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GLOBAL WELL-POSEDNESS OF A SYSTEM FROM QUANTUM HYDRODYNAMICS FOR SMALL DATA

CORENTIN AUDIARD

Abstract. This article describes a joint work of the author with B.Haspot on the existence and uniqueness of global solutions for the Euler-Korteweg equations in the special case of quantum hydrodynamics. Our aim here is to sketch how one can construct global small solutions of the Gross-Pitaevskii equation and use the so-called Madelung transform to convert these into solutions without vacuum of the quantum hydrodynamics. A key point is to bound the the solution of the Gross-Pitaevskii equation away from 0, this condition is fulfilled thanks to recent scattering results.

INTRODUCTION

The motion of a general Euler-Korteweg compressible fluid is described by the following system:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t u + (u \cdot \nabla)u + \nabla g(\rho) = \nabla(K(\rho)\Delta\rho + \frac{1}{2}K'(\rho)|\nabla\rho|^2), \\ (\rho, u)_{/t=0} = (\rho_0, u_0). \end{cases} \quad (\text{EK})$$

Here $u = u(t, x) \in \mathbb{R}^d$ stands for the velocity field, $\rho = \rho(t, x) \in \mathbb{R}^+$ is the density and $g'(\rho) = \frac{1}{\rho}p'(\rho)$ with p the pressure. The function $K(\rho)$ is the capillary coefficient, it is assumed smooth and positive on some subinterval of \mathbb{R}^+ . The local well-posedness for $(\rho_0, u_0) \in H^s \times (1 + H^{s+1})$, $s > d/2 + 1$ was obtained in [3]. The continuation criterion of the solution involved some Gronwall type condition and the boundedness of ρ away from 0.

We deal here with the global well-posedness of the system in the specific case $K(\rho) = \frac{\kappa}{\rho}$, $\kappa \in \mathbb{R}^{+*}$ for which the equations rewrite as the system of quantum hydrodynamics

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t u + (u \cdot \nabla)u + \nabla g(\rho) = 2\kappa \nabla \left(\frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right), \\ (\rho, u)_{/t=0} = (\rho_0, u_0). \end{cases} \quad (\text{QHD})$$

When the velocity $u = \nabla \theta$ is irrotational, the (inverse) Madelung transform $\psi = \sqrt{\rho}e^{i\frac{\theta}{2\sqrt{\kappa}}}$ allows formally to rewrite the Euler-Korteweg system as a nonlinear Schrödinger equation:

$$\begin{cases} 2i\sqrt{\kappa}\partial_t \psi + 2\kappa\Delta\psi = g(|\psi|^2)\psi, \\ \psi(0, \cdot) = \psi_0. \end{cases} \quad (0.1)$$

When $g = \rho - 1$ (that is, $p = \rho^2/2$), we obtain the Gross-Pitaevskii equation, and we will study its version with normalized coefficients

$$\begin{cases} i\partial_t \psi + \Delta\psi = (|\psi|^2 - 1)\psi, \\ \psi(0, \cdot) = \psi_0. \end{cases} \quad (\text{GP})$$

For a more general discussion on the link between (EK) and (0.1) we refer to the survey article [7]. Since the well-posedness of (0.1) is well-known in many cases, a natural idea is to use the existence of solutions to (0.1) in order to deduce the well-posedness of (QHD). The main issue is that the transform $\psi \mapsto (\rho, u)$ does

not have a clear sense when ψ vanishes. This issue was overcome in [1] where the authors used a procedure to convert global solutions of (0.1) in global weak solutions of (QHD) (see also a simpler argument in [7]). However no uniqueness can be obtained from this method. Using rather solutions of the Gross-Pitaevskii equation seems natural as its energy is

$$\mathcal{E}(\psi) = \int_{\mathbb{R}^d} |\nabla \psi|^2 + \frac{1}{2}(|\psi|^2 - 1)^2 dx,$$

and thus tends to bound small solutions (in the energy space) away from 0.

This argument is somewhat limited by the fact that the energy space is not embedded in L^∞ , so that the global well-posedness result from [9] does not provide solutions without vacuum. Actually, even taking small smoother initial data is not sufficient as a growth of norms in higher Sobolev spaces can occur.

Our approach with B.Haspot¹ in [2] takes advantage of a series of results [13, 14, 15] on the scattering of (GP) in order to construct solutions that do not vanish. Provided they are smooth enough, we can then directly work on the equation (QHD) in order to prove uniqueness. Our main results are the following (see section 1.2 for the definition of U):

THEOREM 0.1. — *Let $u_0 = \nabla \theta_0$, $\rho_0 \in L^\infty$ with $\rho_0 \geq c > 0$, $\psi_0 = \sqrt{\rho_0} e^{i\theta_0}$. **Existence for $d \geq 4$.** For any $s > d/2 - 1/6$, there exists $\delta > 0$ such that if:*

$$\|U^{-1} \operatorname{Re}(\varphi_0) + i \operatorname{Im}(\varphi_0)\|_{H^s \cap B_{a',2}^s} + \|\widehat{\varphi}_0\|_{L^1} < \delta, \quad \frac{1}{a'} = \frac{1}{2} + \frac{1}{3d}$$

then there exists a global weak solution of the system (QHD) that satisfies:

$$\sup_{x,t} |\rho - 1| \leq \frac{1}{2}, \quad \rho \in 1 + L^\infty(H^s(\mathbb{R}^d)) \quad \text{and} \quad u \in L^\infty(H^{s-7/6}(\mathbb{R}^d)).$$

Existence for $d = 3$. *If $\varphi_0 = \sqrt{\rho_0} e^{i\theta_0} - 1$ is such that $\varphi_0 \in H^s$ with $s > 4/3$ and $\widehat{\varphi}_0 \in L^1$. Then there exists $\delta > 0$ such that if*

$$\int_{\mathbb{R}^3} \langle x \rangle^2 (|\nabla \rho_0|^2 + |u_0|^2) + \langle x \rangle^2 (\sqrt{\rho_0} \cos \theta_0 - 1)^2 dx + \|\widehat{\varphi}_0\|_{L^1} + \|\varphi_0\|_{H^s} < \delta, \quad (0.2)$$

then there exists a global weak solution (ρ, u) of the system (QHD) that satisfies

$$\sup_{x,t} |\rho - 1| \leq 1/2, \quad \rho \in 1 + L_{loc}^\infty(H^s(\mathbb{R}^3)) \quad \text{and} \quad u \in L_{loc}^\infty(H^{s-7/6}(\mathbb{R}^3)).$$

Uniqueness: *If in addition $\varphi_0 \in H^s(\mathbb{R}^d)$, $s > d/2 + 1$ then the global solution satisfies $(\rho - 1, u) \in L_{loc}^\infty(H^s(\mathbb{R}^d)) \times (L_{loc}^\infty(H^{s-1}(\mathbb{R}^d)) \cap L_{loc}^2(B_{\frac{2d}{d-2},2}^{s-1}(\mathbb{R}^d)))$ and is unique in this space.*

Organisation of the paper: In section 1, we make a (very non exhaustive) review of the scattering theory for Schrödinger equations, including a description of the important results of Gustafson, Nakanishi and Tsai and their key argument on space-time resonances in dimension 3. Section 2 describes the construction of global solutions to (GP) that remain bounded away from 0. We sketch in section 3 how to construct solutions to (QHD) from solutions of (GP) thanks to the Madelung transform. Finally, we mention three open questions and shortly discuss the reachability of one of them.

The detailed arguments can be found in [2, 15].

Notations: We denote by H^s the usual Sobolev spaces and by L^p the Lebesgue space. We use the same notation for a Fourier multiplier U and its symbol $U(\xi)$. $B_{p,2}^s$ is the Besov space with norm $(\sum_{j \geq 0} 2^{2sj} \|\varphi_j u\|_p^2)^{1/2}$ where $\varphi_j(\xi)$ is a dyadic smooth partition of the frequency space. For X a Banach space, we denote $L_t^p X = L^p([0, \infty[, X)$ and $L_T^p X = L^p([0, T], X)$. $L^{p,q}$ refers to the Lorentz space.

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1. SCATTERING FOR THE GROSS-PITAIEVSKII EQUATION

1.1. **The Schrödinger equation and space-time resonances.** We start with some background on the scattering for the nonlinear Schrödinger equation:

$$i\partial_t u + \Delta u = f(u, \bar{u}), \quad x \in \mathbb{R}^d. \quad (\text{NLS})$$

The solution is said to scatter to u_∞ in L^2 when

$$\|e^{-it\Delta}u(t) - u_\infty\|_{L^2} \rightarrow_{t \rightarrow \infty} 0.$$

Except for the special case where $f = \pm|u|^\alpha u$ which has a very special machinery (far from being exhaustive [12, 6, 17, 8]), this kind of result often requires various assumptions on the smoothness, localization and smallness of the initial data, and relies strongly on dispersive and Strichartz estimates:

$$\begin{aligned} \|e^{it\Delta}u_0\|_{L^p} &\lesssim \frac{\|u_0\|_{L^{p'}}}{t^{d(1/2-1/p)}}, \quad p \geq 2, \\ \|e^{it\Delta}u_0\|_{L^p L^q} &\lesssim \|u_0\|_{L^2}, \quad \frac{2}{p} + \frac{d}{q} = \frac{d}{2}, \\ \left\| \int_0^t e^{i(t-s)\Delta} f(s) ds \right\|_{L^p L^q} &\lesssim \|f\|_{L^{p_1} L^{q_1}}, \quad \frac{2}{p_1} + \frac{d}{q_1} = \frac{d}{2}. \end{aligned}$$

When f is a quadratic nonlinearity, it is well known that scattering holds for $d \geq 4$ (Strauss [18]). A simple way to see this is setting $m(t) = \sup_{[0,t]} s^{d/6} \|u(s)\|_{L^3}$, we have (using $d/3 > 1$)

$$\left\| \int_1^t e^{i(t-s)\Delta} u^2 ds \right\|_{L^3} \lesssim \int_1^t \frac{\|u\|_{L^3}^2}{(t-s)^{d/6}} ds \lesssim \int_1^t \frac{m(s)^2}{(t-s)^{d/6} s^{d/3}} ds \lesssim \frac{1}{t^{d/6}} m(t)^2,$$

so that closed estimates can be obtained that imply in turn the convergence of $\int_0^\infty e^{-is\Delta} u^2 ds$.

For $d = 3$, the situation is much more intricate as the decay is not strong enough to apply the above argument. Well-posedness for small data in weighted spaces was obtained by Hayashi and Naumkin [16] in the case $f(u) = \lambda u^2 + \mu \bar{u}^2$, while no global well-posedness nor blow up is known for the case $f(u) = |u|^2$. New insights were brought on this issue with the concept of space-time resonance, developed independently in [11] and [14, 15].

Let us give a short description of the idea in the (not so) simple case $f(u, \bar{u}) = u^2$: rather than u , one might consider $\tilde{u} = e^{-it\Delta}u$, \tilde{u} is solution of

$$\tilde{u}(t) = u_0 - i \int_0^t e^{-is\Delta} (e^{is\Delta} \tilde{u})(e^{is\Delta} \tilde{u}) ds \quad (1.1)$$

$$\Leftrightarrow \widehat{\tilde{u}}(t) = \widehat{u_0} - i \int_0^t \int_{\mathbb{R}^d} e^{is\Phi(\xi, \eta)} \widehat{\tilde{u}}(\eta, t) \widehat{\tilde{u}}(\xi - \eta, t) d\eta ds. \quad (1.2)$$

with $\Phi(\xi, \eta) = |\xi|^2 - |\eta|^2 - |\xi - \eta|^2$. The existence of global solutions reduces to the construction of a fixed point to (1.1) in a suitable functional space, and thus estimate the integral in (1.2). Basically, one might try to use non-stationary phase arguments by integrating by parts in t , but Φ vanishes when $\eta \perp \xi - \eta$. Nevertheless, integration by parts in the η variable can be fructuous too, this idea leads to the following definition:

DEFINITION 1.1. — *The time resonant set is $\mathcal{T} = \{(\eta, \xi) : \Phi(\eta, \xi) = 0\}$. The space resonant set is $\mathcal{S} = \{(\eta, \xi) : \partial_\eta \Phi(\eta, \xi) = 0\}$. The space-time resonant set is $\mathcal{R} = \mathcal{T} \cap \mathcal{S}$.*

In our example, Φ vanishes iff $\eta \perp \xi - \eta$, while $\partial_\eta \Phi$ vanishes if $\eta = \xi - \eta$. This implies for the space-time resonant set

$$\mathcal{T} \cap \mathcal{S} = \{(\xi, \eta) : \Phi(\xi, \eta) = 0\} \cap \{(\xi, \eta) : \partial_\eta \Phi(\xi, \eta) = 0\} = \{\xi = \eta = 0\}.$$

Thus using a convenient frequency partition, there are actually no resonances. On the opposite this clarifies why it is hard to handle the nonlinearity $|u|^2$, indeed in this case $\Phi(\xi, \eta) = |\xi|^2 + |\eta|^2 - |\xi - \eta|^2$ and it is easily checked that $\mathcal{T} \cap \mathcal{S} = \{\xi = 0\}$ which is of dimension 3.

Space-time resonances were introduced by Gustafson et al [15] to obtain the scattering for the Gross-Pitaevskii equation, independently Germain et al used it to give a new proof of scattering for (NLS) with $f(u, \bar{u}) = u^2$ before tackling global existence for the water waves problem [10]. This strategy has had since various applications for the study in long time of dispersive equations.

1.2. The Gross-Pitaevskii equation, $d \geq 4$. In the case of the Gross-Pitaevskii equation (GP), the equation for $\varphi = \psi - 1$ reads

$$i\partial_t \varphi + \Delta \varphi - 2\text{Re}(\varphi) = (|\varphi|^2 + 2\bar{\varphi} + \varphi)\varphi, \quad (1.3)$$

the nonlinearity contains quadratic and cubic terms, including the “bad” nonlinearity $|u|^2$, moreover the linear part $\Delta - 2\text{Re}$ is slightly modified compared to the usual Schrödinger equation, this small change can not be neglected for long time issues. Let us start with the “diagonalization” of $\Delta - 2\text{Re}$:

PROPOSITION 1.2. — *Let $U = \sqrt{-\Delta/(2 - \Delta)}$, $H = \sqrt{-\Delta(2 - \Delta)}$. If $\varphi = \varphi_1 + i\varphi_2$ is solution of (1.3), then $v = U^{-1}\varphi_1 + i\varphi_2$ satisfies*

$$i\partial_t v - Hv = 3(Uv_1)^2 + v_2^2 + |\varphi|^2 Uv_1 + iU^{-1}(2(Uv_1 v_2) + |\varphi|^2 v_2). \quad (1.4)$$

This diagonalization creates a singularity on the imaginary part of the nonlinearity (using rather $v = \varphi_1 + iU\varphi_2$ raises the same kind of problem), this can be resolved thanks to the following normal form:

PROPOSITION 1.3 ([14]). — *Let $z = v + \frac{U^{-1}}{(2 - \Delta)}|\varphi|^2$. If φ is solution of (1.3) then z satisfies*

$$i\partial_t z - Hz = 2\varphi_1^2 + |\varphi|^2 \varphi_1 - i\frac{U^{-1}\text{div}}{2 - \Delta}(4\varphi_1 \nabla \varphi_2 + \nabla |\varphi|^2 \varphi_2) = \mathcal{N}(\varphi). \quad (1.5)$$

The gain here is that the singular multiplier U^{-1} is now compensated by the div operator. Now that (GP) has been reduced to a more standard nonlinear Schrödinger equation, it is essential to check that the group e^{itH} satisfies dispersion and Strichartz estimates similar to those of $e^{it\Delta}$.

PROPOSITION 1.4 ([13]). — *(Dispersion and Strichartz estimates)*
For $2 \leq q \leq \infty$, $\sigma(q) = 1/2 - 1/q$,

$$\|e^{-itH}\varphi\|_{B_{q,2}^s} \lesssim \frac{1}{t^{d\sigma(q)}} \|U^{(d-2)\sigma(q)}\varphi\|_{B_{q',2}^s} \quad (1.6)$$

For $j = 1, 2$, (p_j, q_j) admissible, $2/p_j + d/q_j = d/2$,

$$\left\{ \begin{array}{l} \|e^{-itH}\varphi\|_{L^{p_j} B_{q_j,2}^s} \lesssim \|U^{(d-2)\sigma(q_j)/2}\varphi\|_{L^2}, \\ \left\| \int_0^t e^{-i(t-s)H}\psi(s)ds \right\|_{L^{p_1} B_{q_1,2}^s} \lesssim \|U^{(d-2)(\sigma(q_1)+\sigma(q_2))/2}\psi\|_{L^{p'_2} B_{q'_2,2}^s}. \end{array} \right. \quad (1.7)$$

As can be expected from our previous discussion, scattering in dimension $d \geq 4$ does not raise serious difficulties.

LEMMA 1.5. — *Let $d \geq 4$, $s > d/2 - 1$, for $\|z_0\|_{H^s}$ small enough the problem (1.5) has a unique solution in $X = L_t^\infty H^s \cap L_t^2 B_{q,2}^s$, $1/q = 1/2 - 1/d$. Furthermore if $s > d/2 - 2/3$, $z_0 \in H^s \cap B_{a',2}^s$, $\frac{1}{a'} = \frac{1}{2} + \frac{1}{3d}$, $\frac{1}{q} = \frac{1}{2} - \frac{2}{3d}$, there exists $\varepsilon_0 > 0$ such that*

$$\forall \varepsilon \leq \varepsilon_0, \|z_0\|_{H^s \cap B_{a',2}^s} \leq \varepsilon \Rightarrow \sup_{t \geq 0} t^{1/3} \|z(t)\|_{B_{q,2}^{s-1/3}} \lesssim \varepsilon. \quad (1.8)$$

Idea of the proof of lemma 1.5.

Existence in X . It suffices to solve in X

$$z(t) = e^{-itH} z_0 - i \int_0^t e^{-i(t-\tau)H} \mathcal{N}(\varphi(\tau)) d\tau.$$

Using standard product estimates and a fixed point argument, we have

$$\|\varphi\|_{B_{q,2}^s} \lesssim \|z\|_{B_{q,2}^s}, \quad \|\varphi\|_{H^s} \lesssim \|z\|_{H^s}.$$

this allows to handle φ in the nonlinearity as if it were z . The Strichartz estimates (1.7) give the smallness of $\|e^{-itH} z_0\|_X$. Similarly the nonlinear part is estimated thanks to Strichartz estimates and product rules. For example

$$\left\| \int_0^t e^{-i(t-s)H} \varphi_1^2 ds \right\|_{L_t^2 B_{q,2}^s} \lesssim \|\varphi_1^2\|_{L_t^2 B_{q',2}^s} \lesssim \|\varphi_1\|_{L_t^\infty H^s} \|\varphi_1\|_{L_t^2 B_q^s} \leq \|\varphi_1\|_X^2 \lesssim \|z\|_X^2.$$

The term $U^{-1} \Delta \langle \nabla \rangle^{-2} (|\varphi|^2 \varphi_2)$ is handled thanks to the embedding $\|\varphi\|_{L^4 B_{p,2}^s} \lesssim \|\varphi\|_X$:

$$\begin{aligned} \|U^{-1} \Delta \langle \nabla \rangle^{-2} (|\varphi|^2 \varphi_2)\|_{L^2 B_{q'}^s} &\lesssim \| |\varphi|^2 \varphi_2 \|_{L^2 B_{q'}^s} \lesssim \| |\varphi|^2 \|_{L^2 H^s} \|\varphi\|_{L^\infty H^s} \\ &\lesssim \|\varphi\|_{L^4 B_p^s}^2 \|\varphi\|_{L^\infty H^s} \\ &\lesssim \|Z\|_X^3. \end{aligned}$$

The other terms can be dealt similarly, this gives

$$\|e^{-itH} z_0 - i \int_0^t e^{-i(t-\tau)H} \mathcal{N}(\varphi(\tau)) d\tau\|_{X_0} \leq \|z_0\|_{H^s} + \|z\|_X^2 + \|z\|_X^3.$$

The existence and uniqueness of the solution then follows from a fixed point argument.

Decay estimate. If we assume now $s > d/2 - 2/3$, $z_0 \in H^s \cap B_{a',2}^s$, the linear part is easily estimated thanks to the dispersion estimate (1.6)

$$\|e^{-itH} z_0\|_{B_{q,2}^{s-1/3}} \lesssim \|e^{-itH} z_0\|_{B_{a,2}^s} \lesssim \frac{\|z_0\|_{B_{a',2}^s}}{t^{d(1/2-1/a)}} = \frac{\|z_0\|_{B_{a',2}^s}}{t^{1/3}}.$$

For the Duhamel term $\int_0^t e^{-i(t-s)H} \mathcal{N}(\varphi) ds$, we set $m(t) = \sup_{0 \leq \tau \leq t} \tau^{1/3} \|Z\|_{B_{q,2}^s}$. Using the boundedness of $\|z\|_{H^s}$, the dispersion and product laws in Besov spaces one can prove

$$\begin{aligned} \left\| \int_0^t e^{i(t-\tau)H} \mathcal{N}(\varphi) ds \right\|_{B_{q,2}^s} &\lesssim \int_0^t \frac{\|\mathcal{N}(\varphi)(\tau, \cdot)\|_{B_{q',2}^s}}{(t-\tau)^{2/3}} d\tau \lesssim \int_0^t \frac{m(\tau)^2}{(t-\tau)^{2/3} \tau^{2/3}} d\tau \\ &\text{(the constants in } \lesssim \text{ depend of } \|z\|_X) \\ &= \frac{m(t)^2}{t^{1/3}} \int_0^1 \frac{1}{(1-\tau)^{2/3} \tau^{2/3}} d\tau \lesssim \frac{m(t)^2}{t^{1/3}}. \end{aligned}$$

□

Remark 1.6. — Actually, the critical case $s = d/2 - 1$ is treated in [13], with a proof using less standard product rules that take advantage of the low frequency gain in the Strichartz estimates (1.7).

1.3. The Gross-Pitaevskii equation, $d = 3$. The main result in dimension 3 reads as follows:

THEOREM 1.7 ([15]). — *There exists $\delta > 0$ such that for $\varphi_0 = \psi_0 - 1 \in H^1$ satisfying*

$$\int_{\mathbb{R}^3} \langle x \rangle^2 (|\operatorname{Re}(u_0)|^2 + |\nabla u_0|^2) dx < \delta,$$

the global solution of (1.3) satisfies $v = \varphi_1 + iU\varphi_2 \in C(\mathbb{R}, H^1/\langle x \rangle)$ and scatters, i.e. there exists $v_\infty \in H^1/\langle x \rangle$ such that

$$\|v(t) - e^{-itH}v_\infty\|_{H^1} = O_\infty(t^{-1/2}), \quad \|\langle x \rangle(e^{itH}v - v_\infty)\|_{H^1} \xrightarrow{t \rightarrow \infty} 0.$$

The full proof is very technical thus we will simply highlight a few key points. The argument of the previous section fails because the decay in time is not strong enough and thus the notion of space-time resonances is essential. For the operator e^{-itH} and a nonlinearity $|\varphi|^2$, the phase in (1.2) is

$$\Phi(\xi, \eta) = H(\xi) - H(\eta) + H(\eta - \xi), \quad \text{with } H(\xi) = |\xi|\sqrt{2 + |\xi|^2}.$$

Setting $\tilde{H}(r) = r\sqrt{2 + r^2}$ we see that the space-time resonant is

$$\begin{aligned} \mathcal{T} \cap \mathcal{S} &= \{(\eta, \xi) : \Phi(\xi, \eta) = 0, \partial_\eta \Phi(\xi, \eta) = 0\} \\ &= \{(\eta, \xi), \frac{\eta}{|\eta|} - \frac{\eta - \xi}{|\eta - \xi|} = 0, \tilde{H}'(|\eta|) - \tilde{H}'(|\eta - \xi|) = 0, \Phi(\xi, \eta) = 0\} \\ &= \{\xi = 0\}. \end{aligned}$$

In order to compensate this issue, Gustafson et al introduced the following modified normal form based on bilinear Fourier multiplier:

$$\begin{aligned} v &= \varphi_1 + iU\varphi_2, \quad Z = v + B_1(\varphi_1, \varphi_1) + B_2(\varphi_2, \varphi_2), \\ \widehat{B}_2(w, w) &= \int_{\mathbb{R}^d} \frac{1}{2 + |\eta|^2 + |\xi - \eta|^2} \widehat{w}(\eta) \widehat{w}(\xi - \eta) d\eta, \quad B_1 = -B_2, \end{aligned}$$

PROPOSITION 1.8. — *The new variable Z satisfies the following equation*

$$i\partial_t Z - HZ = UB(v, v) + C(v) + Q(v) = N(\varphi), \quad (1.9)$$

where B involves bilinear Fourier multipliers, and C, Q contain cubic and quartic terms.

The key point is that the multiplier $U(\xi)$ in factor of the quadratic terms is sufficient to compensate the resonances because it vanishes at $\xi = 0$. The problem is then reduced to prove that $Z(t)$ remains small in some space and the main issue is actually to control $\|xe^{itH}Z\|_{H^1}$. Indeed if we consider for example a quadratic term $\mathcal{B}(v, \bar{v})$, where \mathcal{B} is a bilinear multiplier of symbol b

$$\mathcal{F}\left(xe^{itH} \int_0^t e^{-i(t-s)H} \mathcal{B}(v, \bar{v})(s) ds\right) = \nabla_\xi \int_0^t e^{-is\Phi(\xi, \eta)} b(\eta, \xi - \eta) \widehat{v} e^{itH} \widehat{v} d\eta,$$

when the derivative hits $e^{-is\Phi(\xi, \eta)}$ this causes both a loss of derivatives and a loss of decay. An integration by part in the η or s variable may compensate this loss (and should be possible due to the fact that b vanishes at $\xi = 0$), however it requires extra care because b is not smooth. The key to this issue is a bilinear Fourier multiplier estimate with losses. While for a standard product the Strichartz estimate and Hölder give

$$\left\| \int e^{isH} u v ds \right\|_{L^2} \lesssim \|u\|_{L^{p_1} L^{q_1}} \|v\|_{L^{p_2} L^{q_2}}, \quad \frac{1}{p_1} + \frac{1}{p_2} + 3\left(\frac{1}{q_1} + \frac{1}{q_2}\right) = 2 + \frac{3}{2},$$

Gustafson et al prove

$$\begin{aligned} \left\| \int e^{isH} \mathcal{B}(u, v) ds \right\|_{L^2} &\lesssim \|b\|_{B^s} \|u\|_{L^{p_1} L^{q_1}} \|v\|_{L^{p_2} L^{q_2}}, \\ &\frac{1}{p_1} + \frac{1}{p_2} + d\left(\frac{1}{q_1} + \frac{1}{q_2}\right) = 2 + \frac{3}{2} - s, \end{aligned}$$

where the loss in the Strichartz estimate is compensated by the weak smoothness B^s (we do not detail the definition of this space) which roughly speaking allows to spare $3/2 - s$ derivatives compared to the usual Coifman-Meyer condition. We refer to lemma 10.1 in [15] for more details.

Using this result, the “null structure” of the quadratic terms and after a careful

geometric study of the space-time resonant set, Gustafson et al obtain the following result, which essentially implies theorem 1.7 thanks to a bootstrap argument:

THEOREM 1.9 ([15]). — *Set*

$$\|Z\|_{X_T} = \|Z\|_{L_T^\infty H^1} + \|U^{-1/6}Z\|_{L_T^2 W^{1,6}} \|x e^{itH}Z\|_{L_T^\infty H^1},$$

the solution of (1.9) satisfies the a priori estimate

$$\|Z\|_{X_T} \lesssim \|\langle x \rangle Z_0\|_{H^1} + \|Z\|_{X_T}^2 + \|Z\|_{X_T}^6.$$

2. THE EXISTENCE OF GLOBAL SOLUTIONS TO (GP) BOUNDED AWAY FROM $\mathbf{0}$

The aim of this section is to construct solutions of (1.3) such that $\|\varphi\|_{L_{x,t}^\infty}$ remains small, thus providing a solution of (GP) $\psi = 1 + \varphi$ that does not vanish.

The idea is fairly simple: for short time $t \leq 1$, φ is the solution of a nonlinear Schrödinger equation, and thus is easily controlled provided the initial data is small and smooth enough, for long time the decay suffices to provide bounds. The next sections detail a bit more the argument.

2.1. Short time control. Of course, one might assume that $\|\varphi_0\|_{H^s}$ is small enough for some $s > d/2$, so that the continuity of the flow directly gives

$$\|\varphi\|_{L^\infty([0,1] \times \mathbb{R}^d)} \lesssim \|\varphi\|_{L^\infty([0,1], H^s)} \lesssim \|\varphi_0\|_{H^s}.$$

The following lemma allows to deal with lower regularity:

LEMMA 2.1. — *Consider the following nonlinear Schrödinger equation:*

$$\begin{cases} i\partial_t u + \Delta u = F(\operatorname{Re}(u), \operatorname{Im}(u)), \\ u|_{t=0} = u_0, \end{cases}$$

where F is a polynomial of order n . Then if $u_0 \in H^s$, $s > d/2 - 1/(2(n-1))$ and the solution exists on $[0, T]$:

$$\left\| \int_0^t e^{i(t-s)\Delta} F ds \right\|_{C_T H^{s+1/2}} \lesssim \|u\|_{L^\infty H^s \cap L^2 H^{s,q}}^{p-1} \left(1 + \|u\|_{L^\infty H^s \cap L^2 H^{s,q}}\right), \quad \frac{2}{p} + \frac{d}{q} = \frac{d}{2}.$$

Remark 2.2. — As pointed out in [5], the gain of smoothness on the integral term for the Schrödinger equation is relatively well-known. Nevertheless proposition 4.1 in [5] seems to be the only self-contained reference for lemma 2.1. Note also the reference only treats the nonlinearity $|u|^p u$ in dimension at most 3 (with some restriction on (s, p) due to the rules of fractional differentiation), but the same proof applies to our case.

COROLLARY 2.3. — *For $d/2 - 1/4 < s < d/2$, there exists $\delta > 0$ such that if $\|\varphi_0\|_{H^s} + \|\widehat{\varphi_0}\|_{L^1} < \delta$, then the solution of (1.3) is defined for $t \in [0, 1]$ and satisfies $\|\varphi\|_{L^\infty([0,1] \times \mathbb{R}^d)} < 1$.*

Proof. — The nonlinearity is cubic, so that the critical space is $H^{d/2-1}$. The problem in H^s , $s > d/2 - 1/4$ is subcritical thus from the usual fixed point argument, we can get existence for $t \in [0, 1]$ provided $\|u_0\|_{H^s}$ is small enough, the L^∞ bound is then consequence of the Duhamel formula, the obvious estimate $\|e^{it\Delta}\varphi_0\|_\infty \leq \|\widehat{\varphi_0}\|_{L^1}/(2\pi)^d$, and lemma 2.1. \square

2.2. Long time control.

The case $d = 3$. If $d = 3$, we can directly use the scattering of the solution with the dispersion estimates. Recall that $v = \varphi_1 + iU\varphi_2$, thus we need to estimate $U^{-1}v$. Splitting smoothly $v = v_l + v_h$ between low and high frequencies we get

$$\|U^{-1}v\|_{L^\infty} \lesssim \|U^{-1}v\|_{W^{1,6}} \lesssim \|U^{-1}v_l\|_{L^6} + \|\nabla v_h\|_{L^6}.$$

High frequencies can be handled with the dispersion estimate and the generalized Hölder's inequality $\|\nabla v_h\|_{L^6} \lesssim \frac{\|\nabla e^{itH}v_h\|_{L^{6/5,2}}}{t} \lesssim \frac{\|\langle x \rangle e^{itH}\nabla v_h\|_{L^2}}{t}$ while for low

frequencies we may simply write $\|U^{-1}v_l\|_{L^6} \lesssim \|v_l\|_{L^2}$ (actually a more careful analysis, using interpolation between a bound on $\|U^{-2}v_l\|_{L^6}$ and decay of $\|U^{-1/3}v_l\|_{L^6}$ would give a $O(t^{-3/5})$ decay). Since the estimate from theorem 1.9 ensures the smallness of $\|\langle x \rangle e^{itH}v\|_{L_t^\infty H^1}$ we get the expected L^∞ bound.

The case $d \geq 4$. This is a direct consequence of the estimate (1.8) and the embedding $B_{q,2}^{s-1/3} \hookrightarrow L^\infty$. The conclusion reads as follows:

PROPOSITION 2.4. — *Let $s > d/2 - 1/4$. There exists $\delta > 0$ such that if*

- (1) For $d = 3$, $(\int_{\mathbb{R}^d} \langle x \rangle^2 (|\operatorname{Re}(\varphi_0)|^2 + |\nabla \varphi_0|^2) dx)^{1/2} + \|\widehat{\varphi_0}\|_{L^1} + \|\varphi_0\|_{H^s} < \delta$,
- (2) For $d \geq 4$, $\|U^{-1}\operatorname{Re}(\varphi_0) + i\operatorname{Im}(\varphi_0)\|_{H^s \cap B_{a',2}^s} + \|\widehat{\varphi_0}\|_{L^1} < \delta$,
 $1/a' = 1/2 + 1/(3d)$,

then the solution of (1.3) is global and $\|\varphi\|_{L_{x,t}^\infty} < 1$.

3. WELL-POSEDNESS FOR (QHD)

3.1. Existence. Let $\varphi_0 \in H^s$, $s > 4/3$ satisfying the smallness conditions of theorem 0.1 and φ the associated solution. Using regularized initial data $\varphi_{0,n}$ (one may choose mollifiers such that the smallness conditions are preserved) and the propagation of regularity for Schrödinger equations, the Madelung transform applied to the associated solutions φ_n gives global solutions $(\rho_{0,n}, u_{0,n}) = \left(|1 + \varphi_{0,n}|^2, \operatorname{Im} \left(\frac{(1 + \varphi_{0,n}) \nabla \varphi_{0,n}}{|1 + \varphi_{0,n}|^2} \right) \right)$ to (QHD) such that $\rho_{0,n}$ does not vanish. From the continuity of the flow for the Gross-Pitaevskii equation, we have for any $T > 0$ $\|\varphi_n - \varphi\|_{C_T H^s} \rightarrow_n 0$. Using standard rules of product and composition in Sobolev spaces this implies $\|\varphi_n - \varphi\|_{C_T H^s} + \|u_n - u\|_{C_T H^{s-7/6}} \rightarrow_n 0$. This convergence and the uniform boundedness of $\rho_n + 1/\rho_n$ is sufficient to pass to the limit in the distributional sense in (QHD).

3.2. Uniqueness. In order to avoid using the inverse Madelung transform, we rather follow the approach from [4] where the authors introduce an extended system: set $L = \sqrt{\kappa} \ln(\rho)$, $w = \nabla L$, $z = u + iw$, then (L, z) satisfies

$$\begin{cases} \partial_t L + u \cdot w + \sqrt{\kappa} \operatorname{div} u = 0, \\ \partial_t z + \frac{1}{2} \nabla(z \cdot z) + i\sqrt{\kappa} \nabla \operatorname{div} z = -\nabla e^{L/\sqrt{\kappa}}. \end{cases} \quad (3.1)$$

The key a priori estimate reads as follows (it was derived in [3] with different norms):

PROPOSITION 3.1. — *For $d \geq 3$, $s > d/2$, if (L_1, z_1) , (L_2, z_2) are two solutions of (3.1), then $(\delta L, \delta z) = (L_1 - L_2, z_1 - z_2)$ satisfies the following estimate*

$$\begin{aligned} & \|\delta L, \delta z\|_{L_T^\infty L^2} \\ & \lesssim \|\delta L(0), \delta z(0)\|_{L^2} \cdot \exp(CT + C\|Dz_1\|_{L_T^2 B_{2d/(d-2),2}^{s-1}} + \|Dz_2\|_{L_T^2 B_{2d/(d-2),2}^{s-1}}). \end{aligned}$$

The derivation of this estimate when u is not a potential is rather involved, but in our case it simply relies on an energy estimate on $\sqrt{\rho_1} \delta z$, indeed it can be checked that $\tilde{z} = \sqrt{\rho_1} \delta z$ satisfies the following system

$$\begin{aligned} \partial_t \tilde{z} + u_1 \cdot \nabla \tilde{z} + i\kappa \nabla \operatorname{div} \tilde{z} &= -\sqrt{\rho_1} \left(\nabla (e^{L_1/\sqrt{\kappa}} - e^{L_2/\sqrt{\kappa}}) + \tilde{z} \cdot \nabla z_2 \right) - \operatorname{div} u_1 \frac{\tilde{z}}{2} \\ & \quad + i \frac{\sqrt{\kappa} \nabla^2 \sqrt{\rho}}{\sqrt{\rho}} \cdot \tilde{z} - i\sqrt{\kappa} (\nabla \tilde{z} - Dz) \cdot \frac{1}{2} \nabla \ln \rho_1. \end{aligned}$$

Here $\nabla z = \partial_i z_j$, $Dz = (\nabla z)^t$. The right hand side may seem complicated, but taking the scalar product with \tilde{z} and integrating in space, the second line vanishes while the first line contains no derivative of \tilde{z} .

Remark 3.2. — Note that this estimate gives a *conditional uniqueness* result: we only obtain uniqueness amongst functions such that $(\rho, u) \in (1 + L^\infty H^{s+1} \cap L^2_{\text{loc}} B^s_{2d/(d-2), 2}) \times L^\infty_t H^s \cap L^2_{\text{loc}} B^s_{2d/(d-2), 2}$, $s > d/2 + 1$, a condition which is satisfied by our solutions.

PERSPECTIVES

There are at least three natural questions that arise:

- (1) Is there a sharp smallness condition on the initial data?
- (2) Is it possible to obtain scattering in dimension 2?
- (3) Can scattering be obtained for the quasi-linear system (EK)?

Question 3 may actually be the most reachable at least when d is large. Indeed the strategy of mixing energy estimate with dispersion proved to be very efficient for the global well-posedness of gravity water waves [11], and the situation for (EK) is quite similar as energy estimates (see [3]) and dispersive estimates for the linearized problem are also available.

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