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ON THE DISTANCE BETWEEN HOMOTOPY CLASSES IN $W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)$

ITAI SHAFRIR

Abstract. For every $p \in (1,\infty)$ there is a natural notion of topological degree for maps in $W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)$ which allows us to write that space as a disjoint union of classes,

$$W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1) = \bigcup_{d\in\mathbb{Z}} \mathcal{E}_d.$$

For every pair $d_1, d_2 \in \mathbb{Z}$, we show that the distance

$$\mathrm{Dist}_{W^{1/p,p}}(\mathcal{E}_{d_1},\mathcal{E}_{d_2}) := \sup_{f\in\mathcal{E}_{d_1}} \inf_{g\in\mathcal{E}_{d_2}} d_{W^{1/p,p}}(f,g)$$

equals the minimal $W^{1/p,p}$-energy in $\mathcal{E}_{d_1-d_2}$. In the special case $p=2$ we deduce from the latter formula an *explicit* value: $\mathrm{Dist}_{W^{1/2,2}}(\mathcal{E}_{d_1},\mathcal{E}_{d_2}) = 2\pi|d_2-d_1|^{1/2}$.

1. Introduction

For any $1 < p < \infty$ consider the space $W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)$ consisting of the measurable functions $f : \mathbb{S}^1 \to \mathbb{R}^2$ satisfying $f(x) \in \mathbb{S}^1$ a.e. and

$$|f|_{W^{1/p,p}} := \left(\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \frac{|f(x)-f(y)|^p}{|x-y|^2}\,dxdy\right)^{1/p} < \infty. \tag{1.1}$$

Although the functions in $W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)$ are not necessarily continuous, a notion of topological degree does apply to maps in this space, based on the density of $C^\infty(\mathbb{S}^1;\mathbb{S}^1)$ in $W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)$. This is a special case of the concept of topological degree for maps in VMO, that was developed by Brezis and Nirenberg [7] (following a suggestion of L. Boutet de Monvel and O. Gabber [3, Appendix]). It is natural to use this degree to decompose the space into disjoint classes $\{\mathcal{E}_d\}_{d\in\mathbb{Z}}$ and then to define the "minimal energy" in each class, via the semi-norm in (1.1), that is

$$\sigma_p(d) := \inf_{f\in\mathcal{E}_d} |f|_{W^{1/p,p}}\,. \tag{1.2}$$

A lower bound for $\sigma_p(d)$ follows from the following result of Bourgain, Brezis and Mironescu [1] who proved that there exists a positive constant C_p such that

$$|\deg f| \leqslant C_p |f|^p_{W^{1/p,p}}\,,\ \forall f \in W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1). \tag{1.3}$$

Therefore,

$$\sigma_p(d) \geqslant \left(\frac{|d|}{C_p}\right)^{1/p},\ \forall d \in \mathbb{Z}. \tag{1.4}$$

In fact, a generalization of (1.3) to the space $W^{N/p,p}(\mathbb{S}^N;\mathbb{S}^N)$, $N \geqslant 2$, was also proved in [1] (see [2, 9] for refinements of this formula).

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In the special case $p = 2$ an explicit formula for $\sigma_2(d)$ is available, namely,

$$\sigma_2(d) = 2\pi|d|^{1/2}\,. \tag{1.5}$$

An easy way to establish (1.5) is by using the expansion of $f \in W^{1/2,2}(\mathbb{S}^1;\mathbb{S}^1)$ to Fourier series, $f(e^{\imath\theta}) = \sum_{n=-\infty}^{\infty} a_n e^{\imath n\theta}$. Indeed, combining the two well-known formulas (see e.g. [4]):

$$|f|^2_{W^{1/2,2}} = 4\pi^2 \sum_{n=-\infty}^{\infty} |n||a_n|^2 \quad\text{and}\quad \deg f = \sum_{n=-\infty}^{\infty} n|a_n|^2$$

yields the inequality $4\pi^2|\deg f| \leqslant |f|^2_{W^{1/2,2}}$, for every $f \in W^{1/2,2}(\mathbb{S}^1;\mathbb{S}^1)$, while equality occurs, e.g., for $f_d(z) = z^d$.

The distance function $\operatorname{dist}_{W^{1/p,p}}(f,g) = |f-g|_{W^{1/p,p}}$ induces two natural notions of distance between any pair of classes $\mathcal{E}_{d_1}, \mathcal{E}_{d_2}$:

$$\operatorname{dist}_{W^{1/p,p}}(\mathcal{E}_{d_1},\mathcal{E}_{d_2}) := \inf_{f\in\mathcal{E}_{d_1}} \inf_{g\in\mathcal{E}_{d_2}} d_{W^{1/p,p}}(f,g)\,, \tag{1.6}$$

and

$$\operatorname{Dist}_{W^{1/p,p}}(\mathcal{E}_{d_1},\mathcal{E}_{d_2}) := \sup_{f\in\mathcal{E}_{d_1}} \inf_{g\in\mathcal{E}_{d_2}} d_{W^{1/p,p}}(f,g)\,. \tag{1.7}$$

Both quantities in (1.6)–(1.7) were studied in [5]. Regarding $\operatorname{dist}_{W^{1/p,p}}$ the picture is completely clear; it was shown in [5] (by a similar argument to the one used in [7] in the case $p = 2$) that $\operatorname{dist}_{W^{1/p,p}}(\mathcal{E}_{d_1},\mathcal{E}_{d_2}) = 0$ for all $d_1, d_2 \in \mathbb{Z}$, for every $p \in (1,\infty)$. On the other hand, for $\operatorname{Dist}_W^{1/p,p}$ only partial results were obtained. While the upper bound

$$\operatorname{Dist}_{W^{1/p,p}}(\mathcal{E}_{d_1},\mathcal{E}_{d_2}) \leqslant c_2(p)|d_2-d_1|^{1/p},\ \forall d_1,d_2\in\mathbb{Z} \tag{1.8}$$

was proved in [5, Thm. 3, item 2], estimates for the lower bound were obtained only under some restrictions on p and/or d_1, d_2. As an example, it was proved in [5, Prop. 7.3] that

$$\operatorname{Dist}_{W^{1/2,2}}(\mathcal{E}_{d_1},\mathcal{E}_{d_2}) = 2\pi|d_2-d_1|^{1/2}, \text{ for } d_2 > d_1 \geqslant 0. \tag{1.9}$$

In the present paper we give a precise formula for $\operatorname{Dist}_{W^{1/p,p}}(\mathcal{E}_{d_1},\mathcal{E}_{d_2})$, that in the special case $p = 2$ yields the *explicit* formula (1.9) for *all* d_1, d_2.

Theorem 1.1. — *For every $p \in (1,\infty)$ and all $d_1, d_2 \in \mathbb{Z}$ we have*

$$\operatorname{Dist}_{W^{1/p,p}}(\mathcal{E}_{d_1},\mathcal{E}_{d_2}) = \sigma_p(d_2-d_1)\,. \tag{1.10}$$

In particular, there exist two positive constants $c_1(p) < c_2(p)$ such that

$$c_1(p)|d_2-d_1|^{1/p} \leqslant \operatorname{Dist}_{W^{1/p,p}}(\mathcal{E}_{d_1},\mathcal{E}_{d_2}) \leqslant c_2(p)|d_2-d_1|^{1/p},\ \forall d_1,d_2\in\mathbb{Z}. \tag{1.11}$$

Formula (1.11) provides a positive answer to Open Problem 2 from [5] in the case of dimension $N = 1$. It is an immediate consequence of (1.10), (1.4) and (1.8). Note also that (1.10) confirms the symmetry property, $\operatorname{Dist}_{W^{1/p,p}}(\mathcal{E}_{d_1},\mathcal{E}_{d_2}) = \operatorname{Dist}_{W^{1/p,p}}(\mathcal{E}_{d_2},\mathcal{E}_{d_1})$, which is not clear a priori from the definition (1.7) (thus providing support for a positive answer to [5, Open Problem 1]).

In the case $p = 2$ we obtain easily by combining (1.10) with (1.5):

Corollary 1.2. — *We have*

$$\operatorname{Dist}_{W^{1/2,2}}(\mathcal{E}_{d_1},\mathcal{E}_{d_2}) = 2\pi|d_2-d_1|^{1/2},\ \forall d_1,d_2\in\mathbb{Z}. \tag{1.12}$$

Remark 1.3. — Using a similar argument to the one used in the proof of Proposition 4.1 below, it is easy to see that

$$\sigma_p^p(d) \leqslant |d|\sigma_p^p(1),\ \forall d \in \mathbb{Z}. \tag{1.13}$$

It follows that we may take $c_2(p) = \sigma_p(1)$ in (1.11). While for $p = 2$ equality holds in (1.13) (by (1.5)), we do not know whether this is the case for other values of p.

The upper bound in (1.10) is the easier assertion. It follows from a slight modification of the argument used in the proof of item *2* of [5, Theorem 3], that is, the estimate (1.8). The proof of the lower bound in (1.10) is much more involved; it uses some arguments introduced in [6] to prove a lower bound for $\mathrm{Dist}_{W^{1,1}(\Omega;\mathbb{S}^1)}$ where Ω is either a bounded domain in $\mathbb{R}^N$ or a smooth compact manifold, e.g., $\Omega = \mathbb{S}^1$ (for the special case $W^{1,1}(\mathbb{S}^1;\mathbb{S}^1)$, a slightly different argument was used earlier in [5]). In particular, as in [5, 6] we make use of "zig-zag"-type functions in order to construct functions in $\mathcal{E}_{d_1}$ that are "relatively hard to approximate" by functions in $\mathcal{E}_{d_2}$. This is the content of Proposition 1.4 below, whose proof requires some new tools due to the nonlocal character of the $W^{1/p,p}$-energy. In order to state it we need to introduce some notation.

We start with a notation for arcs in $\mathbb{S}^1$. For every $\alpha < \beta$ let

$$\mathcal{A}(\alpha,\beta) = \{e^{i\theta}\,;\,\theta \in (\alpha,\beta)\},\ \mathcal{A}(\alpha,\beta] = \{e^{i\theta}\,;\,\theta \in (\alpha,\beta]\}$$
$$\text{and } \mathcal{A}[\alpha,\beta] = \{e^{i\theta}\,;\,\theta \in [\alpha,\beta]\}. \tag{1.14}$$

For any $n \geqslant 1$ we divide $\mathbb{S}^1$ to $2n$ arcs by setting

$$I_{2j} = \mathcal{A}\big(2j\pi/n, (2j+1)\pi/n\big] \text{ and } I_{2j+1} = \mathcal{A}\big((2j+1)\pi/n, (2j+2)\pi/n\big], \tag{1.15}$$

for $j = 0, 1, \dots, n-1$. Define $\widetilde{T}_n = \widetilde{T}_n^{(\alpha)} \in \mathrm{Lip}(\mathbb{S}^1;\mathbb{S}^1)$ with $\deg \widetilde{T}_n = 1$ by $\widetilde{T}_n(e^{\imath\theta}) = e^{\imath\tau_n(\theta)}$, with τ_n defined on $[0, 2\pi]$ by setting $\tau_n(0) = 0$ and

$$\tau_n'(\theta) = \begin{cases} n^\alpha & \theta \in (2j\,\pi/n, (2j+1)\,\pi/n] \\ -(n^\alpha - 2) & \theta \in ((2j+1)\,\pi/n, (2j+2)\,\pi/n] \end{cases},\ j = 0, 1, \dots, n-1, \tag{1.16}$$

where α is any number satisfying

$$\begin{cases} \alpha \in (1 - 1/p, 1) & \text{if } p \geqslant 2 \\ \alpha \in (1/p, 1) & \text{if } 1 < p < 2 \end{cases}. \tag{1.17}$$

We fix a value of α satisfying (1.17). A useful property of $\widetilde{T}_n$ is

$$d_{\mathbb{S}^1}(x, \widetilde{T}_n(x)) \leqslant \frac{\pi}{n^{1-\alpha}},\quad x \in \mathbb{S}^1, \tag{1.18}$$

where $d_{\mathbb{S}^1}$ denotes the geodesic distance in $\mathbb{S}^1$. The next proposition gives a partial analogue of [6, Prop. 1.3] to the $W^{1/p,p}$-setting.

Proposition 1.4. — *For any $d_1 \neq 0$ let $f(z) = z^{d_1}$ and define for each $n \geqslant 1$, $f_n(z) = \widetilde{T}_n \circ f \in \mathcal{E}_{d_1}$. Then, for every $d_2 \in \mathbb{Z}$ the sequence $\{f_n\}$ satisfies*

$$\lim_{n\to\infty} \inf_{g\in\mathcal{E}_{d_2}} d_{W^{1/p,p}}(f_n, g) = \sigma_p(d_2 - d_1). \tag{1.19}$$

It is clear that Proposition 1.4 implies the inequality "$\geqslant$" in (1.10) when $d_1 \neq 0$ (as we shall see in Section 4 below, the case $d_1 = 0$ is trivial).

The paper is organized as follows. In Section 2 we prove some technical results needed for the proof of our main results. Section 3 is devoted to the proof of a key lemma, essential to the proof of Proposition 1.4. Finally, the proofs of Proposition 1.4 and Theorem 1.1 are given in Section 4.

2. Preliminaries

We recall the following elementary result (see [6, Lemma 5.2]):

Lemma 2.1. — *Let z_1 and z_2 be two points in $\mathbb{S}^1$ satisfying, for some $\varepsilon \in (0,\pi/2)$,*

$$d_{\mathbb{S}^1}(z_1,z_2) \in (\varepsilon, \pi-\varepsilon). \tag{2.1}$$

If the vectors $v_1, v_2 \in \mathbb{R}^2$ satisfy

$$v_j \perp z_j,\ j=1,2, \tag{2.2}$$

then

$$|v_1 - v_2| \geqslant (\sin\varepsilon)|v_j|,\ \ j=1,2. \tag{2.3}$$

The intuition beyond the above result is quite simple. Informally speaking, if the points $z_1, z_2 \in \mathbb{S}^1$ are neither close to each other nor close to being antipodal points, then it is impossible for a pair of nonzero vectors, v_1 and v_2, in the tangent spaces of $\mathbb{S}^1$ at z_1 and z_2, respectively, to be "almost parallel" to each other. The next lemma can be viewed as a "discrete" version of Lemma 2.1, where tangent vectors are replaced by chords.

Lemma 2.2. — *For any $\varepsilon \in (0,\pi/2)$ and every four points $z_1, z_2, w_1, w_2 \in \mathbb{S}^1$ such that*

$$\text{either } z_1\overline{w}_1, z_2\overline{w}_2 \in \mathcal{A}(\varepsilon, \pi-\varepsilon) \quad \text{or} \quad z_1\overline{w}_1, z_2\overline{w}_2 \in \mathcal{A}(\pi+\varepsilon, 2\pi-\varepsilon)\,,$$

we have:

$$|(z_1-w_1)-(z_2-w_2)|^2 \geqslant (\sin^2\varepsilon)\max\left\{|z_1-z_2|^2, |w_1-w_2|^2\right\}\,. \tag{2.4}$$

Proof. — Without loss of generality assume that $z_1\overline{w}_1, z_2\overline{w}_2 \in \mathcal{A}(\varepsilon,\pi-\varepsilon)$ and write $z_j = e^{\imath\varphi_j}$ and $w_j = e^{\imath\psi_j}$ with $\varphi_j - \psi_j \in (\varepsilon, \pi-\varepsilon)$, $j=1,2$. We may also assume that $z_1 \neq z_2$ and $w_1 \neq w_2$; otherwise the result is clear. We have

$$\begin{aligned} z_1 - z_2 &= e^{\imath\varphi_1} - e^{\imath\varphi_2} = 2\imath \sin\left(\frac{\varphi_1-\varphi_2}{2}\right) e^{\imath(\varphi_1+\varphi_2)/2}\,,\\ w_1 - w_2 &= e^{\imath\psi_1} - e^{\imath\psi_2} = 2\imath \sin\left(\frac{\psi_1-\psi_2}{2}\right) e^{\imath(\psi_1+\psi_2)/2}\,. \end{aligned}$$

Therefore,

$$(z_1-z_2)\cdot\overline{w_1-w_2} = |z_1-z_2||w_1-w_2|\tau \exp\imath\big((\varphi_1-\psi_1)/2 + (\varphi_2-\psi_2)/2\big)\,, \tag{2.5}$$

with $\tau \in \{-1,1\}$. Since by our assumption $(\varphi_1-\psi_1)/2 + (\varphi_2-\psi_2)/2 \in (\varepsilon,\pi-\varepsilon)$, we get from (2.5) that an argument of $(z_1-z_2)\cdot\overline{w_1-w_2}$ lies in either the interval $(\varepsilon,\pi-\varepsilon)$ (if $\tau=1$) or $(\pi+\varepsilon, 2\pi-\varepsilon)$ (if $\tau=-1$). In any case, an argument lies in $(\varepsilon, 2\pi-\varepsilon)$, whence

$$|(z_1-w_1)-(z_2-w_2)|^2 \geqslant |z_1-z_2|^2 + |w_1-w_2|^2 - 2(\cos\varepsilon)|z_1-z_2||w_1-w_2|\,,$$

and (2.4) follows. □

We will also need the following result about Lipschitz self-maps of $\mathbb{S}^1$.

Lemma 2.3. — *Let $k \in Lip[0,2\pi]$ with Lipschitz constant L such that $k(0) = k(2\pi)$. Define $K : \mathbb{S}^1 \to \mathbb{S}^1$ by $K(e^{\imath\theta}) = e^{\imath k(\theta)}$, $\theta \in [0,2\pi]$. Then,*

$$\|K\|_{Lip} := \sup_{\substack{x,y\in\mathbb{S}^1\\ x\neq y}} \frac{|K(x)-K(y)|}{|x-y|} \leqslant \max\{1,L\}\,. \tag{2.6}$$

Proof. — For any pair $\theta_1 \neq \theta_2$ in $[0,2\pi)$ we have

$$\frac{|K(e^{\imath\theta_2})-K(e^{\imath\theta_1})|}{|e^{\imath\theta_2}-e^{\imath\theta_1}|} = \left|\frac{\sin\big((k(\theta_2)-k(\theta_1))/2\big)}{\sin\big((\theta_2-\theta_1)/2\big)}\right|$$
$$\leqslant \sup\left\{\frac{|\sin\theta|}{\sin t}\,;\, t\in(0,\pi/2],\ |\theta|\leqslant Lt\right\}. \tag{2.7}$$

Fix any $t \in (0,\pi/2]$. We distinguish two cases: either $Lt \leqslant \pi/2$ or $Lt > \pi/2$. In the first case we have

$$\sup\left\{\frac{|\sin\theta|}{\sin t}\,;\,|\theta|\leqslant Lt\right\} = \frac{\sin(Lt)}{\sin t} \leqslant \max\{L,1\}. \tag{2.8}$$

Indeed, if $L \leqslant 1$ then clearly $\sin(Lt) \leqslant \sin(t)$. On the other hand, if $L > 1$ then we use the fact that the function $g(t) = \sin(Lt) - L\sin t$ satisfies $g(0) = 0$ and $g'(t) = L(\cos(Lt) - \cos t) \leqslant 0$ for $0 \leqslant t \leqslant Lt \leqslant \pi/2$. In the second case (where we must have $L > 1$),

$$\sup\left\{\frac{|\sin\theta|}{\sin t}\,;\,|\theta|\leqslant Lt\right\} = \frac{1}{\sin t} < \frac{1}{\sin\big(\pi/(2L)\big)} < L, \tag{2.9}$$

where the last inequality follows from the easily verified fact that the function $h(L) := L\sin\big(\pi/(2L)\big)$ satisfies $h(1) = 1$ and $h'(L) > 0$ on $[1,\infty)$. The conclusion (2.6) clearly follows from (2.8)–(2.9). □

3. A key lemma

It will be useful to introduce the following notation for $f \in W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)$ and $A \subset \mathbb{S}^1\times\mathbb{S}^1$,

$$E_p(f;A) := \iint_A \frac{|f(x)-f(y)|^p}{|x-y|^2}\,dxdy\,,$$

so in particular $E_p(f;\mathbb{S}^1\times\mathbb{S}^1) = |f|^p_{W^{1/p,p}}$.

The next lemma is the main ingredient in the proof of Proposition 1.4.

Lemma 3.1. — *Let $u,\widetilde{u},v \in W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)\cap C(\mathbb{S}^1;\mathbb{S}^1)$, $\varepsilon \in (0,\pi/20)$, and*

$$\begin{aligned} C_\varepsilon^+ &= \{x\in\mathbb{S}^1;\ (v/\widetilde{u})(x)\in\mathcal{A}[-\varepsilon,\varepsilon]\}\,,\\ C_\varepsilon^- &= \{x\in\mathbb{S}^1;\ (v/\widetilde{u})(x)\in\mathcal{A}[\pi-\varepsilon,\pi+\varepsilon]\}\,,\\ C_\varepsilon &= C_\varepsilon^+\cup C_\varepsilon^-\,,\\ D_\varepsilon &= \mathbb{S}^1\times\mathbb{S}^1\setminus\big((C_\varepsilon^+\times C_\varepsilon^+)\cup(C_\varepsilon^-\times C_\varepsilon^-)\big)\,. \end{aligned} \tag{3.1}$$

Assume that

$$|u(x)-\widetilde{u}(x)|\leqslant\varepsilon,\ \forall x\in\mathbb{S}^1, \tag{3.2}$$

and let $\deg(u) = d_1$, $\deg(v) = d_2$. *Then, for some constant* $c_1 = c_1(p) > 0$ *we have, for* $\varepsilon \leqslant \varepsilon_0(p)$,

$$E_p(v - \widetilde{u}; D_\varepsilon) \geqslant (1 - c_1\varepsilon^{1/2})\sigma_p^p(d_2 - d_1) - c_1\varepsilon^{-p/2}E_p(u; (\mathbb{S}^1 \setminus C_\varepsilon) \times \mathbb{S}^1) - c_1\varepsilon^{p/2}E_p(u; \mathbb{S}^1 \times \mathbb{S}^1). \tag{3.3}$$

Proof. — Note first that (3.2) implies that $\deg(\widetilde{u}) = \deg(u) = d_1$. Hence, setting $w := v/u = v\,\overline{u}$ and $\widetilde{w} := v/\widetilde{u}$, we have $\deg(\widetilde{w}) = \deg(w) = d_2 - d_1$. Consider the map

$$W := \overline{u}(v - \widetilde{u}) + 1 = w + (1 - \widetilde{u}/u). \tag{3.4}$$

Since

$$W(x) - W(y) = \overline{u}(x)\{(v(x) - \widetilde{u}(x)) - (v(y) - \widetilde{u}(y))\} + (\overline{u}(x) - \overline{u}(y))(v(y) - \widetilde{u}(y))\,,$$

the triangle inequality yields,

$$|W(x) - W(y)| \leqslant |(v(x) - \widetilde{u}(x)) - (v(y) - \widetilde{u}(y))| + |1 - \widetilde{w}(y)||u(x) - u(y)|. \tag{3.5}$$

Interchanging between x and y gives

$$|W(x) - W(y)| \leqslant |(v(x) - \widetilde{u}(x)) - (v(y) - \widetilde{u}(y))| + |1 - \widetilde{w}(x)||u(x) - u(y)|. \tag{3.6}$$

By (3.5)–(3.6) we have

$$|W(x) - W(y)| \leqslant |(v(x) - \widetilde{u}(x)) - (v(y) - \widetilde{u}(y))| + 2|u(x) - u(y)|, \quad (x, y) \in \mathbb{S}^1 \times \mathbb{S}^1\,, \tag{3.7}$$

and

$$|W(x) - W(y)| \leqslant |(v(x) - \widetilde{u}(x)) - (v(y) - \widetilde{u}(y))| + \varepsilon|u(x) - u(y)|, \quad (x, y) \in (C_\varepsilon^+ \times \mathbb{S}^1) \cup (\mathbb{S}^1 \times C_\varepsilon^+). \tag{3.8}$$

Note that by (3.1) D_ε can be written as a disjoint union,

$$D_\varepsilon = ((\mathbb{S}^1 \setminus C_\varepsilon) \times \mathbb{S}^1) \cup (C_\varepsilon \times (\mathbb{S}^1 \setminus C_\varepsilon)) \cup (C_\varepsilon^+ \times C_\varepsilon^-) \cup (C_\varepsilon^- \times C_\varepsilon^+). \tag{3.9}$$

Next we will use the following elementary inequality:

$$(a + b)^p \leqslant (1 + \eta)^p a^p + (1 + 1/\eta)^p b^p\,, \quad \forall a, b, \eta, p > 0\,. \tag{3.10}$$

For the proof of (3.10) it suffices to notice that $a + b \leqslant (1 + \eta)a$ when $\eta a \geqslant b$, while $a + b < (1 + 1/\eta)b$ when $\eta a < b$. By (3.9) and (3.10), applied to (3.7)–(3.8) with $\eta = \sqrt{\varepsilon}$, we obtain

$$E_p(v - \widetilde{u}; D_\varepsilon) \geqslant \frac{E_p(W; D_\varepsilon)}{(1 + \sqrt{\varepsilon})^p} - 2(2/\sqrt{\varepsilon})^p E_p\big(u; \mathbb{S}^1 \times (\mathbb{S}^1 \setminus C_\varepsilon)\big) - 2\varepsilon^{p/2}E_p(u; C_\varepsilon^+ \times C_\varepsilon^-). \tag{3.11}$$

By (3.2), $|W - w| = |1 - \widetilde{u}/u| = |u - \widetilde{u}| \leqslant \varepsilon$ in $\mathbb{S}^1$. Hence

$$\big||W| - 1\big| \leqslant |W - w| \leqslant \varepsilon \ \text{ in } \mathbb{S}^1, \tag{3.12}$$

and also

$$|\widetilde{w} - w| = |\widetilde{u} - u| \leqslant \varepsilon \ \text{ in } \mathbb{S}^1. \tag{3.13}$$

Consider the map $\widetilde{W} := W/|W|$, which thanks to (3.12) belongs to $W^{1/p,p}(\mathbb{S}^1; \mathbb{S}^1)$. Furthermore, again by (3.12),

$$|\widetilde{W} - w| \leqslant |\widetilde{W} - W| + |W - w| \leqslant 2\varepsilon \text{ in } \mathbb{S}^1, \tag{3.14}$$

implying in particular that

$$\deg(\widetilde{W}) = d_2 - d_1. \tag{3.15}$$

Combining (3.14) with (3.13) yields

$$|\widetilde{W} - \widetilde{w}| \leqslant 3\varepsilon \text{ and } d_{\mathbb{S}^1}(\widetilde{W}, \widetilde{w}) \leqslant 6\varepsilon \text{ in } \mathbb{S}^1. \tag{3.16}$$

From (3.12) we get in particular that $|W| \geqslant 1-\varepsilon$, whence, using the identity

$$|z_1 - z_2|^2 = (|z_1| - |z_2|)^2 + |z_1|\cdot|z_2| \left|\frac{z_1}{|z_1|} - \frac{z_2}{|z_2|}\right|^2, \quad \forall z_1, z_2 \in \mathbb{C} - \{0\},$$

we get that

$$|W(x) - W(y)| \geqslant (1-\varepsilon)|\widetilde{W}(x) - \widetilde{W}(y)|, \quad \forall x, y \in \mathbb{S}^1. \tag{3.17}$$

Plugging (3.17) in (3.11) yields

$$\begin{aligned} E_p(v - \widetilde{u}; D_\varepsilon) \geqslant \Big(\frac{1-\varepsilon}{1+\sqrt{\varepsilon}}\Big)^p E_p(\widetilde{W}; D_\varepsilon) - 2(2/\sqrt{\varepsilon})^p E_p(u; \mathbb{S}^1 \times (\mathbb{S}^1 \setminus C_\varepsilon)) \\ - 2\varepsilon^{p/2} E_p(u; C_\varepsilon^+ \times C_\varepsilon^-). \end{aligned} \tag{3.18}$$

By (3.16) and (3.1) we have

$$\begin{aligned} C_\varepsilon^+ \subset \widetilde{C}_\varepsilon^+ &:= \{x \in \mathbb{S}^1; \widetilde{W}(x) \in \mathcal{A}[-7\varepsilon, 7\varepsilon]\} \\ C_\varepsilon^- \subset \widetilde{C}_\varepsilon^- &:= \{x \in \mathbb{S}^1; \widetilde{W}(x) \in \mathcal{A}[\pi - 7\varepsilon, \pi + 7\varepsilon]\} \end{aligned}. \tag{3.19}$$

Therefore,

$$\widetilde{D}_\varepsilon := \mathbb{S}^1 \times \mathbb{S}^1 \setminus \big((\widetilde{C}_\varepsilon^+ \times \widetilde{C}_\varepsilon^+) \cup (\widetilde{C}_\varepsilon^- \times \widetilde{C}_\varepsilon^-)\big) \subset D_\varepsilon. \tag{3.20}$$

For each $\delta \in (0, \pi/2)$ we define (as in [6]) the map $K_\delta : \mathbb{S}^1 \to \mathbb{S}^1$ by $K_\delta(e^{\imath\theta}) = e^{\imath k_\delta(\theta)}$ where $k_\delta : [0, 2\pi] \to [0, 2\pi]$ is given by

$$k_\delta(\theta) := \begin{cases} 0, & \text{if } \theta \in (0, \delta) \cup [2\pi - \delta, 2\pi] \\ \pi(\theta - \delta)/(\pi - 2\delta), & \text{if } \theta \in (\delta, \pi - \delta) \\ \pi, & \text{if } \theta \in [\pi - \delta, \pi + \delta] \\ \pi + \pi(\theta - \pi - \delta)/(\pi - 2\delta), & \text{if } \theta \in [\pi + \delta, 2\pi - \delta) \end{cases}. \tag{3.21}$$

Clearly $K_\delta \in \mathrm{Lip}(\mathbb{S}^1; \mathbb{S}^1)$ and $\deg(K_\delta) = 1$. Since $\|k_\delta'\|_\infty = \pi/(\pi - 2\delta)$ we have by Lemma 2.3,

$$\big|K_\delta(e^{\imath\theta_2}) - K_\delta(e^{\imath\theta_1})\big| \leqslant \left(\frac{\pi}{\pi - 2\delta}\right) \big|e^{\imath\theta_2} - e^{\imath\theta_1})\big|, \quad \forall \theta_1, \theta_2 \in [0, 2\pi].$$

Therefore, $w_1 := K_{7\varepsilon} \circ \widetilde{W}$ satisfies $\deg(w_1) = \deg(\widetilde{W}) = d_2 - d_1$ and

$$|w_1(x) - w_1(y)| \leqslant \left(\frac{\pi}{\pi - 14\varepsilon}\right) |\widetilde{W}(x) - \widetilde{W}(y)|, \ \forall x, y \in \mathbb{S}^1. \tag{3.22}$$

By definition of σ_p, (3.22) and the definition of $K_{7\varepsilon}$ (see (3.21)) it follows, using also (3.20) and the fact that w_1 is constant on $\widetilde{C}_\varepsilon^+$ and $\widetilde{C}_\varepsilon^-$, that

$$\begin{aligned} \sigma_p^p(d_2 - d_1) &\leqslant E_p(w_1; \mathbb{S}^1 \times \mathbb{S}^1) = E_p(w_1; \widetilde{D}_\varepsilon) \\ &\leqslant \left(\frac{\pi}{\pi - 14\varepsilon}\right)^p E_p(\widetilde{W}; \widetilde{D}_\varepsilon) \leqslant \left(\frac{\pi}{\pi - 14\varepsilon}\right)^p E_p(\widetilde{W}; D_\varepsilon). \end{aligned} \tag{3.23}$$

Plugging (3.23) in (3.18) yields (3.3), for large enough c_1. □

4. Proof of Theorem 1.1

We begin with the upper bound for $\mathrm{Dist}_{W^{1/p,p}}$:

Proposition 4.1. — *For every $d_1, d_2 \in \mathbb{Z}$ we have*

$$\mathrm{Dist}_{W^{1/p,p}}(\mathcal{E}_{d_1}, \mathcal{E}_{d_2}) \leqslant \sigma_p(d_2 - d_1). \tag{4.1}$$

Proof. — Let $f \in \mathcal{E}_{d_1}$ and $\varepsilon > 0$ be given. We need to prove the existence of $g \in \mathcal{E}_{d_2}$ satisfying

$$|f - g|^p_{W^{1/p,p}} \leqslant \sigma_p^p(d_1 - d_2) + \varepsilon. \tag{4.2}$$

By [5, Lemma 2.2] every map in $W^{1/p,p}(\mathbb{S}^1; \mathbb{S}^1)$ can be approximated by a sequence $\{f_n\} \subset C^\infty(\mathbb{S}^1; \mathbb{S}^1)$ such that each f_n is constant near some point. Therefore, without loss of generality we may assume that the given f satisfies $f \equiv 1$ in $\mathcal{A}(\pi - \delta, \pi + \delta)$ for some small $\delta > 0$. By definition of $\sigma_p(d_2 - d_1)$ there exists $h \in \mathcal{E}_{d_2 - d_1}$ satisfying

$$|h|^p_{W^{1/p,p}} \leqslant \sigma_p^p(d_2 - d_1) + \varepsilon. \tag{4.3}$$

By the density result mentioned above, we may assume that $h \equiv 1$ in $\mathcal{A}(-\eta, \eta)$, for some small $\eta > 0$. Next we invoke the invariance of $|\cdot|_{W^{1/p,p}}$ with respect to Möbius transformations $\mathcal{M}$ that send $\mathbb{S}^1$ to itself (see [8]) to get that

$$|h|_{W^{1/p,p}} = |h \circ \mathcal{M}|_{W^{1/p,p}}. \tag{4.4}$$

For each $n \geqslant 1$ let $\mathcal{M}_n$ be the unique Möbius transformation that sends the ordered triple (with respect to the positive orientation on $\mathbb{S}^1$) $(e^{\imath(\pi+1/n)}, 1, e^{\imath(\pi-1/n)})$ to the ordered triple $(e^{-\imath\eta}, 1, e^{\imath\eta})$. Hence $\mathcal{M}_n$ is a self map of $\mathbb{S}^1$ satisfying $\mathcal{M}_n(\mathcal{A}(\pi + 1/n, 3\pi - 1/n)) = \mathcal{A}(-\eta, \eta)$. Set $h_n = h \circ \mathcal{M}_n$. Clearly $\deg h_n = \deg h = d_2 - d_1$ and by (4.4) and (4.3), for each n, $|h_n|^p_{W^{1/p,p}} = |h|^p_{W^{1/p,p}} \leqslant \sigma_p^p(d_2 - d_1) + \varepsilon$ and

$$\{x \in \mathbb{S}^1\,;\, h_n(x) \neq 1\} \subset \mathcal{A}(\pi - 1/n, \pi + 1/n)\,.$$

For every n set $g_n = f h_n \in \mathcal{E}_{d_2}$. By construction it is clear that for $n > 1/\delta$ we have $g_n - f = f(h_n - 1) = h_n - 1$ on $\mathbb{S}^1$. Therefore, (4.2) holds with $g = g_n$ for such n. □

The main ingredient in the proof of the lower bound for $\mathrm{Dist}_{W^{1/p,p}}$ is Proposition 1.4.

Proof of Proposition 1.4. — Clearly it suffices to consider $d_2 \neq d_1$ with $d_1 > 0$. Let a small $\varepsilon > 0$ be given. In view of the upper bound of Proposition 4.1, it suffices to show that there exists $N(\varepsilon)$ such that (for every sufficiently small ε):

$$|f_n - g|^p_{W^{1/p,p}} \geqslant \sigma_p^p(d_2 - d_1) - \varepsilon^{1/3}, \quad \forall g \in \mathcal{E}_{d_2}, \forall n \geqslant N(\varepsilon). \tag{4.5}$$

Fix any $g \in \mathcal{E}_{d_2}$. By density of smooth maps in $W^{1/p,p}(\mathbb{S}^1; \mathbb{S}^1)$ we may assume that $g \in C^\infty(\mathbb{S}^1; \mathbb{S}^1)$. Clearly it suffices to consider n for which

$$|f_n - g|^p_{W^{1/p,p}} \leqslant \sigma_p^p(d_2 - d_1). \tag{4.6}$$

Consider the map

$$H_n := \bar{f}(g - f_n) + 1 = h + (1 - \bar{f} f_n)\,. \tag{4.7}$$

Put $N_1(\varepsilon) := \left[(\pi/\varepsilon)^{1/(1-\alpha)}\right] + 1$. By (1.18) we deduce that

$$|f_n - f| \leqslant \varepsilon \text{ on } \mathbb{S}^1,\ \forall n \geqslant N_1(\varepsilon). \tag{4.8}$$

For such n we may apply Lemma 3.1 with $u=f, \widetilde{u}=f_n$ and $v=g$ to get that

$$|g-f_n|^p_{W^{1/p,p}} \geqslant (1-c_1\varepsilon^{1/2})\sigma_p^p(d_2-d_1)-c_1\varepsilon^{-p/2}E_p(f;(\mathbb{S}^1\setminus C_\varepsilon^{(n)})\times\mathbb{S}^1)-c_1\gamma_{d_1}\varepsilon^{p/2}, \tag{4.9}$$

where for each $d\in\mathbb{Z}$ we denote

$$\gamma_d := |z^d|^p_{W^{1/p,p}}\,, \tag{4.10}$$

and where

$$C_\varepsilon^{(n)} = \{x\in\mathbb{S}^1\,;\,(\bar{f}_n g)(x)\in\mathcal{A}[-\varepsilon,\varepsilon]\cup\mathcal{A}[\pi-\varepsilon,\pi+\varepsilon]\}.$$

In order to conclude via (4.9) we need to bound the term $E_p(f;(\mathbb{S}^1\setminus C_\varepsilon^{(n)})\times\mathbb{S}^1)$. We claim that there exists $C=C(p,d_1,d_2)$ such that for some $\beta>0$ there holds

$$E_p(f;(\mathbb{S}^1\setminus C_\varepsilon^{(n)})\times\mathbb{S}^1)\leqslant\frac{C}{\varepsilon}n^{-\beta}. \tag{4.11}$$

We may write $\mathbb{S}^1\setminus C_\varepsilon^{(n)} = A_{\varepsilon,+}^{(n)}\cup A_{\varepsilon,-}^{(n)}$ where $A_{\varepsilon,+}^{(n)}=\{x\in\mathbb{S}^1\,;\,(\bar{f}_n g)(x)\in\mathcal{A}(\varepsilon,\pi-\varepsilon)\}$ and $A_{\varepsilon,-}^{(n)}=\{x\in\mathbb{S}^1\,;\,(\bar{f}_n g)(x)\in\mathcal{A}(\pi+\varepsilon,2\pi-\varepsilon)\}$. Next we write $\mathbb{S}^1$ as a disjoint union of the $2nd_1$ arcs given by

$$\tilde{I}_k=\mathcal{A}\Big(\frac{k\pi}{nd_1},\frac{(k+1)\pi}{nd_1}\Big]\,,\ k=0,1,\dots,2nd_1-1.$$

By the definition of f_n we have (for large n) for all $x\neq y$ in $\tilde{I}_k$:

$$\frac{d_{\mathbb{S}^1}(f_n(x),f_n(y))}{d_{\mathbb{S}}^1(x,y)}=\begin{cases} n^\alpha d_1 & k \text{ is even}\\ (n^\alpha-2)d_1 & k \text{ is odd}\end{cases}. \tag{4.12}$$

We use these arcs to write $A_{\varepsilon,+}^{(n)}=\bigcup\limits_{k=0}^{2nd_1-1}J_{k,+}$ where $J_{k,+}=A_{\varepsilon,+}^{(n)}\cap\tilde{I}_k$. Using the following basic relation between the geodesic and Euclidean distances in $\mathbb{S}^1$,

$$\Big(\frac{2}{\pi}\Big)d_{\mathbb{S}^1}(x,y)\leqslant|x-y|\leqslant d_{\mathbb{S}^1}(x,y),\quad\forall x,y\in\mathbb{S}^1\,, \tag{4.13}$$

we deduce from (4.12) that

$$\frac{|f_n(x)-f_n(y)|^p}{|x-y|^2}\geqslant C_1n^{\alpha p}|x-y|^{p-2}\,,\quad\text{for all } x\neq y \text{ in } J_{k,+}, \tag{4.14}$$

for some constant $C_1=C_1(p,d_1)$. Applying (2.4) with $z_1=f_n(x), z_2=f_n(y), w_1=g(x)$ and $w_2=g(y)$ to the L.H.S. of (4.14), and then integrating over $J_{k,+}\times J_{k,+}$ yields

$$\iint_{J_{k,+}\times J_{k,+}}\frac{|(f_n(x)-g(x))-(f_n(y)-g(y))|^p}{|x-y|^2}\,dx\,dy\geqslant$$
$$C_1(\sin^p\varepsilon)n^{\alpha p}\iint_{J_{k,+}\times J_{k,+}}|x-y|^{p-2}\,dx\,dy,\quad k=0,1,\dots,2nd_1-1\,. \tag{4.15}$$

Next, we can also write $A^{(n)}_{\varepsilon,-} = \bigcup_{k=0}^{2nd_1-1} J_{k,-}$ where $\{J_{k,-}\}_{k=0}^{2nd_1-1}$ are defined analogously to $\{J_{k,+}\}_{k=0}^{2nd_1-1}$. The same computation that led to (4.15) gives

$$\iint_{J_{k,-}\times J_{k,-}} \frac{|(f_n(x)-g(x))-(f_n(y)-g(y))|^p}{|x-y|^2}\,dx\,dy \geqslant \\ C_1(\sin^p\varepsilon)n^{\alpha p}\iint_{J_{k,-}\times J_{k,-}} |x-y|^{p-2}\,dx\,dy, \quad k=0,1,\dots,2nd_1-1\,. \tag{4.16}$$

Summing over all indices in (4.15)–(4.16) and taking into account (4.6) yields

$$\sum_{k=0}^{2nd_1-1}\iint_{J_{k,-}\times J_{k,-}} |x-y|^{p-2}\,dx\,dy+\sum_{k=0}^{2nd_1-1}\iint_{J_{k,+}\times J_{k,+}} |x-y|^{p-2}\,dx\,dy \leqslant \frac{C_2}{(n^\alpha\sin\varepsilon)^p}\,. \tag{4.17}$$

Next we treat separately the cases $p\geqslant 2$ and $1<p<2$.

Case I: $p\geqslant 2$

The key tool in treating this case is the following elementary inequality:

$$\iint_{A\times A} |x-y|^a\,dx\,dy \geqslant \kappa_a|A|^{a+2}\,,\ \forall A\subset\mathbb{S}^1,\,\forall a\geqslant 0, \tag{4.18}$$

for some constant $\kappa_a>0$. [Obviously we consider only measurable subsets of $\mathbb{S}^1$ and $|A|$ denotes the one dimensional Hausdorff measure of A]. To verify (4.18) we first note that for any measurable set $A\subset\mathbb{R}$ the set

$$B:=\{x\in A\,;\,|x|\geqslant|A|/4\}\,,$$

satisfies $|B|\geqslant|A|/2$ (here $|C|$ stands for the Lebesgue measure of $C\subset\mathbb{R}$). It follows that

$$\int_A |x|^a\,dx\geqslant\int_B |x|^a\,dx\geqslant|B|(|A|/4)^a\geqslant\tilde{c}_a|A|^{a+1},\ \forall a\geqslant 0. \tag{4.19}$$

Since (4.19) is clearly invariant w.r.t translations, we deduce that also

$$\int_A |x-y|^a\,dx\geqslant\tilde{c}_a|A|^{a+1}\,,\ \forall A\subset\mathbb{R},\,\forall y\in\mathbb{R},\,\forall a\geqslant 0\,,$$

and an additional integration yields

$$\iint_{A\times A} |x-y|^a\,dx\,dy\geqslant\tilde{c}_a|A|^{a+2}\,,\ \forall A\subset\mathbb{R},\,\forall a\geqslant 0\,. \tag{4.20}$$

Switching from $\mathbb{S}^1$ to $\mathbb{R}$, using (4.13), enables us to deduce (4.18) from (4.20).

Applying (4.18) to $A=J_{k,\pm}$ and $a=p-2$ gives

$$\iint_{J_{k,\pm}\times J_{k,\pm}} |x-y|^{p-2}\,dx\,dy\geqslant\kappa_{p-2}|J_{k,\pm}|^p\,. \tag{4.21}$$

Plugging (4.21) in (4.17) yields

$$\sum_{k=0}^{2nd_1-1}(|J_{k,+}|^p+|J_{k,-}|^p)\leqslant\frac{C_3}{(n^\alpha\sin\varepsilon)^p}\,. \tag{4.22}$$

By Hölder inequality and (4.22) we obtain,

$$\left|\mathbb{S}^1\backslash C_\varepsilon^{(n)}\right| = \sum_{k=0}^{2nd_1-1} (|J_{k,+}|+|J_{k,-}|) \leqslant (4nd_1)^{1-1/p}\frac{C_3^{1/p}}{n^\alpha \sin\varepsilon} \leqslant \frac{C_4}{\varepsilon}n^{1-1/p-\alpha}. \quad (4.23)$$

Finally, by (4.23) we get

$$E_p(f;(\mathbb{S}^1\backslash C_\varepsilon^{(n)})\times\mathbb{S}^1) \leqslant 2\pi\left|\mathbb{S}^1\setminus C_\varepsilon^{(n)}\right| \sup_{\substack{x,y\in\mathbb{S}^1\\ x\neq y}} \frac{|f(x)-f(y)|^p}{|x-y|^2} \leqslant \frac{C_5}{\varepsilon}n^{1-1/p-\alpha}, \quad (4.24)$$

which gives (4.11) with $\beta=\alpha-(1-1/p)>0$ (by (1.17)).

Case II: $1<p<2$

Treating this case requires another elementary inequality, namely,

$$\iint_{A\times A}\frac{dx\,dy}{|x-y|^b} \geqslant \lambda_b\left(\iint_{A\times\mathbb{S}^1}\frac{dx\,dy}{|x-y|^b}\right)^2, \ \forall A\subset\mathbb{S}^1,\ \forall b\in(0,1), \quad (4.25)$$

for some $\lambda_b>0$. To confirm (4.25) we first notice that $\int_{\mathbb{S}^1}\frac{dy}{|x-y|^b}:=\eta=\eta(b)$, $\forall x\in\mathbb{S}^1$, and thus

$$\iint_{A\times\mathbb{S}^1}\frac{dx\,dy}{|x-y|^b}=\eta|A|\,, \text{ for every measurable } A\subset\mathbb{S}^1. \quad (4.26)$$

Finally, by (4.26)

$$\iint_{A\times A}\frac{dx\,dy}{|x-y|^b} \geqslant \frac{1}{2^b}|A|^2 = \frac{1}{2^b\eta^2}\left(\iint_{A\times\mathbb{S}^1}\frac{dx\,dy}{|x-y|^b}\right)^2,$$

and (4.25) follows with $\lambda_b=\frac{1}{2^b\eta^2}$.

Next we turn to the proof of (4.11) in this case. Clearly

$$E_p(f;(\mathbb{S}^1\setminus C_\varepsilon^{(n)})\times\mathbb{S}^1) \leqslant C_6\sum_{k=0}^{2nd_1-1}\left(\iint_{J_{k,-}\times\mathbb{S}^1}|x-y|^{p-2}\,dx\,dy+\iint_{J_{k,+}\times\mathbb{S}^1}|x-y|^{p-2}\,dx\,dy\right). \quad (4.27)$$

Applying the Cauchy-Schwarz inequality to (4.27) and using (4.25) (with $A=J_{k,\pm}$ and $b=2-p$) and (4.17) yields

$$E_p(f;(\mathbb{S}^1\setminus C_\varepsilon^{(n)})\times\mathbb{S}^1) \leqslant C_6(4nd_1)^{1/2}\frac{C_2^{1/2}}{\lambda_{2-p}^{1/2}(n^\alpha\sin\varepsilon)^{p/2}} \leqslant \frac{C_7}{\varepsilon}n^{(1-\alpha p)/2},$$

and (4.11) follows in this case as well, with $\beta=(\alpha p-1)/2>0$ (see (1.17)).

Choosing $N(\varepsilon)\geqslant N_1(\varepsilon)$ (see (4.8)) such that, in addition,

$$Cn^{-\beta}\leqslant\varepsilon^{1+p},\ \forall n\geqslant N(\varepsilon),$$

we get from (4.9) and (4.11) that for $n\geqslant N(\varepsilon)$ there holds,

$$|g-f_n|^p_{W^{1/p,p}} \geqslant (1-c_1\varepsilon^{1/2})\sigma_p^p(d_2-d_1)-c_1\varepsilon^{p/2}(1+\gamma_{d_1}) \geqslant \sigma_p^p(d_2-d_1)-\varepsilon^{1/3},$$

for ε sufficiently small (using $p/2>1/2$), and (4.5) follows. $\square$

We can now give the proof of our main result Theorem 1.1.

Proof of Theorem 1.1. — In view of (4.1) of Proposition 4.1, it suffices to prove that

$$\mathrm{Dist}_{W^{1/p,p}}(\mathcal{E}_{d_1}, \mathcal{E}_{d_2}) \geqslant \sigma_p(d_2 - d_1), \quad \forall d_1, d_2 \in \mathbb{Z}. \tag{4.28}$$

In case $d_1 \neq 0$, (4.28) follows from Proposition 1.4. In the remaining (easy) case $d_1 = 0$, we can take the constant function $f = 1$ that satisfies $d^p_{W^{1/p,p}}(f, g) = |g|^p_{W^{1/p,p}} \geqslant \sigma^p_p(d_2)$ for all $g \in \mathcal{E}_{d_2}$. □

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