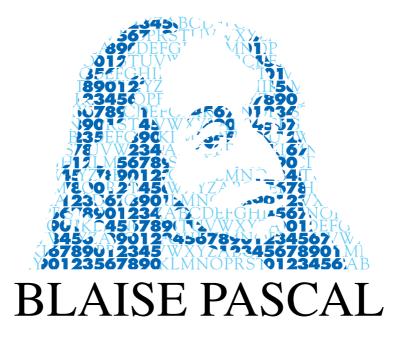
# ANNALES MATHÉMATIQUES



Quang-Tu Bui

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## Injectivity radius of manifolds with a Lie structure at infinity

## Quang-Tu Bui

#### Abstract

Using Lie groupoids, we prove that the injectivity radius of a manifold with a Lie structure at infinity is positive. This relies on the integrability of the corresponding Lie algebroid, a well-known result that we prove explicitly by regarding manifolds with corners as particular instances of orbifolds.

Le rayon d'injectivité des variétés munies d'une structure de Lie à l'infini

#### Résumé

À l'aide des groupoïdes de Lie, on montre que le rayon d'injectivité d'une variété munie d'une structure de Lie à l'infini est strictement positif. La démonstration s'appuie sur l'intégrabilité de l'algébroïde de Lie correspondant, un résultat bien connu que l'on établit directement en regardant les variétés à coins comme des cas particuliers d'orbifolds.

#### 1. Introduction

Manifolds with a Lie structure at infinity were introduced by Ammann, Lauter and Nistor in [1], forming a class of non-compact complete Riemannian manifolds of infinite volume. In the same article, they conjectured that the injectivity radius of a (connected) manifold with Lie structure at infinity is positive. In this paper, we give a proof of this conjecture using the associated groupoid given by [5] and [6]. Together with the results from [1], this implies that manifolds with a Lie structure at infinity are of bounded geometry. In particular, the hypothesis of positive injectivity radius in [2] is now automatically satisfied, as well as in [3], where it is used to obtain uniform parabolic Schauder estimates. Bounded geometry also yields uniform elliptic Schauder estimates, see [4] for a recent application in this direction.

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#### 2. Manifolds with corners seen as orbifolds

The notion of manifolds with corners is central in the definition of Lie structure at infinity. Referring to [7, 8, 9, 10] for further details, we will quickly review this notion by putting emphasis on an important fact which does not appear to be widely known or used: manifolds with corners are a particular type of orbifolds [9, Exercice 1.6.2]. This point of view will turn out to be especially useful in Section 4.

First, on an open set  $\mathcal{U}$  of  $\mathbb{R}^n_k \subset \mathbb{R}^n$ , recall [9, §1.2] that a continuous function  $f:\mathcal{U} \to \mathbb{R}$  is *smooth* if it is smooth on the interior  $\mathring{\mathcal{U}}$  of  $\mathcal{U}$  seen as subset of  $\mathbb{R}^n$  and if for any compact subset  $K \subset \mathcal{U}$ , f is bounded on  $K \cap \mathring{\mathcal{U}}$  together with all its partial derivatives. By Seeley extension theorem, this is equivalent to requiring that  $f = \widetilde{f}|_{\mathcal{U}}$  for  $\widetilde{f} \in C^{\infty}(\widetilde{\mathcal{U}})$  with  $\widetilde{\mathcal{U}}$  an open set of  $\mathbb{R}^n$  such that  $\mathcal{U} = \widetilde{\mathcal{U}} \cap \mathbb{R}^n_k$ . There is yet another way [9, Exercice 1.6.2], not as known, to describe the space of smooth functions on such an open set  $\mathcal{U}$ . To describe it, let  $\Gamma_k \cong (\mathbb{Z}_2)^k$  be the finite group generated by the reflections  $r_i : \mathbb{R}^n \to \mathbb{R}^n$  given by

$$r_i(x_1,\ldots,x_n) = (x_1,\ldots,x_{i-1},-x_i,x_{i+1},\ldots,x_n)$$
 for  $i \in \{1,\ldots,k\}$ .

If  $q: \mathbb{R}^n \to \mathbb{R}^n/\Gamma_k$  is the quotient map and W is an open set of  $\mathbb{R}^n/\Gamma_k$ , then set

$$C^{\infty}(\mathcal{W}) := \{ f : \mathcal{U} \to \mathbb{R} : q^* f \in C^{\infty}(q^{-1}(\mathcal{W})) \}.$$

Thus, smooth functions on W correspond to smooth functions on  $q^{-1}(W)$  which are  $\Gamma_k$ -invariant, which is the usual notion of smoothness on orbifolds.

**Lemma 2.1** ([9, Exercice 1.6.2]). The homeomorphism  $\psi : \mathbb{R}^n/\Gamma_k \to \mathbb{R}^n_k$  defined by

$$\psi(x_1,\ldots,x_n)=(x_1^2,\ldots,x_k^2,x_{k+1},\ldots,x_n)$$

induces an isomorphism

$$\psi^*: \quad C^{\infty}(\mathcal{U}) \quad \to \quad C^{\infty}(\psi^{-1}(\mathcal{U}))$$

$$f \qquad \mapsto \qquad \psi^* f$$

for  $\mathcal{U}$  an open set of  $\mathbb{R}^n_k$ .

*Proof.* Notice first that  $\psi$  is well-defined since  $x_1^2, \ldots, x_k^2, x_{k+1}, \ldots, x_n$  are  $\Gamma_k$ -invariant smooth functions. If  $f \in C^{\infty}(\mathcal{U})$ , then clearly  $\psi^* f \in C^{\infty}(\psi^{-1}(\mathcal{U}))$ . Moreover,  $\psi^* f \equiv 0$  if and only if  $f \equiv 0$ , so  $\psi^*$  is injective. To see it is surjective, let  $f \in C^{\infty}(\psi^{-1}(\mathcal{U}))$  be given. Thus, it can be thought as a  $\Gamma_k$ -invariant smooth function on  $q^{-1}(\psi^{-1}(\mathcal{U}))$ . From this point of view, f is even in  $x_i$  for  $i \leq k$ , so its Taylor series at  $x_i = 0$  is of the form

$$\sum_{i=0}^{\infty} a_{ij} x_i^{2j} \tag{2.1}$$

with  $a_{ij}$  smooth and invariant with respect to the reflexions  $\{r_1, \ldots, r_k\} \setminus \{r_i\}$ . If we set  $u_i := x_i^2$ , then

$$\frac{\partial}{\partial u_i} = \frac{1}{2x_i} \frac{\partial}{\partial x_i},$$

so we see that  $\frac{\partial}{\partial u_i}$  sends  $C^{\infty}(\mathbb{R}^n/\Gamma_k)$  onto itself. This means that

$$(\psi_* f)(u_1, \dots, u_k, x_{k+1}, \dots, x_n) := f(\sqrt{u_1}, \dots, \sqrt{u_k}, x_{k+1}, \dots, x_n)$$

is an element of  $C^{\infty}(\mathcal{U})$  such that  $f = \psi^*(\psi_* f)$ , showing that the map  $\psi^*$  is surjective.  $\square$ 

Lemma 2.1 shows that  $\mathbb{R}^n_k$  can be equivalently replaced by  $\mathbb{R}^n/\Gamma_k$  as a local model to define manifolds with corners. Now, a continuous map  $f:\mathcal{U}_1\to\mathcal{U}_2$  between open sets  $\mathcal{U}_1$  and  $\mathcal{U}_2$  of  $\mathbb{R}^{n_1}_{k_1}$  and  $\mathbb{R}^{n_2}_{k_2}$  is said to be *smooth* if it is of the form  $f(x)=(f_1(x),\ldots,f_{n_2}(x))$  with  $f_i\in C^\infty(\mathcal{U}_1)$  for each i. It is a diffeomorphism if it is a homeomorphism and it is smooth together with its inverse  $f^{-1}:\mathcal{U}_2\to\mathcal{U}_1$ . On a Hausdorff paracompact space X, a chart with corners is a continuous map  $\phi:\mathcal{U}\to\mathbb{R}^n_k$  for some n and k inducing a homeomorphism between an open set  $\mathcal{U}$  of X and an open set of  $\mathbb{R}^n_k$ . An atlas on X is an open cover  $X=\bigcup_{i\in I}\mathcal{U}_i$  together with charts with corners

$$\phi_i:\mathcal{U}_i\to\mathbb{R}^n_k$$

such that for each  $i, j \in I$  with  $\mathcal{U}_i \cap \mathcal{U}_j \neq \emptyset$ ,

$$\phi_i \circ \phi_j^{-1} : \phi_j(\mathcal{U}_i \cap \mathcal{U}_j) \to \phi_i(\mathcal{U}_i \cap \mathcal{U}_j)$$

is a diffeomorphism. We say in this case that  $\phi_i$  and  $\phi_j$  are *compatible*. Furthermore, two atlas are equivalent if their charts are compatible and we refer to an equivalence class of atlas as a  $C^{\infty}$ -structure with corners. It induces a natural sheaf of smooth functions on X. A Hausdorff paracompact space X equipped with a  $C^{\infty}$ -structure with corners is called a t-manifold in [9, Definition 1.6.1]. In particular, by Lemma 2.1, a t-manifold can be seen as an orbifold locally modelled on  $\mathbb{R}^n/\Gamma_k$ .

To be a manifold with corners, we will follow [9] and require that one more condition be satisfied. If  $(x_1, \ldots, x_n)$  are the standard coordinates on  $\mathbb{R}^n_k$ , set

$$\partial_{\ell} \mathbb{R}_{k}^{n} = \{ p \in \mathbb{R}_{k}^{n} : x_{i}(p) = 0 \text{ for exactly } \ell \text{ of the first } k \text{ indices} \}.$$

If  $f:\mathcal{U}_1\to\mathcal{U}_2$  is diffeomorphism between two open sets  $\mathcal{U}_1$  and  $\mathcal{U}_2$  of  $\mathbb{R}^{n_1}_{k_1}$  and  $\mathbb{R}^{n_2}_{k_2}$ , then, by [9, Corollary 1.5.1], f induces a diffeomorphism between  $\partial_\ell\mathbb{R}^{n_1}_{k_1}\cap\mathcal{U}_1$  and  $\partial_\ell\mathbb{R}^{n_2}_{k_2}\cap\mathcal{U}_2$  for each  $\ell$ . Thus, if X a t-manifold, we can consider the subset

$$\partial_{\ell}X = \{p \in X : \phi_i(p) \in \partial_{\ell}\mathbb{R}^n_{k_i} \text{ for each chart } (\mathcal{U}_i, \phi_i) \text{ with } p \in \mathcal{U}_i\}.$$

A boundary hypersurface of X is then the closure of a connected component of  $\partial_1 X$ .

**Definition 2.2.** A manifold with corners is a t-manifold M such that all its boundary hypersurfaces are embedded, that is, if  $H_i$  is a boundary hypersurface of M, then there exists  $\rho_i \in C^{\infty}(M)$  such that  $\rho_i(p) \ge 0$  for all  $p \in M$ ,  $H_i = \rho_i^{-1}(0)$  and  $d\rho_i$  is nowhere zero on  $H_i$ . In this case, we say that  $\rho_i$  is a boundary defining function for  $H_i$ .

Using Lemma 2.1, a compact manifold with corners can be seen as a special instance of a good orbifold, that is, an orbifold covered by a smooth manifold.

**Proposition 2.3.** If M is a compact manifold with corners, there exist a smooth closed manifold  $\widetilde{M}$ , a finite group  $\Gamma_m \cong (\mathbb{Z}_2)^m$  acting smoothly and properly discontinuously on  $\widetilde{M}$  and a homeomorphism  $M \cong \widetilde{M}/\Gamma_m$ , which, combined with the quotient map  $q: \widetilde{M} \to \widetilde{M}/\Gamma_m$  yields a map  $\widetilde{q}: \widetilde{M} \to M$  which induces an isomorphism between smooth functions on M and  $\Gamma_m$ -invariant smooth functions on  $\widetilde{M}$ . In other words, M is diffeomorphic to the good orbifold  $\widetilde{M}/\Gamma_m$ .

*Proof.* Let  $H_1, \ldots, H_\ell$  be the boundary hypersurfaces of M and let  $\rho_1, \ldots, \rho_\ell$  be corresponding boundary defining functions. Let  $M^{\frac{1}{2}}$  be the manifold with corners which, as a topological space, is M, but with algebra of smooth functions given by smooth functions on the interior of M admitting smooth expansions at  $H_i$  for  $i \in \{1, \ldots, \ell\}$  in integer powers of  $\rho_i^{\frac{1}{2}}$  (instead of  $\rho_i$ ). In particular, the boundary hypersurfaces  $H_1^{\frac{1}{2}}, \ldots, H_\ell^{\frac{1}{2}}$  of  $M^{\frac{1}{2}}$  are obtained from  $H_1, \ldots, H_\ell$  through the same construction. Thus,

$$C^{\infty}(M) \subset C^{\infty}(M^{\frac{1}{2}})$$

and  $C^{\infty}(M)$  corresponds to the smooth functions in  $C^{\infty}(M^{\frac{1}{2}})$  with Taylor series at  $H_i^{\frac{1}{2}}$  only involving even powers of  $(\rho_i^{\frac{1}{2}})$  for each i. Taking two copies of  $M^{\frac{1}{2}}$  and gluing them along a maximal subset of disjoint boundary hypersurfaces, we obtain a compact manifold with corners  $M_1^{\frac{1}{2}}$  with at most  $\ell-1$  boundary hypersurfaces and a reflection  $r_1:M_1^{\frac{1}{2}}\to M_1^{\frac{1}{2}}$  interchanging the two copies of  $M^{\frac{1}{2}}$  in  $M_1^{\frac{1}{2}}$ . Repeating this operation at most  $m\leq \ell$  times, we get a sequence of manifolds with corners  $M_i^{\frac{1}{2}}$  with  $M_{i+1}^{\frac{1}{2}}$  the double of  $M_i^{\frac{1}{2}}$  and with  $\widetilde{M}:=M_m^{\frac{1}{2}}$  a closed manifold coming with a finite groupe  $\Gamma_m\cong (\mathbb{Z}_2)^m$  acting smoothly and properly discontinuously on  $\widetilde{M}$  and generated by the reflections corresponding to each doubling performed to obtain  $\widetilde{M}$ . Let  $q:\widetilde{M}\to\widetilde{M}/\Gamma_m$  be the quotient map. Of course,  $\widetilde{M}/\Gamma_k$  is naturally homeomorphic to  $M^{\frac{1}{2}}$ , and combining with the homeomorphism  $M^{\frac{1}{2}}\cong M$  given by the identity map, we see that in terms of smooth charts on  $\widetilde{M}$  and charts with corners on M, the composite map  $\widetilde{q}:\widetilde{M}\to\widetilde{M}/\Gamma_m\cong M$  can be described in local coordinates by maps of the form

$$\widetilde{q}(x_1,\ldots,x_k,x_{k+1},\ldots,x_n) = (x_1^2,\ldots,x_k^2,x_{k+1},\ldots,x_n)$$

for  $k \leq \ell$ . By Lemma 2.1, this shows that  $C^{\infty}(M)$  corresponds to  $\Gamma_m$ -invariant smooth functions on  $\widetilde{M}$ .

A *b-vector field* on a manifold with corners M is a smooth vector field tangent to all the boundary hypersurfaces of M. We denote by  $\mathcal{V}_b(M)$  the space of b-vector fields on M. It is a Lie subalgebra of the space of smooth vector fields on M. In terms of the good orbifold of Proposition 2.3, b-vector fields admits the following description.

**Proposition 2.4.** With respect to the map  $\tilde{q}$  of Proposition 2.3, a b-vector field on a compact manifold with corners M corresponds to a  $\Gamma_m$ -invariant smooth vector field on  $\tilde{M}$ , that is,  $\tilde{q}$  induces a bijection  $\tilde{q}^*: V_b(M) \to C^\infty(\tilde{M}; T\tilde{M})_{\Gamma_m}$ . In particular, the  $\Gamma_m$ -invariance implies that these vector fields are tangent to the smooth codimension 1 hypersurfaces  $\tilde{q}^{-1}(H_i)$  for  $i \in \{1, \ldots, \ell\}$ .

*Proof.* It suffices to check locally that the map  $\psi : \mathbb{R}^n \to \mathbb{R}^n_k$  of Lemma 2.1 induces a bijection between b-vector fields on  $\mathbb{R}^n_k$  and smooth  $\Gamma_k$ -invariant vector fields on  $\mathbb{R}^n$ . If we set  $u_i = x_i^2$ , then in terms of the coordinates  $(u_1, \ldots, u_k, x_{k+1}, \ldots, x_n)$  on  $\mathbb{R}^n_k$  and the coordinates  $(x_1, \ldots, x_n)$  on  $\mathbb{R}^n$ , we have that

$$u_i \frac{\partial}{\partial u_i} = \frac{x_i}{2} \frac{\partial}{\partial x_i} \quad \text{for } i \le k.$$
 (2.2)

Now, a b-vector field is locally of the form

$$\xi = \sum_{i=1}^{k} a_i u_i \frac{\partial}{\partial u_i} + \sum_{i=k+1}^{n} a_i \frac{\partial}{\partial x_i}$$

with  $a_i$  smooth functions. Since the reflection  $r_i$  of Lemma 2.1 is such that

$$(r_i)_* \left( x_i \frac{\partial}{\partial x_i} \right) = x_i \frac{\partial}{\partial x_i}$$

for  $i \le k$ , it follows from (2.2) and Lemma 2.1 that

$$(r_i)_*(\psi^*\xi) = \psi^*\xi \quad \forall i \leq k,$$

so that the map  $\psi$  induces the claimed identification.

#### 3. Lie groupoids and Lie structures at infinity

Following [1] and [11], we recall some definitions and facts.

**Definition 3.1.** A *groupoid* is a small category G in which every morphism is invertible.

The objects of the category are also called *units*, and the set of units is denoted by  $G^{(0)}$ . The set of morphisms is denoted by  $G^{(1)}$ . The range and domain maps are denoted respectively  $r, d: G^{(1)} \to G^{(0)}$ . The multiplication operator  $\mu$  is defined on the set of composable pairs of morphisms by:

$$\mu: G^{(2)} := G^{(1)} \times_{G^{(0)}} G^{(1)} = \{(g,h): d(g) = r(h)\} \to G^{(1)}.$$

The inversion operation is a bijection  $\iota: g \mapsto g^{-1}$  of  $G^{(1)}$ . The identity morphisms give an inclusion  $u: x \mapsto \mathrm{id}_x$  of  $G^{(0)}$  into  $G^{(1)}$ .

**Definition 3.2** ([11, Definition 3]). An almost differentiable groupoid  $G = (G^{(0)}, G^{(1)}, d, r, \mu, u, \iota)$  is a groupoid such that  $G^{(0)}$  and  $G^{(1)}$  are manifolds with corners, the structural maps  $d, r, \mu, u, \iota$  are differentiable, and the domain map d is a submersion.

Consequently, for an almost differentiable groupoid,  $\iota$  is a diffeomorphism,  $r = d \circ \iota$  is a submersion and each fiber  $G_x = d^{-1}(x) \subset G^{(1)}$  is a smooth manifold whose dimension n is constant on each connected component of  $G^{(0)}$ .

Following the convention in [5, p. 578], we require  $G^{(0)}$  and  $d^{-1}(x)$  to be Hausdorff (for all  $x \in G^{(0)}$ ), but not necessarily  $G^{(1)}$  to avoid excluding important cases.

From now on, *Lie groupoid* will stand for almost differentiable groupoid. A Lie groupoid is called *d-simply connected* if its *d*-fibers  $G_x = d^{-1}(x)$  are simply connected ([5]).

**Definition 3.3.** A *Lie algebroid A* over a manifold with corners M is a vector bundle A over M, together with a Lie algebra structure on the space  $\Gamma(A)$  of smooth sections of A and a bundle map  $\rho: A \to TM$ , called the *anchor map*, extended to a map  $\rho_{\Gamma}: \Gamma(A) \to \Gamma(TM)$  between sections of these bundles, such that

(1) 
$$\rho_{\Gamma}([X,Y]) = [\rho_{\Gamma}(X), \rho_{\Gamma}(Y)]$$

(2) 
$$[X, fY] = f[X, Y] + (\rho_{\Gamma}(X)f)Y$$

for any smooth sections X and Y of A and any smooth function f on M.

There is a Lie algebroid A(G) associated to a Lie groupoid G, constructed as follows: let  $T_{\text{vert}}G = \ker d_* = \bigcup_{x \in G^{(1)}} TG_x \subset TG^{(1)}$  be the vertical bundle over  $G^{(1)}$ . Then  $A(G) = T_{\text{vert}}G|_{G^{(0)}}$  is the structural bundle of the Lie algebroid over  $G^{(0)}$ . The anchor map is given by

$$r_*|_A:A\to TG^{(0)}$$

([2]). The Lie bracket of  $\Gamma(A)$  is the Lie bracket of  $\Gamma(T_{\text{vert}}G)$  restricted to right invariant sections.

**Definition 3.4.** A Lie algebroid A over a manifold with corners M is said to be *integrable* if there exists a Lie groupoid G such that  $G^{(0)} = M$  and A is isomorphic to the Lie algebroid associated to G. G is said to *integrate* A.

*Remark 3.5.* There might be more than one Lie groupoid integrating a Lie algebroid. However, by [5, Lie I], if a Lie algebroid over a smooth manifold is integrable, there is a unique *d*-simply connected Lie groupoid integrating it.

Example 3.6.

- (1) Any Lie group is a Lie groupoid with the set of units being a singleton.
- (2) ([11, Example 4, Section 4]) Let M be a smooth connected manifold. Let  $\widetilde{M}$  be the universal cover of M. Let  $H = (\widetilde{M} \times \widetilde{M})/\pi_1(M)$ . Then H is naturally a d-simply connected Lie groupoid with the set of units being M, and the associated Lie algebroid being id:  $TM \to TM$ . It is called the homotopy groupoid.

We can now recall the definitions and basic properties of manifolds with Lie structures at infinity. For details and proofs, we refer to [1].

**Definition 3.7.** A *structural Lie algebra* of vector fields on a manifold M (possibly with corners) is a subspace  $\mathcal{V} \subset \Gamma(TM)$  of the real vector space of vector fields on M with the following properties:

- (1) V is closed under Lie brackets;
- (2) V is a finitely generated projective  $\Gamma(M)$ -module;
- (3) The vector fields in V are tangent to all faces in M.

The Lie algebra of b-vector fields  $\mathcal{V}_b(M)$  is a structural Lie algebra of vector fields, and any structural Lie algebra is a subspace of  $\mathcal{V}_b(M)$  ([1, Example 2.5]). By the Serre–Swan theorem, given a structural Lie algebra of vector fields  $\mathcal{V}$  on M, there exists a vector bundle  $A = A_{\mathcal{V}} \to M$  such that  $\mathcal{V} \simeq \Gamma(A_{\mathcal{V}})$ , and there exists a natural vector bundle map  $\rho: A_{\mathcal{V}} \to TM$  such that the induced map  $\rho_{\Gamma}: \Gamma(A_{\mathcal{V}}) \to \Gamma(TM)$  is identified with the inclusion map  $\mathcal{V} \subset \Gamma(TM)$ . The vector bundle  $A_{\mathcal{V}}$  is then a Lie algebroid with anchor map  $\rho$ .

**Definition 3.8.** A *Lie structure at infinity* on a smooth manifold  $M_0$  is a pair  $(M, \mathcal{V})$ , where

(1) M is a compact manifold, possibly with corners, and  $M_0$  is the interior of M;

- (2) V is a structural Lie algebra of vector fields on M;
- (3)  $\rho: A_{\mathcal{V}} \to TM$  induces an isomorphism on  $M_0$ , that is,  $\rho|_{M_0}: A|_{M_0} \to TM_0$  is an isomorphism of vector bundles.

**Definition 3.9.** A Riemannian manifold with a Lie structure at infinity is a smooth manifold  $M_0$  with a Lie structure at infinity  $(M, \mathcal{V})$  endowed with a bundle metric g on  $A = A_{\mathcal{V}}$ . In particular, g defines a Riemannian metric on  $M_0$  via the anchor map.

A Riemannian manifold with a Lie structure at infinity has infinite volume ([1, Proposition 4.1]), bounded curvature ([1, Corollary 4.3]) and is complete ([1, Corollary 4.9]). Sufficient conditions for the positivity of the injectivity radius are given in [1, Theorem 4.14] and [1, Theorem 4.17].

## 4. Integrability of a Lie algebroid corresponding to a Lie structure at infinity

The following theorem is due to Debord ([6, Theorem 2], see also [5, Corollary 5.9]).

**Theorem 4.1** (Debord). Every almost injective Lie algebroid over a smooth manifold is integrable.

This has the following implication for Lie structures at infinity.

**Theorem 4.2.** Any Lie algebroid over a manifold with corners associated with a Lie structure at infinity is integrable.

*Proof.* This extension of Theorem 4.1 to manifolds with corners is well-known to experts. However, since no explicit proof seems to be available in the literature, we will provide one for the convenience of the readers.

Let  $(M, \mathcal{V})$  be a Lie structure at infinity of  $M_0$  and  $A = A_{\mathcal{V}}$  be the corresponding structural vector bundle. Since M is compact, we can apply Proposition 2.3, so that there is closed manifold M, a finite group  $\Gamma_m \cong (\mathbb{Z}_2)^m$  acting smoothly and properly discontinuously on  $\widetilde{M}$  and a map  $\widetilde{q} : \widetilde{M} \to M$  inducing a diffeomorphism  $\widetilde{M}/\Gamma_m \cong M$ .

Let  $\widetilde{V}_b = C^{\infty}(\widetilde{M}) \otimes_{C^{\infty}(\widetilde{M})_{\Gamma_m}} \widetilde{q}^* V_b(M) \subset \mathfrak{X}(T\widetilde{M})$  be the pull-back of the structural algebra of b-vector fields. By Proposition 2.4,  $\widetilde{V}_b(M)$  is the space of vector fields on  $\widetilde{M}$  which are tangent to  $q^{-1}(\partial M)$  (the union of some closed submanifolds of  $\widetilde{M}$ ). Since V is contained in  $V_b(M)$ , we can also consider its pull-back  $\widetilde{V} = C^{\infty}(\widetilde{M}) \otimes_{C^{\infty}(\widetilde{M})_{\Gamma_m}} \widetilde{q}^* V \subset \widetilde{V}_b$  to  $\widetilde{M}$ .

Now,  $\widetilde{\mathcal{V}}$  is a finitely generated projective  $C^{\infty}(\widetilde{M})$ -module. To see this, it suffices to show that  $\widetilde{\mathcal{V}}$  is locally free of rank n with  $n = \dim M$ . Given  $p \in \widetilde{M}$ , then since  $\mathcal{V}$  is locally free of rank n, there exist  $v_1, \ldots, v_n \in \mathcal{V}$  which locally and freely span  $\mathcal{V}$ 

near q(p). This means  $\widetilde{\mathcal{V}}$  is locally and freely spanned by  $q^*v_1, \ldots, q^*v_n \in \widetilde{\mathcal{V}}$  near p, showing that  $\widetilde{\mathcal{V}}$  is locally free of rank n as claimed.

By the Serre–Swan theorem, there is a vector bundle  $A_{\widetilde{V}}$  over  $\widetilde{M}$  with the space of smooth sections  $C^{\infty}(\widetilde{M},A_{\widetilde{V}})=\widetilde{V}$ . Clearly the inclusions  $\widetilde{V}\subset\widetilde{V_b}\subset C^{\infty}(\widetilde{M},T\widetilde{M})$  induce an anchor map, so that  $A_{\widetilde{V}}$  is naturally an almost injective Lie algebroid. Similarly, let  ${}^bT\widetilde{M}$  be the Lie algebroid corresponding to  $\widetilde{V_b}$ . By Theorem 4.1 and Remark 3.5, there exists therefore a d-simply connected groupoid  $\widetilde{G}$  integrating  $A_{\widetilde{V}}$ . Each element  $g\in \Gamma_m$  induces an automorphism  $\rho(g):A_{\widetilde{V}}\to A_{\widetilde{V}}$ , and by [5, Lie II], an automorphism on  $\widetilde{G}$ . Hence we have an action of the group  $\Gamma_m$  over  $\widetilde{G}$ .

Of course,  $\widetilde{G}^{(0)}/\Gamma_m = \widetilde{M}/\Gamma_m = M$ . If  $I_p \subset \Gamma_m$  denotes the isotropy group of some  $p \in \widetilde{M}$ , then  $I_p$  is non-trivial if and only if  $\widetilde{q}(p) \in \partial M$ . In this case, notice by Proposition 2.4 that the action of  $I_p$  on  ${}^bT_p\widetilde{M}$  is trivial. By [5, Proposition 1.1], if  $p \in \widetilde{q}^{-1}(H_1 \cap \cdots \cap H_k)$  for a maximal set  $H_1, \ldots, H_k$  of boundary hypersurfaces of M, then  $r(\widetilde{G}_p) \subset \widetilde{q}^{-1}(H_1 \cap \cdots \cap H_k)$  as well. This means that  $I_p$  acts on the restrictions of  $\widetilde{G}$  and  $A_{\widetilde{V}}$  to  $\widetilde{q}^{-1}(H_1 \cap \cdots \cap H_k)$ . Since the action of  $I_p$  is trivial on  $A_{\widetilde{V}}|_{q^{-1}(H_1 \cap \cdots \cap H_k)}$ , it will also be trivial on  $\widetilde{G}|_{q^{-1}(H_1 \cap \cdots \cap H_k)}$  by [5, Lie II]. In particular,  $I_p$  acts trivially on  $\widetilde{G}_p$ . This means that an element of  $\gamma \in \Gamma_m$  either acts trivially on  $\widetilde{G}_p$ , or else sends it diffeomorphically onto  $\widetilde{G}_{\gamma(p)}$ .

Therefore, on the quotient  $\widetilde{G}/\Gamma_m$ , the only corners come from  $\widetilde{G}^{(0)}/\Gamma_m$ . In particular,  $\widetilde{G}/\Gamma_m$  naturally makes sense as a Lie groupoid, yielding the desired d-simply connected Lie groupoid integrating  $(M,\mathcal{V})$ .

# 5. Injectivity radius of a manifold with Lie structure at infinity

Let  $M_0$  be a connected smooth manifold with a Lie structure at infinity  $(M, \mathcal{V})$ . By Theorem 4.2, there exists a d-simply connected groupoid  $G = (M, G^{(1)}, d, r, \mu, u, \iota)$  with units M such that  $A(G) \simeq A_{\mathcal{V}}$  as Lie algebroids over M. Therefore A(G) is equipped with an inner product also noted g. The anchor map is given by  $r_* : A(G) \to TM$ .

We have an isomorphism  $r^*A(G) \simeq T_{\text{vert}}G$  where  $r^*A(G)$  is the pull-back of A(G) via the range map  $r: G \to M$  ([2, (19)]). Explicitly, for  $p \in G$ ,  $(r^*A(G))_p = A(G)_{r(p)} = T_{r(p)}G_{r(p)} \cong T_pG_{d(p)}$ . The vector bundle  $r^*A(G)$  is equipped with a metric induced by the metric g on A(G), hence so is  $T_{\text{vert}}G$ . Therefore each  $G_x$  becomes a Riemannian manifold for all  $x \in M$ .

Let  $G_x^x = \{g \in G_x : r(g) = x\}$ . For  $x \in M_0$ ,  $G_x^x$  is a discrete group since  $T_x G_x^x$  is of dimension 0 (being the kernel of the map  $r_* : A(G)_x \to T_x M_0$ ).

**Lemma 5.1** ([2, p. 733]). If  $A \to TM$  is the Lie algebroid associated with a Lie structure at infinity and G is the corresponding d-simply connected Lie groupoid, then for all  $x \in M_0, r : G_x \to M_0$  is a covering map with group  $G_x^x$ .

*Proof.* By [5, Proposition 1.1], for all  $x \in M_0$ ,  $r(G_x) \subset M_0$ ,  $M_0$  being the leaf of the singular foliation of A passing by x. On the other hand,  $G|_{M_0}$  is the unique d-simply connected Lie groupoid which integrates  $TM_0$ , and therefore it is isomorphic to the homotopy groupoid  $(\widetilde{M}_0 \times \widetilde{M}_0)/\pi_1(M_0)$ . Consequently,  $M_0 = r(G_x)$  for all  $x \in M_0$ .

Now, by definition of a Lie structure at infinity,  $r_*: T_yG_x \to T_{r(y)}M_0$  is an isomorphism. This means that  $r: G_x \to M_0$  is a local diffeomorphism. Moreover,  $g_1, g_2 \in G_x$  with  $r(g_1) = r(g_2)$  if and only if there exists  $h = g_1^{-1}g_2 \in G_x^{-1}$  such that  $g_2 = g_1h$ . That is,  $f: G_x \to M_0$  is a covering map with group  $G_x^{-1}$ .

**Theorem 5.2.** Let  $M_0$  be a connected smooth manifold with a Lie structure at infinity  $(M, \mathcal{V})$ . Then for any Riemannian metric g on A, the injectivity radius of  $(M_0, g)$  is positive.

*Proof.* We prove the theorem by contradiction. Suppose that the injectivity radius of  $(M_0, g)$  is zero. Then, as the curvature is bounded, there is a sequence of geodesic loops  $c_i : [0, a_i] \to M_0$ , parametrized by arc-length, with  $a_i \to 0$ . By compactness of M, we can suppose that  $c_i(0)$  converges to a point  $p \in M$ . We have  $p \in \partial M$  since the injectivity radius is positive in any compact subset of  $M_0$ .

Let *U* be a local chart of *M* containing *p* such that *U* is contractible.

**Lemma 5.3.** There exists a number N > 0 such that  $\forall n > N$ , the loop  $c_n$  is contained in U.

*Proof.* Let  $(x_1,\ldots,x_k,y_1,\ldots,y_l)$  be a set of local coordinates centered at the point p with  $x_i \geq 0$  for all i and  $p = (0,\ldots,0)$ . Let  $g_b = \sum_{i=1}^k \frac{\mathrm{d} x_i^2}{x_i^2} + \sum_{i=1}^l \mathrm{d} y_i^2$  be a local b-metric and  $g_0 = \sum_{i=1}^k \mathrm{d} x_i^2 + \sum_{i=1}^l \mathrm{d} y_i^2$  be a local metric with boundary. Since the structural vector fields are tangential vector fields  $(\mathcal{V} \subset \mathcal{V}_b)$ , taking U smaller if needed, there exist constants C, K > 0 such that  $g \geq Cg_b \geq CKg_0$  in  $U \cap M_0$ . Let  $l^t(c_i), l_b^t(c_i), l_0^t(c_i)$  denote the lengths of the segment  $[c_i(0), c_i(t)]$  (of the geodesic loop  $c_i$ ) with respect to the metric g, the local b-metric  $g_b$  and the local metric with boundary  $g_0$  respectively (suppose that the segment is contained in U). Let  $\varepsilon > 0$  be such that  $B_0(p,\varepsilon) = \{x \in \mathbb{R}_+^k \times \mathbb{R}^l : d_0(x,p) < \varepsilon\} \subset U$  (where  $d_0$  is the distance with respect to the metric  $g_0$ , well-defined on  $B_0(p,\varepsilon)$ ). Since  $a_i \to 0$ , there exists  $N_1$  such that  $a_i < \min(\frac{\varepsilon}{4}, CK\frac{\varepsilon}{4})$  for all  $i > N_1$ . Since  $c_i(0) \to p$ , there exists  $N_2$  such that  $d_0(p,c_i(0)) < \frac{\varepsilon}{4}$  for all  $i > N_2$ . Let  $N = \max(N_1,N_2)$ .

Now let n be any number greater than N. Suppose that the loop  $c_n$  is not contained in U. Then it is not contained in  $B_0(p, \frac{\varepsilon}{2})$ . Thus there exists  $t \in [0, a_n]$  minimal such that

 $d_0(c_n(t),p) = \frac{\varepsilon}{2}$ . Then we have  $d_0(c_n(0),c_n(t)) \ge |d_0(c_n(t),p) - d_0(c_n(0),p)| \ge \frac{\varepsilon}{4}$ , which implies  $a_i = l(c_i) \ge l^t(c_i) \ge CKl_0^t(c_i) \ge CKd_0(c_n(0),c_n(t)) \ge CK\frac{\varepsilon}{4}$ , which is a contradiction. Therefore the loop  $c_n$  is contained in U.

The lemma is proven.

Hence, without loss of generality, we can suppose that the loops are contained in U. Denote by  $G=(M,G^{(1)},d,r,\mu,u,\iota)$  the d-simply connected groupoid integrating  $A_V\to TM$ . Since U is contractible, the fundamental class of each loop  $c_i$  is trivial, therefore by Lemma 5.1 we can lift  $c_i$  to a geodesic loop  $\widetilde{c_i}$  in  $G_{c_i(0)}$  (i.e.  $\widetilde{c_i}:[0,a_i]\to r^{-1}(U)\cap G_{c_i(0)}$ ) such that the base points are  $\widetilde{c_i}(0)=\widetilde{c_i}(a_i)=c_i(0)=c_i(a_i)$ . Let  $S(T_{\text{vert}}G)=\{x\in T_{\text{vert}}G:\|x\|=1\}$ . We have a natural projection  $\pi:S(T_{\text{vert}}G)\to G^{(1)}$ . On  $S(T_{\text{vert}}G)$  we have a flow  $\Psi$  which, over each d-fiber  $G_x$  of  $d:G^{(1)}\to G^{(0)}$ , corresponds to the geodesic flow of  $G_x$ . The geodesic loops on  $G_x$  correspond to segments  $[P_i,Q_i]$  of the flow  $\Psi$  on  $S(TG_x)$  (with  $Q_i=\Psi_{a_i}(P_i)$ ). We have two sequences  $P_i=(\widetilde{c_i}(0),\dot{\overline{c_i}}(0))$  and  $Q_i=(\widetilde{c_i}(a_i),\dot{\overline{c_i}}(a_i))$  in  $S(A)\subset S(T_{\text{vert}}G)$ . By compactness of S(A) and M, there exists a subsequence such that  $P_i\to P\in S(TG_p)$  and  $Q_i\to Q\in S(TG_p)$ .

Since  $a_i \to 0$ , we have P = Q. In a local chart, we can write  $(\frac{Q_i - P_i}{a_i}, c_i(0)) \to (w, p)$ . Since  $a_i \to 0$ ,  $w = \dot{\Psi}(P)$ . Since  $P_i, Q_i \in (S(A))_{c_i(0)}$  for all i, w is tangent to the fiber  $S(A)_p = S(TG_p)$ , which is a contradiction (for  $\Psi$  is the geodesic flow over  $G_p$ ).  $\square$ 

Remark 5.4. In [1], a flow  $\Phi$  is defined on S(A) extending the geodesic flow on  $S(TM_0)$ . However,  $\Phi$  itself is not quite a geodesic flow since typically it has fixed points at the boundary. Our approach does not seem to work with this flow. Indeed, to each geodesic loop  $c_i: [0; a_i] \to M_0$ , we have a corresponding segment  $\Phi_i: [0; a_i] \to S(A)$ . By considering a convergent subsequence, the limit of  $(c_i(0), \dot{c}_i(0))$  is a point v contained in  $\partial S(A) = S(A)|_{\partial M}$ . The limit of  $c_i(0)$  is a point  $p = \pi(v)$  in  $\partial M$ . In the notations of [1], we have  $(\pi^\# r_*)(H_v(v)) = 0$  and  $r_*(v) = 0$ . In particular, the flow  $\Phi$  at v is stationary:  $\forall t, \Phi_t(v) = v$ . This, however, is not sufficient to obtain a contradiction, since at the boundary,  $\Phi$  may have some fixed points as mentioned above.

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QUANG-TU BUI Départment de Mathématiques Université du Québec à Montréal C.P. 8888, Succ. Centre-Ville Montréal (Québec) H3C 3P8 Canada buiquangtu1995@gmail.com