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ON THE HALF-TRAJECTORIES OF HOROCYCLIC FLOW ON GEOMETRICALLY INFINITE HYPERBOLIC SURFACES

ADAMOU SAIDOU

Abstract. We study the density of half-horocycles or half-orbits of the horocyclic flow on the unit tangent bundle of geometrically infinite hyperbolic surfaces. In [10] Schapira proved that under some assumptions, both half-horocycles $(h^s v)_{s\geqslant 0}$ and $(h^s v)_{s\leqslant 0}$ are simultaneously dense or not in the nonwandering set of the horocyclic flow. We construct a counterexample, when the assumptions are not satisfied, on a surface of first kind, answering a question of Schapira [10].

Introduction

The horocyclic flow on the unit tangent bundle of a hyperbolic surface has been studied extensively since Hedlund, Hopf, in the thirties. In particular, the density of horocyclic orbits is well understood, and dense orbits are exactly those orbits $(h^s v)_{s\in\mathbb{R}}$ such that v is *horospherical*.

It is an interesting question from the dynamical point of view to ask whether both half-horocycles $(h^s v)_{s\geqslant 0}$ and $(h^s v)_{s\leqslant 0}$ have the same behavior when the full horocycle $(h^s v)_{s\in\mathbb{R}}$ is dense. Positive answers have been obtained in some specific cases. Hedlund answered positively on surface of first kind (see definition in section 2), in the specific case of radial vectors, that is vectors whose (backward) geodesic orbit return infinitely often in a compact set.

In the case of geometrically finite hyperbolic surfaces, Schapira proved that the answer is always positive except for some trivial geometric obstructions. Trivial obstructions consist of those vectors with one half-horocycle which is dense in the nonwanderig set $\mathcal{E}$, and the other which leaves eventually the nonwandering set.

In the case of geometrically infinite hyperbolic surfaces, Schapira extended Hedlund's result and obtained a more general result. Her result is:

Theorem 0.1 (Schapira [10]). — *Let S be a nonelementary oriented hyperbolic surface. Let $v\in T^1S$ be a vector whose full horocycle $(h^s v)_{s\in\mathbb{R}}$ is dense in the nonwandering set $\mathcal{E}$ of the horocyclic flow and such that there exist two constants $\Lambda>0$, $0<\alpha_0<\frac{\pi}{2}$ such that the geodesic ray $(\pi(g^{-t}v))_{t\geqslant 0}$ intersects infinitely many closed geodesics of length at most Λ with an angle of intersection at least α_0. Then both half-orbits $(h^s v)_{s\geqslant 0}$ and $(h^s v)_{s\leqslant 0}$ are simultaneously dense.*

In order to verify optimality of the result, Schapira constructed a counterexample to a completely general result:

Theorem 0.2 (Schapira [10]). — *There exist hyperbolic surfaces whose unit tangent bundle contains a vector v for which no trivial obstruction holds and such that $(h^s v)_{s\geqslant 0}$ is dense in $\mathcal{E}$ $(h^s v)_{s\leqslant 0}$ is not dense in $\mathcal{E}$.*

A question arises after Schapira's results.

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- Is Schapira's counterexample a surface of the first kind?

The aim of this work is to give positive answers to this question. We correct some mistakes in the proof of Theorem 0.2 and provide a counterexample of Schapira's type that lies on a surface of the first kind.

Theorem 0.3. — *There exist hyperbolic surfaces of first kind whose unit tangent bundle contains a vector v such that $(h^s v)_{s\geqslant 0}$ is dense in $\mathcal{E}$ $(h^s v)_{s\leqslant 0}$ is not dense in $\mathcal{E}$.*

This paper is organised as follows. In section 1 we give some tools in hyperbolic geometry used in the next paragraphs. Section 2 is devoted to the proof of Theorem 0.3 were we construct a counterexample of the first kind.

1. Preliminaries

1.1. **Hyperbolic geometry.** A large part of this paragraph is mentionned in [9, 10].

Let $\mathbb{D} = \mathbb{D}(0,1)$ be the hyperbolic disk endowed with the hyperbolic metric $\frac{4dx^2}{(1-|x|^2)^2}$, and o the origin of the disk. Denote by $\pi : T^1\mathbb{D} \to \mathbb{D}$ the canonical projection, and by $\partial\mathbb{D} = S^1$ the boundary at infinity. By abuse of notation we denote by d both the hyperbolic distance on $\mathbb{D}$ and the Sasaki distance on $T^1\mathbb{D}$.

The geodesic flow $(g^t)_{t\in\mathbb{R}}$ acts on $T^1\mathbb{D}$ by moving a vector of a distance t along the geodesic that it defines. The Busemann cocycle is the continuous map defined on $S^1 \times \mathbb{D}^2$ by:

$$\beta_\xi(x,y) = \lim_{z\to\xi}(d(x,z) - d(y,z)).$$

The map $v \in T^1\mathbb{D} \mapsto (v^-, v^+, \beta_{v^-}(\pi(v), o)) \in ((S^1 \times S^1) \setminus Diagonal) \times \mathbb{R}$ is a homeomorphism, where $v^\pm$ are the endpoints of the geodesic $(g^t v)_{t\in\mathbb{R}}$ in S^1 and Δ is the diagonal of $S^1 \times S^1$, The coordinates given by this homeomorphism are called Hopf coordinates. We identify $T^1\mathbb{D}$ with the set of Hopf coordinates $((S^1 \times S^1) \setminus Diagonal) \times \mathbb{R}$.

An isometry of $PSL(2,\mathbb{R})$ is called hyperbolic if it has exactly two fixed points on S^1, parabolic if it fixes one point of S^1, and elliptic otherwise.

The classical identification of $\mathbb{D}$ with $\mathbb{H} = \mathbb{R} \times \mathbb{R}_+^*$ through the homography $z \to i\frac{1+z}{1-z}$ allows to identify the group of isometries preserving orientation of $\mathbb{D}$ with $PSL(2,\mathbb{R})$ acting by homographies on $\mathbb{H}$. This extends to a simply transitive action on $T^1\mathbb{D}$ (or $T^1\mathbb{H}$).

If $\Gamma \subset PSL(2,\mathbb{R})$ is a discrete subgroup without elliptic elements, then the quotient $S = \Gamma\backslash\mathbb{D}$ is a hyperbolic surface and its unit tangent bundle $T^1S = \Gamma\backslash T^1\mathbb{D}$ is identified through the Hopf coordinates with the quotient $\Gamma\backslash(((S^1\times S^1)\backslash\Delta)\times\mathbb{R})$.

The limit set Λ_Γ of the group is defined as $\Lambda_\Gamma = \overline{\Gamma}.o \setminus \Gamma.o \subset S^1$. For all $\xi \in \Lambda_\Gamma$, the set $\Gamma.\xi$ is dense in Λ_Γ.

A horocycle of $\mathbb{D}$ is a level set of a Busemann function. A horoball is a set $\{x \in \mathbb{H}, \beta_\xi(x,y) \leqslant C\}$.

The group Γ is nonelementary if $\#\Lambda_\Gamma = +\infty$. The surface S is said to be of the first kind if $\Lambda_\Gamma = S^1$, and of the second kind otherwise.

1.2. **Horocyclic flow.** A hyperbolic geodesic of $\mathbb{D}$ is a diameter or a half-circle orthogonal to S^1. A vector $v \in T^1\mathbb{D}$ is tangent to a unique geodesic and orthogonal to exactly two horocycles containing its basepoint tangent to S^1 at v^+ and v^-. The set of vectors in $T^1\mathbb{D}$ such that $w^- = v^-$ and whose basepoint belongs to the horocycle tangent to S^1 at v^- and containing $\pi(v)$ is the strong unstable horocycle or strong unstable manifold of v. We denote it by $W^{su}(v) = \{h^s v, s \in \mathbb{R}\}$. The strong stable horocycle is defined similarly. The unstable horocycle flow $(h^s)_{s\in\mathbb{R}}$ acts on $T^1\mathbb{D}$ by moving a vector v of a distance $|s|$ along its strong unstable horocycle. There are two possible orientations for such a flow, and we choose the orientation which corresponds to the right action on $PSL(2,\mathbb{R})$ by multiplication by one-parameter subgroup

$$\left\{ n_s := \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}, s \in \mathbb{R} \right\}$$

on $PSL(2,\mathbb{R})$. This flow makes the vectors turn around their strong unstable horocycle so that the orbit $\{h^s v, s \in \mathbb{R}\}$ is equal to the full strong unstable horocycle.

Moreover, for all $s \in \mathbb{R}$ and $t \in \mathbb{R}$, these geodesic and horocyclic flows satisfy the following fundamental relation

$$g^t \circ h^s = h^{se^t} \circ g^t.$$

Definition 1.1. — Let $(\phi^t)_{t\in\mathbb{R}}$ be a flow acting by homeomorphisms on a topological space X. The nonwandering set of this flow is the set of points $x \in X$ such that for all neighborhoods W of x there exists a sequence $t_n \longrightarrow +\infty$ such that $\phi^{t_n} W \cap W \neq \emptyset$.

Proposition 1.2 (Eberlein[6], Schapira[9]). — (1) *The nonwandering set of the geodesic flow acting on T^1S is*

$$\Omega := \Gamma \backslash \Big(((\Lambda_\Gamma \times \Lambda_\Gamma) \backslash \Delta) \times \mathbb{R} \Big).$$

(2) *The nonwandering set of the horocyclic flow acting on T^1S is*

$$\mathcal{E} := \Gamma \backslash \Big(((\Lambda_\Gamma \times S^1) \backslash \Delta) \times \mathbb{R} \Big).$$

2. Proof of Theorem 0.3

Schapira [10] constructed the counterexample of Theorem 0.2 and asked if it is a counterexample of first kind. Her construction contains some mistakes and lets some questions open. In the following we correct the mistakes and prove that the counterexample we obtain is of the first kind.

Let us recall some definitions.

Let $v \in T^1\mathbb{D}$ be a vector and $v^\pm$ be its endpoints in $\partial\mathbb{D}$. Let $Hor(v) \subset \mathbb{D}$ be the horoball centered at v^- and containing its basepoint $\pi(v)$. The right horoball $Hor^+(v) \subset Hor(v)$ is the set of basepoints of vectors of $\bigcup_{t\geqslant 0} \bigcup_{s\geqslant 0} h^s g^{-t} v$. Similarly, we define the left horoball $Hor^-(v) \subset Hor(v)$ as the other side of $Hor(v)$.

Definition 2.1. — If $v \in T^1\mathbb{D}$ and $\alpha > 0$, the cone of width α around v is the set $\mathcal{C}(v, \alpha)$ of points $x \in Hor(v)$ at (hyperbolic) distance at most α of the geodesic ray $(g^{-t}v)_{t\geqslant 0}$.

Definition 2.2. — A vector $v \in T^1S$ is right horocyclic if it admits a lift $\tilde{v} \in T^1\mathbb{D}$, such that for all $\alpha > 0$ and $D > 0$, the orbit $\Gamma.o$ intersects the right part of the horoball $Hor^+(g^{-D}\tilde{v})$ minus the cone $\mathcal{C}(g^{-D}\tilde{v}, \alpha)$.

A point ξ is right horocyclic if there exists a right horocyclic vector $v \in T^1S$ such that $\xi = v^-$.

A vector v is horospherical if all horoball $Hor^+(g^{-D}v)$ contains infinitelly many points of $\Gamma.o$.

2.1. **Proof of Theorem 0.3.** Let us explain the minor mistake in Schapira's construction. We start with the following construction which is a small modification of the construction in [10] (see Figure 2.1). Let v be the vector in $T^1\mathbb{H}$ such that $v^- = \infty$ and $v^+ = 0$.
Denote by C_n^+ (respectively C_n^-) the half-circle centered at $(2n+1, 0)$ (resp. $(-x_n, 0)$) of radius 1 (resp. R_n) with $R_n = n^3$ and $x_n = R_n + 2\sum_0^{n-1} R_k = n^3 + 2\frac{n^2(n-1)^2}{4} \sim \frac{n^4}{2}$. Denote by D_n^+ (respectively D_n^-) the half-disk centered at $(2n+1, 0)$ (resp. $(-x_n, 0)$) of radius 1 (resp. R_n).

Let Γ be the Schottky free group generated by the family of isometries (γ_n) such that for $n \geqslant 3$, $\gamma_n(C_n^+) = C_n^-$, $\gamma_n(D_n^+)^c = D_n^-$, the axis of γ_n is orthogonal to $C_n^\pm$ and for $n = 1, 2$, $\gamma_1(C_1^+) = C_2^-$ and $\gamma_2(C_2^+) = C_1^-$
Set $S = \Gamma\backslash\mathbb{H}$ be the resultant hyperbolic surface.

In [10] Schapira claimed that the endpoints of the axis of γ_n are the centers of the circles $C_n^\pm$ which is not true. We correct this mistake by determining the equation of these axes.

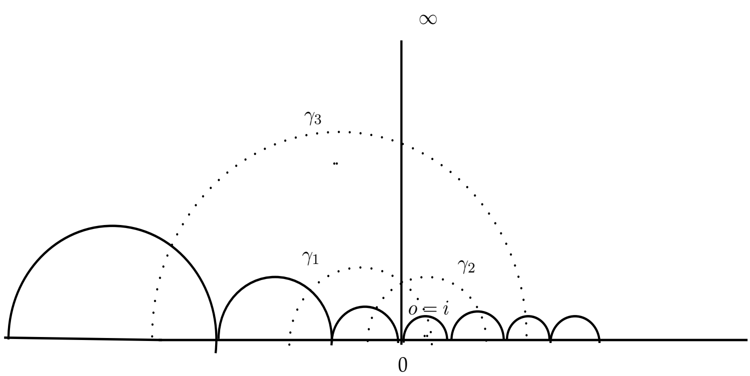

Figure 2.1. Proof of Theorem 0.3

It suffices to prove the following proposition.

Proposition 2.3. — *There exists a family of isometries (γ_n) such that*

(1) *for $n \geqslant 3$, we have $\gamma_n(C_n^+) = C_n^-$, $\gamma_n(D_n^+)^c = D_n^-$, and the axis of γ_n is orthogonal to $C_n^\pm$;*
(2) *$\gamma_1(C_1^+) = C_2^-$ and $\gamma_2(C_2^+) = C_1^-$; and*
(3) *the group generated by the γ_n is a discrete free Schottky group, with limit set equal to S^1.*

Moreover, the limit point $v^+ = +\infty$ is right horospherical and not left horospherical.

Proof. — We start by proving the following lemma. Recall that this lemma is a classical fact well known in hyperbolic geometry.

LEMMA 2.4. — *Given two circles C_n^+ and C_n^- which are not tangent, there is an orientation preserving isometry γ_n such that $\gamma_n(C_n^+) = C_n^-$ and $\gamma_n(D_n^+)^c = D_n^-$. This isometry is unique if we assume that its axis is the common perpendicular line to C_n^+ and C_n^-.*

Proof. —

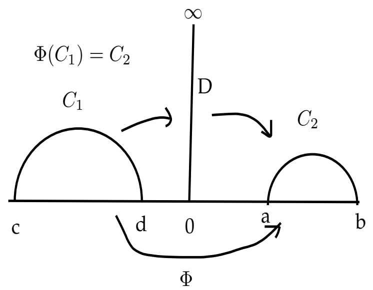

FIGURE 2.2. Proof of Lemma 2.4

Let C_1 and C_2 be two circles which are not tangent and both sides of the vertical line $D = (0, \infty)$. We denote by a, b the extremities of C_1 and by c, d the extremities of C_2. Sending C_2 on C_1 amounts to sending C_2 on D, applying a hyperbolic transformation, and sending D on C_1 (see Figure 2.2). The isometry Φ such that $\Phi(C_2) = C_1$ is represented by the matrix:

$$A_\Phi = \begin{pmatrix} b & \frac{a}{b-a} \\ 1 & \frac{1}{b-a} \end{pmatrix} \begin{pmatrix} e^{\alpha/2} & 0 \\ 0 & e^{-\alpha/2} \end{pmatrix} \begin{pmatrix} \frac{1}{d-c} & \frac{-c}{d-c} \\ -1 & d \end{pmatrix}$$
$$= \begin{pmatrix} \frac{b}{d-c}e^{\alpha/2} - \frac{a}{b-a}e^{-\alpha/2} & \frac{-bc}{d-c}e^{\alpha/2} + \frac{ad}{b-a}e^{-\alpha/2} \\ \frac{1}{d-c}e^{\alpha/2} - \frac{1}{b-a}e^{-\alpha/2} & \frac{-c}{d-c}e^{\alpha/2} + \frac{d}{b-a}e^{-\alpha/2} \end{pmatrix} \in SL(2, \mathbb{R}),$$

for $\alpha \in \mathbb{R}$.

We get $tr(A_\Phi) = \frac{b-c}{d-c}e^{\alpha/2} + \frac{d-c}{b-a}e^{-\alpha/2}$. We choose α such that $|tr(A_\Phi)| > 2$ to get a hyperbolic isometry sending C_1 to C_2. □

Let now prove that the construction of the surface S provides a counterexample.

With the same argument as in [2], v^- can not be left horocyclic. That means the half-horocycle $(h^s v)_{s \leqslant 0}$ is not dense in the nonwandering set $\mathcal{E}$.

In the following we shall prove that for the choices of R_n and x_n we made, v^- is right horocyclic.

By construction the horizontal coordinate of $\gamma_n.o$ goes to $-\infty$ as n goes to $+\infty$. Let us prove that its vertical one goes to $+\infty$ with n.

First we need to determine the axis C_n of γ_n. It is the half-circle centered at a point, say O_n, of coordinates $(\alpha_n, 0)$, and it is orthogonal to C_n^+ at a point, say A_n, of coordinates (a_{n_1}, a_{n_2}), and it is orthogonal to C_n^- at a point B_n of coordinates (b_{n_1}, b_{n_2}) (see Figure 2.3).

We have the following equations:

- The equation of C_n^+ is $(x - (2n+1))^2 + y^2 = 1$.

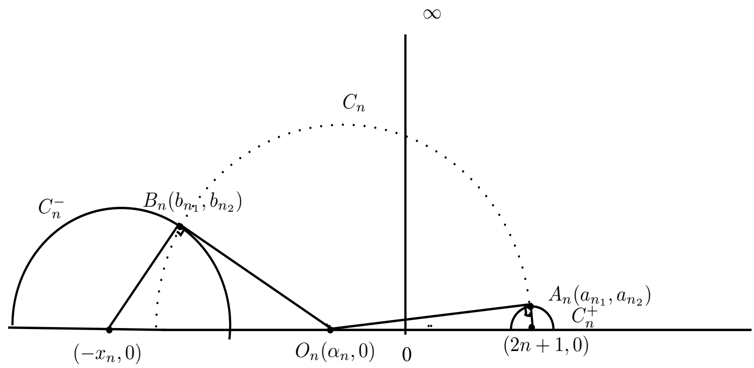

FIGURE 2.3. Proof of Proposition 2.3

- The equation of C_n^- is $(x + x_n)^2 + y^2 = R_n^2$.
- The equation of C_n is: $(x-\alpha_n)^2+y^2 = (b_{n_1}-\alpha_n)^2+b_{n_2}^2 = (a_{n_1}-\alpha_n)^2+a_{n_2}^2$.
- $B_n \in C_n^- \Leftrightarrow (b_{n_1} + x_n)^2 + b_{n_2}^2 = R_n^2$.
- $A_n \in C_n^+ \Leftrightarrow (a_{n_1} - (2n + 1))^2 + a_{n_2}^2 = 1$.

Using the orthogonality of the axis we get:

- $C_n \perp C_n^-$ at $B_n \Leftrightarrow (b_{n_1} + x_n)^2 + 2b_{n_2}^2 + (b_{n_1} - \alpha_n)^2 = (\alpha_n + x_n)^2$.
- $C_n \perp C_n^+$ at $A_n \Leftrightarrow (a_{n_1} - (2n+1))^2 + 2a_{n_2}^2 + (a_{n_1} - \alpha_n)^2 = (2n+1-\alpha_n)^2$.
- $A_n \in C_n$ and $B_n \in C_n \Leftrightarrow (b_{n_1} - \alpha_n)^2 + b_{n_2}^2 = (a_{n_1} - \alpha_n)^2 + a_{n_2}^2$.

We obtain the following system of five equations with five unknowns α_n, b_{n_1}, b_{n_2}, a_{n_1} and a_{n_2}.

$$\begin{cases} (b_{n_1} + x_n)^2 + b_{n_2}^2 = R_n^2 \\ (b_{n_1} + x_n)^2 + 2b_{n_2}^2 + (b_{n_1} - \alpha_n)^2 = (\alpha_n + x_n)^2 \\ (b_{n_1} - \alpha_n)^2 + b_{n_2}^2 = (a_{n_1} - \alpha_n)^2 + a_{n_2}^2 \\ (a_{n_1} - (2n + 1))^2 + a_{n_2}^2 = 1 \\ (a_{n_1} - (2n + 1))^2 + 2a_{n_2}^2 + (a_{n_1} - \alpha_n)^2 = (2n + 1 - \alpha_n)^2 \end{cases} \tag{2.1}$$

We need to obtain the values of α_n and b_n.

$$\begin{cases} b_{n_2}^2 = R_n^2 - (b_{n_1} + x_n)^2 \\ a_{n_2}^2 = 1 - (a_{n_1} - (2n + 1))^2 \\ 2R_n^2 - (b_{n_1} + x_n)^2 + (b_{n_1} - \alpha_n)^2 = (\alpha_n + x_n)^2 \\ (b_{n_1} - \alpha_n)^2 + R_n^2 - (b_{n_1} + x_n)^2 = (a_{n_1} - \alpha_n)^2 + 1 - (a_{n_1} - (2n + 1))^2 \\ 2 - (a_{n_1} - (2n + 1))^2 + (a_{n_1} - \alpha_n)^2 = (2n + 1 - \alpha_n)^2 \end{cases} \tag{2.2}$$

$$\begin{cases} b_{n_2}^2 = R_n^2 - (b_{n_1} + x_n)^2 \\ a_{n_2}^2 = 1 - (a_{n_1} - (2n + 1))^2 \\ -(b_{n_1} + x_n)^2 + (b_{n_1} - \alpha_n)^2 = (\alpha_n + x_n)^2 - 2R_n^2 \\ -(a_{n_1} - (2n + 1))^2 + (a_{n_1} - \alpha_n)^2 = (2n + 1 - \alpha_n)^2 - 2 \\ (\alpha_n + x_n)^2 - R_n^2 = (2n + 1 - \alpha_n)^2 - 1 \end{cases} \tag{2.3}$$

The last equation yields

$$\alpha_n = \frac{(2n+1)^2 + R_n^2 - (1+x_n^2)}{2(x_n+2n+1)}.$$

We get for n large:

$$b_{n_2} = R_n\sqrt{1 - \frac{1}{(x_n+\alpha_n)}}.$$

With $R_n = n^3$ and $x_n \cong \frac{n^4}{2}$, we obtain:

$$\alpha_n \cong \frac{-n^4}{4} \qquad \text{and} \qquad b_{n_2} \cong R_n = n^3.$$

Consider the point Z_n of coordinates $(2n+1, 1)$ in C_n^+. When n goes to $+\infty$, R_n increases and C_n tends to a vertical line which passes through Z_n, proving that $dist(A_n, Z_n)$ goes to 0 when n goes to $+\infty$. So for n large enough we have $dist(A_n, Z_n) \leqslant 1$.

Let $z = 2n+1+i$ and $z' = i$ be the affixes of Z_n and o respectively. By computation we obtain:

$$\sinh(\frac{1}{2}(dist(o, Z_n))) = \frac{|z-z'|}{2(Im(z)Im(z'))^{1/2}} = \frac{2n+1}{2},$$

which yields that

$$\frac{1}{2}(dist(o, Z_n) = \ln(\frac{2n+1}{2} + \sqrt{(\frac{2n+1}{2})^2+1}) \cong \ln(2n+1).$$

Then we get

$$dist(o, Z_n) \cong 2\ln 2n = 2\ln n + 2\ln 2 \cong 2\ln n.$$

So for n large enough we get

$$dist(0, A_n) \leqslant dist(0, Z_n) + dist(Z_n, A_n) \leqslant 2\ln n + 1.$$

Then from the invariance of distance by the isometry γ_n we have

$$dist(\gamma_n.o, B_n) \leqslant 2\ln n + 1.$$

Using the relation between Euclidian distance and hyperbolic distance on a vertical line we have:

$$Im(\gamma_n.o) \geqslant e^{-2\ln n - 1} Im(B_n) \geqslant \frac{n^3}{n^2}.$$

So $\lim_{n\to+\infty} Im(\gamma_n.o) = +\infty$. But this means the sequence $(R_n)_{n\geqslant 0}$ is as required.

For the last statement we need to prove that the limit set Λ_Γ is S^1.

For all $n \geqslant 1$ denote by $D(\gamma_n)$ and $D(\gamma_n^{-1})$ the domains bounded by the half-circles C_n^+ and C_n^- respectively, such that $\gamma_n(D(\gamma_n^{-1})) = \mathbb{H} - \mathring{D}(\gamma_n)$. Denote by $(\gamma_n, \gamma_n^{-1})_{n\geqslant 1}$ the coding of the group Γ. A product of n letters $s_1...s_n$ in this coding is said to be a reduced word of length n if $n = 1$ or $n > 1$ and $s_i \neq s_{i+1}^{-1}$ for all $1 \leqslant i \leqslant n-1$.

We associate to any reduced word $s_1...s_n$ the set $D(s_1)$ if $n = 1$ or the set $D(s_1, ..., s_n)$ defined by $D(s_1, ..., s_n) = s_1...s_{n-1}D(s_n)$. Notice that if $(s_i)_{i\geqslant 1}$ is

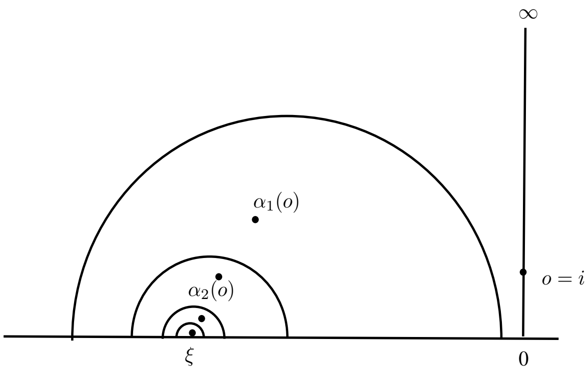

FIGURE 2.4. Proof of Proposition 2.3

a sequence satisfying $s_{i+1} \neq s_i^{-1}$, the sequence of euclidian diameters of the sets $D(s_1, .., s_n)$ converges to 0. We have as in [4]:

$$\Lambda_\Gamma = \bigcap_{n=1}^{+\infty} \bigcup_{\text{reduced words of length } n} \overline{D(s_1, ..., s_n)}.$$

Let ξ be a point in $\partial\mathbb{H}$. Let prove that there exists a sequence of $\alpha_n(o)$ such that $\lim_{n\to\infty} \alpha_n(o) = \xi$. This implies that ξ is in Λ_Γ.

We have to construct a nested family of circles such that the limit of their diameters goes to zero when n goes to $+\infty$. We shall use the following Lemma:

LEMMA 2.5. — *Given a point in $\mathbb{H}$ and a half-plane $D(\gamma_{i_n})$ in $\mathbb{H}$, where γ_{i_n} is an isometry of Γ, there is an isometry in Γ sending this point to the half-plane $D(\gamma_{i_n})$.*

Proof. — We need to prove that if $g \in \Gamma$ is an isometry such that $g = \gamma_{i_1} \cdot \gamma_{i_2} \cdots \gamma_{i_k}$, where $\gamma_{i_j} \in \Gamma$ for $j = 1, ..., k$, and ω is a point in $\mathbb{H}$, then $g(\omega) \in D(\gamma_{i_1}^{-1})$. Remark that

$$(D(\gamma_{i_s}) \cup D(\gamma_{i_s}^{-1})) \cap (D(\gamma_{i_l}) \cup D(\gamma_{i_l}^{-1}) = \emptyset \qquad s, l = 1, ..., k.$$

Using this fact, we have $\gamma_{i_k}(\omega) \in D(\gamma_{i_k}^{-1})$ and $\gamma_{i_{k-1}}\gamma_{i_k}(\omega) \in D(\gamma_{i_{k-1}}^{-1})$ and so on. At the end we get

$$\gamma_{i_1}(\gamma_{i_2}(...(\gamma_{i_{k-1}}(\gamma_{i_k}(\omega)))...)) \in D(\gamma_{i_1}^{-1}).$$

Now, given a point A in $\mathbb{H}$ and a half-plane $D(\gamma_{i_n})$, any isometry $\alpha = \gamma_{i_n}^{-1} \cdot \gamma_{i_l}\gamma_{i_{l-1}} \cdots \gamma_{i_{l+k}}$ satisfiying the same conditions as g sends A to the half-plane $D(\gamma_{i_n})$. □

Consider now the interval $]x, y[$ of $\partial\mathbb{H}$ that contains ξ such that x and y are the extremities of a circle $C_{i_1}^{\epsilon}$, where $\epsilon = +$ or $-$. Let α_0 be the identity, and α_{i_1} the isometry in Γ sending the point o to the interior of the half-circle $C_{i_1}^{\epsilon}$ (see figure 2.4). Let $D(\alpha_{i_1})$ be the interior of the circle $C_{i_1}^{\epsilon}$. At the second step ξ will be in some $D(\alpha_{i_1}, \alpha i_2)$ where α_{i_2} is the isometry sending $\alpha_{i_1}(o)$ to $D(\alpha_{i_1}, \alpha_{i_2})$. We do the same as above at the step n and denote by α_{i_n} the isometry in Γ sending $\alpha_{i_1}....\alpha_{i_{n-1}}(o)$ to $D(\alpha_{i_1}, ..., \alpha_{i_{n-1}}, \alpha_{i_n})$.

This is possible since there is no gap between adjacents half-circles and because of the fact that $D(\alpha_{i_1}, \alpha_{i_2}, ...\alpha_{i_n}) \subset D(\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_{n-1}})$ (see [3]) . The fact that the radius of C_n^- tends to ∞ is not an obstacle to the construction of these isometries. Set $\alpha_i = \alpha_{i_1}$, $\alpha_2 = \alpha_{i_1}\alpha_{i_2}$ and for $n > 2$ let $\alpha_n = \alpha_{i_1}\alpha_{i_2}...\alpha_{i_{n-1}}\alpha_{i_n}$. We construct a sequence $(\alpha_n(o))_{n\geqslant 0}$ and a nested family $D(\alpha_{i_1}, ..., \alpha_{i_{n-1}}, \alpha_{i_n})$ in $\mathbb{H}^2$. As above the diameters of $D(\alpha_{i_1}, ..., \alpha_{i_{n-1}}, \alpha_{i_n})$ goes to 0 as n goes to $+\infty$. Then $\lim_{n\to\infty} \alpha_n(o) = \lim_{n\to\infty} \alpha_{i_1}...\alpha_{i_{n-1}}\alpha_{i_n}(o) = \xi$, proving that ξ is in the limit set of Γ.

As ξ is any point of $\partial\mathbb{H}$ we proved that $\Lambda_\Gamma = \partial\mathbb{H}$. □

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