $SL(2, \mathbb{C})$ -representation, and therefore, it is worth studying RT(M).

The purpose of this paper is to generalize the above result on splices to a certain class of knots. We focus on the character variety X(E(K)) and A-polynomial $A_K(L, M) \in \mathbb{Z}[L, M]$ of a knot K and prove the following main theorem and its corollary.

Theorem 1.1. Suppose that knots K_1 and K_2 in S^3 satisfy the following conditions:

- for any irreducible component $C \subset X(E(K_i))$ (i = 1, 2), either dim C = 0, or dim C = 1 and its image under the map $X(E(K_i)) \rightarrow X(\partial E(K_i))$ is not a point.
- $gcd(A_{K_1}(L, M), A_{K_2}(M, L)) = 1.$

Then $RT(\Sigma(K_1, K_2))$ is a finite set.

Keywords: Reidemeister torsion, *A*-polynomial, character variety, splice, bending, Riley polynomial. 2020 *Mathematics Subject Classification*: 57M27, 57M25, 20C99, 14M99.

Corollary 1.2. For any 2-bridge knots K_1 and K_2 , the set $RT(\Sigma(K_1, K_2))$ is finite.

Curtis [5, 6] defined an $SL(2, \mathbb{C})$ -Casson invariant $\lambda_{SL(2,\mathbb{C})}(M)$ for any homology 3-sphere *M*. Roughly speaking, this invariant counts the number of isolated points of X(M). It is known that $\lambda_{SL(2,\mathbb{C})}(\Sigma(K_1, K_2))$ is vanishing for any K_1, K_2 by Boden and Curtis [2]. By definition, this implies that there are no isolated points in $X(\Sigma(K_1, K_2))$ and any connected component of $X(\Sigma(K_1, K_2))$ has a positive dimension. However by the main theorem $RT(\Sigma(K_1, K_2))$ is a finite set for any knots with the above conditions. In fact, we concretely describe $X(\Sigma(K, K))$ for the cases where *K* is the trefoil knot or figure-eight knot in Section 4.

Recently Abouzaid and Manolescu defined an $SL(2, \mathbb{C})$ -Floer homology and also a full Casson invariant by taking its Euler characteristic in [1]. That is a problem to study a relation with our Reidemeister torsion for a splice.

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2. Character variety, A-polynomial and Reidemeister torsion

2.1. Representation variety and character variety

Let Γ be a finitely generated group. We define the $SL(2, \mathbb{C})$ -representation variety $R(\Gamma)$ of Γ to be the affine algebraic set Hom $(\Gamma, SL(2, \mathbb{C}))$ over \mathbb{C} . Considering the GIT quotient of $R(\Gamma)$ by the action of $SL(2, \mathbb{C})$ by conjugation, one obtains the $SL(2, \mathbb{C})$ -character variety $X(\Gamma) := R(\Gamma)//SL(2, \mathbb{C})$ of Γ (see [9, Section 2] for instance). The character variety $X(\Gamma)$ is again an affine algebraic set and not necessarily irreducible. Let $R^{irr}(\Gamma)$ denote the subset of irreducible representations and $X^{irr}(\Gamma)$ the image of $R^{irr}(\Gamma)$ under the projection $R(\Gamma) \rightarrow X(\Gamma)$. It is known that the induced map $R^{irr}(\Gamma)/SL(2, \mathbb{C}) \rightarrow X^{irr}(\Gamma)$ is bijective.

We focus on the case $\Gamma = \pi_1(M)$ for a connected compact manifold *M* and call $R(M) := R(\pi_1(M))$ (resp. $X(M) := X(\pi_1(M))$) the representation variety (resp. character

variety) of *M*. For instance, the character variety of a torus T^2 is described explicitly as follows: Let λ , μ be generators of $\pi_1(T^2) = \mathbb{Z}^2$ and $\rho \in R(T^2)$. Since λ and μ commute, there exists a representation ρ' such that ρ' is conjugate to ρ and both $\rho'(\lambda)$ and $\rho'(\mu)$ are upper triangular. Considering the (1, 1)-entries of these matrices, one can define the map θ : $R(T^2) \to (\mathbb{C}^{\times})^2 / \sim$ by $\theta(\rho) = (\rho'(\lambda)_{11}, \rho'(\mu)_{11})$, where $(L, M) \sim (L', M')$ if L = L', M = M' or $L^{-1} = L', M^{-1} = M'$.

It is easy to see that this map gives an identification $\theta \colon X(T^2) \to (\mathbb{C}^{\times})^2/\sim$.

The character variety of the complement E(K) of a knot K is complicated in general. However, it is well known that if K is a 2-bridge knot then X(E(K)) does not have an irreducible component of dimension larger than one. More generally, if a 3-manifold M contains no irreducible closed surface and $\partial M \cong T^2$, then dim C = 1 for every irreducible component C of X(M) (see [3, Section 2.4]).

2.2. A-polynomial of knots

We briefly review the *A*-polynomial introduced by Cooper, Culler, Gillet, Long, and Shalen [3] (see also [4]) and a relation with the boundary slopes of knots. For an oriented knot *K*, let $r: X(E(K)) \to X(\partial E(K))$ denote the regular map between affine algebraic sets induced by the inclusion and let $\pi: (\mathbb{C}^{\times})^2 \to (\mathbb{C}^{\times})^2/\sim$ be the natural projection. Here one takes $\lambda, \mu \in \pi_1(E(K))$ as a pair of a longitude λ and a meridian μ . We take λ to be homologically trivial in $H_1(E(K); \mathbb{Z})$. By using these λ and μ one can also identify $\pi_1(\partial E(K))$ with \mathbb{Z}^2 .

For any $[\rho] \in X(E(K))$ one can take $[\rho'] = [r(\rho)]$. To define the *A*-polynomial of a knot, we write *L* for $\rho'(\lambda)_{11}$ and *M* for $\rho'(\mu)_{11}$ as above.

Then, the Zariski closure of $\pi^{-1}(\theta \circ r(X(E(K)))) \subset \mathbb{C}^2$ is an affine algebraic set whose irreducible components are curves C_1, \ldots, C_n and some points. Since $\operatorname{codim} C_j = 1$, the ideal $I(C_j)$ is known to be principal, namely $I(C_j) = (f_j)$ for some $f_j \in \mathbb{C}[L, M]$. It is known that there is $c \in \mathbb{C}$ such that $cf_1(L, M) \cdots f_n(L, M) \in \mathbb{Z}[L, M]$ and its coefficients have no common divisor. The *A*-polynomial $A_K(L, M)$ of *K* is now defined by $A_K(L, M) = cf_1(L, M) \cdots f_n(L, M)$ up to sign, and it is independent of the choice of an orientation of *K*.

Remark 2.1. Since $A_K(L, M)$ has the factor L - 1 coming from abelian representations of $\pi_1(E(K))$, the *A*-polynomial is sometimes defined to be $A_K(L, M)/(L - 1)$. This is not essential in our main theorem due to Lemma 2.2.

Lemma 2.2. If $\theta \circ r(\rho) = (L, 1)$, then L = 1. In particular, the A-polynomial $A_K(L, M)$ does not have the factor M - 1.

Proof. It follows from $r(\rho) = (L, 1)$ that $\rho(\mu)$ is equal to the identity matrix I_2 or $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ up to conjugate. In the case $\rho(\mu) = I_2$, ρ is trivial. In the latter case, $\rho(\lambda)$ is of the form $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ for some $u \in \mathbb{C}$, and hence L = 1.

We next see a relation between the *A*-polynomial and boundary slopes of *K*. The rest of this subsection is devoted to proving Corollary 2.6 which is used in Corollary 1.2, not in Theorem 1.1. Here, $p/q \in \mathbb{Q} \cup \{\infty\}$ is called a *boundary slope* of *K* if there exists a properly embedded incompressible surface *S* in *E*(*K*) such that ∂S is parallel copies of a simple closed curve of slope p/q, namely the homology class of each boundary component of *S* equals $p\mu + q\lambda \in H_1(E(K))$ up to sign. We denote by BS(K) the set of boundary slopes of *K*.

For a polynomial $f(L, M) = \sum_{i,j} a_{ij} L^i M^j \in \mathbb{Z}[L, M]$, the *Newton polygon* N(f) of f is defined by $N(f) = \text{Conv}(\{(i, j) \in \mathbb{Z}^2 \mid a_{ij} \neq 0\})$, where Conv(T) denotes the convex hull of a subset T in \mathbb{R}^2 .

We write by $SS(P) \subset \mathbb{Q} \cup \{\infty\}$ the set of slopes of the sides of a polygon *P*. Note that $SS(N(f)) = \emptyset$ if and only if *f* is a monomial. The set $SS(N(A_K))$ is closely related to BS(K).

Theorem 2.3 ([3, Theorem 3.4]). *The inclusion* $SS(N(A_K)) \subset BS(K)$ *holds for every knot* K.

Let us review some facts about the Minkowski sum. For subsets *T* and *U* of \mathbb{R}^2 , the *Minkowski sum T* + *U* is defined by *T* + *U* = { $t + u \in \mathbb{R}^2 | t \in T, u \in U$ }. One can see that Conv(*T* + *U*) = Conv(*T*) + Conv(*U*), and hence N(fg) = N(f) + N(g). The following proposition is well known and plays a key role in the next lemma.

Proposition 2.4 (see [7, Section 15.1] for example). Let *P* and *Q* be convex polygons. Then $SS(P + Q) = SS(P) \cup SS(Q)$.

For a subset *S* of $\mathbb{Q} \cup \{\infty\}$, we denote by S^{-1} the set $\{s^{-1} \in \mathbb{Q} \cup \{\infty\} \mid s \in S\}$, where we use the convention $0 \cdot \infty = 1$. Also, for a polynomial $f \in \mathbb{Z}[L, M]$, we define $f^T \in \mathbb{Z}[L, M]$ by $f^T(L, M) = f(M, L)$.

Lemma 2.5. Let $f_1, f_2 \in \mathbb{Z}[L, M]$. If $SS(N(f_1)) \cap SS(N(f_2))^{-1} = \emptyset$, then $gcd(f_1, f_2^T)$ is a monomial.

Proof. Let $g = gcd(f_1, f_2^T)$. Then $g \mid f_1$ and $g^T \mid f_2$. By Proposition 2.4, we have $SS(N(g)) \subset SS(N(f_1))$ and $SS(N(g^T)) \subset SS(N(f_2))$. Since $SS(N(g)) = SS(N(g^T))^{-1}$, the assumption implies that $SS(N(g)) = \emptyset$, namely g is a monomial.

Corollary 2.6. If K_1 and K_2 be any 2-bridge knots, then it holds that $gcd(A_{K_1}, A_{K_2}^T) = 1$.

Proof. By [8, Theorem 1(b)], $BS(K_i) \subset 2\mathbb{Z}$ holds. It follows from Theorem 2.3 that $SS(N(A_{K_1})) \cap SS(N(A_{K_2}))^{-1} = \emptyset$, and hence $gcd(A_{K_1}, A_{K_2}^T)$ is a monomial by Lemma 2.5. Here, in general, the *A*-polynomial of a knot *K* is divided by neither *L* nor *M* by definition. Therefore, the monomial must be 1.

2.3. The SL(2, C)-Reidemeister torsion of 3-manifolds

For precise definitions of a Reidemeister torsion, please see Johnson [10], Kitano [12, 13] and Milnor [14, 15] as references.

Let *M* be a 3-manifold and let $\rho \in R(M)$ be an acyclic representation. That is, $C_*(M; \mathbb{C}^2_{\rho})$ is an acyclic chain complex with twisted coefficients.

Then one gets a nonzero complex number $\tau_{\rho}(M) \in \mathbb{C}^{\times}$ for an acyclic chain complex $C_*(M; \mathbb{C}^2_{\rho})$. We call it the $SL(2, \mathbb{C})$ -*Reidemeister torsion* of M for ρ .

Remark 2.7. Throughout this paper, we set $\tau_{\rho}(M) = 0$ if ρ is not acyclic. Then $\tau_{\rho}(M)$ can be regarded as a function on R(M) and also on X(M).

One can use the well-known multiplicativity of the Reidemeister torsion to compute it as below.

Proposition 2.8. Let M be a 3-manifold decomposed into M_1 and M_2 by an embedded torus T^2 . Let $\rho: \pi_1(M) \to SL(2, \mathbb{C})$ be a representation. Suppose that ρ is acyclic on $\pi_1(T^2)$. Then it holds that ρ is acyclic on $\pi_1(M)$ if and only if it is acyclic on both $\pi_1(M_1)$ and $\pi_1(M_2)$. Further in this case it holds that

$$\tau_{\rho}(M) = \tau_{\rho}(M_1)\tau_{\rho}(M_2).$$

One needs the acyclicity of representations to use the above. First we mention the following lemma.

Lemma 2.9. Let ρ be a representation $\pi_1(T^2) \to SL(2, \mathbb{C})$. Then it holds that ρ is acyclic if and only if ρ is not parabolic. Here ρ is said to be parabolic if tr $\rho(x) = 2$ for any $x \in \pi_1(T^2)$.

Proof. First note that for a basis $\{x, y\}$ of $\pi_1(T^2)$ the chain complex $C_*(T^2; \mathbb{C}^2_{\rho})$ is given by

$$0 \to \mathbb{C}^2 \xrightarrow{\partial_2} \mathbb{C}^2 \oplus \mathbb{C}^2 \xrightarrow{\partial_1} \mathbb{C}^2 \to 0,$$

where

$$\partial_2 = \begin{pmatrix} -(\rho(y) - I_2) & \rho(x) - I_2 \end{pmatrix}, \ \partial_1 = \begin{pmatrix} \rho(x) - I_2 \\ \rho(y) - I_2 \end{pmatrix}.$$

We here show that ρ is not parabolic if and only if $H_0(T^2; \mathbb{C}^2_{\rho}) = 0$. If ρ is not parabolic, then there is a basis $\{x, y\}$ such that $\det(\rho(x) - I_2) \neq 0$, and thus $H_0(T^2; \mathbb{C}^2_{\rho}) = 0$.

Conversely, if ρ is parabolic, then $\rho(x)$ and $\rho(y)$ are simultaneously of the form $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ by taking conjugate, and therefore $H_0(T^2; \mathbb{C}^2_{\rho}) \neq 0$.

Next, if $H_0(T^2; \mathbb{C}^2_{\rho}) = 0$, then ρ is acyclic. Indeed, by Kronecker duality (or the universal coefficient theorem) and Poincaré duality, $H_2(T^2; \mathbb{C}^2_{\rho}) \cong H_0(T^2; \mathbb{C}^2_{\rho})$, where $\check{\rho}(\gamma) := {}^t \rho(\gamma)^{-1}$. When ρ is parabolic, so is $\check{\rho}$. It follows from $\chi(T^2) = 0$ that $H_1(T^2; \mathbb{C}^2_{\rho}) = 0$.

3. Proof of the main theorem

Recall that $\Sigma(K_1, K_2)$ denotes the splice. The following lemma is shown in [2, Proof of Corollary 3.3]. We give a proof to be self-contained.

Lemma 3.1. If ρ is irreducible on $\pi_1(\Sigma(K_1, K_2))$, then the restrictions of ρ on $\pi_1(E(K_1))$ and $\pi_1(E(K_2))$ are also irreducible.

Proof. Assume that ρ is reducible on $\pi_1(E(K_1))$. Then we may take ρ as an upper triangular representation on it. Since the longitude λ_1 of K_1 belongs to the commutator subgroup $[\pi_1(E(K_1)), \pi_1(E(K_1))]$, then one can see that L_1 is an upper triangular parabolic matrix as $L_1 = \rho(\lambda_1) = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$.

If $\alpha = 0$, then L_1 is the identity and hence $X_2 = L_1$ is also the identity matrix. This means that ρ must be trivial on $\pi_1(E(K_2))$ and this is a contradiction.

Therefore we may assume $\alpha \neq 0$. Since X_1 commutes with L_1 , then X_1 is also an upper triangular matrix as $X_1 = \begin{pmatrix} \pm 1 & \beta \\ 0 & \pm 1 \end{pmatrix}$ ($\beta \neq 0$). Hence the image $\rho(\pi_1(E(K_1)))$ is an upper triangular subgroup. Since this is an abelian subgroup in $SL(2, \mathbb{C})$, then L_1 must be also the identity. This is a contradiction.

Remark 3.2. By the above arguments, it can be seen that there exists no reducible representation except the trivial representation.

Next we can see the following.

Proposition 3.3. If $\rho: \pi_1(\Sigma(K_1, K_2)) \to SL(2, \mathbb{C})$ be an acyclic representation, then its restriction $\rho|_{\pi_1(T^2)}$ is also acyclic.

Proof. Assume that $\rho|_{\pi_1(T^2)}$ is not acyclic. Consider the homology long exact sequence for

$$0 \to C_*(T^2; \mathbb{C}^2_\rho) \to C_*(E(K_1); \mathbb{C}^2_\rho) \oplus C_*(E(K_2); \mathbb{C}^2_\rho) \to C_*(\Sigma(K_1, K_2); \mathbb{C}^2_\rho) \to 0.$$

Here we simply write ρ for each of $\rho|_{\pi_1(T^2)}$, $\rho|_{\pi_1(E(K_1))}$, and $\rho|_{\pi_1(E(K_2))}$.

Since $C_*(\Sigma(K_1, K_2); \mathbb{C}^2_{\rho})$ is acyclic, we have the exact sequences

$$\begin{split} 0 &\to H_2(T^2; \mathbb{C}^2_{\rho}) \to H_2(E(K_1); \mathbb{C}^2_{\rho}) \oplus H_2(E(K_2); \mathbb{C}^2_{\rho}) \to 0, \\ 0 &\to H_1(T^2; \mathbb{C}^2_{\rho}) \to H_1(E(K_1); \mathbb{C}^2_{\rho}) \oplus H_1(E(K_2); \mathbb{C}^2_{\rho}) \to 0, \\ 0 &\to H_0(T^2; \mathbb{C}^2_{\rho}) \to H_0(E(K_1); \mathbb{C}^2_{\rho}) \oplus H_0(E(K_2); \mathbb{C}^2_{\rho}) \to 0. \end{split}$$

Since ρ is not acyclic on $\pi_1(T^2)$, ρ is parabolic on it by Lemma 2.9. If it is trivial on $\pi_1(T^2)$, it should be trivial on $\pi_1(\Sigma(K_1, K_2))$. Then it is not acyclic on $\Sigma(K_1, K_2)$. For any non-trivial parabolic representation ρ on $\pi_1(T^2)$, it is easy to see

$$H_2(T^2; \mathbb{C}^2_{\rho}) \cong H_0(T^2; \mathbb{C}^2_{\rho}) \cong \mathbb{C}, \ H_1(T^2; \mathbb{C}^2_{\rho}) \cong \mathbb{C}^2$$

by the proof of Lemma 2.9. If ρ is irreducible, then both $\rho|_{\pi_1(E(K_1))}$ and $\rho|_{\pi_1(E(K_2))}$ are irreducible by Lemma 3.1. Then it holds that $H_0(E(K_1); \mathbb{C}^2_{\rho})$ and $H_0(E(K_2); \mathbb{C}^2_{\rho})$ are vanishing. Therefore $H_0(T^2; \mathbb{C}^2_{\rho})$ is vanishing in this case by the above exact sequences. It is contradiction.

Next assume that ρ is reducible. Now we may assume that the image of ρ belongs to the upper triangular subgroup. It is easily seen that the images of the longitudes are trivial I_2 or $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ since the longitudes belong to the commutator subgroup. Therefore the image of each meridian is in the upper triangular parabolic subgroup by the definition of a splice, and thus ρ is abelian. This contradicts the fact that the abelianization of $\pi_1(\Sigma(K_1, K_2))$ is trivial.

Lemma 3.4. Let $f: X \to Y$ be a non-constant regular map between affine algebraic sets X and Y. If X is irreducible and dim X = 1, then $f^{-1}(\{y\})$ is a finite (possibly empty) set for any $y \in Y$.

Proof. The inverse image $f^{-1}(\{y\})$ is a closed subset of *X*, namely $f^{-1}(\{y\})$ is a finite union of irreducible algebraic sets. Since they are proper algebraic subsets of *X*, they are of dimension zero.

The next lemma follows from Lemma 3.4 or Bézout's theorem.

Lemma 3.5. Let $f, g \in \mathbb{C}[L, M]$. Then $\{f = g = 0\} \subset \mathbb{C}^2$ is a finite set if and only if gcd(f, g) = 1.

Using the above lemmas and propositions, we prove the main theorem.

Proof of Theorem 1.1. First note that $gcd_{\mathbb{Z}[L,M]}(f,g) = gcd_{\mathbb{C}[L,M]}(f,g)$ holds for $f,g \in \mathbb{Z}[L,M]$ up to multiplication by elements of \mathbb{C}^{\times} . By Lemma 3.5, the intersection

$$\{(L, M) \in \mathbb{C}^2 \mid A_{K_1}(L, M) = A_{K_2}^T(L, M) = 0\}$$

of the algebraic curves defined by A_{K_1} and $A_{K_2}^T$ is a finite set A. Let us prove that the image of $X(\Sigma(K_1, K_2)) \rightarrow X(E(K_i))$ is a finite set X_i for i = 1, 2. Then Propositions 2.8 and 3.3 complete the proof.

By the definition of the *A*-polynomial, $\theta \circ r_i(X_i) \subset A$. It follows from Lemma 3.4 and the second condition in Theorem 1.1 that $r_i^{-1}(\theta^{-1}(A))$ is a finite set. Thus, X_i is also a finite set.

We next prove Corollary 1.2. Let *K* be a 2-bridge knot. Take and fix a presentation of $\pi_1(E(K))$ and write $\phi(s, t)$ to its Riley polynomial (see Section 4). Then the following lemma is a consequence of [19, Lemma 2].

Lemma 3.6. The coefficient of the leading term of $\phi(s, t) \in \mathbb{Z}[s^{\pm 1}, t]$ with respect to t is a monomial of s.

Proof of Corollary 1.2. It suffices to check that any pair of 2-bridge knots K_1 and K_2 satisfies the conditions in Theorem 1.1. First, Corollary 2.6 implies $gcd(A_{K_1}(L, M), A_{K_2}(M, L)) = 1$. Let *C* be an irreducible component of $X(E(K_i))$.

If *C* consists of reducible representations, then dim C = 1 and $r_i(C) \subset X(\partial E(K_i))$ is not a point. Otherwise, *C* is described by an irreducible factor of the Riley polynomial of K_i , and hence dim C = 1. Assume that $r_i(C)$ is a point $\theta^{-1}(L, M)$. Then the function tr $\rho(\mu)$ is the constant $M + M^{-1}$ on *C*, and thus $s - M \mid \phi(s, t)$. Since $M \neq 0$, this contradicts Lemma 3.6.

We put the following problem.

Problem 3.7. When $RT(\Sigma(K_1, K_2))$ is an infinite set? Or is it always a finite set?

Here we give an observation when dim C > 1 in Theorem 1.1.

Example 3.8. Let *K* be the Montesinos knot M(1/3, 1/3, 1/3, 1/3, 1/2) (see Figure 3.1). Then $\pi_1(E(K))$ has the presentation

$$\left\langle \mu_1, \dots, \mu_5 \middle| \begin{array}{l} \mu_i \mu_{i+1}^{-1} \mu_i^{-1} \mu_{i+1} \mu_i^{-1} = \mu_{i+1} \mu_{i+2}^{-1} \mu_{i+1} \mu_{i+2}^{-1} \mu_{i+1}^{-1} \mu_{i+2} \mu_{i+1}^{-1} & (i = 1, 2, 3) \\ \mu_4 \mu_5^{-1} \mu_4^{-1} \mu_5 \mu_4^{-1} = \mu_5 \mu_1^{-1} \mu_5 \mu_1 \mu_5^{-1} \\ \end{array} \right\rangle.$$

Note that μ_1 is conjugate to μ_2^{-1} , μ_3 , μ_4^{-1} and μ_5 . It follows from [18, Theorem 1] that there is an irreducible component of X(E(K)) with dim ≥ 2 . In fact, we construct a 2-parameter family *C* of representations by a bending (see Section 4) along the sphere *S* intersecting *K* at 4 points illustrated in Figure 3.1.

For $s \in \mathbb{C}^{\times} \setminus \{1\}$, we first define the representation $\rho_s \colon \pi_1(E(K)) \to SL(2, \mathbb{C})$ by $\rho_s(\mu_j) = \begin{pmatrix} s^{-1} \\ s^{2}-1+s^{-2} \\ s^{2}\end{pmatrix}$ if j = 1, 3, 5 and $\rho_s(\mu_j) = \begin{pmatrix} s & 1 \\ 0 \\ s^{-1} \end{pmatrix}$ if j = 2, 4. Note that ρ_s factors through $\pi_1(E(\bar{3}_1)) = \langle x, y | xyx = yxy \rangle$, where $\bar{3}_1$ denotes the right-handed trefoil knot.

Finiteness of the Reidemeister torsion

That is, ρ_s comes from $\bar{\rho}_s: \pi_1(E(\bar{3}_1)) \to SL(2, \mathbb{C})$ by $\bar{\rho}_s(x) = \rho_s(\mu_j)$ if j = 1, 3, 5 and $\bar{\rho}_s(y) = \rho_s(\mu_j)^{-1}$ if j = 2, 4. Here we have $\rho_s(\mu'_1) = \rho_s(\mu_1)$ since $\rho_s(\mu_1) = \rho_s(\mu_5)$. By finding the tangle enclosed by the dotted circle drawn in Figure 3.1 which is a part of $\bar{3}_1$, we can see that the restriction of ρ_s to $\pi_1(S \setminus K)$ is invariant under conjugation by

$$P_{u} = \begin{pmatrix} \left(\frac{s^{2}-1+s^{-2}}{u}\right)^{1/2} & \frac{\left(\frac{s^{2}-1+s^{-2}}{u}\right)^{1/2}-\left(\frac{s^{2}-1+s^{-2}}{u}\right)^{-1/2}}{s-s^{-1}}\\ 0 & \left(\frac{s^{2}-1+s^{-2}}{u}\right)^{-1/2} \end{pmatrix}$$

where $u \in \mathbb{C}^{\times}$ (see Lemma 4.1). Therefore, one obtains representations $\rho_{s,u} \colon \pi_1(E(K)) \to SL(2, \mathbb{C})$ by

$$\rho_{s,u}(\mu_j) = \begin{cases} \rho_s(\mu_j) & \text{if } j = 3, \\ P_u \rho_s(\mu_j) P_u^{-1} & \text{if } j = 1, 2, 4, 5. \end{cases}$$

By the above bending construction, the set $C = \{\rho_{s,u}\}$ is still a 2-parameter family in X(E(K)). Now one can also check it directly

$$\tau_{\rho_{s,u}}(E(K)) = \frac{144s^{-4}(s-1)^8}{-s^{-1}(s-1)^2} = -144(\operatorname{tr} \rho_{s,u}(\mu_2) - 2)^3,$$

and hence $\tau_{\rho_{s,\mu}}(E(K))$ depends only on tr $\rho_{s,\mu}(\mu_2) = s + s^{-1}$.

On the other hand, to be independent of u, it can be explained by the generalized multiplicativity of the Reidemeister torsion to the decomposition $E(K) = M_1 \cup_{S_0} M_2$ along the surface $S_0 = S \cap E(K)$ with 4 boundary components. Although M_1 , M_2 and S_0 are not acyclic, after fixing suitable bases of $H_1(M_1; \mathbb{C}^2_{\rho_{s,u}})$, $H_1(M_2; \mathbb{C}^2_{\rho_{s,u}})$ and $H_1(S_0; \mathbb{C}^2_{\rho_{s,u}})$, we obtain $\tau_{\rho_{s,u}}(E(K)) = \tau_{\rho_{s,u}}(M_1)\tau_{\rho_{s,u}}(M_2)/\tau_{\rho_{s,u}}(S_0)$. By the construction of $\rho_{s,u}$, we see that the value of the right-hand side is independent of u.

Let RT_C be the subset of $RT(\Sigma(K, K))$ consisting of $\tau_{\rho}(\Sigma(K, K))$'s where the restriction of ρ to each E(K) belongs to C. Then RT_C is a finite set even though $C \subset X(\Sigma(K, K))$ is 2-dimensional. Indeed, one can check that tr $\rho_{s,u}(\lambda) = s^{24} + s^{-24}$, and thus there are finitely many solutions (s_1, s_2) of tr $\rho_{s_1,u_1}(\mu) = \text{tr } \rho_{s_2,u_2}(\lambda)$ and tr $\rho_{s_1,u_1}(\lambda) = \text{tr } \rho_{s_2,u_2}(\mu)$. We conclude that there are finitely many possibilities of the value $\tau_{\rho}(\Sigma(K, K)) = \tau_{\rho_{s_1,u_1}}(E(K))\tau_{\rho_{s_2,u_2}}(E(K))$.

Problem 3.9. Can we relax the assumption "either dim C = 0, or dim C = 1 and its image under the map $X(E(K_i)) \rightarrow X(\partial E(K_i))$ is not a point" in Theorem 1.1?



FIGURE 3.1. The Montesinos knot K = M(1/3, 1/3, 1/3, 1/3, 1/2).

4. Computational observation

4.1. Concrete description of $X^{irr}(\Sigma(K_1, K_2))$

In this section we observe some examples of splices. We use Mathematica to compute matrices. Recall that $X^{irr}(M)$ is identified with $R^{irr}(M)/SL(2, \mathbb{C})$. The construction of deformations of a representation used in this section is called a *bending construction* or simply a *bending*. See [11, 17] as a reference.

Here we compute $X^{irr}(\Sigma(K, K))$ for the trefoil knot and the figure-eight knot *K* by using a presentation of a twist knot. Let J(2, 2q) be a twist knot where *q* is a nonzero integer. Please see [16] as a reference for twist knots.

A presentation of $\pi_1(E(J(2, 2q)))$ is given as

$$\pi_1(E(J(2,2q))) = \langle x, y \mid z^q x = y z^q \rangle, \ z = [y, x^{-1}]$$

We take a representation $\rho: \langle x, y \rangle \to SL(2, \mathbb{C})$ from the free group $\langle x, y \rangle$ in $SL(2, \mathbb{C})$ by the correspondence

$$\rho(x) = \begin{pmatrix} s & 1 \\ 0 & 1/s \end{pmatrix}, \ \rho(y) = \begin{pmatrix} s & 0 \\ -t & 1/s \end{pmatrix} \quad (s, t \in \mathbb{C}^{\times}).$$

We use a small letter for a group element and its capital letter for the image of a small letter, like *X* for $\rho(x)$. For $\rho(z^q) = Z^q$, we put the matrix $Z^q = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}$.

We define the Riley polynomial to be $\phi_q(s, t) = z_{11} + (1/s - s)z_{12}$. It can be checked that the previous representation gives an irreducible representation of $\pi_1(E(J(2, 2q)))$ in $SL(2, \mathbb{C})$ if and only if (s, t) satisfies $\phi_q(s, t) = 0$.

It is seen that any $[\rho] \in X(E(J(2, 2q)))$ can be parametrized by

$$\xi = \operatorname{tr} \rho(x) = \operatorname{tr} \rho(y) = s + 1/s,$$

tr
$$\rho(xy) = s^2 + 1/s^2 - t = (s + 1/s)^2 - t - 2 = \xi^2 - t - 2$$

and then by ξ and t.

Here we take other words $\tilde{z} = [x, y^{-1}]$ and $\lambda = \tilde{z}^q z^q$. This *x* gives a meridian of J(2, 2q) and this λ does the corresponding longitude for *x*. Here λ is homologically trivial. Therefore $\langle x, \lambda \rangle$ is the free abelian group of rank 2 and $\rho(x) = X$ commutes with $\rho(\lambda) = L$. We can find another matrix which commutes with *X* and *L* by direct computations.

Lemma 4.1. Any matrix A which commutes with $X = \begin{pmatrix} s & c^2 \\ 0 & 1/s \end{pmatrix}$ ($s \neq \pm 1$, $c \neq 0$) has a form of

$$A = \begin{pmatrix} a & \frac{a-1/a}{s-1/s}c^2\\ 0 & 1/a \end{pmatrix}$$

for some $a \in \mathbb{C}^{\times}$.

Now we consider two copies K_1, K_2 of J(2, 2q) and

$$\pi_1(E(K_1)) = \langle x_1, y_1 | z_1^q x_1 = y_1 z_1^q \rangle, \ z_1 = [y_1, x_1^{-1}],$$

$$\pi_1(E(K_2)) = \langle x_2, y_2 | z_2^q x_2 = y_2 z_2^q \rangle, \ z_2 = [y_2, x_2^{-1}].$$

Further

$$\begin{aligned} \pi_1(\Sigma(K_1, K_2)) &= \pi_1(E(K_1)) *_{\pi_1(T^2)} \pi_1(E(K_2)) \\ &= \left\langle x_1, y_1, x_2, y_2 \mid z_1^q x_1 = y_1 z_1^q, \ z_2^q x_2 = y_2 z_2^q, \ x_1 = \lambda_2, \ \lambda_1 = x_2 \right\rangle. \end{aligned}$$

We consider an irreducible representation $\rho: \pi_1(\Sigma(K_1, K_2)) \to SL(2, \mathbb{C})$. Up to conjugate, we can set that

$$X_1 = \rho(x_1) = \begin{pmatrix} s_1 & 1 \\ 0 & 1/s_1 \end{pmatrix}, \ Y_1 = \rho(y_1) = \begin{pmatrix} s_1 & 0 \\ -t_1 & 1/s_1 \end{pmatrix}.$$

First note that we treat cases of $s_1 \neq \pm 1$. Further we may assume that X_2 is conjugate to $\begin{pmatrix} s_2 & 1 \\ 0 & 1/s_2 \end{pmatrix}$, and Y_2 to $\begin{pmatrix} s_2 & 0 \\ -t_2 & 1/s_2 \end{pmatrix}$ simultaneously, as

$$X_2 = H\begin{pmatrix} s_2 & 1\\ 0 & 1/s_2 \end{pmatrix} H^{-1}, \ Y_2 = H\begin{pmatrix} s_2 & 0\\ -t_2 & 1/s_2 \end{pmatrix} H^{-1}$$

for some $H \in SL(2, \mathbb{C})$.

Here we require the conditions

$$X_1 = L_2, \ L_1 = X_2$$

to get a representation on $\pi_1(\Sigma(K_1, K_2))$. It can be seen that L_1 is an upper triangular matrix and then X_2 is also an upper triangular matrix. By taking

$$H = \begin{pmatrix} c & 0\\ 0 & 1/c \end{pmatrix} \ (c \neq 0),$$

one has

$$X_{2} = H \begin{pmatrix} s_{2} & 1 \\ 0 & 1/s_{2} \end{pmatrix} H^{-1} = \begin{pmatrix} s_{2} & c^{2} \\ 0 & 1/s_{2} \end{pmatrix},$$

$$Y_{2} = H \begin{pmatrix} s_{2} & 0 \\ -t_{2} & 1/s_{2} \end{pmatrix} H^{-1} = \begin{pmatrix} s_{2} & 0 \\ -t_{2}/c^{2} & 1/s_{2} \end{pmatrix}$$

Here L_2 is also an upper triangular matrix and $L_2 = X_1$. Now any $[\rho] = [\rho_1 * \rho_2] \in X(\Sigma(K_1, K_2))$ is corresponding to $(X_1, Y_1, X_2, Y_2) = (X_1, Y_1, L_1, Y_2)$ of the above forms. For $a \in \mathbb{C}^{\times}$ we define A_a by

$$A_a = \begin{pmatrix} a & \frac{a-1/a}{s_1-1/s_1} \\ 0 & 1/a \end{pmatrix}$$

and now consider deformations $[\rho_a] = [(A_a \rho_1 A_a^{-1}) * \rho_2]$ of $[\rho] = [\rho_1 * \rho_2]$ as

$$(A_a X_1 A_a^{-1}, A_a Y_1 A_a^{-1}, X_2, Y_2) = (X_1, A_a Y_1 A_a^{-1}, X_2, Y_2).$$

Lemma 4.2. It holds that $A_a L_1 A_a^{-1} = L_1$.

Proof. We prove $A_a L_1 A_a^{-1} = L_1$. We may assume $s \neq \pm 1$. Here we take eigenvectors $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{u}_2 \in \mathbb{C}^2$ for X_1 such that $X_1 \mathbf{u}_1 = s_1 \mathbf{u}_1, X_1 \mathbf{u}_2 = s_1^{-1} \mathbf{u}_2$. Since $X_1 L_1 = L_1 X_1$, one has

$$X_1 L_1 \mathbf{u}_2 = L_1 X_1 \mathbf{u}_2$$

= $L_1 s_1^{-1} \mathbf{u}_2$
= $s_1^{-1} L_1 \mathbf{u}_2$.

Hence there exists a nonzero constant γ such that $L_1 \mathbf{u}_2 = \gamma \mathbf{u}_2$. This means that L_1 has also $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{C}^2$ as eigenvectors. By similar arguments for $A_a X_1 = X_1 A_a$, one sees A_a has $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{C}^2$ as eigenvectors. Therefore it is seen that X_1, L_1, A_a are simultaneously diagonalizable and in particular $A_a L_1 A_a^{-1} = L_1$.

By the above lemma, one can see $A_a \rho_1 A_a^{-1} = \rho_1$ on the subgroup $\pi_1(T^2)$ generated by $\{x_1, l_1\} = \{x_2, l_2\}$ and then $\rho_a = (A_a \rho_1 A_a^{-1}) * \rho_2$ gives an irreducible representation of $\pi_1(\Sigma(K_1, K_2))$.

Further if $a \neq 1$, then $A_a Y_1 A_a^{-1} \neq Y_1$. This implies $\rho_a \neq \rho \in R(\Sigma(K_1, K_2))$. It can be seen by the following computations. First one sees that

$$tr(\rho_1 * \rho_2(y_1 x_2)) = tr(Y_1 X_2)$$
$$= s_1 s_2 + \frac{1}{s_1 s_2} - c^2 t_1$$

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On the other hand, one sees that

$$\operatorname{tr}\left((A_a\rho_1 A_a^{-1}) * \rho_2)(y_1 x_2)\right) = \operatorname{tr}(A_a Y_1 A_a^{-1} X_2)$$
$$= s_1 s_2 + \frac{1}{s_1 s_2} + \left\{\frac{(s_2 - \frac{1}{s_2})}{(s_1 - \frac{1}{s_1})}\left(\frac{1}{a^2} - 1\right) - \frac{c^2}{a^2}\right\} t_1.$$

Therefore we can find one character

$$[\rho] \mapsto \operatorname{tr} \rho(y_1 x_2)$$

which is not constant on $X(\Sigma(K_1, K_2))$ and we know $X(\Sigma(K_1, K_2))$ has at least one dimension near $[\rho]$.

Proposition 4.3. $X(\Sigma(K_1, K_2))$ has just one dimension near $[\rho]$.

Proof. Take and fix any $[\rho] = [\rho_1 * \rho_2] \in X(\Sigma(K_1, K_2))$. It is seen that the character variety $X(\Sigma(K_1, K_2))$ has at least one dimension near $[\rho]$ by a bending construction.

Consider another one parameter family

$$\{[\rho_u]\}_u = \{[\rho_{1,u} * \rho_{2,u}]\}_u \subset X(\Sigma(K_1, K_2))$$

such that $[\rho_0] = [\rho]$. Here recall there exist only finitely many quadruples $\{(s_1, t_1, s_2, t_2)$'s} for this fixed $[\rho] = [\rho_1 * \rho_2] \in X(\Sigma(K_1, K_2))$ by the proof of the main theorem. Then we may assume that $[\rho_u] = [\rho_{1,u} * \rho_2]$ and $[\rho_{1,u}] = [\rho_1] \in X(K_1)$ for any *u*. Hence one gets $[\rho_u] = [\rho_{1,u} * \rho_2] = [(B\rho_1 B^{-1}) * \rho_2]$, where $B \in SL(2, \mathbb{C})$. Because *B* must commute with X_1 and L_1 , then *B* has a similar form as A_a in Lemma 4.1. Therefore this is a bending construction and the dimension of deformations is one.

4.2. q = 1 Case

Here we put q = 1. This J(2, 2) is the trefoil knot. We write again

$$X = \begin{pmatrix} s & 1 \\ 0 & 1/s \end{pmatrix}, \ Y = \begin{pmatrix} s & 0 \\ -t & 1/s \end{pmatrix}$$

and

$$Z = [Y, X^{-1}] = \begin{pmatrix} 1 - s^2t & \frac{1}{s} - s(1+t) \\ -\frac{t}{s} + st(1+t) & 1 + (2 - \frac{1}{s^2})t + t^2 \end{pmatrix},$$

$$\widetilde{Z} = [X, Y^{-1}] = \begin{pmatrix} 1 - (-2 + s^2)t + t^2 & \frac{-1 + s^2 - t}{s} \\ \frac{t(1 - s^2 + t)}{s} & 1 - \frac{t}{s^2} \end{pmatrix},$$

$$ZX - YZ = \begin{pmatrix} 0 & -1 + 1/s^2 + s^2 - t \\ s(-1 + 1/s^2 + s^2 - t) & 0 \end{pmatrix}$$

The condition that (s, t) gives a representation is $-1 + 1/s^2 + s^2 - t = 0$. On the other hand,

$$\phi_1(s,t) = w_{11} + (1/s - s)w_{12}$$

= 1 - s²t + (1/s - s)(1/s - s(1 + t))
= 1 - s²t + 1/s² - 1 - t - 1 + s² + s²t
= -1 + 1/s² + s² - t
= \xi² - 3 - t,

where m = s + 1/s.

Hence in the case of the trefoil knot, one sees

$$t = \xi^2 - 3$$

and $X^{irr}(E(J(2, 2)))$ is given by

$$\{(\xi, t) \in \mathbb{C}^2 \mid t = \xi^2 - 3, \ t \neq 0\}.$$

If t = 0, then the corresponding representation is not irreducible.

Remark 4.4. If s = 1, that is, $\xi = 2$, then the chain complex is not acyclic. In the other cases, $\tau_{\rho}(E(J(2, 2))) = 2$.

Compute

$$L = \widetilde{Z}Z$$

$$= \begin{pmatrix} 1 - t^{2} + s^{4}t^{2} - t^{3} + \frac{t(1+t)}{s^{2}} - s^{2}t(1+t+t^{2}) & \frac{(1+s^{2})t(1+t+s^{4}(1+t)-s^{2}(3+3t+t^{2}))}{s^{3}} \\ \frac{t^{2}(1+s^{6}-s^{2}t-s^{4}t)}{s^{3}} & 1 - t^{2} + \frac{t^{2}}{s^{4}} - t^{3} + s^{2}t(1+t) - \frac{t(1+t+t^{2})}{s^{2}} \end{pmatrix}.$$

By putting $t = 1 - (1/s^2 + s^2)$, one obtains

$$L = \begin{pmatrix} -s^6 & -\frac{(1+s^2+s^4)(1+s^6)}{s^5} \\ 0 & -1/s^6 \end{pmatrix}$$

and

$$tr(L) = -s^6 - 1/s^6 = -T_6(m).$$

Here $T_6(x) = x^6 - 6x^4 + 9x^2 - 2$ is the normalized Chebyshev polynomial of degree 6. Remark that $T_6(x)$ has the property $T_6(2\cos\theta) = 2\cos 6\theta$.

By relations $x_1 = \lambda_2$, $\lambda_1 = x_2$, one has

$$tr(X_1) = tr(L_2), tr(L_1) = tr(X_2).$$

By putting $\xi_1 = s_1 + 1/s_1$, $\xi_2 = s_2 + 1/s_2$, one obtains

$$\xi_1 = -T_6(\xi_2), \ -T_6(\xi_1) = \xi_2.$$

Hence we obtain only one equation

$$\xi = -T_6(-T_6(\xi)) = -T_6(T_6(\xi)).$$

This equation $\xi + T_6(T_6(\xi)) = 0$ is a polynomial equation of degree 36 with distinct 36 roots as follows:

$$-2 = 2\cos\pi, \ 2\cos\frac{k\pi}{35} \ (k = 1, 3, \dots, 33), \ 2\cos\frac{k\pi}{37} \ (k = 1, 3, \dots, 35).$$

It is seen that they are the roots as

$$T_{6}(T_{6}(-2)) = T_{6}(T_{6}(2\cos\pi))$$

= 2 cos 36\pi = 2,
$$T_{6}\left(T_{6}\left(2\cos\frac{k\pi}{35}\right)\right) = 2\cos\frac{36k\pi}{35} = -2\cos\frac{k\pi}{35},$$

$$T_{6}\left(T_{6}\left(2\cos\frac{k\pi}{37}\right)\right) = 2\cos\frac{36k\pi}{37} = -2\cos\frac{k\pi}{37}.$$

Further one easily sees that $\xi = -2$ does not give a representation on the splice. The roots $2 \cos \frac{k\pi}{35}(k = 1, 3, ..., 33)$ are corresponding to the condition $s_1^{36} = s_1$ coming from matrix equations $L_1 = X_2$ and $L_2 = X_1$.

It can be seen that there exists a k such that $tr(\rho(x_1)) = 2 \cos \frac{k\pi}{35}$ and $tr(\rho(x_2)) = -T_6(2 \cos \frac{k\pi}{35})$ for any $[\rho] \in X^{irr}(\Sigma(K_1, K_2))$. On the other hand, the roots $2 \cos \frac{k\pi}{37}(k = 1, 3, ..., 35)$ are corresponding to the condition $s_1^{36} = s_1^{-1}$ coming from equations $L_1 = X_2^{-1}$ and $L_2 = X_1$. They give representations of the splicing of 3_1 and its mirror image, not 3_1 .

Take $[\rho] = [\rho_1 * \rho_2] \in X^{irr}(\Sigma(K_1, K_2))$ and identify it with (X_1, Y_1, X_2, Y_2) . Consider

$$A_a = \begin{pmatrix} a & \frac{a-1/a}{s_1-1/s_1} \\ 0 & 1/a \end{pmatrix},$$

where $a \in \mathbb{C}^{\times}$, $s_1, s_2 \in \mathbb{C}^{\times}$ are satisfying $s_1 + 1/s_1 = \xi_1, s_2 + 1/s_2 = \xi_2$ and $\xi_1 = -T_6(T_6(\xi_1)), \xi_2 = -T_6(\xi_1)$. In this case, one gets

$$\operatorname{tr} \left((A_a \rho_1 A_a^{-1}) * (\rho_2)(y_1 x_2) \right)$$

= $\operatorname{tr}(X_2 A_a Y_1 A_a^{-1})$
= $s_1 s_2 + \frac{1}{s_1 s_2} - \frac{c^2}{a^2} (s_1^2 + 1/s_1^2 - 1) + \frac{(1 - a^2)(s_2 - 1/s_2)}{(s_1 - 1/s_1)} (s_1^2 + 1/s_1^2 - 1).$

Here *c* is determined by $X_2 = L_1$, namely

$$c^{2} = -\frac{(1+s_{1}^{2}+s_{1}^{4})(1+s_{1}^{4})}{s_{1}^{5}}.$$

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4.3. q = -1 Case

We put q = -1. This J(2, -2) is the figure-eight knot. In this case the Riley polynomial $\phi_{-1}(s, t)$ is given by

$$\phi_{-1}(s,t) = t^2 - (s^2 + 1/s^2 - 3)t - s^2 - 1/s^2 + 3$$
$$= t^2 - (\xi^2 - 5)t - \xi^2 + 5,$$

where $\xi = s + 1/s$.

Then the irreducible representation part of $X^{irr}(E(J(2, 4)))$ is

$$\{(\xi,t)\in\mathbb{C}^2\mid t^2-(\xi^2-5)t-\xi^2+5=0,\ t\neq 0\}.$$

Under the same notations, one obtains

$$\xi_1 = \xi_2^4 - 5\xi_2^2 + 2, \ \xi_1^4 - 5\xi_1^2 + 2 = \xi_2.$$

Hence we obtain only one equation

$$\xi = \xi^{16} - 20\xi^{14} + 158\xi^{12} - 620\xi^{10} + 1244\xi^8 - 1190\xi^6 + 487\xi^4 - 60\xi^2 - 2\xi^{10} + 1244\xi^8 - 1190\xi^6 + 487\xi^4 - 60\xi^2 - 2\xi^{10} + 1244\xi^8 - 1190\xi^6 + 487\xi^4 - 60\xi^2 - 2\xi^{10} + 1244\xi^8 - 1190\xi^6 + 487\xi^4 - 60\xi^2 - 2\xi^{10} + 1244\xi^8 - 1190\xi^6 + 487\xi^4 - 60\xi^2 - 2\xi^{10} + 1244\xi^8 - 1190\xi^6 + 487\xi^4 - 60\xi^2 - 2\xi^{10} + 1244\xi^8 - 1190\xi^6 + 487\xi^4 - 60\xi^2 - 2\xi^{10} + 1244\xi^8 - 1190\xi^6 + 487\xi^4 - 60\xi^2 - 2\xi^{10} + 1244\xi^8 - 1190\xi^6 + 487\xi^4 - 60\xi^2 - 2\xi^{10} + 1244\xi^8 - 1190\xi^6 + 487\xi^4 - 60\xi^2 - 2\xi^{10} + 1244\xi^8 - 1190\xi^6 + 487\xi^4 - 60\xi^2 - 2\xi^{10} + 1244\xi^8 - 1190\xi^6 + 487\xi^4 - 60\xi^2 - 2\xi^{10} + 1244\xi^8 - 1190\xi^6 + 487\xi^4 - 60\xi^2 - 2\xi^{10} + 1244\xi^8 - 1190\xi^6 + 487\xi^4 - 60\xi^2 - 2\xi^{10} + 124\xi^8 - 1190\xi^6 + 487\xi^8 - 1190\xi^6 + 487\xi^8 - 1190\xi^6 + 487\xi^8 - 1190\xi^6 + 487\xi^8 - 1190\xi^6 + 124\xi^8 - 1190\xi^6 + 124\xi^6 - 110\xi^6 + 124\xi^6 - 110\xi^6 + 120\xi^6 +$$

and 16 roots $v_0 = -2$, $v_i \neq \pm 1$ (i = 1, ..., 15). For any v_i ($i \neq 0$), we can take a bending construction to do deformations in $X^{irr}(\Sigma(J(2, -2), J(2, -2)))$.

Remark 4.5. In this case, *t* is a root of $t^2 - (v_i^2 - 5)t - v_i^2 + 5 = 0$.

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