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Complex Analytic Geometry / *Géométrie analytique complexe*

On the local univalence of nondegenerate holomorphic vector fields

Sur l'univalence locale des champs de vecteurs holomorphes non-dégénérés

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Abstract. We prove that, in all dimensions, germs of nondegenerate holomorphic vector fields on complex manifolds are univalent in the sense of Palais (semicomplete in the sense of Rebelo), this is, that there exist neighborhoods of their singular points where all their solutions are single-valued. This implies that, in stark contrast with the degenerate case, all germs of nondegenerate holomorphic vector fields give local models for complete holomorphic vector fields on complex manifolds (albeit possibly non-Hausdorff ones).

Résumé. On prouve que, en toute dimension, tout germe de champs de vecteurs holomorphe singulier non-dégénéré sur une variété est univalent au sens de Palais (semicomplet au sens de Rebelo): en restriction à un voisinage convenable du point singulier, ses solutions n'ont pas de multivaluation. Ceci implique que, à la différence du cas dégénéré, un germe de champ de vecteurs holomorphe non-dégénéré est le modèle local d'un champ de vecteurs holomorphe complet sur une variété complexe (pas nécessairement séparée).

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1. Introduction

If X is a vector field in a neighborhood of 0 in $\mathbf{R}^n$, there is a manifold M, a complete vector field Y on M and a point p in M such that the germ of Y at p is given by the germ of X at 0: in the real realm, the singularities of complete vector fields are arbitrary. This is no longer true for holomorphic vector fields. The solutions of a holomorphic vector field on a complex manifold may present multivaluedness ($t^{-1/n}$, which is multivalued if $n \geq 2$, is a solution of $-\frac{1}{n}z^{n+1}\partial/\partial z$), and this will prevent it from being embeddable into a complete vector field. Holomorphic vector fields on complex manifolds that do not present multivaluedness are called *univalent* (following Palais) or *semicomplete* (following Rebelo); semicompleteness is a necessary condition for a vector field to be embeddable into a complete one. (Precise definitions will be given in Section 2.) This notion may be germified: the germs of complete vector fields around their singular points are semicomplete germs, and not arbitrary ones. Moreover, there are germs which are not semicomplete (like those of the form $(z^n + \cdots)\partial/\partial z$ at 0 in $\mathbf{C}$ for $n \geq 3$ [5, Proposition 3.1]). This makes it possible to study complete holomorphic vector fields locally, and poses the problem of understanding semicomplete germs.

Singularities of holomorphic vector fields with a degenerate linear part have very seldomly semicomplete germs. For instance, in dimension two, Rebelo proved that all germs of semicomplete vector fields with an isolated singularity have a nontrivial second jet [5], and his works, either by himself or in collaboration with Ghys, have provided us with a very comprehensive study of semicomplete vector fields in a neighborhood of an isolated fixed point with a degenerate linear part; in particular, they have shown that their germs fit into countably many explicit and simple orbital normal forms [2, Théorème A and Proposition 3.16], [6, Théorème 4.1]. Results and conjectures for higher dimensions point in the same direction [7].

In contrast, it follows from some of the well-established features of germs of nondegenerate vector fields that they are, for the greater part, semicomplete. Recall that a vector field with a nondegenerate singularity on $(\mathbf{C}^n, 0)$ is said to belong to the *Siegel domain* if, in $\mathbf{C}$, the convex hull of the eigenvalues of its linear part at the singular point contains 0 (it is said to belong to the *strict* Siegel domain if this convex hull contains 0 in its interior); it is otherwise said to belong to the *Poincaré domain*. Germs in the Poincaré domain may be redressed to their Poincaré–Dulac normal forms (see [3, Theorem 5.5]). As observed in [6, Lemme 3.1] for dimension two and in [8] for higher dimensions, germs of vector fields in the Poincaré domain are always semicomplete (as vector fields on $\mathbf{C}^n$, their Poincaré–Dulac normal forms are actually complete vector fields). For germs in the Siegel domain, a consequence of Bryuno's theorem [1] is that most vector fields in the Siegel domain are linearizable, and are thus semicomplete.

Some works address directly the problem of semicompleteness for nondegenerate germs. Rebelo proved that, for germs of nondegenerate vector fields in the Siegel domain in dimension two, there are no obstructions for semicompleteness from the orbital point of view, that all such germs are semicomplete up to multiplication by a nonvanishing holomorphic function [6, Théorème A]. In [8, Theorem 2], Reis proved that the same was true for many vector fields in higher dimensions, including all vector fields in dimension three in the strict Siegel domain (however, from dimension five onwards, there are open sets in the space of linear parts where the hypothesis of her result are not satisfied).

The aim of this Note is to settle the problem of semicompleteness for nondegenerate germs:

Theorem 1. *Let X be a holomorphic vector field defined on a neighborhood of* 0 *in* $\mathbf{C}^n$*, and having a nondegenerate singularity at* 0*. There exists a neighborhood of* 0 *in restriction to which the vector field is semicomplete (the germ of X at* 0 *is semicomplete).*

General results due to Palais [4, Chapter III, Theorem IX] imply that, for a univalent holomorphic vector field X on the complex manifold M, there exists a complex manifold N endowed with an embedding of M and a complete vector field Y on N extending X (although the manifold N need not be a Hausdorff one). From this and Theorem 1 we obtain:

Corollary 2. *Let X be a holomorphic vector field defined on a neighborhood of* 0 *in* $\mathbf{C}^n$*, having a singularity at* 0 *with nondegenerate linear part. There exists a (possibly non-Hausdorff) complex manifold M, a complete holomorphic vector field Y on M and a point p in M such that the germ of Y at p is, in a suitable chart, given by the germ of X at* 0*.*

It would be interesting to establish if the manifold in this corollary can always be supposed to be a Hausdorff one, or to exhibit an example where this is not possible.

2. Preliminaries

Unless otherwise stated, all manifolds will be supposed to be complex, all vector fields holomorphic, and so on. A holomorphic *semiglobal flow* or *maximal local action of* **C** on the manifold M is a couple (Ω,Φ) where Ω is an open subset of $\mathbf{C}\times M$ containing $\{0\}\times M$ such that $\Omega\cap(\mathbf{C}\times\{p\})$ is connected for every $p\in M$, and $\Phi:\Omega\to M$ is a holomorphic map such that

- $\Phi(0,p)=p$ for all $p\in M$;
- if (s,p), $(t,\Phi(s,p))$ and $(t+s,p)$ are in Ω, $\Phi(t,\Phi(s,p))=\Phi(t+s,p)$;
- for each $p\in M$, for $\Omega_p=\{t\mid(t,p)\in\Omega\}$, the map $\phi_p:\Omega_p\to\mathbf{C}\times M$ given by $t\mapsto(t,\Phi(t,p))$ is a proper one.

We recover the notion of holomorphic flow/action of **C** when $\Omega=\mathbf{C}\times M$, where the third condition is a superfluous one. A semiglobal flow induces a vector field X on M, given by $X(p)=\frac{d}{dt}\,\Phi(t,p)\big|_{t=0}$. A vector field is said to be *univalent* or *semicomplete* if it arises in this way. This notion is due to Palais, who conceived it in the more general setting of Lie algebras of vector fields on manifolds [4, Chapter III, Definitions VI and VII]; it was rediscovered and intensely studied by Rebelo in the context of holomorphic vector fields [5, Définition 2.3]. Complete vector fields are semicomplete, and semicomplete vector fields remain so when restricted to open subsets. *Semicomplete germs* of holomorphic vector fields are those that have a semicomplete representative; by the previous observations, this is a well-defined notion. Germs of complete vector fields around their singular points are necessarily semicomplete ones.

If X is a vector field on a manifold M and L is a one-dimensional orbit of X, the *time form* of X on L is the holomorphic 1-form ω_L on L such that $\omega_L(X|_L)\equiv 1$ (it need not be exact). Rebelo gave the the following criterion for semicompleteness [6, Proposition 2.1]: *the vector field X on the manifold M is semicomplete if for every one-dimensional orbit L of X, every path $\gamma:[0,1]\to L$ for which $\int_\gamma\omega_L=0$ is closed.*

3. Proof of Theorem 1

Let X be a vector field on $(\mathbf{C}^k,0)$ with a nondegenerate singularity. Up to multiplying X by a constant, we will suppose that its linear part has no purely imaginary eigenvalues. Order the (not necessarily different) k eigenvalues of the linear part of X at 0 as $\lambda_1,\dots,\lambda_m,\mu_1,\dots,\mu_n$ so that $\Re(\lambda_i)>0$ and $\Re(\mu_i)<0$.We will suppose that $m>0$, but we do not exclude the case $n=0$ (the proof is nevertheless written with the case $n\neq 0$ in mind). There are two submanifolds invariant by X tangent to the subspaces associated to the above decomposition, as it follows from the Invariant Manifold Theorem (see, for instance, [3, Theorem 7.1]). Through a holomorphic change of coordinates these submanifolds can be redressed onto the corresponding linear subspaces, and there are thus coordinates $(z_1,\dots,z_m,w_1,\dots,w_n)$ of $\mathbf{C}^k$ around 0 where

$$X=\sum_{i=1}^m F_i(z,w)\frac{\partial}{\partial z_i}+\sum_{j=1}^n G_j(z,w)\frac{\partial}{\partial w_j},$$

with $F_i(0,w)\equiv 0$, $G_i(z,0)\equiv 0$. We can furthermore suppose that the linear part of X is in its Jordan normal form, that $F_i=\lambda_i z_i+\epsilon_i z_{i+1}+f_i(z,w)$, $G_i=\mu_i z_i+\delta_i z_{i+1}+g_i(z,w)$, with ϵ_i and δ_i real numbers (that vanish if $\lambda_i\neq\lambda_{i+1}$), f_i and g_i holomorphic functions with vanishing first jets at 0.

Let $\Re(X)$ be the real vector field given by the real part of X (the one whose local flow gives the local complex flow of X restricted to real time).[1] By derivating along $\Re(X)$, we have that

$$\Re(X)\cdot\sum_{i=1}^{m}|z_i|^2 = 2\sum_{i=1}^{m}\Re(\lambda_i)|z_i|^2+\epsilon_i\Re\left(z_i\overline{z_{i+1}}\right)+\Re\left(z_i\overline{f_i}\right), \tag{1}$$

$$\Re(X)\cdot\sum_{i=1}^{n}|w_i|^2 = 2\sum_{i=1}^{n}\Re(\mu_i)|w_i|^2+\delta_i\Re\left(w_i\overline{w_{i+1}}\right)+\Re\left(w_i\overline{g_i}\right). \tag{2}$$

By means of a linear change of coordinates making ϵ_i and δ_i as small as necessary, we can a priori suppose that the quadratic term in the right-hand side of (1) is strictly positive away from $0\times\mathbf{C}^n$, and that that of (2) is strictly negative away from $\mathbf{C}^m\times 0$. By conveniently scaling the variables we may suppose that X is defined in the product $U = B^m\times B^n$ of unit balls of the corresponding dimensions, and that, within U, expressions (1) and (2) are, respectively, strictly positive away from $0\times\mathbf{C}^n$ and strictly negative away from $\mathbf{C}^m\times 0$.

Let us prove that X is semicomplete in U. We will use the criterion due to Rebelo mentioned at the end of the previous section. Let $\gamma:[0,1]\to U$ be a path taking values in the one-dimensional orbit L such that $\int_\gamma\omega_L = 0$. We affirm that γ is a closed path. Let $Q(z,w)=\sum_{i=1}^{m}|z_i|^2$. Let $r<1$ be such that $Q\circ\gamma(t)\le r^2$ for all t, and let $K=\{(z,w)\in U \mid Q(z,w)\le r^2\}$, ∂K its boundary. By pushing γ positively along the orbits of $\Re(X)$, we can deform γ, through a homotopy with fixed endpoints, into the concatenation of three smooth paths: a first one, ρ_1, going from $\gamma(0)$ along an orbit of $\Re(X)$ up to ∂K; a second one, σ, contained in the real one-dimensional set $L\cap\partial K$; and a third one, ρ_2, going to $\gamma(1)$ following negatively an orbit of $\Re(X)$. This is possible because, within U, every orbit of $\Re(X)$ starting in K intersects ∂K in positive time: on the one hand, by (1) and (2), the orbits of $\Re(X)$ that start in K cannot exit U without intersecting ∂K and, on the other, by the transversality of $\Re(X)$ with the level sets of Q that follows from (1), the orbits of $\Re(X)$ cannot accumulate to subsets of K, and cannot remain indefinitely within K. Since ρ_1 is part of an orbit of $\Re(X)$ and $\omega_L(\Re(X))\in\mathbf{R}$, $\int_{\rho_1}\omega_L\in\mathbf{R}$, and the same happens for ρ_2. Up to a homotopy with fixed endpoints within $L\cap\partial K$, we may suppose that σ is either constant or transverse to the orbits of $\Re(X)$. In the second case, $\Im(\omega_L(\sigma'(t)))\neq 0$ for every t and thus $\Im(\int_\sigma\omega_L)\neq 0$. This implies that $\int_{\rho_1*\sigma*\rho_2}\omega_L$ (which equals $\int_\gamma\omega_L$) does not vanish, for it has a nontrivial imaginary part. Thus, σ is constant, and γ is homotopic to $\rho_1*\rho_2$. But ρ_1 and ρ_2 go along the same orbit of $\Re(X)$, and since $\int_{\rho_1*\rho_2}\omega_L = 0$, $\rho_1*\rho_2$ is a closed path. This establishes that γ is closed, and finishes the proof of Theorem 1. □

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[1] If $z_j = x_j + iy_j$, the real part of $\sum_j F_j\partial/\partial z_j$ is $\sum_j(\Re(F_j)\partial/\partial x_j+\Im(F_j)\partial/\partial y_j)$.