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# WEYL FORMULAE FOR THE ROBIN LAPLACIAN IN THE SEMICLASSICAL LIMIT

AYMAN KACHMAR, PIERIG KERAVAL, AND NICOLAS RAYMOND

**Abstract.** This paper is devoted to establish semiclassical Weyl formulae for the Robin Laplacian on smooth domains in any dimension. Theirs proofs are reminiscent of the Born-Oppenheimer method.

## 1. Introduction

1.1. **Context and motivations.** For $d \geqslant 2$, let us consider an *open bounded connected* subset of $\mathbb{R}^d$ denoted by $\Omega$ with a $\mathcal{C}^3$ connected boundary $\Gamma = \partial\Omega$ and for which the standard tubular coordinates are well defined (see Section 2.2). On this domain, we consider the Robin Laplacian $\mathcal{L}_h$ defined as the self-adjoint operator associated with the closed quadratic form defined on $H^1(\Omega)$ by the formula

$$\forall u \in H^1(\Omega)\,, \qquad \mathcal{Q}_h(u) = \int_\Omega |h\nabla u|^2 \,\mathrm{d}\mathbf{x} - h^{\frac{3}{2}} \int_\Gamma |u|^2 \,\mathrm{d}\Gamma,$$

where $\mathrm{d}\Gamma$ is the surface measure of the boundary and $h > 0$ is the semiclassical parameter. The domain of the operator $\mathcal{L}_h$ is given by

$$\mathsf{Dom}(\mathcal{L}_h) = \{u \in H^2(\Omega) : \mathbf{n} \cdot h^{\frac{1}{2}} \nabla u = -u \text{ on } \Gamma\},$$

where $\mathbf{n}$ is the inward pointing normal to the boundary. Note that, by a usual trace theorem, the traces of $u$ and $\nabla u$ are well-defined as elements of $H^{\frac{3}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$, respectively.

The aim of this paper is to quantify the number of non positive eigenvalues created by the Robin condition in the semiclassical limit $h \to 0$. The estimate of the non positive spectrum of the Robin Laplacian in the semiclassical limit (or equivalently in the strong coupling limit) has given rise to many contributions (in various geometric contexts) in the last years (see [14, 6, 7, 20, 8]). Negative eigenvalues of the operator $\mathcal{L}_h$ have eigenfunctions localized near the boundary of the domain thereby serving as edge states. One of the most characteristic results is established in [20] and states that the $n$-th eigenvalue of $\mathcal{L}_h$ is approximated, modulo $\mathcal{O}(h^2)$, by the $n$-th eigenvalue of the effective Hamiltonian acting on the boundary

$$\mathcal{L}_h^{\mathsf{eff}} = -h + h^2 \mathcal{L}^\Gamma - h^{\frac{3}{2}} \kappa, \tag{1.1}$$

where $\mathcal{L}^\Gamma$ is the Laplace-Beltrami operator on $\Gamma$ and where $\kappa$ is the mean curvature. The approximation of the eigenfunctions of $\mathcal{L}_h$ via those of the effective Hamiltonian is obtained in [7] for the two dimensional situation.

The above $\mathcal{O}(h^2)$ was dependent on the considered eigenvalue. In the present paper, we remove this dependence and deduce Weyl formulae. In two dimensions, the problem of deriving a strengthened effective Hamiltonian was also tackled to

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investigate semiclassical tunneling in presence of symmetries in [8]. Moreover, in [8, Section 7], as a byproduct of the strategy developed there (which was initially inspired by [17, 5] or [21]), Weyl formulae are established in two dimensions. The present paper is an extension of these results to any dimension and it proves, in an appropriate energy window, a *uniform* approximation of $\mathsf{sp}(\mathcal{L}_h)$ by the spectrum of a slight perturbation of (1.1).

1.2. **Results.** For $\lambda \in \mathbb{R}$, we denote by

$$\mathsf{N}(\mathcal{L}_h, \lambda) = \mathrm{Tr}\Big(\mathbf{1}_{(-\infty,\lambda]}(\mathcal{L}_h)\Big),$$

the number of eigenvalues $\mu_n(h)$ of $\mathcal{L}_h$ below the energy level $\lambda$. Let us now state our main two theorems that relate the counting functions of $\mathcal{L}_h$ and $\sqrt{h}\mathcal{L}^\Gamma - \kappa$ in the semiclassical limit.

Theorem 1.1. — *We have the following Weyl estimate for the low lying eigenvalues:*

$$\forall E \in \mathbb{R}\,, \quad \mathsf{N}\left(\mathcal{L}_h, -h + Eh^{\frac{3}{2}}\right) \underset{h\to 0}{\sim} \mathsf{N}\left(h^{\frac{1}{2}}\mathcal{L}^\Gamma - \kappa, E\right).$$

Theorem 1.2. — *We have the following Weyl estimate for the non positive eigenvalues:*

$$\mathsf{N}(\mathcal{L}_h, 0) \underset{h\to 0}{\sim} \mathsf{N}\left(h\mathcal{L}^\Gamma, 1\right).$$

*Remark 1.3.* — Note that we have the classical Weyl estimates (see for instance [24, Theorem 14.11]):

$$\mathsf{N}\left(h^{\frac{1}{2}}\mathcal{L}^\Gamma - \kappa, E\right) \underset{h\to 0}{\sim} \frac{1}{\left(2\pi h^{\frac{1}{4}}\right)^{d-1}} \mathsf{Vol}_{T^*\Gamma}\{(s,\sigma) \,:\, |\sigma|_g^2 - \kappa(s) \leqslant E\}, \tag{1.2}$$

$$\mathsf{N}\left(h\mathcal{L}^\Gamma, 1\right) \underset{h\to 0}{\sim} \frac{1}{\left(2\pi h^{\frac{1}{2}}\right)^{d-1}} \mathsf{Vol}_{T^*\Gamma}\{(s,\sigma) \,:\, |\sigma|_g^2 \leqslant 1\}. \tag{1.3}$$

Note that they remain true if $E$ and $1$ are replaced by $E + o(1)$ and $1 + o(1)$ respectively.

*Remark 1.4.* — Let us notice here that these results are proved in the case of a $\mathcal{C}^3$ bounded and connected boundary. The connectedness is actually not necessary but avoids to consider each connected component separately. For Theorem 1.1, the boundedness of $\Gamma$ is not necessary either (bounds on the curvature are enough), but allows a lighter presentation. We refer to [20] where such geometric assumptions are accurately described.

*Remark 1.5.* — The proof we give to Theorem 1.2 uses the classical Weyl law in the interior of the domain $\Omega$. This law requires that the domain $\Omega$ is bounded.

1.3. **Strategy of the proofs.** In Section 2, we show that the interior of $\Omega$ does not contribute to the creation of non positive spectrum (the Laplacian is non negative inside $\Omega$). We quantify this thanks to classical Agmon estimates and reduce the investigation to a Robin Laplacian on a thin neighborhood of the boundary (see Proposition 2.2). In Section 3, by using an idea from the Born-Oppenheimer context, we derive *uniform* effective Hamiltonians (see Theorem 3.1) whose eigenvalues simultaneously describe the eigenvalues of $\mathcal{L}_h$ less than $-\varepsilon_0 h$ (for $\varepsilon_0 > 0$ as small as we want). In particular, we show that the effectiveness of the reduction to a boundary operator is determined by the estimate of the *Born-Oppenheimer correction*. This correction is an explicit quantity related to dimension one. It appears in physics and, for instance, in the contributions [16, 18, 22, 19, 23] in the context of time evolution or in the works [11, 15, 13] related to resonance and eigenvalue problems (see also the review [10] where both aspects are considered). Let also mention here [1] dealing with the semiclassical counting function in the Born-Oppenheimer approximation (in a pseudo-differential context). Our strategy gives rise to a rather short proof (which does not even require approximations of the eigenfunctions) and displays a uniformity in the spectral estimates that implies the Weyl formula of Theorem 1.1. In Section 4, we establish Theorem 1.2. Note that the proof of Theorem 1.2 does not follow from a reduction to the effective Hamiltonian but uses a variational argument as the one in [3]. This argument is based on a decomposition of the operator via a rough partition of the unity and a separation of variables.

## 2. The Robin Laplacian near the boundary

### 2.1. **Reduction near the boundary via Agmon estimates.**

The eigenfunctions (with negative eigenvalues) of the initial operator $\mathcal{L}_h$ are localized near the boundary since the Laplacian is non negative inside the domain. This localization is quantified by the following proposition (the proof of which is a direct adaptation of the case in dimension two, see [7] and also [9]).

Proposition 2.1. — *Let $\epsilon_0 \in (0,1)$ and $\alpha \in (0, \sqrt{\epsilon_0})$. There exist constants $C > 0$ and $h_0 \in (0,1)$ such that, for $h \in (0, h_0)$, if $u_h$ is a normalized eigenfunction of $\mathcal{L}_h$ with eigenvalue $\mu \leqslant -\epsilon_0 h$, then,*

$$\int_\Omega \left(|u_h(\mathbf{x})|^2 + h|\nabla u_h(\mathbf{x})|^2\right) \exp\left(\frac{2\alpha\,\mathrm{dist}(\mathbf{x},\Gamma)}{h^{\frac{1}{2}}}\right) \mathrm{d}\mathbf{x} \leqslant C.$$

Given $\delta \in (0, \delta_0)$ (with $\delta_0 > 0$ small enough), we introduce the $\delta$-neighborhood of the boundary

$$\mathcal{V}_\delta = \{\mathbf{x} \in \Omega \ : \ \mathrm{dist}(\mathbf{x},\Gamma) < \delta\}, \tag{2.1}$$

and the quadratic form, defined on the variational space

$$W_\delta = \{u \in H^1(\mathcal{V}_\delta) \ : \ u(\mathbf{x}) = 0\,, \quad \text{for all } \mathbf{x} \in \Omega \text{ such that } \mathrm{dist}(\mathbf{x},\Gamma) = \delta\},$$

by the formula

$$\forall u \in W_\delta\,, \qquad \mathcal{Q}_h^{\{\delta\}}(u) = \int_{\mathcal{V}_\delta} |h\nabla u|^2 \,\mathrm{d}\mathbf{x} - h^{\frac{3}{2}} \int_\Gamma |u|^2 \,\mathrm{d}\Gamma.$$

Note again that the trace of $u$ is well-defined by a classical trace theorem. Let us denote by $\mu_n^{\{\delta\}}(h)$ the $n$-th eigenvalue of the corresponding operator $\mathcal{L}_h^{\{\delta\}}$. It is then standard to deduce from the min-max principle and the Agmon estimates of Proposition 2.1 the following proposition (see [9]).

Proposition 2.2. — *Let $\epsilon_0 \in (0,1)$ and $\alpha \in (0, \sqrt{\epsilon_0})$.There exist constants $C > 0$, $h_0 \in (0,1)$ such that, for all $h \in (0, h_0)$, $\delta \in (0, \delta_0)$, $n \geqslant 1$ such that $\mu_n(h) \leqslant -\epsilon_0 h$,*

$$\mu_n^{\{\delta\}}(h) \leqslant \mu_n(h) + C \exp\left(-\alpha\delta h^{-\frac{1}{2}}\right). \tag{2.2}$$

*Moreover, we have, for all $n \geqslant 1$, $h > 0$, and $\delta \in (0, \delta_0)$,*

$$\mu_n(h) \leqslant \mu_n^{\{\delta\}}(h).$$

2.2. **Description of the boundary coordinates.** Let $\iota$ denote the embedding of $\Gamma$ in $\mathbb{R}^d$ and $g$ the induced metrics on $\Gamma$. $(\Gamma, g)$ is a $\mathcal{C}^3$ Riemmanian manifold, which we orientate according to the ambient space. Let us introduce the map $\Phi : \Gamma \times (0, \delta) \to \mathcal{V}_\delta$ defined by the formula

$$\Phi(s,t) = \iota(s) + t\mathbf{n}(s),$$

which we assume to be injective. The transformation $\Phi$ is a $\mathcal{C}^3$ diffeomorphism for $\delta \in (0, \delta_0)$ and $\delta_0$ is sufficiently small. The induced metrics on $\Gamma \times (0, \delta)$ is given by

$$G = g \circ (\mathsf{Id} - tL(s))^2 + \mathrm{d}t^2,$$

where $L(s) = -d\mathbf{n}_s$ is the second fondamental form of the boundary at $s$. We also define the mean curvature:

$$\kappa(s) = \operatorname{Tr} L(s).$$

2.3. **The Robin Laplacian in boundary coordinates.** For all $u \in L^2(\mathcal{V}_{\delta_0})$, we define the pull-back function

$$\widetilde{u}(s,t) := u(\Phi(s,t)). \tag{2.3}$$

For all $u \in H^1(\mathcal{V}_{\delta_0})$, we have

$$\int_{\mathcal{V}_{\delta_0}} |u|^2 \,\mathrm{d}\mathbf{x} = \int_{\Gamma \times (0,\delta_0)} |\widetilde{u}(s,t)|^2 \,\tilde{a} \,\mathrm{d}\Gamma \,\mathrm{d}t, \tag{2.4}$$

$$\int_{\mathcal{V}_{\delta_0}} |\nabla u|^2 \,\mathrm{d}\mathbf{x} = \int_{\Gamma \times (0,\delta_0)} \left[\langle \nabla_s \widetilde{u}, \tilde{g}^{-1} \nabla_s \widetilde{u}\rangle + |\partial_t \widetilde{u}|^2\right] \tilde{a} \,\mathrm{d}\Gamma \,\mathrm{d}t. \tag{2.5}$$

where

$$\tilde{g} = \left(\mathsf{Id} - tL(s)\right)^2,$$

and $\tilde{a}(s,t) = |\tilde{g}(s,t)|^{\frac{1}{2}}$. Here $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product in $\mathbb{R}^d$ and $\nabla_s$ is the differential on $\Gamma$ seen through the metrics $g$ (by the Riesz representation theorem). In other words, $\nabla_s \widetilde{u}$ is the vector of $\mathbb{R}^d$ belonging to the tangent space to $\Gamma$ at $s$ and satisfying $g(\nabla_s \widetilde{u}, v) = d_s \widetilde{u}(v)$, for all $v$ in the tangent space at $s$.

The operator $\mathcal{L}_h^{\{\delta\}}$ is expressed in $(s,t)$ coordinates as

$$\mathcal{L}_h^{\{\delta\}} = -h^2 \tilde{a}^{-1} \nabla_s (\tilde{a} \tilde{g}^{-1} \nabla_s) - h^2 \tilde{a}^{-1} \partial_t (\tilde{a} \partial_t),$$

acting on $L^2(\tilde{a} \,\mathrm{d}\Gamma \,\mathrm{d}t)$. In these coordinates, the Robin condition becomes

$$h^2 \partial_t u = -h^{\frac{3}{2}} u \quad \text{on} \quad t = 0.$$

We introduce, for $\delta \in (0, \delta_0)$,

$$\begin{aligned}
&\widetilde{\mathcal{V}}_\delta = \{(s,t) \ : \ s \in \Gamma \text{ and } 0 < t < \delta\},\\
&\widetilde{W}_\delta = \{u \in H^1(\widetilde{\mathcal{V}_\delta}) \ : \ u(s,\delta) = 0\},\\
&\widetilde{\mathcal{D}}_\delta = \{u \in H^2(\widetilde{\mathcal{V}_\delta}) \cap \widetilde{W}_\delta \ : \ \partial_t u(s,0) = -h^{-\frac{1}{2}} u(s,0)\},\\
&\widetilde{\mathcal{Q}}_h^{\{\delta\}}(u) = \int_{\widetilde{\mathcal{V}_\delta}} \left(h^2 \langle \nabla_s u, \tilde{g}^{-1} \nabla_s u\rangle + |h\partial_t u|^2\right) \tilde{a} \,\mathrm{d}\Gamma \,\mathrm{d}t - h^{\frac{3}{2}} \int_\Gamma |u(s,0)|^2 \,\mathrm{d}\Gamma,\\
&\widetilde{\mathcal{L}}_h^{\{\delta\}} = -h^2 \tilde{a}^{-1} \nabla_s (\tilde{a} \tilde{g}^{-1} \nabla_s) - h^2 \tilde{a}^{-1} \partial_t (\tilde{a} \partial_t).
\end{aligned} \tag{2.6}$$

We now take
$$\delta = h^{\rho}, \tag{2.7}$$
and write simply $\widetilde{\mathcal{L}}_h$ for $\widetilde{\mathcal{L}}_h^{\{\delta\}}$. The operator $\widetilde{\mathcal{L}}_h$ with domain $\widetilde{\mathcal{D}}_\delta$ is the self-adjoint operator defined via the closed quadratic form $\widetilde{V}_\rho \ni u \mapsto \widetilde{\mathcal{Q}}_h(u)$ by Friedrich's theorem.

2.4. **The rescaled operator.** Let us now take advantage of the homogeneity of the transverse Robin Laplacian $-h^2\tilde{a}^{-1}\partial_t(\tilde{a}\partial_t)$ with boundary condition $\partial_t u(s,0) = -h^{-\frac{1}{2}}u(s,0)$, where $s$ is considered as a parameter. Near the boundary, this operator looks like $-h^2\partial_t^2$ with boundary condition $\partial_t u(s,0) = -h^{-\frac{1}{2}}u(s,0)$ and we see that the rescaling $t = h^{\frac{1}{2}}\tau$ allows to erase the dependence on $h$ in the boundary condition. That is why, we introduce the rescaling
$$(\sigma,\tau) = (s, h^{-\frac{1}{2}}t),$$
the new semiclassical parameter $\hbar = h^{\frac{1}{4}}$ and the new weights
$$\widehat{a}(\sigma,\tau) = \tilde{a}(\sigma, h^{\frac{1}{2}}\tau)\,, \qquad \widehat{g}(\sigma,\tau) = \tilde{g}(\sigma, h^{\frac{1}{2}}\tau). \tag{2.8}$$
We consider rather the operator
$$\widehat{\mathcal{L}}_\hbar = h^{-1}\widetilde{\mathcal{L}}_h, \tag{2.9}$$
acting on $L^2(\widehat{a}\,\mathrm{d}\Gamma\,\mathrm{d}\tau)$ and expressed in the coordinates $(\sigma,\tau)$. As in (2.6), we let
$$\begin{aligned}
&\widehat{\mathcal{V}}_T = \{(\sigma,\tau) \ :\ \sigma\in\Gamma \text{ and } 0<\tau<T\},\\
&\widehat{W}_T = \{u\in H^1(\widehat{\mathcal{V}}_T) \ :\ u(\sigma,T)=0\},\\
&\widehat{\mathcal{D}}_T = \{u\in H^2(\widehat{\mathcal{V}}_T)\cap \widehat{W}_T \ :\ \partial_\tau u(\sigma,0) = -u(\sigma,0)\},\\
&\widehat{\mathcal{Q}}_\hbar^T(u) = \int_{\widehat{\mathcal{V}}_T}\Big(\hbar^4\langle\nabla_\sigma u, \widehat{g}^{-1}\nabla_\sigma u\rangle + |\partial_\tau u|^2\Big)\widehat{a}\,\mathrm{d}\Gamma\,\mathrm{d}\tau - \int_\Gamma |u(\sigma,0)|^2\,\mathrm{d}\Gamma,\\
&\widehat{\mathcal{L}}_\hbar^T = -\hbar^4\widehat{a}^{-1}\nabla_\sigma(\widehat{a}\widehat{g}^{-1}\nabla_\sigma) - \widehat{a}^{-1}\partial_\tau\widehat{a}\partial_\tau.
\end{aligned} \tag{2.10}$$
In what follows, we let $T = \hbar^{-1}$ (or equivalently $\rho = \frac{1}{4}$) and write $\widehat{\mathcal{Q}}_\hbar$ for $\widehat{\mathcal{Q}}_\hbar^T$.

## 3. A variational Born-Oppenheimer reduction

The aim of this section is to prove the following result (that implies Theorem 1.1).

Theorem 3.1. — *For $\varepsilon_0\in(0,1)$, $h>0$, we let*
$$\mathcal{N}_{\varepsilon_0,h} = \{n\in\mathbb{N}^* : \mu_n(h)\leqslant -\varepsilon_0 h\}.$$
*There exist positive constants $h_0, C_+, C_-$ such that, for all $h\in(0,h_0)$,*
$$\forall n\in\mathcal{N}_{\varepsilon_0,h}\,,\quad \mu_n^-(h)\leqslant\mu_n(h)\,,\qquad \text{and}\qquad \forall n\geqslant 1\,,\qquad \mu_n(h)\leqslant\mu_n^+(h), \tag{3.1}$$
*where $\mu_n^\pm(h)$ is the $n$-th eigenvalue of $\mathcal{L}_h^{\mathsf{eff},\pm}$ defined by*
$$\mathcal{L}_h^{\mathsf{eff},+} = -h + (1 + C_+h^{\frac{1}{2}})h^2\mathcal{L}^\Gamma - \kappa h^{\frac{3}{2}} + C_+h^2,$$
*and*
$$\mathcal{L}_h^{\mathsf{eff},-} = -h + (1 - C_-h^{\frac{1}{2}})h^2\mathcal{L}^\Gamma - \kappa h^{\frac{3}{2}} - C_-h^2.$$

Before writing the proof of Theorem 3.1, we discuss some of its consequences in the spirit of the Born-Oppenheimer method. In the sequel, $\mathcal{L}^{\mathsf{eff}} = -h + h^2\mathcal{L}^{\Gamma} - h^{\frac{3}{2}}\kappa$ is the effective Hamiltonian introduced in (1.1). The sequence of eigenvalues of $\mathcal{L}^{\mathsf{eff}}$ is denoted by $\big(\mu_n^{\mathsf{eff}}(h)\big)_{n\geqslant 1}$. We can compare the eigenvalues of the operator $\mathcal{L}_h$ and those of the effective Hamiltonian, but with a rather bad error term.

Corollary 3.2. — *Let $\varepsilon_0 \in (0,1)$. There exist positive constants $h_0, C$ such that, for all $h \in (0,h_0)$,*

$$\forall n \in \mathcal{N}_{\varepsilon_0,h}\,, \quad \big|\mu_n(h) - \mu_n^{\mathsf{eff}}(h)\big| \leqslant Ch^{\frac{3}{2}}.$$

The proof of Corollary 3.2 is essentially the same as the one of the following refined corollary related to energy levels below $-h + Eh^{\frac{3}{2}}$. This is connected to the works in [13, 23].

Corollary 3.3. — *For $E \in \mathbb{R}$, we let*

$$\mathcal{N}_h(E) := \{n \in \mathbb{N}^* : \mu_n(h) \leqslant -h + Eh^{\frac{3}{2}}\},$$

*There exist $C > 0$, $h_0 > 0$ such that, for all $n \in \mathcal{N}_h(E)$ and $h \in (0,h_0)$,*

$$|\mu_n(h) - \mu_n^{\mathsf{eff}}(h)| \leqslant Ch^2.$$

*Proof.* — Given $E \in \mathbb{R}$ and $\varepsilon_0 \in (0,1)$, for $h$ small enough, we have $-h+Eh^{\frac{3}{2}} \leqslant -\varepsilon_0 h$. In particular, we have, for $h$ small enough and for all $n \in \mathcal{N}_h(E)$,

$$\mu_n^-(h) \leqslant \mu_n(h) \leqslant \mu_n^+(h).$$

Then, by elementary considerations on the quadratic forms, for all $E \in \mathbb{R}$, there exist $C > 0$, $h_0 > 0$ such that, for all $\psi \in \operatorname{range} \mathbb{1}_{(-\infty,-h+Eh^{\frac{3}{2}})}(\mathcal{L}_h^{\mathsf{eff},-})$,

$$h^2\mathcal{Q}^{\Gamma}(\psi) \leqslant Ch^{\frac{3}{2}}\|\psi\|^2,$$

and then

$$\mathcal{Q}_h^{\mathsf{eff}}(\psi) = \mathcal{Q}_h^{\mathsf{eff},-}(\psi) + C_- h^{\frac{1}{2}} h^2\mathcal{Q}^{\Gamma}(\psi) + C_- h^2\|\psi\|^2 \leqslant \mathcal{Q}_h^{\mathsf{eff},-}(\psi) + Ch^2\|\psi\|^2,$$

where $\mathcal{Q}^{\Gamma}$, $\mathcal{Q}_h^{\mathsf{eff},\pm}$ and $\mathcal{Q}_h^{\mathsf{eff}}$ are the quadratic forms of $\mathcal{L}^{\Gamma}$, $\mathcal{L}_h^{\mathsf{eff},\pm}$ and $\mathcal{L}_h^{\mathsf{eff}}$, respectively.

By the min-max principle, we deduce that, for all $n \in \mathcal{N}_h(E)$,

$$\mu_n^{\mathsf{eff}}(h) \leqslant \mu_n^-(h) + Ch^2 \leqslant \mu_n(h) + Ch^2.$$

In particular, we have, for all $n \in \mathcal{N}_h(E)$,

$$\mu_n^{\mathsf{eff}}(h) \leqslant -h + Eh^{\frac{3}{2}} + Ch^2.$$

In the same way, we have, for all $\psi \in \operatorname{range} \mathbb{1}_{(-\infty,-h+Eh^{\frac{3}{2}}+Ch^2)}(\mathcal{L}_h^{\mathsf{eff}})$,

$$\mathcal{Q}_h^{\mathsf{eff},+}(\psi) \leqslant \mathcal{Q}_h^{\mathsf{eff}}(\psi) + \tilde{C}h^2\|\psi\|^2,$$

and we get, for all $n \in \mathcal{N}_h(E)$,

$$\mu_n(h) \leqslant \mu_n^+(h) \leqslant \mu_n^{\mathsf{eff}}(h) + \tilde{C}h^2.$$

□

3.1. **Proof of Theorem 1.1.** Let us now explain how we can deduce Theorem 1.1. By the first inequality in (3.1), we have

$$\mathsf{N}\left(\mathcal{L}_h, -h + Eh^{\frac{3}{2}}\right) \leqslant \mathsf{N}\left(\mathcal{L}_h^{\mathsf{eff},-}, -h + Eh^{\frac{3}{2}}\right).$$

This becomes, for an appropriate constant $\tilde{C}_- > 0$,

$$\mathsf{N}\left(\mathcal{L}_h, -h + Eh^{\frac{3}{2}}\right) \leqslant \mathsf{N}\left(h^{\frac{1}{2}}\mathcal{L}^\Gamma - \kappa, E + \tilde{C}_- h^{\frac{1}{2}}\right).$$

It remains to apply the usual semiclassical Weyl estimate for the counting function in the right-hand-side. We get, by using the second inequality in (3.1),

$$\mathsf{N}\left(\mathcal{L}_h^{\mathsf{eff},+}, -h + Eh^{\frac{3}{2}}\right) \leqslant \mathsf{N}\left(\mathcal{L}_h, -h + Eh^{\frac{3}{2}}\right),$$

and we deduce the semiclassical lower bound for the counting function in the same way.

3.2. **Strategy of the proof of Theorem 3.1.** Let us briefly explain the main lines of the proof of Theorem 3.1.

i. From the localization estimates of Proposition 2.2, we only have to consider $\mathcal{L}_h$ restricted to a thin neighborhood of the boundary (say of size $\delta = h^{\frac{1}{4}}$), that is the operator $\widetilde{\mathcal{L}}_h$. Due to scaling considerations, we may even work with $\widehat{\mathcal{L}}_\hbar$.
ii. The operator $\widehat{\mathcal{L}}_\hbar$ is partially semiclassical. Since the effective semiclassical variable is $\sigma$, it is natural to consider (an approximation of) the operator acting in the variable $\tau$ with $\sigma$ considered as a parameter (see Section 3.3).
iii. Since the important quantity in the investigation is the mean curvature $\kappa$, it is easier to keep only the main term in the expansion of the metrics induced by the tubular coordinates (see Section 3.4). We get an approximated operator $\widehat{\mathcal{L}}_\hbar^{\mathsf{app}}$
iv. Finally, we want to prove upper and lower bounds on the eigenvalues of $\widehat{\mathcal{L}}_\hbar^{\mathsf{app}}$. For the upper bounds, we insert almost explicit test functions (quasi tensor products of functions on the boundary and of the groundstate of the transverse operator) in the quadratic form. For the lower bound, we use the spectral decomposition of the transverse operator to decompose $\widehat{\mathcal{L}}_\hbar^{\mathsf{app}}$ into two orthogonal components, modulo some remainders involving the commutator between the transverse groundsate (depending on $\sigma$) and the tangential derivative $\nabla_\sigma$.

3.3. **The corrected Feshbach projection.** Let us introduce

$$\mathcal{H}_{\kappa(\sigma),\hbar} = \mathcal{H}_B^{\{T\}},$$

with

$$B = h^{\frac{1}{2}}\kappa(\sigma) = \hbar^2\kappa(\sigma)$$

and where $\mathcal{H}_B^{\{T\}}$ is defined in (A.9). We introduce for $\sigma \in \Gamma$ the Feshbach projection $\Pi_\sigma$ on the normalized groundstate of $\mathcal{H}_{\kappa(\sigma),\hbar}$, denoted by $v_{\kappa(\sigma),\hbar}$,

$$\Pi_\sigma \psi = \langle \psi, v_{\kappa(\sigma),\hbar}\rangle_{L^2((0,T),(1-B\tau)\,\mathrm{d}\tau)} v_{\kappa(\sigma),\hbar}.$$

We also let

$$\Pi_\sigma^\perp = \mathsf{Id} - \Pi_\sigma$$

and

$$f(\sigma) = \langle \psi, v_{\kappa(\sigma),\hbar} \rangle_{L^2((0,T),(1-B\tau)\,\mathrm{d}\tau)}, \tag{3.2}$$

$$R_\hbar(\sigma) = \|\nabla_\sigma v_{\kappa(\sigma),\hbar}\|^2_{L^2((0,T),\,(1-B\tau)\,\mathrm{d}\tau)}, \tag{3.3}$$

The quantity $R_\hbar$ is sometimes called "Born-Oppenheimer correction". It measures the commutation defect between $\nabla_\sigma$ and $\Pi_\sigma$.

*Remark 3.4.* — In a first approximation, one could try to use the projection on $v_{0,\hbar}$, but one would lose the uniformity in our estimates. Note that the idea to consider a corrected Feshbach projection appears in many different contexts: WKB analysis (see for instance [2, Sections 2.4 & 3.2], [7] and [8]), norm resolvent convergence (see for instance [12, Section 4.2]) or space/time adiabatic limits (see [22, Chapter 3]). The reader might also consider to read [13, p. 35], where the authors tackle a similar problem of finding a corrected Feshbach projection in order to decouple the variable of their fiber operator and the semiclassical variable ($h$ is called $\varepsilon$ in their paper and they call "horizontal variable" our variable $s$). In particular, they emphasize that, without such a correction, the decoupling appears modulo a remainder $\mathcal{O}(h)$ larger than the gap between the eigenvalues.

3.4. **Approximation of the metrics.** In this section, we introduce an approximated quadratic form by approximating first the metrics. For that purpose, let us introduce the approximation of the weight:

$$\tilde{m}(s,t) = 1 - t\kappa(s)\,, \qquad \kappa(s) = \mathsf{Tr}\, L(s).$$

We have

$$|\tilde{a}(s,t) - \tilde{m}(s,t)| \leqslant Ct^2.$$

Let us now state two elementary lemmas.

Lemma 3.5. — *We have the estimate, for all $\psi \in \widehat{W}_T$,*

$$\left| \int_{\widehat{\mathcal{V}}_T} |\partial_\tau \psi|^2 \widehat{a}\,\mathrm{d}\Gamma\,\mathrm{d}\tau - \int_{\widehat{\mathcal{V}}_T} |\partial_\tau \psi|^2 \widehat{m}\,\mathrm{d}\Gamma\,\mathrm{d}\tau \right| \\ \leqslant C\hbar^4 \int_\Gamma |f(\sigma)|^2\,\mathrm{d}\Gamma + C\hbar^2 \int_{\widehat{\mathcal{V}}_T} |\partial_\tau \Pi_\sigma^\perp \psi|^2\,\mathrm{d}\Gamma\,\mathrm{d}\tau,$$

*where $\widehat{m}(\sigma,\tau) = \tilde{m}(\sigma,\hbar^2\tau)$.*

*Proof.* — We have

$$\left| \int_{\widehat{\mathcal{V}}_T} |\partial_\tau \psi|^2 \widehat{a}\,\mathrm{d}\Gamma\,\mathrm{d}\tau - \int_{\widehat{\mathcal{V}}_T} |\partial_\tau \psi|^2 \widehat{m}\,\mathrm{d}\Gamma\,\mathrm{d}\tau \right| \leqslant C\hbar^4 \int_{\widehat{\mathcal{V}}_T} \tau^2 |\partial_\tau \psi|^2\,\mathrm{d}\Gamma\,\mathrm{d}\tau.$$

Then, we use an orthogonal decomposition to get

$$\begin{aligned} &\left| \int_{\widehat{\mathcal{V}}_T} |\partial_\tau \psi|^2 \widehat{a}\,\mathrm{d}\Gamma\,\mathrm{d}\tau - \int_{\widehat{\mathcal{V}}_T} |\partial_\tau \psi|^2 \widehat{m}\,\mathrm{d}\Gamma\,\mathrm{d}\tau \right| \\ &\quad \leqslant \tilde{C}\hbar^4 \left( \int_{\widehat{\mathcal{V}_T}} \tau^2 |\partial_\tau \Pi_\sigma \psi|^2\,\mathrm{d}\Gamma\,\mathrm{d}\tau + \int_{\widehat{\mathcal{V}}_T} \tau^2 |\partial_\tau \Pi_\sigma^\perp \psi|^2\,\mathrm{d}\Gamma\,\mathrm{d}\tau \right) \\ &\quad \leqslant C\hbar^4 \int_\Gamma |f(\sigma)|^2 \left( \int_0^T \tau^2 |\partial_\tau v_{\kappa(\sigma),\hbar}|^2\,\mathrm{d}\tau \right) \mathrm{d}\Gamma + C\hbar^2 \int_{\widehat{\mathcal{V}}_T} |\partial_\tau \Pi_\sigma^\perp \psi|^2\,\mathrm{d}\Gamma\,\mathrm{d}\tau, \end{aligned}$$

where we have used that $T = \hbar^{-1}$ for the orthogonal component. The result then follows from the Agmon estimates in one dimension (Proposition A.6). $\square$

LEMMA 3.6. — *We have the estimate, for all $\psi \in \widehat{W}_T$,*

$$\left| \int_{\widehat{\mathcal{V}}_T} \langle \nabla_\sigma \psi, \widehat{g}^{-1} \nabla_\sigma \psi \rangle \widehat{a} \, \mathrm{d}\Gamma \, \mathrm{d}\tau - \int_{\widehat{\mathcal{V}}_T} \langle \nabla_\sigma \psi, \nabla_\sigma \psi \rangle \widehat{m} \, \mathrm{d}\Gamma \, \mathrm{d}\tau \right|$$
$$\leqslant C \int_\Gamma \left( \hbar^2 \| \nabla_\sigma f(\sigma) \|^2 + \hbar R_\hbar(\sigma) |f(\sigma)|^2 \right) \mathrm{d}\Gamma + C\hbar \int_{\widehat{\mathcal{V}}_T} \| \nabla_\sigma \Pi_\sigma^\perp \psi \|^2 \, \mathrm{d}\Gamma \, \mathrm{d}\tau.$$

*Proof.* — First, we write

$$\left| \int_{\widehat{\mathcal{V}}_T} \langle \nabla_\sigma \psi, \widehat{g}^{-1} \nabla_\sigma \psi \rangle \widehat{a} \, \mathrm{d}\Gamma \, \mathrm{d}\tau - \int_{\widehat{\mathcal{V}}_T} \langle \nabla_\sigma \psi, \nabla_\sigma \psi \rangle \widehat{m} \, \mathrm{d}\Gamma \, \mathrm{d}\tau \right|$$
$$\leqslant \int_{\widehat{\mathcal{V}}_T} \| \nabla_\sigma \psi \|^2 |\widehat{a} - \widehat{m}| \, \mathrm{d}\Gamma \, \mathrm{d}\tau + \int_{\widehat{\mathcal{V}}_T} |\langle \nabla_\sigma \psi, (\widehat{g}^{-1} - \mathsf{Id}) \nabla_\sigma \psi \rangle | \widehat{a} \, \mathrm{d}\Gamma \, \mathrm{d}\tau$$
$$\leqslant C \int_{\widehat{\mathcal{V}}_T} (\hbar^4 \tau^2 + \hbar^2 \tau) \| \nabla_\sigma \psi \|^2 \, \mathrm{d}\Gamma \, \mathrm{d}\tau.$$

Then, by an orthogonal decomposition, we get

$$\left| \int_{\widehat{\mathcal{V}}_T} \langle \nabla_\sigma \psi, \widehat{g}^{-1} \nabla_\sigma \psi \rangle \widehat{a} \, \mathrm{d}\Gamma \, \mathrm{d}\tau - \int_{\widehat{\mathcal{V}}_T} \langle \nabla_\sigma \psi, \nabla_\sigma \psi \rangle \widehat{m} \, \mathrm{d}\Gamma \, \mathrm{d}\tau \right|$$
$$\leqslant C \int_{\widehat{\mathcal{V}}_T} (\hbar^4 \tau^2 + \hbar^2 \tau) \| \nabla_\sigma \Pi_\sigma \psi \|^2 \, \mathrm{d}\Gamma \, \mathrm{d}\tau + C\hbar \int_{\widehat{\mathcal{V}}_T} \| \nabla_\sigma \Pi_\sigma^\perp \psi \|^2 \, \mathrm{d}\Gamma \, \mathrm{d}\tau,$$

where we used $T = \hbar^{-1}$ on the orthogonal part.

Finally, we use the naive inequality

$$\| \nabla_\sigma \Pi_\sigma \psi \|^2 \leqslant 2 \left( \| \nabla_\sigma f(\sigma) \|^2 |v_{\kappa(\sigma),\hbar}|^2 + \| \nabla_\sigma v_{\kappa(\sigma),\hbar} \|^2 |f(\sigma)|^2 \right),$$

and the conclusion again follows from Agmon estimates. $\square$

Let us now introduce the approximated quadratic form

$$\widehat{\mathscr{Q}}_\hbar^{\mathsf{app}}(\psi) = \int_{\widehat{\mathcal{V}}_T} \left( \hbar^4 \| \nabla_\sigma \psi \|^2 + |\partial_\tau \psi|^2 \right) \widehat{m} \, \mathrm{d}\Gamma \, \mathrm{d}\tau - \int_\Gamma |\psi(\sigma, 0)|^2 \, \mathrm{d}\Gamma. \tag{3.4}$$

The sense of this approximation is quantified by the following lemma (that is a consequence of Lemmas 3.5 and 3.6).

LEMMA 3.7. — *We have, for all $\psi \in \widehat{W}_T$,*

$$\left| \widehat{\mathscr{Q}}_\hbar(\psi) - \widehat{\mathscr{Q}}_\hbar^{\mathsf{app}}(\psi) \right|$$
$$\leqslant C\hbar^4 \int_\Gamma |f(\sigma)|^2 \, \mathrm{d}\sigma + C\hbar^2 \int_{\widehat{\mathcal{V}}_T} |\partial_\tau \Pi_\sigma^\perp \psi|^2 \, \mathrm{d}\Gamma \, \mathrm{d}\tau$$
$$+ C \int_\Gamma \left( \hbar^6 \| \nabla_\sigma f(\sigma) \|^2 + \hbar^5 R_\hbar(\sigma) |f(\sigma)|^2 \right) \mathrm{d}\Gamma + C\hbar^5 \int_{\widehat{\mathcal{V}}_T} \| \nabla_\sigma \Pi_\sigma^\perp \psi \|^2 \, \mathrm{d}\Gamma \, \mathrm{d}\tau.$$

3.5. **Upper bound.** The following proposition provides an upper bound of the quadratic form on a subspace.

Proposition 3.8. — *There exist $C > 0$, $\hbar_0 > 0$ such that, for all $\psi \in \widehat{\mathcal{D}}_T$ and $\hbar \in (0, \hbar_0)$, we have*

$$\widehat{\mathcal{Q}}_\hbar(\Pi_\sigma \psi) \leqslant \int_\Gamma \hbar^4(1 + C\hbar^2)\|\nabla_\sigma f(\sigma)\|^2 \,\mathrm{d}\Gamma \\ + \int_\Gamma \left(\lambda_1(\mathcal{H}_{\kappa(\sigma),\hbar}) + C\hbar^4 + \hbar^4(1 + C\hbar)R_\hbar(\sigma)\right)|f(\sigma)|^2 \,\mathrm{d}\Gamma.$$

*Proof.* — First, we use Lemma 3.7. Then, we are reduced to estimates on the approximated quadratic form. By writing $\Pi_\sigma \psi = f(\sigma) v_{\kappa(\sigma),\hbar}$ and considering the derivative of this product, we get

$$\begin{aligned}\widehat{\mathcal{Q}}^{\mathsf{app}}_\hbar(\Pi_\sigma \psi) &= \int_{\widehat{\mathcal{V}}_T} \left(\hbar^4\|\nabla_\sigma \Pi_\sigma \psi\|^2 + |\partial_\tau \Pi_\sigma \psi|^2\right)\widehat{m} \,\mathrm{d}\Gamma \,\mathrm{d}\tau - \int_\Gamma |\Pi_\sigma \psi(\sigma, 0)|^2 \,\mathrm{d}\Gamma \\ &= \int_\Gamma \left(\hbar^4\|\nabla_\sigma f(\sigma)\|^2 + \left(\hbar^4 R_\hbar(\sigma) + q_{\kappa(\sigma),\hbar}(v_{\kappa(\sigma),\hbar})\right)|f(\sigma)|^2\right) \,\mathrm{d}\Gamma \\ &\quad + 2\hbar^4 \int_\Gamma f(\sigma) \left\langle \nabla_\sigma f(\sigma), \int_0^T v_{\kappa(\sigma),\hbar} \nabla_\sigma v_{\kappa(\sigma),\hbar} \widehat{m} \,\mathrm{d}\tau \right\rangle \,\mathrm{d}\Gamma.\end{aligned}$$

where $q_{\kappa(\sigma),\hbar}$ is the quadratic form associated with $\mathcal{H}_{\kappa(\sigma),\hbar}$. By definition, we have

$$q_{\kappa(\sigma),\hbar}(v_{\kappa(\sigma),\hbar}) = \lambda_1(\mathcal{H}_{\kappa(\sigma),\hbar}).$$

Then we notice from the normalization of $v_{\kappa(\sigma),\hbar}$ that

$$\nabla_\sigma \left(\int_0^T |v_{\kappa(\sigma),\hbar}|^2 \widehat{m} \,\mathrm{d}\tau\right) = 0,$$

and since $\nabla_\sigma B = \hbar^2 \nabla_\sigma \kappa(\sigma)$, we have

$$\int_0^T v_{\kappa(\sigma),\hbar} \nabla_\sigma v_{\kappa(\sigma),\hbar} \widehat{m} \,\mathrm{d}\tau = \mathcal{O}(\hbar^2).$$

This implies the estimate:

$$\left|\hbar^4 \int_\Gamma f(\sigma) \left\langle \nabla_\sigma f(\sigma), \int_0^T v_{\kappa(\sigma),\hbar} \nabla_\sigma v_{\kappa(\sigma),\hbar} \widehat{m} \,\mathrm{d}\tau \right\rangle \,\mathrm{d}\Gamma\right| \\ \leqslant C\hbar^6 \int_\Gamma \left(|f(\sigma)|^2 + \|\nabla_\sigma f(\sigma)\|^2\right) \,\mathrm{d}\Gamma, \quad (3.5)$$

and the conclusion follows. □

3.6. **Lower bound.** Let us now establish the following lower bound of the quadratic form.

Proposition 3.9. — *There exist $C>0$, $\hbar_0>0$ such that, for all $\psi \in \widehat{\mathcal{D}}_T$ and $\hbar \in (0,\hbar_0)$, we have*

$$\begin{aligned}
&\widehat{\mathcal{Q}}_\hbar(\psi)\\
&\quad \geqslant \int_\Gamma \left(\hbar^4(1-C\hbar^2)\|\nabla_\sigma f(\sigma)\|^2 + \left(\lambda_1(\mathcal{H}_{\kappa(\sigma),\hbar}) - C(\hbar^4+\hbar^2 R_\hbar(\sigma)\right)|f(\sigma)|^2\right) \mathrm{d}\Gamma\\
&\qquad + \int_\Gamma \hbar^4(1-C\hbar)\|\nabla_\sigma \Pi_\sigma^\perp \psi\|^2_{L^2(\widehat{m}\,\mathrm{d}\tau)}\,\mathrm{d}\Gamma\\
&\qquad + \int_\Gamma \left((1-C\hbar^2)\lambda_2(\mathcal{H}_{\kappa(\sigma),\hbar}) - C(\hbar^6+\hbar^2 R_\hbar(\sigma))\right)\|\Pi_\sigma^\perp\psi\|^2_{L^2(\widehat{m}\,\mathrm{d}\tau)}\,\mathrm{d}\Gamma.
\end{aligned}$$

*Proof.* — The proof will be done in a few steps.

i. First, we use Lemma 3.7 to write

$$\begin{aligned}
\widehat{\mathcal{Q}}_\hbar(\psi) \geqslant{}& \int_{\widehat{\mathcal{V}}_T} \hbar^4\|\nabla_\sigma\psi\|^2\widehat{m}\,\mathrm{d}\Gamma\,\mathrm{d}\tau + \int_\Gamma q_{\kappa(\sigma),\hbar}(\psi)\,\mathrm{d}\Gamma\\
&- C\int_\Gamma \left(\hbar^6\|\nabla_\sigma f(\sigma)\|^2 + \hbar^4(1+\hbar R_\hbar(\sigma))|f(\sigma)|^2\right)\mathrm{d}\Gamma\\
&- C\hbar^5\int_{\widehat{\mathcal{V}}_T}\|\nabla_\sigma\Pi_\sigma^\perp\psi\|^2\widehat{m}\,\mathrm{d}\Gamma\,\mathrm{d}\tau - C\hbar^2\int_{\widehat{\mathcal{V}}_T}|\partial_\tau\Pi_\sigma^\perp\psi|^2\widehat{m}\,\mathrm{d}\Gamma\,\mathrm{d}\tau. \quad (3.6)
\end{aligned}$$

ii. On one hand, we get, by using an orthogonal decomposition, for each $\sigma\in\Gamma$,

$$q_{\kappa(\sigma),\hbar}(\psi) = q_{\kappa(\sigma),\hbar}(\Pi_\sigma\psi) + q_{\kappa(\sigma),\hbar}(\Pi_\sigma^\perp\psi).$$

Then, we get, by using the min-max principle,

$$\begin{aligned}
&\int_\Gamma q_{\kappa(\sigma),\hbar}(\psi)\,\mathrm{d}\Gamma - C\hbar^2\int_{\widehat{\mathcal{V}}_T}|\partial_\tau\Pi_\sigma^\perp\psi|^2\widehat{m}\,\mathrm{d}\Gamma\,\mathrm{d}\tau\\
&\geqslant \int_\Gamma q_{\kappa(\sigma),\hbar}(\Pi_\sigma\psi)\,\mathrm{d}\Gamma + (1-C\hbar^2)\int_\Gamma q_{\kappa(\sigma),\hbar}(\Pi_\sigma^\perp\psi)\,\mathrm{d}\Gamma \quad (3.7)\\
&\geqslant \int_\Gamma \left(\lambda_1(\mathcal{H}_{\kappa(\sigma),\hbar})|f(\sigma)|^2 + (1-C\hbar^2)\lambda_2(\mathcal{H}_{\kappa(\sigma),\hbar})\|\Pi_\sigma^\perp\psi\|^2_{L^2(\widehat{m}\,\mathrm{d}\tau)}\right)\mathrm{d}\Gamma.
\end{aligned}$$

On the other hand, we also have

$$\|\nabla_\sigma\psi\|^2_{L^2(\widehat{m}\,\mathrm{d}\tau)} = \|\Pi_\sigma\nabla_\sigma\psi\|^2_{L^2(\widehat{m}\,\mathrm{d}\tau)} + \|\Pi_\sigma^\perp\nabla_\sigma\psi\|^2_{L^2(\widehat{m}\,\mathrm{d}\tau)}. \quad (3.8)$$

iii. Then, we estimate the commutator:

$$\begin{aligned}
[\nabla_\sigma,\Pi_\sigma]\,\psi ={}& \langle\psi,\nabla_\sigma v_{\kappa(\sigma),\hbar}\rangle_{L^2(\widehat{m}\,\mathrm{d}\tau)} v_{\kappa(\sigma),\hbar} + \langle\psi, v_{\kappa(\sigma),\hbar}\rangle_{L^2(\widehat{m})\,\mathrm{d}\tau}\nabla_\sigma v_{\kappa(\sigma),\hbar}\\
&- \hbar^2\nabla_\sigma\kappa(\sigma)\left(\int_0^T \psi v_{\kappa(\sigma),\hbar}\tau\,\mathrm{d}\tau\right)v_{\kappa(\sigma),\hbar}.
\end{aligned}$$

We get, thanks to the Cauchy-Schwarz inequality and Agmon estimates (see Proposition A.6),

$$\|[\Pi_\sigma,\nabla_\sigma]\,\psi\|_{L^2(\widehat{m}\,\mathrm{d}\tau)} \leqslant \left(2R_\hbar(\sigma)^{\frac12} + C\hbar^2\right)\|\psi\|_{L^2(\widehat{m}\,\mathrm{d}\tau)}. \quad (3.9)$$

Then, we write

$$\Pi_\sigma\nabla_\sigma\psi = \nabla_\sigma f(\sigma)v_{\kappa(\sigma),\hbar} + f(\sigma)\nabla_\sigma v_{\kappa(\sigma),\hbar} + [\Pi_\sigma,\nabla_\sigma]\,\psi. \quad (3.10)$$

Let us recall the following classical inequality:

$$\forall\, a,b\in\mathbb{C}^{n-1}, \forall\,\varepsilon\in(0,1),\ \|a+b\|^2\geqslant(1-\varepsilon)\|a\|^2-\varepsilon^{-1}\|b\|^2.$$

We take $\varepsilon=\hbar^2$, $a=\nabla_\sigma f(\sigma)v_{\kappa(\sigma),\hbar}$ and $b=f(\sigma)\nabla_\sigma v_{\kappa(\sigma),\hbar}+[\Pi_\sigma,\nabla_\sigma]\,\psi$. We get, from (3.9) and (3.10),

$$\int_0^T\|\Pi_\sigma\nabla_\sigma\psi\|^2\widehat{m}\,\mathrm{d}\tau\geqslant(1-\hbar^2)\|\nabla_\sigma f(\sigma)\|^2 \\ -C\hbar^{-2}\big(R_\hbar(\sigma)+\mathcal{O}(\hbar^4)\big)\big(|f(\sigma)|^2+\|\Pi_\sigma^\perp\psi\|^2_{L^2(\widehat{m}\,\mathrm{d}\tau)}\big). \quad (3.11)$$

In the same way, we get

$$\int_0^T\|\Pi_\sigma^\perp\nabla_\sigma\psi\|^2\widehat{m}\,\mathrm{d}\tau\geqslant(1-\hbar^2)\|\nabla_\sigma\Pi_\sigma^\perp\psi\|^2_{L^2(\widehat{m}\,\mathrm{d}\tau)} \\ -C\hbar^{-2}\big(R_\hbar(\sigma)+\mathcal{O}(\hbar^4)\big)\big(|f(\sigma)|^2+\|\Pi_\sigma^\perp\psi\|^2_{L^2(\widehat{m}\,\mathrm{d}\tau)}\big). \quad (3.12)$$

iv. Now we use (3.6), (3.7), (3.8) and the estimates (3.11), (3.12) and the conclusion follows.

□

3.7. **Derivation of the effective Hamiltonians.** We can now end the proof of Theorem 3.1.

i. We apply Proposition A.5 to get

$$\lambda_1(\mathcal{H}_{\kappa(\sigma),\hbar})=-1-\kappa(\sigma)\hbar^2+\mathcal{O}(\hbar^4),$$

and we use Lemmas A.1, A.3 to deduce that there exist positive constants $\hbar_0$ and $C$ such that, for all $\hbar\in(0,\hbar_0)$,

$$\lambda_2(\mathcal{H}_{\kappa(\sigma),\hbar})\geqslant-C\hbar\geqslant-\frac{\varepsilon_0}{2}.$$

Then we notice, thanks to Lemma A.7, that the Born-Oppenheimer correction satisfies $R_\hbar(\sigma)=\mathcal{O}(\hbar^4)$.

ii. As a consequence of Proposition 3.8, there exists $C_+>0$ such that, for all $\psi\in\widehat{\mathcal{D}}_T$ and $\hbar$ small enough,

$$\widehat{\mathcal{Q}}_\hbar(\Pi_\sigma\psi)\leqslant\widehat{\mathcal{Q}}_\hbar^{\mathsf{eff},+}(f),$$

where, for all $f\in H^1(\Gamma)$,

$$\widehat{\mathcal{Q}}_\hbar^{\mathsf{eff},+}(f)=\int_\Gamma\big(\hbar^4(1+C_+\hbar^2)\|\nabla_\sigma f\|^2+\big(-1-\kappa(\sigma)\hbar^2+C_+\hbar^4\big)\,|f|^2\big)\,\mathrm{d}\Gamma.$$

For $n\geqslant1$, let

$$G_{n,\hbar}=\Big\{fv_{\kappa(\sigma),\hbar}\in\widehat{\mathcal{D}}_T\ :\ f\in F_{n,\hbar}\Big\},$$

where $F_{n,\hbar}\subset H^1(\Gamma)$ is the eigenspace of the operator $\widehat{\mathcal{L}}_\hbar^{\mathsf{eff},+}$ associated with the eigenvalues $\Big(\widehat{\mu}_k^{\mathsf{eff},+}(\hbar)\Big)_{1\leqslant k\leqslant n}$. We have $\dim G_{n,\hbar}=n$ and, for all $\psi\in G_{n,\hbar}$,

$$\widehat{\mathcal{Q}}_\hbar(\psi)\leqslant\widehat{\mu}_n^{\mathsf{eff},+}(\hbar)\|\psi\|^2_{L^2(\widehat{a}\,\mathrm{d}\Gamma\,\mathrm{d}\tau)},$$

so that, by application of the min-max principle,

$$\widehat{\mu}_n(\hbar) \leqslant \widehat{\mu}_n^{\mathsf{eff},+}(\hbar).$$

iii. For $\varepsilon_0 \in (0,1)$, thanks to Proposition 3.9, there exists $C_- > 0$ such that, for all $\psi \in \widehat{\mathcal{D}}_T$ and $\hbar$ small enough,

$$\widehat{\mathcal{Q}}_\hbar(\psi) \geqslant \widehat{\mathcal{Q}}_\hbar^{\mathsf{eff},-}(f) - \frac{\varepsilon_0}{2}\|\Pi_\sigma^\perp \psi\|^2_{L^2(\widehat{m}\,\mathrm{d}\Gamma\,\mathrm{d}\tau)},$$

where, for all $f \in H^1(\Gamma)$,

$$\widehat{\mathcal{Q}}_\hbar^{\mathsf{eff},-}(f) = \int_\Gamma \left(\hbar^4(1 - C_-\hbar^2)\|\nabla_\sigma f\|^2 + \left(-1 - \kappa(\sigma)\hbar^2 - C_-\hbar^4\right)|f|^2\right) \mathrm{d}\Gamma.$$

We consider the quadratic form defined, for $(f,\varphi) \in H^1(\Gamma) \times \widehat{V}_T$, by

$$\widehat{\mathcal{Q}}_\hbar^{\mathsf{tens}}(f,\varphi) = \widehat{\mathcal{Q}}_\hbar^{\mathsf{eff},-}(f) - \frac{\varepsilon_0}{2}\|\varphi\|^2_{L^2(\widehat{m}\,\mathrm{d}\Gamma\,\mathrm{d}\tau)}.$$

By application of the min-max principle (see also [21, Chapter 13]), we have the comparison of the Rayleigh quotients:

$$\widehat{\mu}_n(\hbar) \geqslant \widehat{\mu}_n^{\mathsf{tens}}(\hbar).$$

Note that the spectrum of $\widehat{\mathcal{L}}_\hbar^{\mathsf{tens}}$ lying below $-\varepsilon_0$ is discrete and coincides with the spectrum of $\widehat{\mathcal{L}}_\hbar^{\mathsf{eff},-}$. Then, for all $n \in \mathcal{N}_{\varepsilon_0,h}$, $\widehat{\mu}_n^{\mathsf{tens}}(\hbar)$ is the $n$-th eigenvalue of $\widehat{\mathcal{L}}_\hbar^{\mathsf{tens}}$ and its satisfies $\widehat{\mu}_n^{\mathsf{tens}}(\hbar) = \widehat{\mu}_n^{\mathsf{eff},-}(\hbar)$.

iv. Finally, we apply Proposition 2.2 to compare $\widehat{\mu}_n(\hbar)$ and $\mu_n(h)$.

## 4. Asymptotic counting formula for the non positive eigenvalues

This section is devoted to the proof of Theorem 1.2. For that purpose we prove an upper bound in Proposition 4.1 and a lower bound in Proposition 4.2.

The philosophy behind the proof of Theorem 1.2 is different compared to that of Theorem 1.1. In fact, our proof of Theorem 1.1 follows from the derivation of an *effective Hamiltonian* that describes all the eigenvalues below the energy level $-\varepsilon_0 h$, for an arbitrary $\varepsilon_0 \in (0,1)$. Since this proof breaks for $\varepsilon_0 = 0$, we follow a different approach by comparing the eigenvalue counting functions of the Robin Laplacian on the open domain $\Omega$ and the Laplace-Beltrami operator on the boundary $\partial\Omega$. A key point is to isolate the contribution of "bulk eigenvalues" through the classical Weyl formula.

Proposition 4.1. — *There exist $C, h_0 > 0$ such that for all $h \in (0,h_0)$,*

$$\mathsf{N}(\mathcal{L}_h, 0) \leqslant (1 + o(1))\mathsf{N}\left(h\mathcal{L}^\Gamma, 1\right).$$

*Proof.* — Consider a quadratic partition of the unity $(\chi_{j,h})_{j=1,2}$ in $\overline{\Omega}$ satisfying

$$\sum_{j=1}^2 \chi_{j,h}^2 = 1\,, \quad \sum_{j=1}^2 |\nabla\chi_{j,h}|^2 \leqslant Ch^{-2\rho},$$

and

$$\mathsf{supp}\,\chi_{1,h} \subset \{\mathsf{dist}(x,\partial\Omega) < h^\rho\}.$$

For all $u \in H^1(\Omega)$, the following "IMS" localization formula holds (cf. [4])

$$\begin{aligned}\mathcal{Q}_h(u) &= \mathcal{Q}_h(\chi_{1,h}u) + \mathcal{Q}_h(\chi_{2,h}u) - h^2 \sum_{j=1}^{2} \|u\nabla\chi_{j,h}\|^2 \\ &\geqslant \mathcal{Q}_h(\chi_{1,h}u) + \mathcal{Q}_h(\chi_{2,h}u) - Ch^{2-2\rho}\|u\|^2.\end{aligned} \tag{4.1}$$

Let $\Delta^{\mathsf{Dir}}$ be the Dirichlet Laplacian on $\Omega$ with domain $H^2(\Omega)\cap H^1_0(\Omega)$. In the sequel, $\delta = h^\rho$, $\mathcal{V}_\delta$ is the domain introduced in (2.1) and $\mathcal{L}_h^{\{\delta\}}$ is the Robin Laplacian $\mathcal{L}_h$ on the domain

$$\mathcal{D}_h^{\{\delta\}} = \{u \in H^2(\mathcal{V}_\delta) \ : \ \mathbf{n}\cdot h^{\frac12}\nabla u = -u \text{ on } \Gamma,\ u = 0 \text{ on } \operatorname{dist}(x,\Gamma) = \delta)\}.$$

Since $\chi_{1,h}u$ and $\chi_{2,h}u$ are in the form domains of the operators $\mathcal{L}_h^{\{\delta\}}$ and $-\Delta^{\mathsf{Dir}}$ respectively, and the mapping $L^2(\Omega) \ni u \mapsto (\chi_{1,h}u, \chi_{2,h}u) \in L^2(\mathcal{V}_\delta)\oplus L^2(\Omega)$ is an isometry, we get by the min-max principle (cf. [3]):

$$\mathsf{N}\left(\mathcal{L}_h, 0\right) \leqslant \mathsf{N}\left(\mathcal{L}_h^{\{\delta\}}, Ch^{2-2\rho}\right) + \mathsf{N}\left(-h^2\Delta^{\mathsf{Dir}}, Ch^{2-2\rho}\right). \tag{4.2}$$

Now, we estimate $\mathcal{Q}_h(u)/\|u\|^2$ for all $u \in \mathcal{D}_{h,\rho}\setminus\{0\}$ by using the boundary coordinates (see Section 2.3 and especially (2.6)) and a rough Taylor expansion of the metrics:

$$\frac{\mathcal{Q}_h(u)}{\|u\|^2_{L^2(\mathcal{V}_\delta)}} \geqslant (1 - Ch^\rho)\frac{\widetilde{\mathcal{Q}}_h^{\mathsf{tens}}(\widetilde{u})}{\|\widetilde{u}\|^2_{L^2(\widetilde{\mathcal{V}}_\delta)}},$$

where

$$\|v\|^2_{L^2(\widetilde{\mathcal{V}}_\delta)} = \int_{\widetilde{\mathcal{V}}_\delta} |v|^2 \,\mathrm{d}\Gamma\,\mathrm{d}t,$$

and

$$\widetilde{\mathcal{Q}}_h^{\mathsf{tens}}(v) = \int_{\widetilde{\mathcal{V}}_\delta} \left(h^2\langle\nabla_s v, \nabla_s v\rangle + |h\partial_t v|^2\right)\mathrm{d}\Gamma\,\mathrm{d}t - h^{\frac32}\int_\Gamma |v(s,0)|^2\,\mathrm{d}\Gamma.$$

The quadratic form $\widetilde{\mathcal{Q}}_h^{\mathsf{tens}}$ defines a self-adjoint operator $\widetilde{\mathcal{L}}_h^{\mathsf{tens}}$ on $\widetilde{\mathcal{V}_\delta}$ with Robin condition at $t = 0$ and Dirichlet condition at $t = \delta = h^\rho$. Thanks to the min-max principle, we deduce that, for $h$ sufficiently small, the following comparison between the eigenvalues of $\mathcal{L}_h^{\{\delta\}}$ and $\widetilde{\mathcal{L}}_h^{\mathsf{tens}}$ holds:

$$\mu_n^{\{\delta\}}(h) \geqslant (1 - Ch^\rho)\mu_n\left(\widetilde{\mathcal{L}}_h^{\mathsf{tens}}\right).$$

Consequently, we have $\mathsf{N}\left(\mathcal{L}_h^{\{\delta\}}, Ch^{2-2\rho}\right) \leqslant \mathsf{N}\left(\widetilde{\mathcal{L}}_h^{\mathsf{tens}}, \widetilde{C}h^{2-2\rho}\right)$ for a new constant $\widetilde{C} > 0$. Plugging this into (4.2), we get:

$$\mathsf{N}\left(\mathcal{L}_h, 0\right) \leqslant \mathsf{N}\left(\widetilde{\mathcal{L}}_h^{\mathsf{tens}}, \widetilde{C}h^{2-2\rho}\right) + \mathsf{N}\left(-h^2\Delta^{\mathsf{Dir}}, Ch^{2-2\rho}\right). \tag{4.3}$$

Then, by using the usual Weyl formula for the Dirichlet Laplacian, we get for $h$ sufficiently small

$$\mathsf{N}\left(-h^2\Delta^{\mathsf{Dir}}, Ch^{2-2\rho}\right) \leqslant Ch^{-d\rho}, \tag{4.4}$$

and it remains to analyze $\mathsf{N}\left(\widetilde{\mathcal{L}}_h^{\mathsf{tens}}, Ch^{2-2\rho}\right)$. The operator $\widetilde{\mathcal{L}}_h^{\mathsf{tens}}$ is in a tensorial form and it has a Hilbertian decomposition by using the Hilbertian basis of the eigenfunctions of the transverse Robin Laplacian $\mathcal{L}_h^{\mathsf{trans}} := h^2D_t^2$ acting on $L^2((0,h^\rho), \mathrm{d}t)$, with Robin condition at $t = 0$ and Dirichlet condition at $t = h^\rho$.

Let $\left(\lambda_n^{\mathsf{trans}}(h)\right)$ be the increasing sequence of the eigenvalues of the transverse operator counting multiplicities. The spectral decomposition of the transverse operator yields

$$\mathsf{sp}\left(\widetilde{\mathcal{L}}_h^{\mathsf{tens}}\right) = \bigcup_{n=1}^{\infty} \left(h^2 \mathsf{sp}(\mathcal{L}^\Gamma) + \lambda_n^{\mathsf{trans}}(h)\right).$$

We will show that only the first eigenvalue $\lambda_1^{\mathsf{trans}}(h)$ contributes to the spectrum of $\widetilde{\mathcal{L}}_h^{\mathsf{tens}}$ below $Ch^{2-2\rho}$. We know that the second eigenvalue of the transverse operator $h^2 D_t^2$ is of order $h^{2\rho}$ (see Lemma A.2, with $T = h^{\rho-\frac{1}{2}}$). Since $\rho \in \left(0, \frac{1}{2}\right)$, we get $h^{2\rho} \gg h^{2-2\rho}$ and thus we have only to consider the first transverse eigenvalue whose asymptotic expansion is $-h + \mathcal{O}(h^\infty)$, by Lemma A.1. We get, for $h$ sufficiently small,

$$\mathsf{N}\left(\widetilde{\mathcal{L}}_h^{\mathsf{tens}}, \widetilde{C}h^{2-2\rho}\right) \leqslant \mathsf{N}\left(h\mathcal{L}^\Gamma, 1 + 2\widetilde{C}h^{1-2\rho}\right) \underset{h\to 0}{\sim} \mathsf{N}\left(h\mathcal{L}^\Gamma, 1\right). \tag{4.5}$$

We deduce the upper bound by combining (4.3), (4.4), (4.5), (1.3) and taking $\rho$ small enough. ☐

Proposition 4.2. — *There exist $C, h_0 > 0$ such that for all $h \in (0, h_0)$,*

$$\mathsf{N}\left(\mathcal{L}_h, 0\right) \geqslant (1 + o(1))\mathsf{N}\left(h\mathcal{L}^\Gamma, 1\right).$$

*Proof.* — To find the lower bound, we just have to bound the quadratic form $\mathcal{Q}_h$ on an appropriate subspace. We consider $\rho \in \left(0, \frac{1}{2}\right)$. We first notice that, for $u$ such that $\mathsf{supp}\, u \subset \widetilde{\mathcal{V}}_{h^\rho}$,

$$\mathcal{Q}_h(u) \leqslant (1 + Ch^\rho)\widetilde{\mathcal{Q}}_h^{\mathsf{tens}}(\widetilde{u}).$$

We apply this inequality to the space spanned by functions in the form $\widetilde{u}(s,t) = f_{h,n}(s)u_h(t)$ where the $f_{h,n}$ are the eigenfunctions of $h^2\mathcal{L}^\Gamma + \lambda(h)$ associated with non positive eigenvalues and $u_h$ is the first eigenfunction of the transverse Robin Laplacian with eigenvalue $\lambda(h) = -h + \mathcal{O}(h^\infty)$. The conclusion again follows from the min-max principle and the fact that $\mathsf{N}\left(h\mathcal{L}^\Gamma, 1 + \mathcal{O}(h^\infty)\right) \underset{h\to 0}{\sim} \mathsf{N}\left(h\mathcal{L}^\Gamma, 1\right)$. ☐

## Appendix A. Reminders about Robin Laplacians in one dimension

The aim of this section is to recall a few spectral properties related to the Robin Laplacian in dimension one. Most of them have been established in [7] or [8].

### A.1. **On a half line.**

As simplest model, we start with the operator, acting on $L^2(\mathbb{R}_+)$, defined by

$$\mathcal{H}_0 = -\partial_\tau^2 \tag{A.1}$$

with domain

$$\mathsf{Dom}(\mathcal{H}_0) = \{u \in H^2(\mathbb{R}_+) \ : \ u'(0) = -u(0)\}. \tag{A.2}$$

Note that this operator is associated with the quadratic form

$$V_0 \ni u \mapsto \int_0^{+\infty} |u'(\tau)|^2\, d\tau \, - |u(0)|^2,$$

with $V_0 = H^1(0, +\infty)$.

The spectrum of this operator is $\{-1\}\cup[0,\infty)$. The eigenspace of the eigenvalue $-1$ is generated by the $L^2$-normalized function

$$u_0(\tau)=\sqrt{2}\,\exp(-\tau)\,. \tag{A.3}$$

We will also consider this operator in a bounded interval $(0,T)$ with $T$ sufficiently large and Dirichlet condition at $\tau=T$.

A.2. **On an interval.** Let us consider $T\geqslant 1$ and the self-adjoint operator acting on $L^2(0,T)$ and defined by

$$\mathcal{H}_0^{\{T\}}=-\partial_\tau^2, \tag{A.4}$$

with domain,

$$\mathsf{Dom}(\mathcal{H}_0^{\{T\}})=\{u\in H^2(0,T)\ :\ u'(0)=-u(0)\quad\text{and}\quad u(T)=0\}. \tag{A.5}$$

The spectrum of the operator $\mathcal{H}_0^{\{T\}}$ is purely discrete and consists of a strictly increasing sequence of eigenvalues denoted by $\left(\lambda_n\left(\mathcal{H}_0^{\{T\}}\right)\right)_{n\geqslant1}$. This operator is associated with the quadratic form

$$V_0^{\{T\}}\ni u\mapsto\int_0^T|u'(\tau)|^2\,d\tau\,-\,|u(0)|^2,$$

with $V_0^{\{T\}}=\{v\in H^1(0,T)\,|\,v(T)=0\}$.
The next lemma gives the localization of the two first eigenvalues $\lambda_1\left(\mathcal{H}_0^{\{T\}}\right)$ and $\lambda_2\left(\mathcal{H}_0^{\{T\}}\right)$ for large values of $T$.

Lemma A.1. — *As $T\to+\infty$ we have*

$$\lambda_1(\mathcal{H}_0^{\{T\}})=-1+4\big(1+o(1)\big)\exp\big(-2T\big)\quad\text{and}\quad\lambda_2(\mathcal{H}_0^{\{T\}})\geqslant0. \tag{A.6}$$

*Let us now discuss the estimates of the next eigenvalues.*

Lemma A.2. — *For all $T>1$ and $n\geqslant2$,*

$$\left(\frac{(2n-3)\pi}{2T}\right)^2<\lambda_n(\mathcal{H}_0^{\{T\}})<\left(\frac{(n-1)\pi}{T}\right)^2.$$

*Proof.* — Let $w\geqslant0$ and $\lambda=-w^2$ be a non-positive eigenvalue of the operator $\mathcal{H}_0^{\{T\}}$ with an eigenfunction $u$. We have,

$$-\,u''=\lambda u\quad\text{in }(0,T)\,,\quad u'(0)=-u(0)\,,\quad u(T)=0. \tag{A.7}$$

If $w=0$ and $T>1$, then $u=0$ is the unique solution of (A.7). Thus, $w>0$ and

$$u(\tau)=A\cos(w\tau)+B\sin(w\tau), \tag{A.8}$$

for some constants $A\in\mathbb{R}$ and $B\in\mathbb{R}$ that depend on $T$. The boundary conditions satisfied by $u$ yield that $A=-Bw$, $\cos(wT)\neq0$ and

$$\tan(wT)=w.$$

Thus $w$ is a fixed point of the $\pi/T$-periodic function $x\mapsto\tan(xT)$. Obviously, there exist infinitely many solutions, at least one solution in every interval $(-\frac{\pi}{2T},\frac{\pi}{2T})+\frac{k\pi}{T}$, $k=0,\pm1,\cdots$. Since we are interested in the positive solutions, we specialize first into the interval $(-\frac{\pi}{2T},\frac{\pi}{2T})$. Define the function $g(x)=\tan(xT)-x$. Clearly, $x=0$ is a zero of this function in the interval $(-\frac{\pi}{2T},\frac{\pi}{2T})$. It is the unique zero of $g$ in this interval since $g'(x) = T(1+\tan^2(xT)) - 1 > 0$ for $T > 1$. Thus, the smallest $w > 0$ that satisfies $g(w) = 0$ does live in the interval $(\frac{\pi}{2T}, \frac{\pi}{T})$, which is $\sqrt{-\lambda_2(\mathcal{H}_0^{\{T\}})}$. The next positive zero of $g$, $\sqrt{-\lambda_3(\mathcal{H}_0^{\{T\}})}$, lives in the interval $(\frac{\pi}{2T}, \frac{\pi}{T}) + \frac{\pi}{T}$, etc. $\square$

A.3. **In a weighted space.** Let $B \in \mathbb{R}$, $T > 0$ such that $|B|T < \frac{1}{3}$. Consider the self-adjoint operator, acting on $L^2\big((0,T);(1-B\tau)\,\mathrm{d}\tau\big)$ and defined by

$$\mathcal{H}_B^{\{T\}} = -(1-B\tau)^{-1}\partial_\tau(1-B\tau)\partial_\tau = -\partial_\tau^2 + B(1-B\tau)^{-1}\partial_\tau, \tag{A.9}$$

with domain

$$\mathsf{Dom}(\mathcal{H}_B^{\{T\}}) = \{u \in H^2(0,T) \ :\ u'(0) = -u(0) \quad \text{and} \quad u(T) = 0\}. \tag{A.10}$$

The operator $\mathcal{H}_B^{\{T\}}$ is the Friedrichs extension in $L^2\big((0,T);(1-B\tau)\,\mathrm{d}\tau\big)$ associated with the quadratic form defined for $u \in V_h^{\{T\}}$, by

$$q_B^{\{T\}}(u) = \int_0^T |u'(\tau)|^2(1-B\tau)\,\mathrm{d}\tau - |u(0)|^2.$$

The operator $\mathcal{H}_B^{\{T\}}$ is with compact resolvent. The strictly increasing sequence of the eigenvalues of $\mathcal{H}_B^{\{T\}}$ is denoted by $(\lambda_n(\mathcal{H}_B^{\{T\}})_{n\in\mathbb{N}^*}$. It is easy to compare the spectra of $\mathcal{H}_B^{\{T\}}$ and $\mathcal{H}_0^{\{T\}}$ as $B$ goes to 0.

Lemma A.3. — *There exist $T_0, C > 0$ such that for all $T \geqslant T_0$, $B \in \big(-\frac{1}{3T}, \frac{1}{3T}\big)$ and $n \in \mathbb{N}^*$ we have*

$$\Big|\lambda_n(\mathcal{H}_B^{\{T\}}) - \lambda_n(\mathcal{H}_0^{\{T\}})\Big| \leqslant C|B|T\Big(\,\big|\lambda_n(\mathcal{H}_0^{\{T\}})\big| + 1\Big).$$

Then we notice that, for all $T > 0$, the family $\Big(\mathcal{H}_B^{\{T\}}\Big)_B$ is analytic for $B$ small enough. More precisely, we have

Lemma A.4. — *There exist $T_0 > 0$ such that for all $T \geqslant T_0$, the two functions $\big(-\frac{1}{3T}, \frac{1}{3T}\big) \ni B \mapsto \lambda_1\Big(\mathcal{H}_B^{\{T\}}\Big)$ and $\big(-\frac{1}{3T}, \frac{1}{3T}\big) \mapsto u_B^{\{T\}}$ are analytic. Here $u_B^{\{T\}}$ is the corresponding positive and normalized eigenfunction $\lambda_1\Big(\mathcal{H}_B^{\{T\}}\Big)$.*

The next proposition states a two-term asymptotic expansion of the eigenvalue $\lambda_1(\mathcal{H}_B^{\{T\}})$.

Proposition A.5. — *There exist $T_0 > 0$ and $C > 0$ such that for all $T \geqslant T_0$ and all $B \in \big(-\frac{1}{3T}, \frac{1}{3T}\big)$ we have*

$$\Big|\lambda_1(\mathcal{H}_B^{\{T\}}) - (-1-B)\Big| \leqslant CB^2 + Ce^{-T/2}.$$

We have also a decay estimate of $u_B^{\{T\}}$ that is a classical consequence of Proposition A.5, the fact that the Dirichlet problem on $(0,T)$ is positive and of Agmon estimates.

Proposition A.6. — *There exist $T_0 > 0$, $\alpha > 0$ and $C > 0$ such that for all $T \geqslant T_0$ and all $B \in \big(-\frac{1}{3T}, \frac{1}{3T}\big)$ we have*

$$\|e^{\alpha\tau}u_B^{\{T\}}\|_{H^1\big((0,T);(1-B\tau)\,\mathrm{d}\tau\big)} \leqslant C.$$

LEMMA A.7. — *There exist $C > 0$ and $T_0 > 0$ such that for all $T \geqslant T_0$ and all $B \in \left(-\frac{1}{3T}, \frac{1}{3T}\right)$ we have*

$$\left|\partial_B \lambda_1 \left(\mathcal{H}_B^{\{T\}}\right)\right| \leqslant C, \tag{A.11}$$

$$\|\partial_B \tilde{u}_B^{\{T\}}\|_{L^2((0,T),\,\mathrm{d}\tau)} \leqslant C. \tag{A.12}$$

*where* $\tilde{u}_B^{\{T\}} = (1 - B\tau)^{\frac{1}{2}} u_B^{\{T\}}$.

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