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Abstract

We generalize the results in [3] giving a reduction algorithm allowing to compute the index of seaweed subalgebras of classical simple Lie algebras. We thus are able to obtain the index of some interesting families of seaweed subalgebras and to give new examples of large classes of Frobenius Lie algebras among them.

Indice des sous-algèbres biparaboliques d'une algèbre de Lie classique

Résumé

Nous généralisons les résultats de [3] en donnant un algorithme de réduction permettant de calculer l'indice des sous-algèbres biparaboliques d'une algèbre de Lie simple classique. Nous obtenons ainsi l'indice d'une famille intéressante des sous-algèbres biparaboliques et nous donnons de nouveaux exemples de grandes classes de sous-algèbres de Frobenius.

1. **Introduction**

Let $\mathfrak{g}$ be a Lie algebra of an algebraic complex Lie group G and $\mathfrak{g}^*$ the dual space. For $f \in \mathfrak{g}^*$, we denote by $\mathfrak{g}_f$ the stabilizer of f for the coadjoint action. Recall that the index of $\mathfrak{g}$ is the minimal dimension of stabilizers for the coadjoint action,

$$\operatorname{ind}(\mathfrak{g}) = \min\{\dim \mathfrak{g}_f \mid f \in \mathfrak{g}^*\}$$

Lie algebras with index zero are called Frobenius Lie algebra and are of special interest stemming from their connection to the classical Yang–Baxter equation (CYBE). In [1], Belavin and Drinfel'd showed that this family provides solutions to the CYBE.

Throughout the paper, all considered Lie groups and Lie algebras are algebraic defined over the complex field. For any pair of integers (r, s), we denote by $r[s]$ the remainder of Euclidean division of r by s and by $r \wedge s$ the greatest common divisor of r and s. For $\underline{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$, we set $|\underline{a}| := a_1 + \cdots + a_k$.

In [7], Dergachev and Kirillov introduced the notion of seaweed subalgebras in the case of $\mathfrak{gl}(n)$, where they exhibited a method for computing the index of such algebras. This notion was generalized for arbitrary reductive Lie algebras in [10]. In the case of classical Lie algebras, a seaweed subalgebra may be parametrized by two compositions of positives integers (see Section 2 for a precise description).

Keywords: Index of a Lie algebra, Frobenius Lie algebra, Seaweed Subalgebra, Meander graph.
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Let $n \in \mathbb{N}^\times$, $\underline{a} = (a_1, \ldots, a_k)$ and $\underline{b} = (b_1, \ldots, b_t)$ be two compositions verifying $|\underline{a}| \leq n$ and $|\underline{b}| \leq n$. We associate to the pair $(\underline{a}, \underline{b})$ a unique seaweed subalgebra of $\mathfrak{sp}(2n)$ (resp. $\mathfrak{so}(2n+1)$, $\mathfrak{so}(2n)$) which we denote by $\mathfrak{q}_n^C(\underline{a} \mid \underline{b})$ (resp. $\mathfrak{q}_n^B(\underline{a} \mid \underline{b})$, $\mathfrak{q}_n^D(\underline{a} \mid \underline{b})$), and all seaweed subalgebras of $\mathfrak{sp}(2n)$ (resp. $\mathfrak{so}(2n+1)$, $\mathfrak{so}(2n)$) are thus obtained up to conjugation by the connected adjoint group of $\mathfrak{sp}(2n)$ (resp. $\mathfrak{so}(2n+1)$, $\mathfrak{so}(2n)$). When $|\underline{a}| = |\underline{b}| = n$, a unique seaweed subalgebra of $\mathfrak{gl}(n)$ may also be associated to the pair $(\underline{a}, \underline{b})$ (up to conjugation by the connected adjoint group of $\mathfrak{gl}(n)$) which we denote by $\mathfrak{q}^A(\underline{a} \mid \underline{b})$ (see [2], [11] and [12]).

For $\underline{a} = (a_1, \ldots, a_k) \in \mathbb{N}^k$ and $\underline{b} = (b_1, \ldots, b_t) \in \mathbb{N}^t$ such that $|\underline{a}| \leq n$ and $|\underline{b}| \leq n$, we denote by $\widetilde{\underline{a}}$ (resp. $\widetilde{\underline{b}}$) the sequence obtained from $\underline{a}$ (resp. $\underline{b}$) by removing null terms and we put $\mathfrak{q}^A(\underline{a} \mid \underline{b}) = \mathfrak{q}^A(\widetilde{\underline{a}} \mid \widetilde{\underline{b}})$ if $|\underline{a}| = |\underline{b}| = n$ and $\mathfrak{q}_n^I(\underline{a} \mid \underline{b}) = \mathfrak{q}_n^I(\widetilde{\underline{a}} \mid \widetilde{\underline{b}})$, $I = B,\ C \ or\ D$.

For $n > 1$, let Ξ_n be the set of pairs of compositions $(\underline{a} = (a_1, \ldots, a_k), \underline{b} = (b_1, \ldots, b_t))$ which verify: $|\underline{a}| = n$, $|\underline{b}| = n-1$ and $a_k > 1$ or $|\underline{b}| = n$, $|\underline{a}| = n-1$ and $b_t > 1$.

In [7], Dergachev and Kirillov associated to each seaweed subalgebra $\mathfrak{q}^A(\underline{a} \mid \underline{b})$ of $\mathfrak{gl}(n)$ a graph, called the meander of $\mathfrak{q}^A(\underline{a} \mid \underline{b})$ and denoted by $\Gamma^A(\underline{a} \mid \underline{b})$, it is constructed in the following way: we place n consecutive points on a horizontal line, called vertices of $\Gamma^A(\underline{a} \mid \underline{b})$ and numbered from 1 up to n. Next, we connect by an arc, below (resp. above) this line, each pair of distinct vertices of the form $(a_1+\cdots+a_{i-1}+j, a_1+\cdots+a_i-j+1)$, $1 \leq j \leq a_i$, $1 \leq i \leq k$ (resp. $(b_1+\cdots+b_{i-1}+j, b_1+\cdots+b_i-j+1)$, $1 \leq j \leq b_i$, $1 \leq i \leq t$). The authors gave a formula for the index of $\mathfrak{q}^A(\underline{a} \mid \underline{b})$ in terms of the connected components of this graph. This result was generalised to the case of $\mathfrak{sp}(2n)$ in two different ways in [6] and [11], and to the case of $\mathfrak{so}(n)$ in [11] and [12], where the authors associated to each subalgebra $\mathfrak{q}_n^I(\underline{a} \mid \underline{b})$ a meander denoted by $\Gamma_n^I(\underline{a} \mid \underline{b})$, $I = B,\ C$ or D. When $I = B,\ C$ or $I = D$ and $(\underline{a} \mid \underline{b}) \notin \Xi_n$, the meander $\Gamma_n^I(\underline{a} \mid \underline{b})$ verifies $\Gamma_n^I(\underline{a} \mid \underline{b}) = \Gamma^A(a_1, \ldots, a_k, 2(n-|\underline{a}|), a_k, \ldots, a_1 \mid b_1, \ldots, b_t, 2(n-|\underline{b}|), b_t, \ldots, b_1)$. When $I = D$ and $(\underline{a} \mid \underline{b}) \in \Xi_n$, the meander $\Gamma_n^D(\underline{a} \mid \underline{b})$, the construction of which we recall in Section 4, has two arcs crossing each other ([11] and [12]).

In certain particular cases, algebraic formulas for the index of seaweed subalgebras have been obtained. The first one was given by Elashvili in [9], where he showed that $\operatorname{ind} \mathfrak{q}^A(a, b \mid a+b) = a \wedge b$, for any $(a,b) \in (\mathbb{N}^\times)^2$. In [5], the authors showed that $\operatorname{ind} \mathfrak{q}^A(a, b, c \mid a+b+c) = (a+b) \wedge (b+c)$ for any $(a,b,c) \in (\mathbb{N}^\times)^3$. In [3], we proved the two previous formulas in a different manner and we generalised Elashvili's result by giving an algebraic formula for the index of seaweed subalgebras of the form $\mathfrak{q}^A(\underbrace{a, \ldots, a}_{m}, b \mid ma+b)$ for any $(a,b,m) \in (\mathbb{N}^\times)^3$ (see Section 4). For the general case, the index problem seems to be hard. In particular, the classification of Frobenius seaweed subalgebras remains an open question.

In [6], the authors gave a formula for the index of seaweed subalgebras $\mathfrak{q}_n^C(a, b \mid c)$ when $|a+b-c| = 1$ or 2, allowing them to determine the family of Frobenius subalgebras which are of the form $\mathfrak{q}_n^C(a, b \mid c)$, $(a, b, c) \in (\mathbb{N}^\times)^3$. In this paper, we give a formula for the index of seaweed subalgebras $\mathfrak{q}_n^C(a, b \mid c)$ (resp. $\mathfrak{q}_n^B(a, b \mid c)$, $\mathfrak{q}_n^D(a, b \mid c)$), $(a, b, c) \in (\mathbb{N}^\times)^3$ (resp. $(a, b, c) \in (\mathbb{N}^\times)^3$, $(a, b, c) \in (\mathbb{N}^\times)^3$ and $b > 1$ if $a + b = n$) (see Theorems 3.22 and 4.12). More precisely, we show the following theorem:

Theorem 1.1. *Let $a, b, c, n \in \mathbb{N}^\times$ be such that $s := \max(a + b, c) \leq n$. Set $p = (a + b) \wedge (b + c)$ and $r = |a + b - c|$, then*

(1) (a) *If $p > r$, we have* $\operatorname{ind} \mathfrak{q}_n^B(a, b \mid c) = \operatorname{ind} \mathfrak{q}_n^C(a, b \mid c) = p - [\frac{r+1}{2}] + n - s$

(b) *If $p \leq r$, we have*

$$\begin{aligned}\operatorname{ind} \mathfrak{q}_n^B(a, b \mid c) &= \operatorname{ind} \mathfrak{q}_n^C(a, b \mid c)\\ &= \begin{cases} [\frac{r}{2}] + n - s & \textit{if } p \textit{ and } r \textit{ have the same parity} \\ [\frac{r}{2}] - 1 + n - s & \textit{otherwise} \end{cases}\end{aligned}$$

(2) *Let $\Gamma_n^D(a, b \mid c)$ be the meander of $\mathfrak{q}_n^D(a, b \mid c)$. Then*

(a) *If $((a, b), c) \notin \Xi_n$, we have* $\operatorname{ind} \mathfrak{q}_n^D(a, b \mid c) = \operatorname{ind} \mathfrak{q}_n^C(a, b \mid c) + \epsilon$*, where ϵ is given by:*

$$\epsilon = \begin{cases} 0 & \textit{if } r \textit{ is even} \\ 1 & \textit{if } r \textit{ is odd, } s = n \textit{ and the vertices } n \textit{ and } n+1 \textit{ belong to the same} \\ & \textit{segment of } \Gamma_n^D(a, b \mid c) \\ -1 & \textit{otherwise} \end{cases}$$

(b) *If $((a, b), c) \in \Xi_n$, we have* $\operatorname{ind} \mathfrak{q}_n^D(a, b \mid c) = |(a \wedge n) - 2|$

Consequently, we classify the Frobenius subalgebras of this family (see Corollaries 3.23 and 4.13).

For a seaweed subalgebra $\mathfrak{q}^A(\underline{a} \mid \underline{b})$ of $\mathfrak{gl}(n)$, we denote

$$\Psi[\mathfrak{q}^A(\underline{a} \mid \underline{b})] = \begin{cases} \operatorname{ind} \mathfrak{q}^A(\underline{a} \mid \underline{b}) & \text{if the vertex } n \text{ belongs to a segment of } \Gamma^A(\underline{a} \mid \underline{b}) \\ \operatorname{ind} \mathfrak{q}^A(\underline{a} \mid \underline{b}) - 2 & \text{otherwise} \end{cases}$$

Note that the index of seaweed subalgebras can be computed by the inductive formula obtained by Panyushev in [10]. In [3] we gave another formula allowing to compute the index of seaweed subalgebras $\mathfrak{q}^A(\underline{a}, \underline{b})$ in the case of $\mathfrak{gl}(n)$. In the present work, we generalise this result to the cases of seaweed subalgebras $\mathfrak{q}_n^C(\underline{a}, \underline{b})$, $\mathfrak{q}_n^B(\underline{a}, \underline{b})$ and $\mathfrak{q}_n^D(\underline{a}, \underline{b})$ (see Theorems 3.12, 4.6 and 4.9). More precisely, we first show that we may reduce to case $\underline{a} = (t)$ and $|\underline{b}| \leq t \leq n$, $t \in \mathbb{N}^\times$. Next, we give the following two theorems:

Theorem 1.2. *Let $t \in \mathbb{N}^{\times}$ and $\underline{a} = (a_1, \ldots, a_k)$ be a composition verifying $|\underline{a}| \leq t \leq n$. Let $a_{k+1} = t - |\underline{a}|$, $d_i = (a_1 + \ldots a_{i-1}) - (a_{i+1} + \cdots + a_{k+1})$, $1 \leq i \leq k$ and $\mathfrak{q}_n(t \mid \underline{a}) := \mathfrak{q}_n^B(t \mid \underline{a})$, $\mathfrak{q}_n^C(t \mid \underline{a})$ or $\mathfrak{q}_n(t \mid \underline{a}) := \mathfrak{q}_n^D(t \mid \underline{a})$ if $(t \mid \underline{a}) \notin \Xi_n$.*

(1) *For any $1 \leq i \leq k$ such that $d_i \neq 0$ and any $\alpha \in \mathbb{Z}$ such that $a_i + \alpha|d_i| \geq 0$, we have*

$$\operatorname{ind} \mathfrak{q}_n(t \mid \underline{a}) = \operatorname{ind} \mathfrak{q}_{n+\alpha|d_i|}(t + \alpha|d_i| \mid a_1, \ldots, a_{i-1}, a_i + \alpha|d_i|, a_{i+1}, \ldots, a_k)$$

In particular, we have

$$\operatorname{ind} \mathfrak{q}_n(t \mid \underline{a}) = \operatorname{ind} \mathfrak{q}_{n-a_i+a_i[|d_i|]}(t - a_i + a_i[|d_i|] \mid a_1, \ldots, a_{i-1}, a_i[|d_i|], a_{i+1}, \ldots, a_k)$$

(2) *For any $1 \leq i \leq k$ such that $d_i = 0$, we have*

$$\operatorname{ind} \mathfrak{q}_n(t \mid \underline{a}) = a_i + \operatorname{ind} \mathfrak{q}_{n-a_i}(t - a_i \mid a_1, \ldots, a_{i-1}, a_{i+1} \ldots, a_k)$$

Theorem 1.3. *Let $\underline{a} = (a_1, \ldots, a_k)$ be a composition verifying $1 \leq |\underline{a}| = n - 1$ $(i.e.\ (n \mid \underline{a}) \in \Xi_n)$. Let $\underline{a}' = (a'_1, \ldots, a'_k) = (a_1, \ldots, a_{k-1}, a_k+1)$, $d_k = -(a'_1 + \cdots + a'_{k-1})$ and $d_i = (a'_1 + \cdots + a'_{i-1}) - (a'_{i+1} + \cdots + a'_k)$, $1 \leq i \leq k - 1$.*

(1) *For any $1 \leq i \leq k$ such that $d_i \neq 0$ and any $\alpha \in \mathbb{Z}$ such that $a'_i + \alpha|d_i| \geq 0$, we have*

$$\operatorname{ind} \mathfrak{q}_n^D(n \mid a_1, \ldots, a_k) = \Psi[\mathfrak{q}^A(n + \alpha|d_i| \mid a'_1, \ldots, a'_{i-1}, a'_i + \alpha|d_i|, a'_{i+1}, \ldots, a'_k)]$$

In particular, if $t_i = a'_i - a'_i[|d_i|]$, we have

$$\operatorname{ind} \mathfrak{q}_n^D(n \mid a_1, \ldots, a_k) = \Psi[\mathfrak{q}^A(n - t_i \mid a'_1, \ldots, a'_{i-1}, a'_i[|d_i|], a'_{i+1} \ldots, a'_k)]$$

(2) *For any $1 \leq i \leq k$ such that $d_i = 0$, we have*

$$\operatorname{ind} \mathfrak{q}_n^D(n \mid a_1, \ldots, a_k) = a_i + \Psi[\mathfrak{q}^A(n - a_i \mid a_1, \ldots, a_{i-1}, a_{i+1} \ldots, a_k)]$$

As a consequence of these two theorems, we give new families of Frobenius seaweed subalgebras of $\mathfrak{sp}(2n)$ and $\mathfrak{so}(n)$ (see Lemma 3.16 and Theorem 4.15). Finally, we describe a relationship between Frobenius seaweed subalgebras of $\mathfrak{sl}(n)$ and Frobenius seaweed subalgebras of $\mathfrak{so}(2n)$ of the form $\mathfrak{q}_n^D(\underline{a} \mid \underline{b})$ where $(\underline{a} \mid \underline{b}) \in \Xi_n$. Consequently, we deduce that for any $n \geq 1$ and for any pair $(\underline{a} \mid \underline{b}) \in \Xi_{2n+1}$, $\mathfrak{q}_{2n+1}^D(\underline{a} \mid \underline{b})$ cannot be a Frobenius subalgebra. Also, we prove that the number of Frobenius seaweed subalgebras $\mathfrak{q}_{2n}^D(\underline{a} \mid \underline{b})$, where $(\underline{a} \mid \underline{b}) \in \Xi_{2n}$, is exactly double the number of Frobenius seaweed subalgebras of $\mathfrak{sl}(n)$ (see Theorem 4.14).

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2. Generalities on seaweed subalgebras

Let $\boldsymbol{G}$ be an algebraic Lie group, $\mathfrak{g}$ its Lie algebra and $\mathfrak{g}^*$ the dual space. Via the coadjoint representation, $\mathfrak{g}$ and $\boldsymbol{G}$ act on $\mathfrak{g}^*$ in the following way:

$$(x.f)(y) = f([y,x]), \quad x, y \in \mathfrak{g} \text{ and } f \in \mathfrak{g}^*$$
$$(x.f)(y) = f(\operatorname{Ad} x^{-1} y), \quad x \in \boldsymbol{G}, y \in \mathfrak{g} \text{ and } f \in \mathfrak{g}^*$$

For $f \in \mathfrak{g}^*$, let $\boldsymbol{G}_f$ be the stabilizer of f under this action and $\mathfrak{g}_f$ its Lie algebra:

$$\boldsymbol{G}_f = \{x \in \boldsymbol{G} \; ; \; f(\operatorname{Ad} x^{-1} y) = f(y), \; y \in \mathfrak{g}\}$$
$$\mathfrak{g}_f = \{x \in \mathfrak{g} \; ; \; f([x,y]) = 0, \; y \in \mathfrak{g}\}$$

We call the index of $\mathfrak{g}$, denoted by $\operatorname{ind} \mathfrak{g}$, the integer defined by:

$$\operatorname{ind} \mathfrak{g} = \min\{\dim \mathfrak{g}_f \; , \; f \in \mathfrak{g}^*\}$$

When $\operatorname{ind} \mathfrak{g} = 0$, $\mathfrak{g}$ is called Frobenius Lie algebra.

Suppose that $\mathfrak{g}$ is a semisimple Lie algebra. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, $\Delta \subset \mathfrak{h}^*$ the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$, $\pi := \{\alpha_1, \dots, \alpha_n\}$ a set of simple roots numbered, when $\mathfrak{g}$ is simple, in accordance with Bourbaki [4]. Then $\Delta = \Delta^+ \cup \Delta^-$ where Δ^+ is the set of positive roots relative to π and $\Delta^- = -\Delta^+$. For $\alpha \in \Delta$, let $\mathfrak{g}_\alpha := \{x \in \mathfrak{g}; \; [h,x] = \alpha(h)x, \; h \in \mathfrak{h}\}$, it is a 1-dimensional vector space.

For any subset $\pi' \subset \pi$, let $\Delta^+_{\pi'} = \Delta^+ \cap \mathbb{N}\pi'$ where $\mathbb{N}\pi'$ denote the set of linear combinations with coefficients in $\mathbb{N}$ of the elements of π', $\Delta^-_{\pi'} = -\Delta^+_{\pi'}$ and $\mathfrak{n}^\pm_{\pi'} = \bigoplus_{\alpha \in \Delta^\pm_{\pi'}} \mathfrak{g}_\alpha$.

Definitions 2.1. *Let (π', π'') a pair of subsets of π, the subalgebra $\mathfrak{q}_{\pi',\pi''} := \mathfrak{n}^+_{\pi'} \oplus \mathfrak{h} \oplus \mathfrak{n}^-_{\pi''}$ is called a standard seaweed subalgebra of $\mathfrak{g}$.*

We call seaweed subalgebra of $\mathfrak{g}$ any subalgebra $\boldsymbol{G}$-conjugate to a standard seaweed subalgebra of $\mathfrak{g}$.

If $\pi' = \pi$ or $\pi'' = \pi$, any subalgebra $\boldsymbol{G}$-conjugate to $\mathfrak{q}_{\pi',\pi''}$ is called a parabolic subalgebra of $\mathfrak{g}$.

Until now, we suppose $\mathfrak{g}$ simple. Let $\pi' \subset \pi$. We set $\mathcal{S}_{\pi'} := (i_1, i_2 - i_1, \dots, i_k - i_{k-1}, n+1-i_k)$, $\mathcal{T}_{\pi'} := (i_1, i_2 - i_1, \dots, i_k - i_{k-1})$, if $\pi' = \{\alpha_{i_1}, \dots, \alpha_{i_k}\}$ and $\mathcal{S}_\emptyset := (n+1)$, $\mathcal{T}_\emptyset := \emptyset$. So, $\mathcal{S}_{\pi'}$ is a composition of $n+1$ and $\mathcal{T}_{\pi'}$ is a composition of an integer $t \leq n$.

Suppose $\mathfrak{g} = \mathfrak{sl}(n+1)$, we associate to each pair $(\underline{a}, \underline{b})$ of compositions of $n+1$ the seaweed subalgebra $\mathfrak{q}_s^A(\underline{a} \mid \underline{b}) := \mathfrak{q}_{\pi',\pi''}$ such that $\underline{a} = \mathcal{S}_{\pi\setminus\pi'}$ and $\underline{b} = \mathcal{S}_{\pi\setminus\pi''}$. The subalgebra $\mathfrak{q}^A(\underline{a} \mid \underline{b}) := \mathfrak{q}_s^A(\underline{a} \mid \underline{b}) \oplus \mathbb{C}I_{n+1}$, where I_{n+1} is the identity matrix of order $n+1$, is a seaweed subalgebra of $\mathfrak{gl}(n+1)$ verifiying $\operatorname{ind}\mathfrak{q}^A(\underline{a} \mid \underline{b}) = \operatorname{ind}\mathfrak{q}_s^A(\underline{a} \mid \underline{b}) + 1$. All seaweed subalgebras of $\mathfrak{gl}(n+1)$ are thus obtained (up to conjugation by the connected adjoint group of $\mathfrak{gl}(n+1)$). The subalgebra $\mathfrak{q}^A(\underline{a} \mid \underline{b})$ is a parabolic subalgebra if and only if $\underline{a} = (n+1)$ or $\underline{b} = (n+1)$. Let $(e_1, \ldots, e_n)$ be the canonical basis of $\mathbb{C}^n$, $\mathscr{V} = \{V_0 = \{0\} \subsetneq V_1 \subsetneq \cdots \subsetneq V_m = \mathbb{C}^n\}$ and $\mathscr{W} = \{\mathbb{C}^n = W_0 \supsetneq W_1 \supsetneq \cdots \supsetneq W_t = \{0\}\}$ the two flags in $\mathbb{C}^n$ such that $V_i = \langle e_1, \ldots, e_{a_1+\cdots+a_i}\rangle$, $1 \le i \le m$, and $W_i = \langle e_{b_1+\cdots+b_i+1}, \ldots, e_n\rangle$, $1 \le i \le t-1$. Then $\mathfrak{q}^A(\underline{a}|\underline{b})$ is the stabilizer of the pair of flags $(\mathscr{V}, \mathscr{W})$ in $\mathfrak{gl}(n)$.

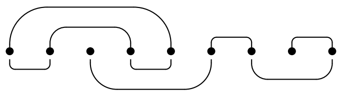
Suppose $\mathfrak{g} = \mathfrak{sp}(2n)$ (resp. $\mathfrak{so}(2n+1), \mathfrak{so}(2n)$), we associate to any pair of compositions $(\underline{a}, \underline{b})$ verifiying $|\underline{a}| \le n$ and $|\underline{b}| \le n$, the seaweed subalgebra $\mathfrak{q}_n^I(\underline{a} \mid \underline{b}) := \mathfrak{q}_{\pi',\pi''},\ I = C$ (resp. $B,\ D$) of $\mathfrak{sp}(2n)$ (resp. $\mathfrak{so}(2n+1), \mathfrak{so}(2n)$) such that $\underline{a} = \mathcal{T}_{\pi\setminus\pi'}$ and $\underline{b} = \mathcal{T}_{\pi\setminus\pi''}$. Up to conjugation by the connected adjoint group of $\mathfrak{sp}(2n)$ (resp. $\mathfrak{so}(2n+1), \mathfrak{so}(2n)$), all seaweed subalgebras of $\mathfrak{sp}(2n)$ (resp. $\mathfrak{so}(2n+1), \mathfrak{so}(2n)$) are thus obtained. The subalgebra $\mathfrak{q}_n^I(\underline{a} \mid \underline{b}),\ I = C$ (resp. $B,\ D$) is a parabolic subalgebra of $\mathfrak{sp}(2n)$ (resp. $\mathfrak{so}(2n+1), \mathfrak{so}(2n)$) if and only if $\underline{a} = \emptyset$ or $\underline{b} = \emptyset$. Let again $(e_1, \ldots, e_n)$ be the canonical basis of $\mathbb{C}^n$. In the case where $\mathfrak{g} = \mathfrak{sp}(n)$ and n is even (resp. $\mathfrak{g} = \mathfrak{so}(n)$ and n is odd), we endow $\mathbb{C}^n$ with the antisymmetric (resp. symmetric) bilineair form $\langle\cdot,\cdot\rangle$ defined by $\langle e_i, e_{n+1-j}\rangle = \delta_{i,j}$ $1 \le i, j \le n, i+j \le n+1$. Let $(\underline{a}, \underline{b})$ be a pair of compositions verifiying $|\underline{a}| \le [\frac{n}{2}]$ and $|\underline{b}| \le [\frac{n}{2}]$, $\mathscr{V} = \{V_0 = \{0\} \subsetneq V_1 \subsetneq \cdots \subsetneq V_m\}$ and $\mathscr{W} = \{W_0 \supsetneq W_1 \supsetneq \cdots \supsetneq W_t = \{0\}\}$ the two flags of isotropic subspaces such that $V_i = \langle e_1, \ldots, e_{a_1+\cdots+a_i}\rangle$, $1 \le i \le m$, and $W_i = \langle e_{n-(b_1+\cdots+b_{t-i})+1}, \ldots, e_n\rangle$, $0 \le i \le t-1$. Then $\mathfrak{q}^C_{[\frac{n}{2}]}(\underline{a}|\underline{b})$ (resp. $\mathfrak{q}^B_{[\frac{n}{2}]}(\underline{a}|\underline{b})$) is the stabilizer of the pair $(\mathscr{V}, \mathscr{W})$ in $\mathfrak{sp}(n)$ (resp. $\mathfrak{so}(n)$). The case $\mathfrak{g} = \mathfrak{so}(n)$ and n is even is more complicated: if $|a| = \frac{n}{2}$, $a_m > 1$ and $|b| = \frac{n}{2} - 1$ (resp. $|b| = \frac{n}{2}$, $b_t > 1$ and $|a| = \frac{n}{2} - 1$), then $\mathfrak{q}^D_{\frac{n}{2}}(\underline{a}|\underline{b})$ is the stabilizer of the pair $(\mathscr{V}, \mathscr{W}')$ (resp. $(\mathscr{V}', \mathscr{W})$) in $\mathfrak{so}(n)$ where $\mathscr{W}'$ (resp. $\mathscr{V}'$) is obtained from $\mathscr{W}$ (resp. $\mathscr{V}$) by replacing W_0 (resp. V_m) by $W_0' = \{e_{\frac{n}{2}}, e_{\frac{n}{2}+2}, \ldots, e_n\}$ (resp. $V_m' = \{e_1, \ldots, e_{\frac{n}{2}-1}, e_{\frac{n}{2}+1}\}$). Otherwise, $\mathfrak{q}^D_{\frac{n}{2}}(\underline{a}|\underline{b})$) is always the stabilizer of the pair $(\mathscr{V}, \mathscr{W})$ (see [8]).

Let $\underline{a} = (a_1, \ldots, a_k)$ a composition of an integer $n \in \mathbb{N}^\times$, we set $I_i = [a_1 + \cdots + a_{i-1} + 1, \ldots, a_1 + \cdots + a_{i-1} + a_i] \cap \mathbb{N},\ 1 \le i \le k$ and we associate to $\underline{a}$ the involution $\theta_{\underline{a}}$ of $\{1, \ldots, n\}$, defined by $\theta_{\underline{a}}(x) = 2(a_1 + \cdots + a_{i-1}) + a_i - x + 1, x \in I_i, 1 \le i \le k$.

Let $(\underline{a}, \underline{b})$ a pair of compositions of an integer $n \in \mathbb{N}^\times$, we associate to the seaweed subalgebra $\mathfrak{q}^A(\underline{a} \mid \underline{b})$ of $\mathfrak{gl}(n)$ a graph denoted by $\Gamma^A(\underline{a} \mid \underline{b})$ and called meander of $\mathfrak{q}^A(\underline{a} \mid \underline{b})$, whose vertices are n consecutive points on a horizontal line, numberd $1, 2, \ldots, n$. It is constructed by the following way: we connect by un arc, below (resp. above) the horizontal line, each pair of distinct vertices of $\Gamma(\underline{a} \mid \underline{b})$ of the form $(x, \theta_{\underline{a}}(x))$

(resp.$(x, \theta_{\underline{b}}(x))$), $x \in \{1, \ldots, n\}$. A connected composant of $\Gamma^A(\underline{a} \mid \underline{b})$ is either a cycle or a segment (see [2]).

Example 2.2.

$$\Gamma^A(2,4,3 \mid 5,2,2) =$$ 

Theorem 2.3 ([7]). *Let $\mathfrak{q}^A(\underline{a} \mid \underline{b})$ be a seaweed subalgebra of $\mathfrak{gl}(n)$ and $\Gamma^A(\underline{a} \mid \underline{b})$ the associated meander, we have*

$$\operatorname{ind} \mathfrak{q}^A(\underline{a} \mid \underline{b}) = 2 \times (\textit{number of cycles}) + \textit{number of segments}$$

Lemma 2.4 ([3]). *Let $\underline{a} = (a_1, \ldots, a_k)$ and $\underline{b} = (b_1, \ldots, b_t)$ be two compositions such that $|\underline{b}| = |\underline{a}| = n$ and set $\underline{a}^{-1} = (a_k, \ldots, a_1)$, we have*

$$\operatorname{ind} \mathfrak{q}^A(\underline{a} \mid \underline{b}) = \operatorname{ind} \mathfrak{q}^A(2n \mid \underline{a}^{-1}, \underline{b})$$

Theorem 2.5 ([3]). *Let $a, b, c, d, n \in \mathbb{N}^\times$ such that $a + b = c + d = n$, we have*

(1) $\operatorname{ind} \mathfrak{q}^A(a, b \mid n) = a \wedge b$

(2) $\operatorname{ind} \mathfrak{q}^A(a, b \mid c, d) = \operatorname{ind} \mathfrak{q}^A(a, b, c \mid n + c) = (a + b) \wedge (b + c)$

Theorem 2.6 ([3]). *Let $\mathfrak{p}^A(a_1, \ldots, a_k) := \mathfrak{q}^A(a_1, \ldots, a_k \mid n)$ be a parabolic subalgebra of $\mathfrak{gl}(n)$. We set $d_k = -(a_1 + \cdots + a_{k-1})$ and $d_i = (a_1 + \cdots + a_{i-1}) - (a_{i+1} + \cdots + a_k)$, $1 \leq i \leq k - 1$.*

(1) *For any $1 \leq i \leq k$ such that $d_i \neq 0$ and any $\alpha \in \mathbb{Z}$ such that $a_i + \alpha|d_i| \geq 0$, we have*

$$\operatorname{ind} \mathfrak{p}^A(a_1, \ldots, a_k) = \operatorname{ind} \mathfrak{p}^A(a_1, \ldots, a_i + \alpha|d_i|, \ldots, a_k)$$

In particular, we have

$$\operatorname{ind} \mathfrak{p}^A(a_1, \ldots, a_k) = \operatorname{ind} \mathfrak{p}^A(a_1, \ldots, a_{i-1}, a_i[|d_i|], a_{i+1} \ldots, a_k)$$

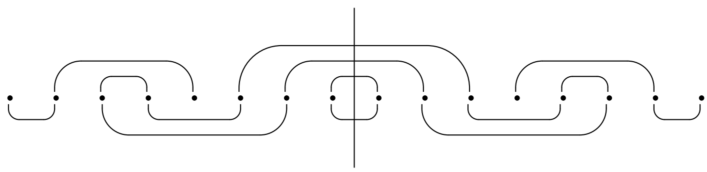
(2) *For any $1 \leq i \leq k$ such that $d_i = 0$, we have*

$$\operatorname{ind} \mathfrak{p}^A(a_1, \ldots, a_k) = a_i + \operatorname{ind} \mathfrak{p}^A(a_1, \ldots, a_{i-1}, a_{i+1} \ldots, a_k)$$

3. Seaweed subalgebras of $sp(2n)$

Let $n \in \mathbb{N}^{\times}$ and $(\underline{a}, \underline{b})$ a pair of compositions verifiying $|\underline{a}| \leq n$ and $|\underline{b}| \leq n$. We associate to the seaweed subalgebra $\mathfrak{q}_n^C(\underline{a} \mid \underline{b})$ of $\mathfrak{sp}(2n)$ the meander $\Gamma_n^C(\underline{a} \mid \underline{b}) := \Gamma^A(\underline{a}' \mid \underline{b}')$, where $\underline{a}' = (a_1, \ldots, a_k, 2(n - |\underline{a}|), a_k, \ldots, a_1)$ and $\underline{b}' = (b_1, \ldots, b_t, 2(n - |\underline{b}|), b_t, \ldots, b_1)$. $\underline{a}'$ and $\underline{b}'$ are two compositions of $2n$, so $\Gamma_n^C(\underline{a} \mid \underline{b})$ has $2n$ vertices numbered from 1 up to $2n$. Let σ the symmetry with respect to the vertical line between the n-th and $(n+1)$-th vertices. By construction, the meander $\Gamma_n^C(\underline{a} \mid \underline{b})$ is σ-stable.

Example 3.1.

$\Gamma_8^C(2, 5 \mid 1, 4) =$ 

Theorem 3.2 ([11]). *Let $\mathfrak{q}_n^C(\underline{a} \mid \underline{b})$ be a seaweed subalgebra of $\mathfrak{sp}(2n)$ and $\Gamma_n^C(\underline{a} \mid \underline{b})$ the associated meander, we have*

$$\operatorname{ind} \mathfrak{q}_n^C(\underline{a} \mid \underline{b}) = \textit{number of cycles} + \frac{1}{2} \times (\textit{number of segments that are not } \sigma\textit{-stable})$$

Corollary 3.3. *Let $\underline{a} = (a_1, \ldots, a_k)$ and $\underline{b} = (b_1, \ldots, b_t)$ be two compositions such that $|\underline{a}| \leq n$ and $|\underline{b}| \leq n$.*

(1) $\operatorname{ind} \mathfrak{q}_n^C(\underline{a} \mid \underline{b}) = \operatorname{ind} \mathfrak{q}_n^C(\underline{b} \mid \underline{a})$

(2) $\operatorname{ind} \mathfrak{q}_n^C(\underline{a} \mid \underline{b}) = \operatorname{ind} \mathfrak{q}_{\max(|\underline{a}|, |\underline{b}|)}^C(\underline{a} \mid \underline{b}) + n - \max(|\underline{a}|, |\underline{b}|)$

(3) $\operatorname{ind} \mathfrak{q}_n^C(\underline{a} \mid \emptyset) = \sum_{1 \leq i \leq k} [\frac{a_i}{2}] + (n - |\underline{a}|)$

(4) *If there exist $1 \leq i \leq k$ and $1 \leq j \leq t$ such that $a_1 + \cdots + a_i = b_1 + \cdots + b_j$, then*

$$\begin{aligned} \operatorname{ind} \mathfrak{q}_n^C(\underline{a} \mid \underline{b}) = \operatorname{ind} \mathfrak{q}^A&(a_1, \ldots, a_i \mid b_1, \ldots, b_j) \\ &+ \operatorname{ind} \mathfrak{q}_{n-(a_1+\cdots+a_i)}^C(a_{i+1}, \ldots, a_k \mid b_{j+1}, \ldots, b_t) \end{aligned}$$

where $\mathfrak{q}^A(a_1, \ldots, a_i \mid b_1, \ldots, b_j)$ is the seaweed subalgebra of $\mathfrak{gl}(a_1 + \cdots + a_i)$ associated to pair $((a_1, \ldots, a_i), (b_1, \ldots, b_j))$. In particular, if $|\underline{a}| = |\underline{b}|$, then

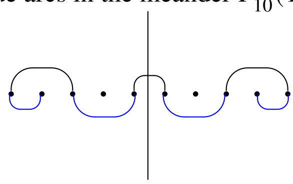
$$\operatorname{ind} \mathfrak{q}_n^C(\underline{a} \mid \underline{b}) = \operatorname{ind} \mathfrak{q}^A(\underline{a} \mid \underline{b}) + \operatorname{ind} \mathfrak{q}_{n-|\underline{a}|}^C(\emptyset \mid \emptyset) = \operatorname{ind} \mathfrak{q}^A(\underline{a} \mid \underline{b}) + n - |\underline{a}|$$

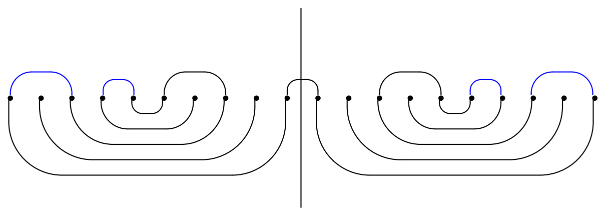
Lemma 3.4. *Let $\underline{a} = (a_1, \ldots, a_k)$, $\underline{b} = (b_1, \ldots, b_t)$ be two compositions such that $|\underline{b}| \leq |\underline{a}| \leq n$ and $\underline{a}^{-1} := (a_k, \ldots, a_1)$, we have*

$$\operatorname{ind} \mathfrak{q}_n^C(\underline{a} \mid \underline{b}) = \operatorname{ind} \mathfrak{q}_{n+|\underline{a}|}^C(2|\underline{a}| \mid \underline{a}^{-1}, \underline{b})$$

Proof. By Corollary 3.3, we may suppose $|\underline{a}| = n$. Set $I = [1, n]$, $I' = [1, |\underline{b}|]$ and $\underline{c} = (a_k, \ldots, a_1, b_1, \ldots, b_t)$. We verify easily that θ_a (resp. θ_b) is the restriction of $\theta_n \theta_c \theta_n$ (resp. $\theta_{n+|\underline{b}|}\theta_c\theta_{n+|\underline{b}|}$) to I (resp. I'). Since meanders $\Gamma^C_{2n}(2n \mid \underline{a}^{-1}, \underline{b})$ and $\Gamma^C_n(\underline{a} \mid \underline{b})$ are σ-stable, so there exists a bijection between sets of connected components of these two meanders which preserves the number of cycles and the number of segments that are not σ-stable (see Example 3.5). So the result follows from Theorem 3.2. □

Example 3.5. Let us consider the pair $((2,3),(3,1))$, the blue arcs in the meander $\Gamma^C_5(2,3 \mid 3,1)$ are replaced with the blue arcs in the meander $\Gamma^C_{10}(10 \mid 3,2,3,1)$.

$\Gamma^C_5(2,3 \mid 3,1) =$ 

$\Gamma^C_{10}(10 \mid 3,2,3,1) =$ 

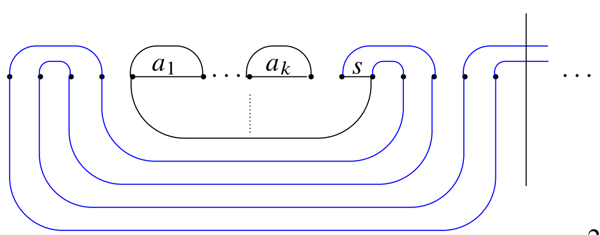
Remark 3.6. In view of Corollary 3.3 and Lemma 3.4, we may reduce the study of the index of seaweed subalgebras of $\mathfrak{sp}(2n)$ to case of seaweed subalgebras of the form $\mathfrak{q}^C_n(n \mid \underline{a})$, where $\underline{a}$ is a composition of an integer less than or equal to n.

Lemma 3.7. *Let $\underline{a} = (a_1, \ldots, a_k)$ a composition verifying $|\underline{a}| \leq n$ and $s = n - |\underline{a}|$. Then for any $t \in \mathbb{N}$, we have*

$$\operatorname{ind} \mathfrak{q}^C_{n+4ts}(n + 4ts \mid \underbrace{2s, \ldots, 2s}_{t}, \underline{a}, \underbrace{2s, \ldots, 2s}_{t}) = \operatorname{ind} \mathfrak{q}^C_n(n \mid \underline{a})$$

In particular, if $\mathfrak{q}^C_n(n \mid \underline{a})$ a Frobenius subalgebra of $\mathfrak{sp}(2n)$, then for any $t \in \mathbb{N}$, $\mathfrak{q}^C_{n+4ts}(n + 4ts \mid \underbrace{2s, \ldots, 2s}_{t}, \underline{a}, \underbrace{2s, \ldots, 2s}_{t})$ is a Frobenius subalgebra of $\mathfrak{sp}(2(n + 4ts))$.

Proof. By the following figure, the result is cleary true for $t = 1$.

$\Gamma^C_{n+4s}(n + 4s \mid 2s, \underline{a}, 2s) =$ 

So the result follows by induction on t. □

Lemma 3.8. *Let $\underline{a} = (a_1, \dots, a_k)$ be a composition verifiying $|\underline{a}| \leq n$. Suppose that there exists $1 \leq i \leq k$ such that $|\underline{a}|_i := a_1 + \cdots + a_i \leq n - |\underline{a}|$. Then we have,*

$$\operatorname{ind} \mathfrak{q}_n^C(n \mid \underline{a})] = \sum_{1\leq j\leq i} \left[\frac{a_j}{2}\right] + \operatorname{ind} \mathfrak{q}^C_{n-2|\underline{a}|_i}(n - 2|\underline{a}|_i \mid a_{i+1}, \dots, a_k)$$

In particular, if $|\underline{a}| \leq [\frac{n}{2}]$, we have

$$\operatorname{ind} \mathfrak{q}_n^C(n \mid \underline{a}) = \sum_{1\leq j\leq k} \left[\frac{a_j}{2}\right] + \left[\frac{n - 2|\underline{a}|}{2}\right]$$

Proof. Observe that the meander $\Gamma_n^C(n \mid \underline{a})$ is a disjoint union of the meanders $\Gamma^C_{2|\underline{a}|_i}(2|\underline{a}|_i \mid a_1, \dots, a_i)$ and $\Gamma^C_{n-2|\underline{a}|_i}(n - 2|\underline{a}|_i \mid a_{i+1}, \dots, a_k)$. On the other hand, we verify that the meander $\Gamma^C_{2|\underline{a}|_i}(2|\underline{a}|_i \mid a_1, \dots, a_i)$ has $\sum_{1\leq j\leq i}[\frac{a_j}{2}]$ cycles and $\sum_{1\leq j\leq i}[\frac{a_j+1}{2}] - \sum_{1\leq j\leq i}[\frac{a_j}{2}]$ segments that are all σ-stable. So the result follows from Theorem 3.2. □

Lemma 3.9. *Let $\underline{a} = (a_1, \dots, a_k)$ be a composition verifiying $|\underline{a}| \leq n$ and set $a_{k+1} = n - |\underline{a}|$.*

(1) *Let $a_{i,j} = (a_1 + \cdots + a_i) - (a_j + \cdots + a_{k+1})$ and $a^{i,j} = (a_{i+1} + \cdots + a_{j-1}) + |a_{i,j}|$, $1 \leq i < j \leq k+1$. We have,*

$$\operatorname{ind} \mathfrak{q}_n^C(n \mid \underline{a}) = \begin{cases} \operatorname{ind} \mathfrak{q}^C_{n+a^{i,j}}(n + a^{i,j} \mid a_1, \dots, a_i, a^{i,j}, a_{i+1}, \dots, a_k) & \text{if } a_{i,j} < 0 \\ \operatorname{ind} \mathfrak{q}^C_{n+a^{i,j}}(n + a^{i,j} \mid a_1, \dots, a_{j-1}, a^{i,j}, a_j, \dots, a_k) & \text{if } a_{i,j} \geq 0 \end{cases}$$

(2) *Let $d_i = (a_1 + \cdots + a_{i-1}) - (a_{i+1} + \cdots + a_{k+1})$, $1 \leq i \leq k$. Suppose there exists $1 \leq i \leq k$ such that $a_i \geq |d_i|$. We have*

(a) $\operatorname{ind} \mathfrak{q}_n^C(n \mid \underline{a})$
$$= \begin{cases} \operatorname{ind} \mathfrak{q}^C_{n+a_i+d_i}(n + a_i + d_i \mid a_1, \dots, a_i, a_i + d_i, a_{i+1}, \dots, a_k) & \text{if } d_i < 0 \\ \operatorname{ind} \mathfrak{q}^C_{n+a_i-d_i}(n + a_i - d_i \mid a_1, \dots, a_{i-1}, a_i - d_i, a_i, \dots, a_k) & \text{if } d_i \geq 0 \end{cases}$$

(b) $\operatorname{ind} \mathfrak{q}_n^C(n \mid \underline{a}) = \operatorname{ind} \mathfrak{q}^C_{n-|d_i|}(n - |d_i| \mid a_1, \dots, a_{i-1}, a_i - |d_i|, a_{i+1}, \dots, a_k)$.

Proof. (1). Suppose $a_{i,j} \leq 0$. It follows from the proof of Lemma 3.4 that there exists a bijection between the sets of connected components of the meanders $\Gamma^C_{a^{i,j}}(a^{i,j} \mid a_{i+1}, \dots, a_{j-1})$ and $\Gamma^C_{2a^{i,j}}(2a^{i,j} \mid a^{i,j}, a_{i+1}, \dots, a_{j-1})$ which preserves the number of cycles and the number of segments that are not σ-stable. This bijection extends naturally to a bijection between the set of connected components of the meander $\Gamma_n^C(n \mid \underline{a})$ and the set of connected components of the meander $\Gamma^C_{n+a^{i,j}}(n + a^{i,j} \mid a_1, \dots, a_i, a^{i,j}, a_{i+1}, \dots, a_k)$

which also preserves the number of cycles and the number of segments that are not σ-stable. The result follows from Theorem 3.4. The case $a_{i,j} \geq 0$ is proved in the same way (see Example 3.10).

(2). (a). It suffices to remark that

$$\begin{cases} a^{i,i+1} = a_{i,i+1} = a_i + d_i & \text{if } d_i < 0 \\ a^{i-1,i} = -a_{i-1,i} = a_i - d_i & \text{if } d_i \geq 0 \end{cases}$$

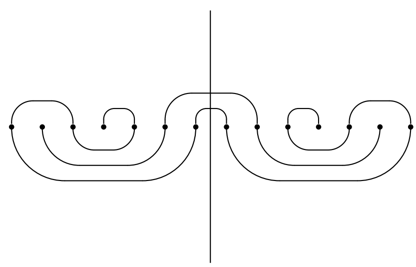
(see Example 3.11).

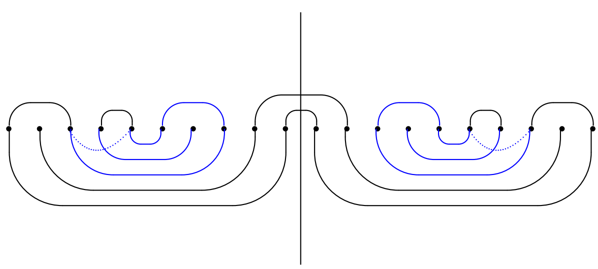
(b). Let $\underline{b} = (b_1, \ldots, b_k)$ such that $b_j = a_j$ if $j \neq i$ and $b_i = a_i - |d_i|$. We verify that $b_{i-1,i+1} = d_i$ and $b^{i-1,i+1} = a_i$. It follows from (1) that we have

$$\operatorname{ind} \mathfrak{q}_n^C(\underline{b} \mid n) = \begin{cases} \operatorname{ind} \mathfrak{q}^C_{n+a_i+d_i}(n + a_i + d_i \mid a_1, \ldots, a_i, a_i + d_i, a_{i+1}, \ldots, a_k) & \text{if } d_i < 0 \\ \operatorname{ind} \mathfrak{q}^C_{n+a_i-d_i}(n + a_i - d_i \mid a_1, \ldots, a_{i-1}, a_i - d_i, a_i, \ldots, a_k) & \text{if } d_i \geq 0 \end{cases}$$

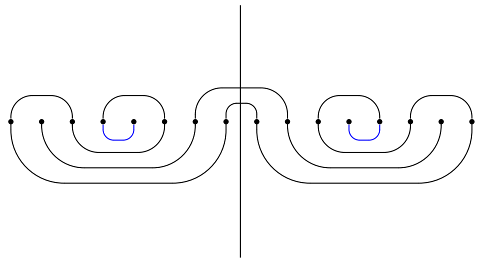
The result follows from (a). □

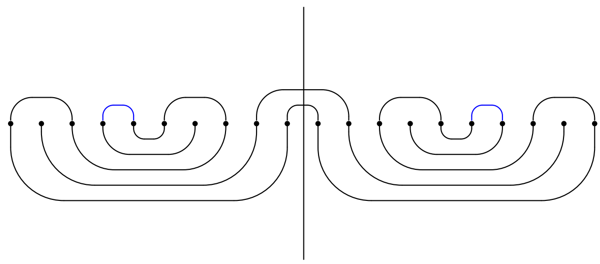
Example 3.10. Let us consider the composition $\underline{a} = (a_1, a_2) = (3, 2)$ and $n = 7$, then $|\underline{a}| = 5$, $a_3 = n - |\underline{a}| = 2$, $a_{1,3} = a_1 - a_3 = 1 > 0$ and $a^{1,3} = a_2 + a_{1,3} = 3$. We have $\operatorname{ind} \mathfrak{q}_7^C(7 \mid 3, 2) = \operatorname{ind} \mathfrak{q}_{10}^C(10 \mid 3, 2, 3)$.

$\Gamma_7^C(7 \mid 3, 2) =$ 

$\Gamma_{10}^C(10 \mid 3, 2, 3) =$ 

Example 3.11. Let us consider the composition $\underline{a} = (a_1, a_2) = (3, 3)$ and $n = 8$, then $|\underline{a}| = 6$, $a_3 = n - |\underline{a}| = 2$, $d_2 = a_1 - a_3 = 1 > 0$ and $a_2 - d_2 = 2$. we have ind $\mathfrak{q}_8(8 \mid 3,3) = \operatorname{ind} \mathfrak{q}_{10}(10 \mid 3,2,3)$. The blue arcs in the meander $\Gamma_8^C(8 \mid 3,3)$ are replaced with the blue arcs in the meander $\Gamma_{10}^C(10 \mid 3,2,3)$.

$\Gamma_8^C(8 \mid 3,3) =$ 

$\Gamma_{10}^C(10 \mid 3,2,3) =$ 

The following theorem is an immediate consequence of Corollary 3.3 and Lemma 3.9.

Theorem 3.12. *Let* $\underline{a} = (a_1, \ldots, a_k)$ *a composition verifying* $|\underline{a}| \leq n$. *We set* $a_{k+1} = n - |\underline{a}|$ *and* $d_i = (a_1 + \cdots + a_{i-1}) - (a_{i+1} + \cdots + a_{k+1}),\ 1 \leq i \leq k$.

(1) *For any* $1 \leq i \leq k$ *and any* $\alpha \in \mathbb{Z}$ *such that* $a_i + \alpha|d_i| \geq 0$, *we have*

$$\operatorname{ind} \mathfrak{q}_n^C(n \mid \underline{a}) = \operatorname{ind} \mathfrak{q}_{n+\alpha|d_i|}^C(n + \alpha|d_i| \mid a_1, \ldots, a_{i-1}, a_i + \alpha|d_i|, a_{i+1}, \ldots, a_k)$$

In particular, for any $1 \leq i \leq k$ *such that* $d_i \neq 0$, *we have*

$$\operatorname{ind} \mathfrak{q}_n^C(n \mid \underline{a}) = \operatorname{ind} \mathfrak{q}_{n-a_i+a_i[|d_i|]}^C(n - a_i + a_i[|d_i|] \mid a_1, \ldots, a_{i-1}, a_i[|d_i|], a_{i+1}, \ldots, a_k)$$

(2) *For any* $1 \leq i \leq k$ *such that* $d_i = 0$, *we have*

$$\operatorname{ind} \mathfrak{q}_n^C(n \mid \underline{a}) = a_i + \operatorname{ind} \mathfrak{q}_{n-a_i}^C(a_1, \ldots, a_{i-1}, a_{i+1} \ldots, a_k)$$

Remark 3.13. The lemma 2.4 of [3] show that if the composition $\underline{a} = (a_1, \ldots, a_k)$ and the integer n verify $|\underline{a}| \leq n$, then there exists $1 \leq i \leq k$ such that $a_i \geq |d_i|$ or $|\underline{a}| \leq [\frac{n}{2}]$.

Remark 3.14. In view of Corollary 3.3, Lemma 3.4, Lemma 3.8 and Remark 3.13, the previous theorem give a reduction algorithm allowing to compute the index of seaweed subalgebras of $\mathfrak{sp}(2n)$.

Example 3.15. Consider the seaweed subalgebra $\mathfrak{q}^C_{200}(15, 185 \mid 17, 61, 117)$ of $\mathfrak{sp}(400)$. By Lemma 3.4, we have

$$\operatorname{ind} \mathfrak{q}^C_{200}(15, 185 \mid 17, 61, 117) = \chi[\mathfrak{q}^C_{400}(400 \mid 185, 15, 17, 61, 117)]$$

Then, by applying successively Theorem 3.12, we have

$$\begin{aligned}
\operatorname{ind} \mathfrak{q}^C_{400}(400 \mid 185, 15, 17, 61, 117) &= \operatorname{ind} \mathfrak{q}^C_{385}(385 \mid 185, 17, 61, 117)\\
&= \operatorname{ind} \mathfrak{q}^C_{369}(369 \mid 185, 1, 61, 117)\\
&= \operatorname{ind} \mathfrak{q}^C_{185}(185 \mid 1, 1, 61, 117)\\
&= \operatorname{ind} \mathfrak{q}^C_{69}(69 \mid 1, 1, 61, 1)\\
&= \operatorname{ind} \mathfrak{q}^C_{9}(9 \mid 1, 1, 1, 1)
\end{aligned}$$

It follows from Lemma 3.8 that we have

$$\begin{aligned}
\operatorname{ind} \mathfrak{q}^C_{400}(400 \mid 185, 15, 17, 61, 117) &= \operatorname{ind} \mathfrak{q}^C_{9}(9 \mid 1, 1, 1, 1)\\
&= 0
\end{aligned}$$

Lemma 3.16. *For any $k \in \mathbb{N}^\times$ and $(\alpha_1, \dots, \alpha_k) \in \mathbb{N}^k$, we set $a_{k+1} = k$, $\underline{a} = (a_1, \dots, a_k)$ the composition defined by $a_i = 1 + \alpha_i(a_{i+1} + \cdots + a_{k+1} - i + 1)$, $1 \le i \le k$ and $r = |\underline{a}| + k$. Then $\mathfrak{q}^C_r(r \mid \underline{a})$ is a Frobenius subalgebra of $\mathfrak{sp}(2r)$.*

Proof. Let $s_i = a_{i+1} + \cdots + a_{k+1} - i + 1$, $1 \le i \le k$, so $a_i[s_i] = 1$. Using the reduction given by Theorem 3.12, we obtain

$$\operatorname{ind} \mathfrak{q}^C_r(r \mid \underline{a}) = \operatorname{ind} \mathfrak{q}^C_{2k-1}(2k - 1 \mid \underbrace{1, \dots, 1}_{k-1})$$

It follows from Lemma 3.8 that we have

$$\operatorname{ind} \mathfrak{q}^C_r(r \mid \underline{a}) = 0 \qquad \square$$

Lemma 3.17. *Let $\underline{a} = (a_1, \dots, a_k)$ and $\underline{b} = (b_1, \dots, b_t)$ be two compositions verifiying $|\underline{b}| \le |\underline{a}| \le n$. Suppose $a_1 > b_1$, we have*

$$\operatorname{ind} \mathfrak{q}^C_n(\underline{a} \mid \underline{b}) = \operatorname{ind} \mathfrak{q}^C_{n-b_1}(a_1 - b_1 - a_1[a_1 - b_1], a_1[a_1 - b_1], a_2, \dots, a_k \mid b_2 \dots, b_t)$$

Proof. It follows from Corollary 3.3 that we may suppose $|\underline{a}| = n$. Using Lemma 3.4 and Theorem 3.12, we have

$$\begin{aligned}
&\operatorname{ind} \mathfrak{q}_n^C(\underline{a} \mid \underline{b})\\
&= \operatorname{ind} \mathfrak{q}_{2n}^C(2n \mid a_k, \ldots, a_1, \underline{b})\\
&= \operatorname{ind} \mathfrak{q}_{2n-b_1}^C(2n - b_1 \mid a_k, \ldots, a_1, b_2, \ldots, b_t)\\
&= \operatorname{ind} \mathfrak{q}_{2n-b_1-a_1+a_1[a_1-b_1]}^C(2n - b_1 - a_1 + a_1[a_1 - b_1] \mid a_k, \ldots, a_2,\\
&\qquad\qquad a_1[a_1 - b_1], b_2, \ldots, b_t)\\
&= \operatorname{ind} \mathfrak{q}_{2n-2b_1}^C(2n - 2b_1 \mid a_k, \ldots, a_2, a_1[a_1 - b_1], a_1 - b_1 - a_1[a_1 - b_1], b_2, \ldots, b_t)\\
&= \operatorname{ind} \mathfrak{q}_{n-b_1}^C(a_1 - b_1 - a_1[a_1 - b_1], a_1[a_1 - b_1], a_2, \ldots, a_k \mid b_2 \ldots, b_t)
\end{aligned}$$ □

Lemma 3.18. *Let $\underline{a} = (a_1, \ldots, a_k)$ and $\underline{b} = (b_1, \ldots, b_t)$ be two compositions verifiying $|\underline{b}| = |\underline{a}| = n$. Suppose $a_1 > b_1$, we have*

$$\operatorname{ind} \mathfrak{q}^A(\underline{a} \mid \underline{b}) = \operatorname{ind} \mathfrak{q}^A(a_1 - b_1 - a_1[a_1 - b_1], a_1[a_1 - b_1], a_2, \ldots, a_k \mid b_2 \ldots, b_t)$$

Proof. This is a direct consequence of the previous lemma and Corollary 3.3 (4). □

Theorem 3.19. *Let $(a, b, n) \in (\mathbb{N}^\times)^3$ such that $b \leq a \leq n$. Then the index of $\mathfrak{q}_n^C(a \mid b)$ is given by*

$$\operatorname{ind} \mathfrak{q}_n^C(a \mid b) = \begin{cases} n & \text{if } a = b \\ \left[\frac{a[a-b]}{2}\right] + \left[\frac{a-b-a[a-b]}{2}\right] + n - a & \text{if } a \neq b \end{cases}$$

Proof. Suppose $a = b$, the result follows from Corollary 3.3 (4). Suppose $a > b$, it follows from Corollary 3.3 that we may suppose that $a = n$. Using Lemma 3.17, we have

$$\operatorname{ind} \mathfrak{q}_n^C(a \mid b) = \operatorname{ind} \mathfrak{q}_{a-b}^C(a - b - a[a - b], a[a - b] \mid \emptyset)$$

Hence we have the result. □

Lemma 3.20. *Let $\mathfrak{q}_n^C(n \mid \underline{a})$ be a seaweed subalgebra of $\mathfrak{sp}(2n)$ where $\underline{a} = (a_1, \ldots, a_k)$ is a composition which verify $|\underline{a}| \leq n$. Set $s = n - |\underline{a}|$ and $\underline{a}' = (a_1, \ldots, a_k, s)$. Let us consider the seaweed subalgebra $\mathfrak{q}^A(n \mid \underline{a}')$ of $\mathfrak{gl}(n)$ associated to the pair $(n, \underline{a}')$. Then there exist $\alpha \in \mathbb{N}$ and a composition $\underline{c} = (c_1, \ldots, c_j)$ verifiying $j \leq k$ and $|\underline{c}| \leq s$ such that*

$$\operatorname{ind} \mathfrak{q}_n^C(n \mid \underline{a}) = \alpha + \operatorname{ind} \mathfrak{q}_{s+|\underline{c}|}^C(s + |\underline{c}| \mid \underline{c})$$
$$\operatorname{ind} \mathfrak{q}^A(n \mid \underline{a}') = \alpha + \operatorname{ind} \mathfrak{q}^A(s + |\underline{c}| \mid \underline{c}, s)$$

Proof. When $2|\underline{a}| \leq n$, it suffices to consider $\alpha = 0$ and $\underline{c} = \underline{a}$. Suppose that $2|\underline{a}| > n$, it follows from [3, Lemma 2.4] that there exists $1 \leq i \leq k$ such that $a_i \geq |d_i|$ (see the definition of $d_i's$ in Theorems 2.6 and 3.12). Using Theorems 2.6 and 3.12, the result follows by induction on $|\underline{a}|$. □

Theorem 3.21. *Let $\underline{a} = (a_1, \ldots, a_k)$ and $\underline{b} = (b_1, \ldots, b_t)$ be two compositions verifying $|\underline{b}| \leq |\underline{a}| = n$ and $s = |\underline{a}| - |\underline{b}|$. Suppose that $k + t < s$, then $\mathfrak{q}_n^C(\underline{a} \mid \underline{b})$ is not a Frobenius subalgebra.*

Proof. Using Lemma 3.4 and Theorem 3.12, we have

$$\operatorname{ind} \mathfrak{q}_n^C(\underline{a} \mid \underline{b}) = \operatorname{ind} \mathfrak{q}_{2n-a_1}^C(2n - a_1 \mid a_k, \ldots, a_2, \underline{b})$$

Suppose that $\mathfrak{q}_n^C(\underline{a} \mid \underline{b})$ is a Frobenius subalgebra of $\mathfrak{sp}(2n)$. It follows from Lemma 3.20 that there exists a composition $\underline{c} = (c_1, \ldots, c_j)$ such that $j \leq k + t - 1 < s - 1$, $|\underline{c}| \leq s$ and $\operatorname{ind} \mathfrak{q}_{s+|\underline{c}|}^C(s + |\underline{c}| \mid \underline{c}) = 0$. Since $|\underline{c}| \leq [\frac{s+|\underline{c}|}{2}]$, it follows from Lemma 3.8 that $c_i = 1,\ 1 \leq i \leq j$ and there exists $\epsilon \in \{0, 1\}$ such that $j = |\underline{c}| = s - \epsilon$. So we deduce that $j \geq s - 1$. We have therefore a contradiction. □

Theorem 3.22. *Let $(a, b, c) \in \mathbb{N}^\times$ and set $n = \max(a + b, c)$, $p = (a + b) \wedge (b + c)$ and $r = |a + b - c|$, then*

(1) *If $p > r$, we have*

$$\operatorname{ind} \mathfrak{q}_n^C(a, b \mid c) = p - \left[\frac{r+1}{2}\right]$$

(2) *If $p \leq r$, we have*

$$\operatorname{ind} \mathfrak{q}_n^C(a, b \mid c) = \begin{cases} [\frac{r}{2}] & \text{if } p \text{ and } r \text{ have the same parity} \\ [\frac{r}{2}] - 1 & \text{otherwise} \end{cases}$$

Proof. Let us consider the case where $c \leq a + b = n$, it follows from Lemma 3.4 and Theorem 3.12 that

$$\operatorname{ind} \mathfrak{q}_n^C(a, b \mid c) = \operatorname{ind} \mathfrak{q}_{2n}^C(2n \mid b, a, c) = \operatorname{ind} \mathfrak{q}_{b+c+r}^C(b + c + r \mid b, c)$$

Suppose that $r = 0$, we deduce from Corollary 3.3 (4) and Theorem 2.5 that

$$\operatorname{ind} \mathfrak{q}_{b+c+r}^C(b + c + r \mid b, c) = \operatorname{ind} \mathfrak{q}^A(b + c \mid b, c) = b \wedge c = p$$

Now, we suppose that $r \neq 0$ and prove the following properties that will be useful to us,

$\mathcal{P}$: Any triple $(x, y, z) \in \mathbb{N}^3$ verifies one of the following conditions,

(i) $x + y \leq z$

(ii) $x \geq y + z$

(iii) $y > |x - z|$

$\mathcal{P}'$: Let $z \in \mathbb{N}^\times$. For any pair $(x, y) \in \mathbb{N}^2$, there exists $(x', y') \in \mathbb{N}^2$ vérifiying

$$\begin{cases} (x' + y') \wedge (y' + z) = (x + y) \wedge (y + z) =: q \\ x' = z \text{ and } y' = q - z & \text{if } q > z \\ x' + y' \leq z & \text{if } q \leq z \end{cases}$$

such that we have

$$\operatorname{ind} \mathfrak{q}^C_{x+y+z}(x + y + z \mid x, y) = \operatorname{ind} \mathfrak{q}^C_{x'+y'+z}(x' + y' + z \mid x', y')$$

The property $\mathcal{P}$ is obvious. Let us prove the property $\mathcal{P}'$ by induction on the sum $x + y$. Suppose that $x + y \leq z$, in particular $q \leq z$. The result is true with $x = x'$ and $y = y'$. Suppose that $x + y > z$, it follows from the property $\mathcal{P}$ that $x \geq y + z$ or $y > |x - z|$. If $x = z$, in particular $q = x + y = y + z > z$, the result is again true with $x = x'$ and $y = y'$. If $x \neq z$, we deduce from Theorem 3.12 that we have

$$\begin{aligned} &\operatorname{ind} \mathfrak{q}^C_{x+y+z}(x + y + z \mid x, y) \\ &\quad = \begin{cases} \operatorname{ind} \mathfrak{q}^C_{x[y+z]+y+z}(x[y + z] + y + z \mid x[y + z], y) & \text{if } x \geq y + z \\ \operatorname{ind} \mathfrak{q}^C_{x+y[|x-z|]+z}(x + y[|x - z|] + z \mid x, y[|x - z|]) & \text{if } y > |x - z| \end{cases} \end{aligned}$$

Remark that $(x[y+z]+y) \wedge (y+z) = (x+y[|x-z|]) \wedge (y[|x-z|]+z) = q$. So it suffices to apply the induction hypothesis to the pair

$$(x_1, y_1) = \begin{cases} (x[y + z], y) & \text{if } x \geq y + z \\ (x, y[|x - z|]) & \text{if } y > |x - z| \end{cases}$$

It follows by the above properties that there exist $(b', c') \in \mathbb{N}^2$ verifiying

$$\begin{cases} (b' + c') \wedge (c' + r) = (b + c) \wedge (c + r) = p \\ b' = r \text{ and } c' = p - r & \text{if } p > r \\ b' + c' \leq r & \text{if } p \leq r \end{cases}$$

such that

$$\operatorname{ind} \mathfrak{q}^C_n(a, b \mid c) = \operatorname{ind} \mathfrak{q}^C_{b'+c'+r}(b' + c' + r \mid b', c')$$

Suppose that $p > r$, in particular $b' = r$ and $c' = p - r$. We deduce from Theorem 3.12 and Lemma 3.8 that we have

$$\operatorname{ind} \mathfrak{q}^C_n(a, b \mid c) = \operatorname{ind} \mathfrak{q}^C_{b'+c'+r}(b' + c' + r \mid b', c')] = c' + \left[\frac{r}{2}\right] = p - \left[\frac{r+1}{2}\right]$$

Suppose that $p \leq r$, in particular $b' + c' \leq r$. It follows from Lemma 3.8 that we have

$$\operatorname{ind} \mathfrak{q}_n^C(a, b \mid c) = \left[\frac{b'}{2}\right] + \left[\frac{c'}{2}\right] + \left[\frac{r - b' - c'}{2}\right]$$

Now we distinguish two cases:

- If p is even, so the integers c', b' and r are of the same parity. In particular, we have

$$\operatorname{ind} \mathfrak{q}_n^C(a, b \mid c) = \begin{cases} [\frac{r}{2}] & \text{if } r \text{ is even} \\ [\frac{r}{2}] - 1 & \text{if } r \text{ is odd} \end{cases}$$

- If p is odd, so there exists two integers of the opposite parity among $c', b'$and $r - b' - c'$. In particular, we have

$$\operatorname{ind} \mathfrak{q}_n^C(a, b \mid c) = \begin{cases} [\frac{r}{2}] - 1 & \text{if } r \text{ is even} \\ [\frac{r}{2}] & \text{if } r \text{ is odd} \end{cases}$$

  Suppose that $a + b \leq c = n$. It follows from Corollary 3.3 that $\operatorname{ind} \mathfrak{q}_n^C(a, b \mid c) = \operatorname{ind} \mathfrak{q}_n^C(c \mid a, b)$, so it suffices to remark that $p = (a + b) \wedge (b + r)$. □

Corollary 3.23. *Let* $(a, b, c) \in \mathbb{N}^\times$*, then* $\mathfrak{q}_n^C(a, b \mid c)$ *is a Frobenius subalgebra of* $\mathfrak{sp}(2n)$ *if and only if* $\max(a + b, c) = n$ *and one of the following conditions holds:*

(1) $r = 1$ *and* $p = 1$

(2) $r = 2$ *and* $p = 1$

(3) $r = 3$ *and* $p = 2$

Theorem 3.24. *Let* $\underline{a} = (a_1, \dots, a_k)$ *and* $\underline{b} = (b_1, \dots, b_t)$ *be two compositions verifiying* $|\underline{b}| < |\underline{a}| = n$ *and set* $s = |\underline{a}| - |\underline{b}|$. *Consider the seaweed subalgebra* $\mathfrak{q}^A(\underline{a} \mid \underline{b}, s)$ *of* $\mathfrak{gl}(n)$ *associated to the pair* $(\underline{a}, (b_1, \dots, b_t, s))$. *Then*

(1) *There exists a composition* $\underline{d}$ *of* s *which verifies*

$$\operatorname{ind} \mathfrak{q}_n^C(\underline{a} \mid \underline{b}) - \operatorname{ind} \mathfrak{q}^A(\underline{a} \mid \underline{b}, s) = \operatorname{ind} \mathfrak{q}_s^C(\underline{d} \mid \emptyset) - \operatorname{ind} \mathfrak{q}^A(\underline{d} \mid s)$$

(2) (a) *Suppose* $\operatorname{ind} \mathfrak{q}^A(\underline{a} \mid \underline{b}, s) = 1$*, then* $\operatorname{ind} \mathfrak{q}_n^C(\underline{a} \mid \underline{b}) = [\frac{s-1}{2}]$
(b) *Suppose* $\operatorname{ind} \mathfrak{q}_n^C(\underline{a} \mid \underline{b}) = 0$*, then* $\operatorname{ind} \mathfrak{q}^A(\underline{a} \mid \underline{b}, s) = [\frac{s+1}{2}]$

Proof. (1). It follows from Lemmas 2.4 and 3.4 that we have

$$\operatorname{ind} \mathfrak{q}^A(\underline{a} \mid \underline{b}, s) = \operatorname{ind} \mathfrak{q}^A(2n \mid \underline{a}^{-1}, \underline{b}, s)$$
$$\operatorname{ind} \mathfrak{q}_n^C(\underline{a} \mid \underline{b}) = \operatorname{ind} \mathfrak{q}_{2n}^C(2n \mid \underline{a}^{-1}, \underline{b})$$

where $\underline{a}^{-1} = (a_k, \dots, a_1)$. By Lemma 3.20, there exist $\alpha \in \mathbb{N}$ and a composition $\underline{c} = (c_1, \dots, c_u)$ verifiying $|\underline{c}| \leq s$ that we have

$$\operatorname{ind} \mathfrak{q}^A(2n \mid \underline{a}^{-1}, \underline{b}, s) = \alpha + \operatorname{ind} \mathfrak{q}^A(s + |\underline{c}| \mid \underline{c}, s)$$
$$\operatorname{ind} \mathfrak{q}_{2n}^C(2n \mid \underline{a}^{-1}, \underline{b}) = \alpha + \operatorname{ind} \mathfrak{q}_{s+|\underline{c}|}^C(s + |\underline{c}| \mid \underline{c})$$

Set

$$\underline{d} = \begin{cases} (s - |\underline{c}|, c_u, \dots, c_1) & \text{if } |\underline{c}| < s \\ (c_u, \dots, c_1) & \text{if } |\underline{c}| = s \end{cases}$$

Since $|\underline{c}| \leq s$, we have $|\underline{c}| + s - 2(c_1 + \cdots + c_{i-1}) \geq 2c_i,\ 1 \leq i \leq u$. By applying Lemmas 3.17 and 3.18 with $a_1 = |\underline{c}| + s - 2(c_1 + \cdots + c_{i-1})$ and $b_1 = c_i,\ i = 1, \dots, u$, we obtain

$$\operatorname{ind} \mathfrak{q}^A(s + |\underline{c}| \mid \underline{c}, s) = \operatorname{ind} \mathfrak{q}^A(\underline{d} \mid s)$$
$$\operatorname{ind} \mathfrak{q}_{s+|\underline{c}|}^C(s + |\underline{c}| \mid \underline{c}) = \operatorname{ind} \mathfrak{q}_s^C(\underline{d} \mid \emptyset)$$

Hence we have the result.

(2). (a). Suppose $\operatorname{ind} \mathfrak{q}^A(\underline{a} \mid \underline{b}, s) = 1$, so $\alpha = 0$ and $\operatorname{ind} \mathfrak{q}^A(\underline{d} \mid s) = 1$. Therefore, we have

$$\operatorname{ind} \mathfrak{q}_n^C(\underline{a} \mid \underline{b}) = \operatorname{ind} \mathfrak{q}_s^C(\underline{d} \mid \emptyset) = \left[\frac{c_1}{2}\right] + \cdots + \left[\frac{c_u}{2}\right] + \left[\frac{s - |\underline{c}|}{2}\right]$$

On the other hand, it follows from Theorem 2.3 that the meander $\Gamma^A(\underline{d} \mid s)$ of the subalgebra $\mathfrak{q}^A(\underline{d} \mid s)$ is a segment, which implies that there are exactly two odd integers among $c_1, \dots, c_u, s - |\underline{c}|$ and s. In particular,

$$\operatorname{ind} \mathfrak{q}_n^C(\underline{a} \mid \underline{b})] = \left[\frac{s-1}{2}\right]$$

(b). Suppose $\operatorname{ind} \mathfrak{q}_n^C(\underline{a} \mid \underline{b}) = 0$. We deduce from Lemma 3.8 that $c_i = 1,\ 1 \leq i \leq u$ and $|\underline{c}| = s - \epsilon,\ \epsilon \in \{0, 1\}$. In particular, we have

$$\operatorname{ind} \mathfrak{q}^A(\underline{a} \mid \underline{b}, s) = \operatorname{ind} \mathfrak{q}^A(\underline{d} \mid s) = \left[\frac{s+1}{2}\right]$$ □

Corollary 3.25. *Let $\underline{a} = (a_1, \dots, a_k)$ and $\underline{b} = (b_1, \dots, b_t)$ be two compositions verifiying $|\underline{b}| \leq |\underline{a}| = n$. Suppose that $s = |\underline{a}| - |\underline{b}| = 1$ or 2, then $\mathfrak{q}_n^C(\underline{a} \mid \underline{b})$ is a Frobenius subalgebra of $\mathfrak{sp}(2n)$ if and only if $\mathfrak{q}^A(\underline{a} \mid \underline{b}, s) \cap \mathfrak{sl}(n)$ is a Frobenius subalgebra of $\mathfrak{sl}(n)$.*

4. Seaweed subalgebras of $\mathfrak{so}(p)$

As we have seen in Section 2, any seaweed subalgebra of $\mathfrak{so}(2n+1)$ (resp. $\mathfrak{so}(2n)$) is conjugate, under the action of the connected adjoint group, to one of $\mathfrak{q}_n^B(\underline{a} \mid \underline{b})$ (resp. $\mathfrak{q}_n^D(\underline{a} \mid \underline{b})$) where $\underline{a}$ and $\underline{b}$ are two compositions such that $|\underline{a}| \leq n$ and $|\underline{b}| \leq n$. We associate to $\mathfrak{q}_n^B(\underline{a} \mid \underline{b})$ the same meander as for $\mathfrak{q}_n^C(\underline{a} \mid \underline{b})$. Moreover, the subalgebras $\mathfrak{q}_n^B(\underline{a} \mid \underline{b})$ and $\mathfrak{q}_n^C(\underline{a} \mid \underline{b})$ have the same index (see [11]). Thus all results obtained in this article for seaweed subalgebras of $\mathfrak{sp}(2n)$ are again valid for seaweed subalgebras of $\mathfrak{so}(2n+1)$.

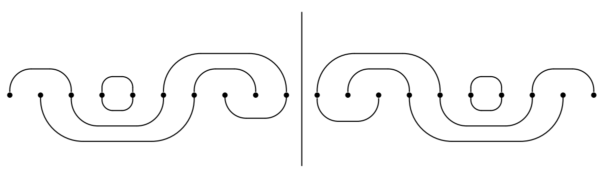
Let Ξ_n be the set of pairs $(\underline{a} = (a_1, \dots, a_k), \underline{b} = (b_1, \dots, b_t))$ verifiying: $|\underline{a}| = n$, $|\underline{b}| = n-1$ and $a_k > 1$ or $|\underline{b}| = n$, $|\underline{a}| = n-1$ and $b_t > 1$.

In [12], Panyushev and Yakimova associated to each seaweed subalgebra $\mathfrak{q}_n^D(\underline{a} \mid \underline{b})$ of $\mathfrak{so}(2n)$ a meander, denoted by $\Gamma_n^D(\underline{a} \mid \underline{b})$, in the following way:

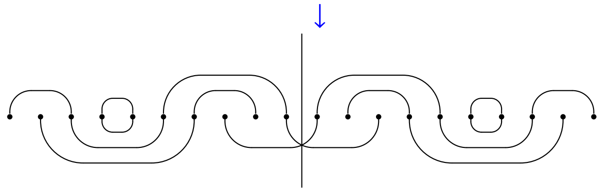
The case $(\underline{a}, \underline{b}) \notin \Xi_n$. As explained in the second section, there exist two subsets π' and π'' of the set of simple roots $\pi = \{\alpha_1, \dots, \alpha_n\}$ such that $\mathfrak{q}_n^D(\underline{a} \mid \underline{b}) = \mathfrak{q}_{\pi', \pi''}$. Up to permutation of α_{n-1} and α_n, we may assume that $|\underline{a}| \neq n-1$ and $|\underline{b}| \neq n-1$ (see [12, Proposition 3.4]). $\Gamma_n^D(\underline{a} \mid \underline{b})$ is the meander associated to the seaweed subalgebra $\mathfrak{q}_n^C(\underline{a} \mid \underline{b})$.

The case $(\underline{a} = (a_1, \dots, a_k), \underline{b} = (b_1, \dots, b_t)) \in \Xi_n$. Suppose that $|\underline{a}| = n$ and set $\underline{b}' := (b_1, \dots, b_{t-1}, b_t + 1)$. Then $\Gamma_n^D(\underline{a} \mid \underline{b})$ is obtained from $\Gamma_n^C(\underline{a} \mid \underline{b}')$ by replacing the arc joining vertices $a_1 + \cdots + a_{k-1} + 1$ and n by an arc joining vertices $a_1 + \cdots + a_{k-1} + 1$ and $n+1$, and the arc joining vertices $n+1$ and $n + a_k$ by an arc joining vertices n and $n + a_k$. It is clear that these new arcs cross each other and they are the only arcs of $\Gamma_n^D(\underline{a} \mid \underline{b})$ which verify this property, they will be called crossed arcs. Moreover, we may check easily that the crossed arcs lie in the same cycle or in two different segments. When $|\underline{b}| = n$, $\Gamma_n^D(\underline{a} \mid \underline{b})$ is just the meander symmetric of $\Gamma_n^D(\underline{b} \mid \underline{a})$ with respect to the horizontal line.

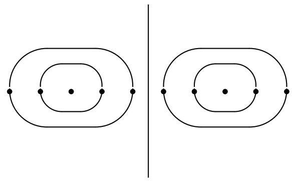
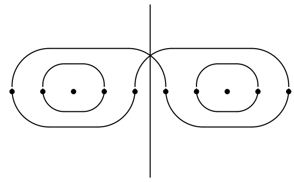
Example 4.1.

$\Gamma_{10}^C(1, 6, 3 \mid 3, 2, 5) =$ 

$\downarrow$

$\Gamma_{10}^D(1, 6, 3 \mid 3, 2, 4) =$ 

Example 4.2.

$\Gamma_5^C(5 \mid 5) =$  $\rightarrow$ $\Gamma_5^D(4 \mid 5) =$ 

Theorem 4.3 ([12]). *Let $\mathfrak{q}_n^D(\underline{a} \mid \underline{b})$ be a seaweed subalgebra of $\mathfrak{so}(2n)$, we have*

$$\operatorname{ind} \mathfrak{q}_n^D(\underline{a} \mid \underline{b}) = \textit{number of cycles} + \frac{1}{2}\ (\textit{number of segments that are not } \sigma\textit{-stable}) + \epsilon$$

where ϵ is given by:

- *If $(\underline{a}, \underline{b}) \notin \Xi_n$, then*

$$\epsilon = \begin{cases} 0 & \textit{if } |\underline{a}| - |\underline{b}| \textit{ is even} \\ 1 & \textit{if } |\underline{a}| - |\underline{b}| \textit{ is odd, } \max(|\underline{a}|, |\underline{b}|) = n \textit{ and the vertices } n \textit{ and } n+1 \\ & \textit{belong to the same segment of } \Gamma_n^D(\underline{a} \mid \underline{b}) \\ -1 & \textit{otherwise} \end{cases}$$

- *If $(\underline{a}, \underline{b}) \in \Xi_n$, then*

$$\epsilon = \begin{cases} -1 & \textit{if the crossed arcs lie in the same cycle} \\ 0 & \textit{otherwise} \end{cases}$$

Remark 4.4. Let us keep the notations of Theorem 4.3. If $(\underline{a}, \underline{b}) \notin \Xi_n$, we have

$$\operatorname{ind} \mathfrak{q}_n^D(\underline{a} \mid \underline{b}) = \operatorname{ind} \mathfrak{q}_n^C(\underline{a} \mid \underline{b}) + \epsilon$$

As for $\mathfrak{sp}(2n)$, we have the following properties analogous to Corollary 3.3 and Lemma 3.4:

Lemma 4.5. *Let $\underline{a} = (a_1, \dots, a_k)$ and $\underline{b} = (b_1, \dots, b_t)$ be two compositions such that $|\underline{b}| \leq |\underline{a}| \leq n$. Set $\underline{a}^{-1} = (a_k, \dots, a_1)$, we have*

(1) $\operatorname{ind} \mathfrak{q}_n^D(\underline{a} \mid \underline{b}) = \operatorname{ind} \mathfrak{q}_n^D(\underline{b} \mid \underline{a})$

(2) $\operatorname{ind} \mathfrak{q}_n^D(\underline{a} \mid \underline{b}) = \operatorname{ind} \mathfrak{q}_{n+|\underline{a}|}^D(2|\underline{a}| \mid \underline{a}^{-1}, \underline{b})$

Theorem 4.6. *Let $t \in \mathbb{N}^{\times}$ and $\underline{a} = (a_1, \ldots, a_k)$ be a composition verifiying $|\underline{a}| \leq t \leq n$ and $(t \mid \underline{a}) \notin \Xi_n$. Set $a_{k+1} = t - |\underline{a}|$ and $d_i = (a_1 + \cdots + a_{i-1}) - (a_{i+1} + \cdots + a_{k+1})$, $1 \leq i \leq k$.*

(1) *For any $1 \leq i \leq k$ such that $d_i \neq 0$ and any $\alpha \in \mathbb{Z}$ such that $a_i + \alpha|d_i| \geq 0$, we have*

$$\operatorname{ind} \mathfrak{q}_n^D(t \mid \underline{a}) = \operatorname{ind} \mathfrak{q}_{n+\alpha|d_i|}^D(t + \alpha|d_i| \mid a_1, \ldots, a_{i-1}, a_i + \alpha|d_i|, a_{i+1}, \ldots, a_k)$$

In particular, we have

$$\operatorname{ind} \mathfrak{q}_n^D(t \mid \underline{a}) = \operatorname{ind} \mathfrak{q}_{n-a_i+a_i[|d_i|]}^D(t - a_i + a_i[|d_i|] \mid a_1, \ldots, a_{i-1}, a_i[|d_i|], a_{i+1}, \ldots, a_k)$$

(2) *For any $1 \leq i \leq k$ such that $d_i = 0$, we have*

$$\operatorname{ind} \mathfrak{q}_n^D(t \mid \underline{a}) = a_i + \operatorname{ind} \mathfrak{q}_{n-a_i}^D(t - a_i \mid a_1, \ldots, a_{i-1}, a_{i+1} \ldots, a_k)$$

Proof. Recall that in this case, we have $\Gamma_n^D(t \mid \underline{a}) = \Gamma_n^C(t \mid \underline{a})$. So, it follows from Theorem 4.3 that

$$\operatorname{ind} \mathfrak{q}_n^D(t \mid \underline{a}) = \operatorname{ind} \mathfrak{q}_n^C(t \mid \underline{a}) + \epsilon$$

where ϵ is given by:

$$\epsilon = \begin{cases} 0 & \text{if } t - |\underline{a}| \text{ is even} \\ 1 & \text{if } t - |\underline{a}| \text{ is odd, } t = n \text{ and the vertices } n \text{ and } n+1 \text{ lie in the same segment} \\ & \text{of } \Gamma_n^D(t \mid \underline{a}) \\ -1 & \text{in the remaining cases} \end{cases}$$

By Theorem 3.12, it remains to verify the condition on the arc joining vertices n and $n+1$. Now, we set

$$(n' \mid t' \mid \underline{a}') := \begin{cases} (n + \alpha|d_i| \mid t + \alpha|d_i| \mid a_1, \ldots, a_{i-1}, a_i + \alpha|d_i|, a_{i+1}, \ldots, a_k) & \\ & \text{if } d_i \neq 0 \text{ and } a_i + \alpha|d_i| \geq 0 \\ (n - a_i \mid t - a_i \mid a_1, \ldots, a_{i-1}, a_{i+1} \ldots, a_k) & \text{if } d_i = 0 \end{cases}$$

In particular, $t' - |\underline{a}'| = t - |\underline{a}|$ and $(t' \mid \underline{a}') \notin \Xi_{n'}$. It follows that $\Gamma_{n'}^D(t' \mid \underline{a}') = \Gamma_{n'}^C(t' \mid \underline{a}')$. Remark that in the case $d_i = 0$, the meander $\Gamma_n^C(t \mid \underline{a})$ is the disjoint union of the meanders $\Gamma_{n'}^C(t' \mid \underline{a}')$ and $\Gamma_{a_i}^C(a_i \mid a_i)$, and in the case $a_i + \alpha|d_i| \geq 0$, the meander $\Gamma_{n'}^C(t' \mid \underline{a}')$ is obtained from $\Gamma_n^C(t \mid \underline{a})$ as explained in the proof of Lemma 3.9. So, we deduce that the arc of $\Gamma_n^C(t \mid \underline{a})$ joining the vertices n and $n+1$ lies in a segment if and only if the arc of $\Gamma_{n'}^C(t' \mid \underline{a}')$ joining the vertices n' and $n'+1$ lies also in a segment. □

Remark 4.7. Let $n \geq 2$ and $(\underline{a} = (a_1, \ldots, a_k), \underline{b} = (b_1, \ldots, b_t)) \in \Xi_n$ such that $|\underline{b}| = n - 1$. Set $\underline{b}' := (b_1, \ldots, b_{t-1}, b_t + 1)$ and consider $\Gamma^A(\underline{a} \mid \underline{b}')$ the meander of the seaweed subalgebra $\mathfrak{q}^A(\underline{a} \mid \underline{b}')$ of $\mathfrak{gl}(n)$ whose vertices are the n first vertices of the meander $\Gamma_n^D(\underline{a} \mid \underline{b})$. So, the crossed arcs of the meander $\Gamma_n^D(\underline{a} \mid \underline{b})$ lie in the same cycle if and only if the last vertex (n-th vertex) of the meander $\Gamma^A(\underline{a} \mid \underline{b}')$ lies in a cycle of $\Gamma^A(\underline{a} \mid \underline{b}')$.

Corollary 4.8. *With the previous notations, we have*

$$\operatorname{ind} \mathfrak{q}_n^D(\underline{a} \mid \underline{b}) = \begin{cases} \operatorname{ind} \mathfrak{q}^A(\underline{a} \mid \underline{b}') & \textit{if the n-th vertex of } \Gamma^A(\underline{a} \mid \underline{b}') \textit{ lies in a segment} \\ \operatorname{ind} \mathfrak{q}^A(\underline{a} \mid \underline{b}') - 2 & \textit{otherwise} \end{cases}$$

In particulier, for $n \geq 1$, we have

$$\operatorname{ind} \mathfrak{q}_n^D(n \mid n-1) = |n-2|$$

For the seaweed subalgebra $\mathfrak{q}^A(\underline{a} \mid \underline{b})$, we put

$$\Psi[\mathfrak{q}^A(\underline{a} \mid \underline{b})] = \begin{cases} \operatorname{ind} \mathfrak{q}^A(\underline{a} \mid \underline{b}) & \text{if the } n\text{-th vertex of } \Gamma^A(\underline{a} \mid \underline{b}) \text{ lies in a segment} \\ \operatorname{ind} \mathfrak{q}^A(\underline{a} \mid \underline{b}) - 2 & \text{otherwise} \end{cases}$$

Theorem 4.9. *Let $\underline{a} = (a_1, \ldots, a_k)$ be a composition verifiying $1 \leq |\underline{a}| = n-1$,i.e. $(n \mid \underline{a}) \in \Xi_n$. We set $\underline{a}' = (a_1', \ldots, a_k') = (a_1, \ldots, a_{k-1}, a_k + 1)$, $d_k = -(a_1' + \cdots + a_{k-1}')$ and $d_i = (a_1' + \cdots + a_{i-1}') - (a_{i+1}' + \cdots + a_k')$, $1 \leq i \leq k-1$.*

(1) *For any $1 \leq i \leq k$ such that $d_i \neq 0$ and any $\alpha \in \mathbb{Z}$ such that $a_i' + \alpha|d_i| \geq 0$, we have*

$$\operatorname{ind} \mathfrak{q}_n^D(n \mid a_1, \ldots, a_k) = \Psi[\mathfrak{q}^A(n + \alpha|d_i| \mid a_1', \ldots, a_{i-1}', a_i' + \alpha|d_i|, a_{i+1}', \ldots, a_k')]$$

In particular, if we set $t_i = a_i' - a_i'[|d_i|]$, then

$$\operatorname{ind} \mathfrak{q}_n^D(n \mid a_1, \ldots, a_k) = \Psi[\mathfrak{q}^A(n - t_i \mid a_1', \ldots, a_{i-1}', a_i'[|d_i|], a_{i+1}' \ldots, a_k')]$$

(2) *For any $1 \leq i \leq k$ such that $d_i = 0$, we have*

$$\operatorname{ind} \mathfrak{q}_n^D(n \mid a_1, \ldots, a_k) = a_i + \Psi[\mathfrak{q}^A(n - a_i \mid a_1, \ldots, a_{i-1}, a_{i+1} \ldots, a_k)]$$

Proof. Let $1 \leq i \leq k$ and set $(n' \mid \underline{a}'')$ the pair given by

$$(n' \mid \underline{a}'') = \begin{cases} (n + \alpha|d_i| \mid a_1', \ldots, a_{i-1}', a_i' + \alpha|d_i|, a_{i+1}', \ldots, a_k') & \text{if } d_i \neq 0 \text{ and } a_i' + \alpha|d_i| \geq 0 \\ (n - a_i \mid a_1, \ldots, a_{i-1}, a_{i+1} \ldots, a_k) & \text{if } d_i = 0 \end{cases}$$

Consider $\Gamma^A(n \mid \underline{a}')$ the meander of $\mathfrak{q}^A(n \mid \underline{a}')$ and $\Gamma^A(n' \mid \underline{a}'')$ the meander of $\mathfrak{q}^A(n' \mid \underline{a}'')$ obtained from $\Gamma^A(n \mid \underline{a}')$ in the manner introduced in [3, Lemma 2.3]. We verify that the last vertex of $\Gamma^A(n \mid \underline{a}')$ lies in a segment of $\Gamma^A(n \mid \underline{a}')$ if and only if the last vertex of $\Gamma^A(n' \mid \underline{a}'')$ also lies in a segment of $\Gamma^A(n' \mid \underline{a}'')$. It follows from Theorem 2.6 that

$$\Psi[\mathfrak{q}^A(n \mid \underline{a}')] = \Psi[\mathfrak{q}^A(n' \mid \underline{a}'')]$$

The result follows immediatly from the previous corollary. □

Remark 4.10. In view of Lemma 4.5, the theorems 4.6 and 1.3 provide a reduction algorithm allowing to compute the index of seaweed subalgebras in the case of $\mathfrak{so}(2n)$.

Example 4.11. Consider the seaweed subalgebra $\mathfrak{q}^D_{335}(218, 15, 102 \mid 33, 301)$ of $\mathfrak{so}(670)$. We verify that $(218, 15, 102 \mid 33, 301) \in \Xi_{335}$. By Lemma 4.5, we have

$$\operatorname{ind} \mathfrak{q}^D_{335}(218, 15, 102 \mid 33, 301) = \operatorname{ind} \mathfrak{q}^D_{670}(670 \mid 102, 15, 218, 33, 301)$$

By applying Theorem 4.6, we have

$$\begin{aligned}
\operatorname{ind} \mathfrak{q}^D_{670}(670 \mid 102, 15, 218, 33, 301) &= \Psi[\mathfrak{q}^A(670 \mid 102, 15, 218, 33, 302)] \\
&= \Psi[\mathfrak{q}^A(452 \mid 102, 15, 33, 302)] \\
&= \Psi[\mathfrak{q}^A(152 \mid 102, 15, 33, 2)] \\
&= \Psi[\mathfrak{q}^A(52 \mid 2, 15, 33, 2)] \\
&= \Psi[\mathfrak{q}^A(22 \mid 2, 15, 3, 2)] \\
&= \Psi[\mathfrak{q}^A(7 \mid 2, 3, 2)] \\
&= 3 + \Psi[\mathfrak{q}^A(4 \mid 2, 2)] \\
&= 3 + \Psi[\mathfrak{q}^A(2 \mid 2)] \\
&= 3 + 2 - 2 \\
&= 3
\end{aligned}$$

Consider the family of seaweed subalgebras of $\mathfrak{so}(2n)$ of the form $\mathfrak{q}^D_n(a, b \mid c)$ where $(a, b, c) \in (\mathbb{N}^\times)^3$. In the case $(a, b \mid c) \notin \Xi_n$, using Theorems 3.22 and 4.3, it is not difficult to obtain a formula for the index of $\mathfrak{q}^D_n(a, b \mid c)$. For the case $(a, b \mid c) \in \Xi_n$, it follows from Lemma 4.5 and Corollary 4.8 that we may suppose $(a, b \mid c) = (a, n - a - 1 \mid n)$. By Theorem 2.5, we have $\operatorname{ind} \mathfrak{q}^A(a, n - a \mid n) = a \wedge n$. It follows from [3, Lemma 3.3] that the last vertex of meander $\Gamma^A(a, n - a \mid n)$ lies in a cycle if and only if $a \wedge n \geq 2$. Thus, we have the following theorem,

Theorem 4.12. *Let $(a, n) \in (\mathbb{N}^\times)^2$ such that $a \leq n - 2$, we have*

$$\operatorname{ind} \mathfrak{q}^D_n(a, n - a - 1 \mid n) = \operatorname{ind} \mathfrak{q}^D_n(a, n - a \mid n - 1) = |(a \wedge n) - 2|$$

Corollary 4.13. *Let $(a,b,c) \in (\mathbb{N}^{\times})^3$ and set $p = (a+b) \wedge (b+c)$, $r = |a+b-c|$ and $q = a \wedge n$. Then $\mathfrak{q}_n^D(a,b \mid c)$ is a Frobenius subalgebra of $\mathfrak{so}(2n)$ if and only if one of the following conditions holds:*

(1) $r = 1$, $q = 2$ *and* $\max(a+b,c) = n$

(2) $r = 1$, $p = 1$ *and* $\max(a+b,c) = n-1$

(3) $r = 2$, $p = 1$ *and* $\max(a+b,c) = n$

(4) $r = 3$, $p = 2$ *and* $\max(a+b,c) = n-1$

Theorem 4.14. *Set $\mathcal{F}_n^A := \{\mathfrak{q}^A(\underline{a} \mid \underline{b}) \subset \mathfrak{gl}(n) \ : \ \operatorname{ind}\mathfrak{q}^A(\underline{a} \mid \underline{b}) = 1\}$ and $\mathcal{F}_n^D := \{\mathfrak{q}_n^D(\underline{a} \mid \underline{b}) \subset \mathfrak{so}(2n) \ : \ (\underline{a} \mid \underline{b}) \in \Xi_n$ and $\operatorname{ind}\mathfrak{q}_n^D(\underline{a} \mid \underline{b}) = 0\}$. Let $\underline{a} = (a_1,\dots,a_m)$ and $\underline{b} = (b_1,\dots,b_t)$ be two compositions of n such that $\mathfrak{q}^A(\underline{a} \mid \underline{b}) \in \mathcal{F}_n^A$. Then the subalgebras $\mathfrak{q}_{2n}^D(2a_1,\dots,2a_m \mid 2b_1,\dots,2b_{t-1},2b_t-1)$ and $\mathfrak{q}_{2n}^D(2a_1,\dots,2a_{m-1}, 2a_m-1 \mid 2b_1,\dots,2b_t)$ belong to $\mathcal{F}_{2n}^D$, and all subalgebras of $\mathcal{F}_{2n}^D$ are thus obtained. Moreover, for any $n \geq 1$, we have*

$$\mathcal{F}_{2n+1}^D = \emptyset \quad \text{and} \quad \sharp\mathcal{F}_{2n}^D = 2\sharp\mathcal{F}_n^A$$

Proof. Let $(\underline{c} = (c_1,\dots,c_m), \underline{d} = (d_1,\dots,d_t)) \in \Xi_n$ such that $|\underline{c}| = n$. Let $\underline{d}' = (d_1,\dots,d_{t-1},d_t+1)$, it is a composition of n. It follows from Corollary 4.8 that $\mathfrak{q}_n^D(\underline{c} \mid \underline{d})$ is a Frobenius subalgebra of $\mathfrak{so}(2n)$ if and only if $\Gamma^A(\underline{c} \mid \underline{d}')$ is a cycle. On the other hand, it follows from [3, Lemma 3.4] that $\Gamma^A(\underline{c} \mid \underline{d}')$ is a cycle if and only if $\operatorname{ind}\mathfrak{q}^A(\underline{c} \mid \underline{d}') = c_1 \wedge \dots \wedge c_m \wedge d_1 \wedge \dots \wedge d_{t-1} \wedge (d_t+1) = 2$. From Theorem 2.3, we see that $\operatorname{ind}\mathfrak{q}^A(\underline{a} \mid \underline{b}) = 1$ if and only if $\Gamma^A(\underline{a} \mid \underline{b})$ is a segment. We deduce moreover, from [3, Lemma 2.6], that the map $\mathfrak{q}^A(\underline{a} \mid \underline{b}) \longmapsto \mathfrak{q}^A(2a_1,\dots,2a_m \mid 2b_1,\dots,2b_t)$ is a bijection from $\mathcal{F}_n^A$ to the set of seaweed subalgebras $\mathfrak{q}^A(\underline{a} \mid \underline{b})$ of $\mathfrak{gl}(2n)$ whose associated meander is a cycle. In particular, $\mathfrak{q}_{2n}^D(2a_1,\dots,2a_m \mid 2b_1,\dots,2b_{t-1},2b_t-1)$ and $\mathfrak{q}_{2n}^D(2a_1,\dots,2a_{m-1},2a_m-1 \mid 2b_1,\dots,2b_t)$ belong to $\mathcal{F}_{2n}^D$, and all subalgebras of $\mathcal{F}_{2n}^D$ are thus obtained. Moreover, the condition $c_1 \wedge \dots \wedge c_m \wedge d_1 \wedge \dots \wedge d_{t-1} \wedge (d_t+1) = 2$ show that $\mathcal{F}_n^D = \emptyset$ when n is an odd integer. □

In [3], we studied the family of seaweed subalgebras of $\mathfrak{gl}(n)$ of the form $\mathfrak{q}^A(n \mid \underbrace{a,\dots,a}_{m}, b)$ where $(a,b,m) \in (\mathbb{N}^{\times})^3$. The index of such a subalgebra is given by the following formula:

$$\operatorname{ind}\mathfrak{q}^A(n \mid \underbrace{a,\dots,a}_{m}, b) = (a \wedge b)\phi_m\left(\frac{a}{a \wedge b}, \frac{b}{a \wedge b}\right)$$

where ϕ_m is the map defined on $I := \{(a, b) \in \mathbb{N}^{\times 2} \mid a \text{ or } b \text{ is odd}\}$ by:

$$\phi_m(a, b) = \begin{cases} [\frac{m}{2}] + 1 & \text{if } a \text{ and } b \text{ are odd} \\ [\frac{m+1}{2}] & \text{if } a \text{ id odd and } b \text{ even} \\ 1 & \text{if } a \text{ is even and } b \text{ odd} \end{cases}$$

Moreover, we explicitly described the meander $\Gamma^A(n \mid \underbrace{a, \dots, a}_{m}, b)$ (see [3, Lemma 3.3]). In particular, we know that the last vertex of $\Gamma^A(n \mid \underbrace{a, \dots, a}_{m}, b)$ lies in a segment if and only if $(a \wedge b) = 1$. Then, we deduce the following theorem:

Theorem 4.15. *With the previous notations, let $(a, b, m) \in (\mathbb{N}^{\times})^3$. We set $n = ma + b + 1$ and $p = a \wedge (b + 1)$. Then $(n, (\underbrace{a, \dots, a}_{m}, b)) \in \Xi_n$ and we have*

(1) $$\operatorname{ind} \mathfrak{q}_n^D(n \mid \underbrace{a, \dots, a}_{m}, b) = \begin{cases} \phi_m(a, b + 1) & \textit{if } p = 1 \\ p\phi_m(\frac{a}{p}, \frac{b+1}{p}) - 2 & \textit{if } p \geq 2 \end{cases}$$

(2) $\mathfrak{q}_n^D(n \mid \underbrace{a, \dots, a}_{m}, b)$ *is a Frobenius subalgebra if and only if $p = 2$ and one of the following conditions holds:*

   (a) *m=1*
   (b) $\frac{a}{2}$ *is even and* $\frac{b+1}{2}$ *odd*
   (c) $\frac{a}{2}$ *is odd,* $\frac{b+1}{2}$ *is even and* $m = 2$

References

[1] Aleksandr A. Belavin and Vladimir G. Drinfel'd. Solutions of the classical Yang-Baxter equation for simple Lie algebras. *Funkts. Anal. Prilozh.*, 16(3):1–29, 1982.

[2] Meher Bouhani. Formes linéaires de type réductif et unipotent. *J. Lie Theory*, 29(1):143–179, 2019.

[3] Meher Bouhani. Sur l'indice des sous-algèbres biparaboliques de $\mathfrak{gl}(n)$. *Bull. Sci. Math.*, 166: article no. 102932 (22 pages), 2021.

[4] Nicolas Bourbaki. *Groupes et algèbres de Lie. Chapitres IV, V et VI: Groupes de Coxeter et systèmes de Tits. Groupes engendrés par des réflexions. Systèmes de racines.*, volume 1337 of *Actualités Scientifiques et Industrielles*. Hermann, 1968.

[5] Vincent Coll, Matthew Hyatt, Colton Magnant, and Hua Wang. Frobenius seaweed Lie algebras II. *J. Gen. Lie Theory Appl.*, 9(1): article no. 1000227 (7 pages), 2015.

[6] Vincent E. Coll, Matthew Hyatt, and Colton Magnant. Symplectic meanders. *Commun. Algebra*, 45(11):4717–4729, 2017.

[7] Vladimir Dergachev and Alexandre A. Kirillov. Index of Lie algebras of seaweed type. *J. Lie Theory*, 10(2):331–343, 2010.

[8] Nabila Djebali. Sous-groupes réductifs canoniques des sous-groupes biparaboliques de SO(n,C) ou de SO(p,q) dont l'algèbre de Lie est quasi-réductive. *J. Lie Theory*, 28(2):434–477, 2018.

[9] Alexander G. Elashvili. On the index of parabolic subalgebras of semi-simple Lie algebras. 1990.

[10] Dmitri I. Panyushev. Inductive formulas for the index of seaweed Lie algebras. *Mosc. Math. J.*, 1(2):221–241, 2001.

[11] Dmitri I. Panyushev and Oksana S. Yakimova. On seaweed subalgebras and Meander graphs in type C. *Pac. J. Math.*, 285(2):485–499, 2016.

[12] Dmitri I. Panyushev and Oksana S. Yakimova. On seaweed subalgebras and Meander graphs in type D. *J. Pure Appl. Algebra*, 222(11):3414–3431, 2018.

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