INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus
Mathématique

Julien Poirier and Nour Seloula

Regularity results for a model in magnetohydrodynamics with imposed pressure

Volume 358, issue 9-10 (2020), p. 1033-1043

Published online: 5 January 2021

https://doi.org/10.5802/crmath.113

Les Comptes Rendus. Mathématique *sont membres du*
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 1778-3569

Comptes Rendus
Mathématique
2020, 358, nº 9-10, p. 1033-1043
https://doi.org/10.5802/crmath.113

Mathematical Physics / *Physique mathématique*

Regularity results for a model in magnetohydrodynamics with imposed pressure

Résultats de régularité pour un modèle en magnétohydrodynamique avec des conditions aux limites sur la pression

Julien Poirier[a] **and Nour Seloula**[a]

[a] Laboratoire de Mathématiques Nicolas Oresme (LMNO). Université de Caen (UMR 6139), 14000 Caen, France

E-mails: julien.poirier@unicaen.fr, nour-elhouda.seloula@unicaen.fr

Abstract. The magnetohydrodynamics (MHD) problem is most often studied in a framework where Dirichlet type boundary conditions on the velocity field is imposed. In this Note, we study the (MHD) system with pressure boundary condition, together with zero tangential trace for the velocity and the magnetic field. In a three-dimensional bounded possibly multiply connected domain, we first prove the existence of weak solutions in the Hilbert case, and later, the regularity in $\boldsymbol{W}^{1,p}(\Omega)$ for $p \geq 2$ and in $\boldsymbol{W}^{2,p}(\Omega)$ for $p \geq 6/5$ using the regularity results for some Stokes and elliptic problems with this type of boundary conditions. Furthermore, under the condition of small data, we obtain the existence and uniqueness of solutions in $\boldsymbol{W}^{1,p}(\Omega)$ for $3/2 < p < 2$ by using a fixed-point technique over a linearized (MHD) problem.

Résumé. La plupart des travaux sur le système de la magnétohydrodynamique (MHD) considèrent une condition aux limites de type Dirichlet pour le champ de vitesses. Dans cette Note, nous étudions le système (MHD) avec une pression donnée au bord, ainsi qu'une trace tangentielle nulle pour la vitesse du fluide et le champ magnétique. Dans un ouvert borné tridimensionnel, éventuellement multiplement connexe, on commence par prouver l'existence de solutions faibles dans le cas Hilbertien, et ensuite, nous montrons la régularité $\boldsymbol{W}^{1,p}(\Omega)$ pour $p \geq 2$ et $\boldsymbol{W}^{2,p}(\Omega)$ pour $p \geq 6/5$ en utilisant les résultats de régularité pour certains problèmes de Stokes avec ce type de conditions aux limites. De plus, pour des données petites, nous démontrons l'existence et l'unicité des solutions dans $\boldsymbol{W}^{1,p}(\Omega)$ pour $3/2 < p < 2$ en utilisant un théorème de point fixe appliqué au problème linéarisé de (MHD).

2020 Mathematics Subject Classification. 35J60,35Q35,35Q60.

Manuscript received 23rd June 2020, revised 7th August 2020, accepted 7th September 2020.

1. Introduction

Let Ω be an open bounded set of space $\mathbb{R}^3$ of class $\mathscr{C}^{1,1}$. In this paper, we consider the following incompressible stationary magnetohydrodynamics (MHD) system: find the velocity field $\boldsymbol{u}$, the pressure P, the magnetic field $\boldsymbol{b}$ and constants α_i such that for $1 \leq i \leq I$:

$$\begin{cases} -\nu\Delta\boldsymbol{u} + (\mathbf{curl}\,\boldsymbol{u})\times\boldsymbol{u} + \nabla P - \kappa(\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{b} = \boldsymbol{f} & \text{and} \quad \operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega, \\ \kappa\mu\,\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{b} - \kappa\,\mathbf{curl}(\boldsymbol{u}\times\boldsymbol{b}) = \boldsymbol{g} & \text{and} \quad \operatorname{div}\boldsymbol{b} = 0 \quad \text{in } \Omega, \\ \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{0} & \text{and} \quad \boldsymbol{b}\times\boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma, \\ P = P_0 \quad \text{on } \Gamma_0 & \text{and} \quad P = P_0 + \alpha_i \quad \text{on } \Gamma_i, \\ \langle\boldsymbol{u}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i} = 0 & \text{and} \quad \langle\boldsymbol{b}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i} = 0, \end{cases} \tag{MHD}$$

where Γ is the boundary of Ω which is possibly multiply connected. Here $\Gamma = \bigcup_{i=0}^{I}\Gamma_i$ where Γ_i are the connected components of Γ with Γ_0 the exterior boundary which contains Ω and all the other boundaries. We denote by $\boldsymbol{n}$ the unit vector normal to Γ. The constants ν, μ and κ are constant kinematic, magnetic viscosity and a coupling number respectively. The vector functions $\boldsymbol{f}$, $\boldsymbol{g}$ and scalar function P_0 are given. In this paper, we assume that $\nu = \mu = \kappa = 1$ for convenience.

We do not assume that Ω is simply-connected but we suppose that there exist J connected open surfaces Σ_j, $1 \leq j \leq J$, called 'cuts', contained in Ω such that each surface Σ_j is an open subset of a smooth manifold. The boundary of each Σ_j is contained in Γ. The intersection $\overline{\Sigma_i}\cap\overline{\Sigma_j}$ is empty for $i \neq j$, and finally the open set $\Omega^\circ = \Omega \setminus \cup_{j=1}^{J}\Sigma_j$ is simply-connected.

Using the identity $\boldsymbol{u}\cdot\nabla\boldsymbol{u} = (\mathbf{curl}\,\boldsymbol{u})\times\boldsymbol{u} + \frac{1}{2}\nabla|\boldsymbol{u}|^2$, the classical nonlinear term $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ in the Navier–Stokes equations is replaced by $(\mathbf{curl}\,\boldsymbol{u})\times\boldsymbol{u}$. The pressure $P = p + \frac{1}{2}|\boldsymbol{u}|^2$ is then the Bernoulli (or dynamic) pressure, where p is the kinematic pressure. The boundary conditions involving the pressure are used in various physical applications. For example, in hydraulic networks, as oil ducts, microfluidic channels or the blood circulatory system. Pressure driven flows occur also in the modeling of the cerebral venous network from three-dimensional angiographic images obtained by magnetic resonance. We note that the (MHD) problem have been extensively studied by many authors. Whereas most of the contributions are often given where Dirichlet type boundary conditions on the velocity field are imposed. At a continuous level, we can refer, for exemple to [5, 18] for the existence and the regularity of the solutions of (MHD) problem, to [2, 4, 10] for the global solvability of (MHD) problem under mixed boundary conditions for the magnetic field. Also in [1, 3], the authors have studied the stationary magnetohydrodynamic equations of electrically and heat conducting fluid. For the discretization approaches of (MHD), a few related contributions include mixed finite elements [13,14,16], discontinuous galerkin finite elements [15] or iterative penalty finite element methods [12] and so on. The boundary condition under the form $P = P_0 + \alpha_i$ on Γ_i, $i = 1, \ldots, I$ was first introduced in [11] for the Stokes and the Navier–Stokes systems in steady hilbertian case. The authors studied the differences $\alpha_i - \alpha_0$, $i = 1 \ldots I$ which represent the unknown pressure drop on inflow and outflow sections Γ_i in a network of pipes. This work is extended to L^p-theory for $1 < p < \infty$ in [8]. In our work, we study the (MHD) problem with pressure boundary condition, together with no tangential flow and no tangential magnetic field on the boundary. Up to our knowledge, with these type of boundary conditions, this work is the first to give a complete L^p-theory for the (MHD) problem not only for large values of $p \geq 2$ but also for small values $3/2 < p < 2$ in $\Omega \subset \mathbb{R}^3$ multiply connected domain with a boundary Γ not necessary connected.

We introduce some notations and functions spaces which are used in this paper. The vector fields and matrix fields as well as the corresponding spaces are denoted by bold font and blackboard bold font respectively. For $1 < p < \infty$, $\boldsymbol{L}^p(\Omega)$ denotes the usuel vector-valued $\boldsymbol{L}^p$-space over Ω. If $p \in [1,\infty)$, p' denotes the conjugate exponent of p, i.e. $\frac{1}{p'} = 1 - \frac{1}{p}$. For $p, r \in [1,\infty)$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{3}$, we introduce the following space

$$\boldsymbol{H}^{r,p}(\mathbf{curl},\Omega) := \left\{\boldsymbol{v} \in \boldsymbol{L}^r(\Omega);\, \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)\right\},$$

equipped with the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{H}}^{r,p}(\mathbf{curl},\,\Omega) = \|\boldsymbol{v}\|_{\boldsymbol{L}^r(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}.$$

The closure of $\mathscr{D}(\Omega)$ in $\boldsymbol{H}^{r,p}(\mathbf{curl},\,\Omega)$ is denoted by $\boldsymbol{H}_0^{r,p}(\mathbf{curl},\,\Omega)$. Its dual space is denoted by $[\boldsymbol{H}_0^{r,p}(\mathbf{curl},\,\Omega)]'$ which can be characterized as follows:

$$\left[\boldsymbol{H}_0^{r,p}(\mathbf{curl},\Omega)\right]' = \left\{\boldsymbol{F} + \mathbf{curl}\boldsymbol{\psi},\ \boldsymbol{F} \in \boldsymbol{L}^{r'}(\Omega),\ \boldsymbol{\psi} \in \boldsymbol{L}^{p'}(\Omega)\right\}. \tag{1}$$

The proof of this characterization is similar to that of [17, Proposition 1.0.5]. Moreover, we have

$$\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r,p}(\mathbf{curl},\Omega)]'} \le \inf_{\boldsymbol{f}=\boldsymbol{F}+\mathbf{curl}\,\boldsymbol{\psi}} \max\left\{\|\boldsymbol{F}\|_{\boldsymbol{L}^{r'}(\Omega)}, \|\boldsymbol{\psi}\|_{\boldsymbol{L}^{p'}(\Omega)}\right\}.$$

Next we introduce the kernel

$$\boldsymbol{K}_N^p(\Omega) = \left\{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \operatorname{div}\boldsymbol{v} = 0,\ \mathbf{curl}\boldsymbol{v} = \boldsymbol{0},\ \boldsymbol{v}\times\boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma\right\}.$$

Thanks to [9, Corollary 4.2], we know that this kernel is of finite dimension and spanned by the functions ∇q_i^N, $1 \le i \le I$, where q_i^N is the unique solution of the problem

$$\begin{cases} -\Delta q_i^N = 0 \quad \text{in } \Omega, \quad q_i^N|_{\Gamma_0} = 0 \quad \text{and} \quad q_i^N|_{\Gamma_k} = \text{constant},\ 1 \le k \le I \\ \langle \partial_n q_i^N, 1\rangle_{\Gamma_k} = \delta_{ik},\ 1 \le k \le I, \quad \text{and} \quad \langle \partial_n q_i^N, 1\rangle_{\Gamma_0} = -1. \end{cases} \tag{2}$$

Moreover, the functions ∇q_i^N, $1 \le i \le I$, belong to $\boldsymbol{W}^{1,q}(\Omega)$ for any $1 < q < \infty$. We will use also the symbol σ to represent a set of divergence free functions. In other words if $\boldsymbol{X}$ is Banach space, then $\boldsymbol{X}_\sigma = \{\boldsymbol{v} \in \boldsymbol{X};\quad \operatorname{div}\boldsymbol{v} = 0 \quad \text{in } \Omega\}$.

2. Weak solutions

The next Theorem 1 deals with the existence of weak solutions for the (MHD) system in the Hilbert case. We use the Schauder Fixed Point Theorem for this purpose. We note that in order to obtain the necessary estimates, the last conditions in (MHD) on the flux through the connected components Γ_i are important. Indeed, let us define the space

$$\boldsymbol{X}_N^p(\Omega) = \left\{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \operatorname{div}\boldsymbol{v} \in L^p(\Omega), \quad \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{L}^p(\Omega) \quad \text{and} \quad \boldsymbol{v}\times\boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma\right\}.$$

It is well known (see [9, Corollary 3.2]) that for any vector $\boldsymbol{v} \in \boldsymbol{X}_N^p(\Omega)$ we have

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} \le C\left(\|\mathbf{curl}\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\operatorname{div}\boldsymbol{v}\|_{L^p(\Omega)} + \sum_{i=1}^{I}\left|\langle \boldsymbol{v}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i}\right|\right), \tag{3}$$

for some constant C depending only on Ω. The same inequality remains valid for tangential vector fields (that is $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ on Γ) with fluxs through the cuts Σ_j. Observe that if $\operatorname{div}\boldsymbol{v} = 0$ in Ω and $\langle \boldsymbol{v}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0$, we have from (3) that the norm $\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}$ is equivalent to the full norm $\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)}$.

Theorem 1. *Let $\boldsymbol{f}, \boldsymbol{g} \in [\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)]'$ and $P_0 \in H^{-\frac{1}{2}}(\Gamma)$ satisfying the compatibility conditions*

$$\forall\ \boldsymbol{v} \in \boldsymbol{K}_N^2(\Omega), \quad \langle \boldsymbol{g}, \boldsymbol{v}\rangle_{\Omega_{6,2}} = 0, \tag{4}$$

$$\operatorname{div}\boldsymbol{g} = 0 \quad in \quad \Omega, \tag{5}$$

where $\langle\cdot,\cdot\rangle_{\Omega_{r,p}}$ *denotes the duality product between* $[\boldsymbol{H}_0^{r,p}(\mathbf{curl},\Omega)]'$ *and* $\boldsymbol{H}_0^{r,p}(\mathbf{curl},\Omega)$. *Then the* (MHD) *problem has at least one weak solution* $(\boldsymbol{u},\boldsymbol{b},P,\boldsymbol{\alpha})\in\boldsymbol{H}^1(\Omega)\times\boldsymbol{H}^1(\Omega)\times L^2(\Omega)\times\mathbb{R}^I$ *and the following estimates hold:*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}+\|\boldsymbol{b}\|_{\boldsymbol{H}^1(\Omega)}+\|P\|_{L^2(\Omega)}\leq C\left(\|\boldsymbol{f}\|_{\left[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)\right]'}+\|\boldsymbol{g}\|_{\left[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)\right]'}+\|P_0\|_{H^{-1/2}(\Gamma)}\right)\quad(6)$$

with $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_I)$ *given by*

$$\alpha_i=\langle\boldsymbol{f},\nabla q_i^N\rangle_{\Omega_{6,2}}-\langle P_0,\nabla q_i^N\cdot\boldsymbol{n}\rangle_\Gamma+\int_\Omega(\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{b}\cdot\nabla q_i^N\,dx-\int_\Omega(\mathbf{curl}\,\boldsymbol{u})\times\boldsymbol{u}\cdot\nabla q_i^N\,dx,\quad(7)$$

where $\langle\cdot,\cdot\rangle_\Gamma$ *denotes the duality product between* $H^{-1/2}(\Gamma)$ *and* $H^{1/2}(\Gamma)$.

Remark 2. The choice of the space $[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'$ for $\boldsymbol{f}$ and $\boldsymbol{g}$ is optimal to study the regularity $\boldsymbol{W}^{1,p}(\Omega)$ with $p\geq 2$. Indeed, for the case $p=2$, unlike the case of Dirichlet type boundary conditions, the space $\boldsymbol{H}^{-1}(\Omega)$ is not suitable for source term in the right hand side to find solutions in $\boldsymbol{H}^1(\Omega)$. Let us analyse the case of $\boldsymbol{f}$, it holds true also for $\boldsymbol{g}$. Since $\boldsymbol{v}\in\boldsymbol{X}_N^p(\Omega)$, then we can firstly consider the duality pairing

$$\langle\boldsymbol{f},\,\boldsymbol{v}\rangle_{\left[\boldsymbol{H}_0^{2,2}(\mathbf{curl},\,\Omega)\right]'\times\boldsymbol{H}_0^{2,2}(\mathbf{curl},\,\Omega)}$$

in view to write an equivalent variational formulation. Then, we must suppose that $\boldsymbol{f}$ belongs to $[\boldsymbol{H}_0^{2,2}(\mathbf{curl},\,\Omega)]'$. But, thanks to (3), $\boldsymbol{v}$ belongs to $\boldsymbol{H}^1(\Omega)\hookrightarrow\boldsymbol{L}^6(\Omega)$. Then, the previous hypothesis on $\boldsymbol{f}$ can be weakened by considering the space $[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\,\Omega)]'$ which is a subspace of $\boldsymbol{H}^{-1}(\Omega)$ and thanks to the characterization of this space (given in (1)), the previous duality is replaced by

$$\langle\boldsymbol{f},\,\boldsymbol{v}\rangle_{\left[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\,\Omega)\right]'\times\boldsymbol{H}_0^{6,2}(\mathbf{curl},\,\Omega)}=\int_\Omega\boldsymbol{F}\cdot\boldsymbol{v}\,d\boldsymbol{x}+\int_\Omega\boldsymbol{\psi}\cdot\mathbf{curl}\,\boldsymbol{v}\,d\boldsymbol{x}.$$

The case $p>2$ can be analyzed in a similar way and this proves that the space $[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'$ with $\frac{1}{r}=\frac{1}{p}+\frac{1}{3}$ is optimal to obtain the regulatity $\boldsymbol{W}^{1,p}(\Omega)$.

3. Regularity of the weak solution

The following Theorems 3 and 4 concern the L^p-regularity of the weak solution. The proof is essentially based on the estimates obtained in the Hilbert case and the Stokes regularity results in [9] and [8]. We note that the Inf-Sup conditions involving the **curl** operator play a fundamental role. It is important to mention that there is no necessary compatibility condition for the data $\boldsymbol{f}$ in order to apply the regularity of the Stokes problem. In fact, by the defintion of the constants α_i in (7), the necessary condition for the existence of solution of the Stokes system is verified.

Theorem 3 (regularity $W^{1,p}(\Omega)$ **with** $p>2$**).** *Let* $p>2$. *Suppose that* $\boldsymbol{f},\boldsymbol{g}\in[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'$, $P_0\in W^{1-\frac{1}{r},r}(\Gamma)$ *satisfying* (5) *and the compatibility conditions*

$$\forall\ \boldsymbol{v}\in\boldsymbol{K}_N^{p'}(\Omega),\quad\langle\boldsymbol{g},\boldsymbol{v}\rangle_{\Omega_{r',p'}}=0.\quad(8)$$

Then the weak solution for the (MHD) *system given by Theorem 1 satisfies*

$$(\boldsymbol{u},\boldsymbol{b},P,\boldsymbol{\alpha})\in\boldsymbol{W}^{1,p}(\Omega)\times\boldsymbol{W}^{1,p}(\Omega)\times W^{1,r}(\Omega)\times\mathbb{R}^I.$$

Moreover, we have the following estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}+\|\boldsymbol{b}\|_{\boldsymbol{W}^{1,p}(\Omega)}+\|P\|_{W^{1,r}(\Omega)}\leqslant C\left(\|\boldsymbol{f}\|_{\left(\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)\right)'}+\|\boldsymbol{g}\|_{\left(\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)\right)'}+\|P_0\|_{W^{1/r',r}(\Gamma)}\right)\quad(9)$$

Next, the existence of a strong solution for more regular data is given in the following Theorem 4.

Theorem 4 (regularity $W^{2,p}$ with $p \geq \frac{6}{5}$). *Let us suppose that Ω is of class $\mathscr{C}^{2,1}$ and $p \geq \frac{6}{5}$. Let $\boldsymbol{f}$, $\boldsymbol{g}$ and P_0 satisfy* (8), (5) *and*

$$\boldsymbol{f} \in \boldsymbol{L}^p(\Omega), \quad \boldsymbol{g} \in \boldsymbol{L}^p(\Omega) \quad \textit{and} \quad P_0 \in W^{1-\frac{1}{p},p}(\Gamma).$$

Then the weak solution $(\boldsymbol{u}, \boldsymbol{b}, P, \boldsymbol{\alpha})$ for the (MHD) system given by Theorem 1 belongs to $\boldsymbol{W}^{2,p}(\Omega) \times \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega) \times \mathbb{R}^I$ and satisfies the following estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\boldsymbol{b}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|P\|_{W^{1,p}(\Omega)} \leqslant C\left(\left\|\boldsymbol{f}\right\|_{\boldsymbol{L}^p(\Omega)} + \left\|\boldsymbol{g}\right\|_{\boldsymbol{L}^p(\Omega)} + \|P_0\|_{W^{1-\frac{1}{p},p}(\Gamma)}\right) \tag{10}$$

4. Linearized MHD system

We consider the following linearized (MHD) system: Find $(\boldsymbol{u}, \boldsymbol{b}, P, c_i)$ such that

$$\begin{cases} -\Delta\boldsymbol{u} + (\mathbf{curl}\,\boldsymbol{w}) \times \boldsymbol{u} + \nabla P - (\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{d} = \boldsymbol{f} & \text{and} \quad \operatorname{div}\boldsymbol{u} = h \quad \text{in } \Omega, \\ \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{b} - \mathbf{curl}(\boldsymbol{u} \times \boldsymbol{d}) = \boldsymbol{g} & \text{and} \quad \operatorname{div}\boldsymbol{b} = 0 \quad \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = 0 \quad \text{and} \quad \boldsymbol{b} \times \boldsymbol{n} = 0 \quad \text{on } \Gamma, \quad P = P_0 \quad \text{on } \Gamma_0 & \text{and} \quad P = P_0 + c_i \quad \text{on } \Gamma_i, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} = 0 & \text{and} \quad \langle \boldsymbol{b} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} = 0, \ 1 \leq i \leq I, \end{cases} \tag{11}$$

where $\boldsymbol{w}$ and $\boldsymbol{d}$ are given. The aim of this section is to study the L^p regularity of the weak solution for the linearized problem (11). We consider the cases $p \geq 2$ and $\frac{3}{2} < p < 2$ for regularity in $W^{1,p}(\Omega)$. These results will be used in the following to show the regularity $W^{1,p}(\Omega)$ of weak solution for the nonlinear (MHD) problem for $3/2 < p < 2$.

Theorem 5. *Let $p \geqslant 2$. Suppose that*

$$\boldsymbol{f}, \boldsymbol{g} \in \left[\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)\right]', P_0 \in W^{1-\frac{1}{r},r}(\Gamma), h \in W^{1,r}(\Omega)$$

satisfying the compatibility conditions (8), (5) *together with $\mathbf{curl}\,\boldsymbol{w} \in \boldsymbol{L}^s(\Omega)$, $\boldsymbol{d} \in \boldsymbol{L}^3_\sigma(\Omega)$ and $\nabla\boldsymbol{d} \in \mathbb{L}^s(\Omega)$ where s is given by*

$$s = \frac{3}{2} \quad \textit{if} \quad 2 \leq p < 3, \quad s > \frac{3}{2} \quad \textit{if} \quad = 3 \quad \textit{and} \quad s = r \quad \textit{if} \quad p > 3. \tag{12}$$

Then, the linearized problem (11) *has a unique solution $(\boldsymbol{u}, \boldsymbol{b}, P, \boldsymbol{c}) \in \boldsymbol{W}^{1,p}(\Omega) \times \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^I$. Moreover, we have the estimate:*

$$\begin{aligned} &\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\boldsymbol{b}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|P\|_{W^{1,r}(\Omega)} \\ &\qquad \leq C\left(1 + \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^s(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{L}^3(\Omega)} + \|\nabla\boldsymbol{d}\|_{\mathbb{L}^s(\Omega)}\right)\left(\left\|\boldsymbol{f}\right\|_{\left[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)\right]'} + \|P_0\|_{W^{1-\frac{1}{r},r}(\Gamma)}\right. \\ &\qquad \left. + \left\|\boldsymbol{g}\right\|_{\left[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)\right]} + \left(1 + \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^s(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{L}^3(\Omega)} + \|\nabla\boldsymbol{d}\|_{\mathbb{L}^s(\Omega)}\right)\|h\|_{W^{1,r}(\Omega)}\right) \end{aligned}$$

and $\boldsymbol{c} = (c_1, \cdots, c_I)$ satisfying for $1 \leq i \leq I$:

$$c_i = \langle \boldsymbol{f}, \nabla q_i^N\rangle_{\Omega_{6,2}} + \langle h - P_0, \nabla q_i^N \cdot \boldsymbol{n}\rangle_\Gamma - \int_\Omega (\mathbf{curl}\,\boldsymbol{w}) \times \boldsymbol{u} \cdot \nabla q_i^n \, dx + \int_\Omega (\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{d} \cdot \nabla q_i^N \, dx. \tag{13}$$

Sketch of the proof

The existence and uniqueness of a weak solution for $p = 2$ follows from the Lax–Milgram Lemma. For $p > 2$, we use the same construction as in [8, Theorem 3.6] with some further changes in order to deal with the magnetic field. Note that the choice of spaces for $\mathbf{curl}\,\boldsymbol{w}$ and $\nabla \boldsymbol{d}$ with s defined in (12) is necessary in order to give sense to the terms $\int_\Omega (\mathbf{curl}\,\boldsymbol{w} \times \boldsymbol{u}) \cdot \boldsymbol{v}\, d\boldsymbol{x}$ and $\int_\Omega (\boldsymbol{u} \cdot \nabla)\boldsymbol{d} \cdot \boldsymbol{v}\, d\boldsymbol{x}$ respectively.

Remark 6. We also need to study the problem where the second equation in (11) is replaced by $\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{b} - \mathbf{curl}(\boldsymbol{u} \times \boldsymbol{d}) + \nabla\tau = \boldsymbol{g}$ in Ω with $\tau = 0$ on Γ. The scalar τ represents the Lagrange multiplier associated with magnetic divergence constraint. This problem appears as the dual problem associated to (MHD) in the study of weak solutions for $p < 2$. Note that, taking the divergence in the above equation, τ is a solution of the following Dirichlet problem:

$$\Delta\tau = \operatorname{div} \boldsymbol{g} \quad \text{in } \Omega \quad \text{and} \quad \tau = 0 \quad \text{on } \Gamma. \tag{14}$$

In particular, if the function $\boldsymbol{g}$ is divergence-free, we have $\tau = 0$. Nevertheless, the introduction of τ will be useful to enforce zero divergence condition over the magnetic field.

Theorem 7. *Let $\frac{3}{2} < p < 2$. Assume that*

$$\boldsymbol{f}, \boldsymbol{g} \in \left[\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)\right]', \; h = 0, \; P_0 \in W^{1-\frac{1}{r},r}(\Gamma)$$

satisfying the compatibility conditions (8), (5) *together with* $\mathbf{curl}\,\boldsymbol{w} \in \boldsymbol{L}^{3/2}(\Omega)$ *and* $\boldsymbol{d} \in \boldsymbol{W}_\sigma^{1,3/2}(\Omega)$. *Then the linearized problem* (11) *has a unique solution* $(\boldsymbol{u}, \boldsymbol{b}, P, \boldsymbol{c}) \in \boldsymbol{W}^{1,p}(\Omega) \times \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^I$ *with* $\boldsymbol{c} = (c_1, \cdots, c_I)$ *given in* (13). *Moreover, we have the following estimates:*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\boldsymbol{b}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\left(1 + \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{3/2}(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,3/2}(\Omega)}\right) \times \left(\|\boldsymbol{f}\|_{\left[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)\right]'} + \|\boldsymbol{g}\|_{\left[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)\right]'} + \|P_0\|_{W^{1-\frac{1}{r},r}(\Gamma)}\right) \tag{15}$$

and

$$\|P\|_{W^{1,r}(\Omega)} \leq C\left(1 + \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{3/2}(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,3/2}(\Omega)}\right)^2 \times \left(\|\boldsymbol{f}\|_{\left[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)\right]'} + \|\boldsymbol{g}\|_{\left[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)\right]'} + \|P_0\|_{W^{1-\frac{1}{r},r}(\Gamma)}\right). \tag{16}$$

Proof. The linearized problem (11) is equivalent to find $(\boldsymbol{u}, \boldsymbol{b}, P, c_i) \in \boldsymbol{W}_\sigma^{1,p}(\Omega) \times \boldsymbol{W}_\sigma^{1,p}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}$ with $\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0}$ and $\boldsymbol{b} \times \boldsymbol{n} = \boldsymbol{0}$ on Γ, $\langle \boldsymbol{u} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} = 0$ and $\langle \boldsymbol{b} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} = 0$, $1 \leqslant i \leqslant I$ such that: For any $(\boldsymbol{v}, \boldsymbol{a}, \theta, \tau) \in \boldsymbol{V}(\Omega)$,

$$\langle \boldsymbol{u}, -\Delta\boldsymbol{v} - (\mathbf{curl}\,\boldsymbol{w}) \times \boldsymbol{v} + (\mathbf{curl}\,\boldsymbol{a}) \times \boldsymbol{d} + \nabla\theta\rangle_{\Omega p^*,p} - \int_\Omega P \operatorname{div} \boldsymbol{v}\, dx + \langle \boldsymbol{b}, \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{a} + \mathbf{curl}(\boldsymbol{v} \times \boldsymbol{d}) + \nabla\tau\rangle_{\Omega p^*,p} = \langle \boldsymbol{f}, \boldsymbol{v}\rangle_{\Omega r',p'} + \langle \boldsymbol{g}, \boldsymbol{a}\rangle_{\Omega r',p'} - \int_\Gamma P_0 \boldsymbol{v} \cdot \boldsymbol{n}\, d\sigma, \tag{17}$$

$$c_i = \langle \boldsymbol{f}, \nabla q_i^N\rangle_{\Omega r',p'} - \int_\Gamma P_0 \nabla q_i^N \cdot \boldsymbol{n}\, d\sigma + \int_\Omega (\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{d} \cdot \nabla q_i^N\, dx - \int_\Omega (\mathbf{curl}\,\boldsymbol{w}) \times \boldsymbol{u} \cdot \nabla q_i^N\, dx, \tag{18}$$

where the space $\boldsymbol{V}(\Omega)$ is defined by:

$$\boldsymbol{V}(\Omega) := \Big\{(\boldsymbol{v}, \boldsymbol{a}, \theta, \tau) \in \boldsymbol{W}^{1,p'}(\Omega) \times \boldsymbol{W}^{1,p'}(\Omega) \times W^{1,(p^*)'}(\Omega) \times W_0^{1,(p^*)'}(\Omega); \; \operatorname{div} \boldsymbol{v} \in W_0^{1,(p^*)'}(\Omega), \quad \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{a} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on} \quad \Gamma, \quad \theta = 0, \quad \text{on} \quad \Gamma_0 \; \text{and} \quad \theta = \text{constant} \quad \text{on} \quad \Gamma_i, \langle \boldsymbol{v} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} = \langle \boldsymbol{a} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} = 0, \quad \forall\, 1 \leqslant i \leqslant I\Big\},$$

with $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{3}$. Since $2 < p' < 3$, for any $(\boldsymbol{F}, \boldsymbol{G}, \phi) \in [\boldsymbol{H}_0^{p^*,p}(\mathbf{curl},\Omega)]' \times [\boldsymbol{H}_0^{p^*,p}(\mathbf{curl},\Omega)]' \perp \boldsymbol{K}_N^p(\Omega) \times W_0^{1,(p^*)'}(\Omega)$, the following problem

$$\begin{cases} -\Delta \boldsymbol{v} - (\mathbf{curl}\,\boldsymbol{w}) \times \boldsymbol{v} + \nabla\theta + (\mathbf{curl}\,\boldsymbol{a}) \times \boldsymbol{d} = \boldsymbol{F} & \text{and } \operatorname{div} \boldsymbol{v} = \phi \quad \text{in } \Omega, \\ \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{a} + \mathbf{curl}(\boldsymbol{v} \times \boldsymbol{d}) + \nabla\tau = \boldsymbol{G} & \text{and } \operatorname{div} \boldsymbol{a} = 0 \quad \text{in } \Omega, \\ \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0}, \quad \boldsymbol{a} \times \boldsymbol{n} = \boldsymbol{0} & \text{and } \tau = 0 \text{ on } \Gamma,\ \theta = 0 \text{ on } \Gamma_0 \text{ and } \theta = \beta_i \text{ on } \Gamma_i, \\ \langle \boldsymbol{v} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} = 0 & \text{and } \langle \boldsymbol{a} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} = 0,\ 1 \le i \le I. \end{cases} \tag{19}$$

has a unique solution $(\boldsymbol{v}, \boldsymbol{a}, \theta, \tau, \boldsymbol{\beta}) \in \boldsymbol{W}^{1,p'}(\Omega) \times \boldsymbol{W}^{1,p'}(\Omega) \times W^{1,(p^*)'}(\Omega) \times W_0^{1,(p^*)'}(\Omega) \times \mathbb{R}^I$ with $\operatorname{div} \boldsymbol{v} \in W_0^{1,(p^*)'}(\Omega)$ and $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_I)$ such that:

$$\beta_i = \left\langle \boldsymbol{F}, \nabla q_i^N \right\rangle_{\Omega_{p^*,p}} + \left\langle (\mathbf{curl}\,\boldsymbol{a}) \times \boldsymbol{d}, \nabla q_i^N \right\rangle_{\Omega_{p^*,p}} - \left\langle (\mathbf{curl}\,\boldsymbol{w}) \times \boldsymbol{v}, \nabla q_i^N \right\rangle_{\Omega_{p^*,p}} + \int_\Gamma \phi \nabla q_i^N \cdot \boldsymbol{n}\, ds.$$

Indeed, thanks to Remark 6, the scalar potential τ is decoupled from the system and is a solution of (14), where the right hand side $\operatorname{div} \boldsymbol{g}$ is replaced by $\operatorname{div} \boldsymbol{G}$. Since $\operatorname{div} \boldsymbol{G}$ belongs to $W^{-1,(p^*)'}(\Omega)$, we deduce the existence and uniqueness of

$$\tau \in W_0^{1,(p^*)'}(\Omega) \quad \text{satisfying} \quad \|\tau\|_{W^{1,r}(\Omega)} \le C \|\boldsymbol{G}\|_{\left[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)\right]'}.$$

With τ known, we set $\boldsymbol{G}' = \boldsymbol{G} - \nabla\tau$ and then the system (19) becomes involving only $\boldsymbol{v}$ and $\boldsymbol{a}$. Since $\boldsymbol{G}'$ belongs to $[\boldsymbol{H}_0^{p^*,p}(\mathbf{curl},\Omega)]'$ and satisfies the compatibility condition (8), thanks to Theorem 5, we have the existence and uniqueness of the pair $(\boldsymbol{v}, \boldsymbol{a})$. Moreover, we know that

$$\begin{aligned} &\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p'}(\Omega)} + \|\boldsymbol{a}\|_{\boldsymbol{W}^{1,p'}(\Omega)} + \|\theta\|_{W^{1,(p^*)'}(\Omega)} \leqslant C\left(1 + \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{3/2}(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,3/2}(\Omega)}\right) \\ &\times \left(\|\boldsymbol{F}\|_{\left[\boldsymbol{H}_0^{p^*,p}(\mathbf{curl},\Omega)\right]'} + \|\boldsymbol{G}\|_{\left[\boldsymbol{H}_0^{p^*,p}(\mathbf{curl},\Omega)\right]'} + \left(1 + \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{3/2}(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,3/2}(\Omega)}\right) \|\phi\|_{W^{1,(p^*)'}(\Omega)} \right). \end{aligned} \tag{20}$$

We note that, from Theorem 5 for $2 < p' < 3$, the value of s is $3/2$. Therefore, using (20), we have

$$\begin{aligned} &\left| \langle \boldsymbol{f}, \boldsymbol{v}\rangle_{\Omega r',p'} + \langle \boldsymbol{g}, \boldsymbol{a}\rangle_{\Omega r',p'} - \int_\Gamma P_0\, \boldsymbol{v} \cdot \boldsymbol{n}\, d\sigma \right| \\ &\leqslant \|\boldsymbol{f}\|_{\left[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)\right]'} \|\boldsymbol{v}\|_{\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)} + \|\boldsymbol{g}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} \|\boldsymbol{a}\|_{\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)} \\ &\qquad + \|P_0\|_{W^{1-\frac{1}{r},r}(\Gamma)} \|\boldsymbol{v} \cdot \boldsymbol{n}\|_{\boldsymbol{W}^{1-\frac{1}{p'},p'}(\Gamma)} \\ &\leqslant C\left(\|\boldsymbol{f}\|_{\left[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)\right]'} + \|\boldsymbol{g}\|_{\left[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)\right]'} + \|P_0\|_{W^{1-\frac{1}{r},r}(\Gamma)} \right) \left(\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p'}(\Omega)} + \|\boldsymbol{a}\|_{\boldsymbol{W}^{1,p'}(\Omega)} \right) \\ &\leqslant C\left(\|\boldsymbol{f}\|_{\left[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)\right]'} + \|\boldsymbol{g}\|_{\left[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)\right]'} + \|P_0\|_{W^{1-\frac{1}{r},r}(\Gamma)} \right) \left(1 + \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)} \right) \\ &\times \left(\|\boldsymbol{F}\|_{\boldsymbol{H}_0^{p^*,p}(\mathbf{curl},\Omega)} + \|\boldsymbol{G}\|_{\boldsymbol{H}_0^{p^*,p}(\mathbf{curl},\Omega)} + \left(1 + \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)} \right) \|\phi\|_{W^{1,(p^*)'}(\Omega)} \right). \end{aligned}$$

We deduce that the linear mapping

$$(\boldsymbol{F}, \boldsymbol{G}, \phi) \to \langle \boldsymbol{f}, \boldsymbol{v}\rangle_{\Omega_{r',p'}} + \langle \boldsymbol{g}, \boldsymbol{a}\rangle_{\Omega_{r',p'}} - \int_\Gamma P_0\, \boldsymbol{v} \cdot \boldsymbol{n}\, d\sigma$$

defines an element $(\boldsymbol{u}, \boldsymbol{b}, P)$ of $\boldsymbol{H}_0^{p^*,p}(\mathbf{curl},\Omega) \times \boldsymbol{H}_0^{p^*,p}(\mathbf{curl},\Omega) \times W^{-1,p^*}(\Omega)$ solution of (17) and satisfies the estimate:

$$\begin{aligned} &\|\boldsymbol{u}\|_{\boldsymbol{H}_0^{p^*,p}(\mathbf{curl},\Omega)} + \|\boldsymbol{b}\|_{\boldsymbol{H}_0^{p^*,p}(\mathbf{curl},\Omega)} + \left(1 + \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{3/2}(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,3/2}(\Omega)}\right)^{-1} \|P\|_{W^{-1,p^*}(\Omega)} \\ &\leqslant C\left(1 + \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{3/2}(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,3/2}(\Omega)}\right) \left(\|\boldsymbol{f}\|_{\left[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)\right]'} + \|\boldsymbol{g}\|_{\left[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)\right]'} + \|P_0\|_{W^{1-\frac{1}{r},r}(\Gamma)} \right). \end{aligned} \tag{21}$$

In order to recover the solution of (11), it stays us to prove that $\boldsymbol{u}, \boldsymbol{b} \in \boldsymbol{W}^{1,p}_{\sigma}(\Omega)$, $P \in W^{1,r}(\Omega)$, that $\langle \boldsymbol{u} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} = 0$, $\langle \boldsymbol{b} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} = 0$ for all $1 \leqslant i \leqslant I$ and to recover the relation (18). To show that $\langle \boldsymbol{u} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} = 0$, we choose $(\mathbf{0}, \mathbf{0}, \theta, 0)$ with $\theta \in W^{1,(p^*)'}(\Omega)$ satisfying $\theta = 0$ on Γ_0 and $\theta = \delta_{ij}$ on Γ_j for all $1 \leqslant j \leqslant I$ and a fixed $1 \leqslant i \leqslant I$. Then:

$$0 = \langle \boldsymbol{u}, \nabla\theta\rangle_{\Omega_{p^*,p}} = \int_{\Omega} \boldsymbol{u} \cdot \nabla\theta \, dx = \int_{\Gamma} \theta \boldsymbol{u} \cdot \boldsymbol{n} \, d\sigma - \int_{\Omega} \operatorname{div} \boldsymbol{u} \, \theta \, dx = \int_{\Gamma_i} \boldsymbol{u} \cdot \boldsymbol{n} \, d\sigma.$$

For the condition $\langle \boldsymbol{b} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} = 0$ for all $1 \leqslant i \leqslant I$, we set

$$\widetilde{\boldsymbol{b}} = \boldsymbol{b} - \sum_{i=1}^{I} \langle \boldsymbol{b} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} \nabla q_i^N.$$

Observe that by the definition of q_i^N, $\widetilde{\boldsymbol{b}}$ is also solution of (17) and satisfies the condition $\langle \widetilde{\boldsymbol{b}} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} = 0$. Next, taking test functions $(\mathbf{0}, \mathbf{0}, \theta, 0)$ and $(\mathbf{0}, \mathbf{0}, 0, \tau)$ with $\theta \in W^{1,(p^*)'}(\Omega)$ and $\tau \in \mathscr{D}(\Omega)$, we respectively recover $\operatorname{div} \boldsymbol{u} = 0$ and $\operatorname{div} \boldsymbol{b} = 0$ in Ω. Besides, since $\boldsymbol{u}, \boldsymbol{b} \in \boldsymbol{H}_0^{p^*,p}(\mathbf{curl}, \Omega)$, we have $\boldsymbol{u}$ and $\boldsymbol{b}$ belong to $\boldsymbol{X}_N^p(\Omega)$. Thanks to (3), we deduce that $\boldsymbol{u}, \boldsymbol{b} \in \boldsymbol{W}^{1,p}(\Omega)$. Thus, the estimate (15) follows from (3) and (21).

Finally, in order to prove that $P \in W^{1,r}(\Omega)$, we take the test functions $(\boldsymbol{v}, \mathbf{0}, 0, 0)$ with $\boldsymbol{v} \in \mathscr{D}(\Omega)$. Then, by De Rham's theorem there exists $P \in L^p(\Omega)$ such that:

$$\nabla P = \boldsymbol{f} + \Delta \boldsymbol{u} - (\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{u} + (\mathbf{curl}\, \boldsymbol{b}) \times \boldsymbol{d} \quad \text{in } \Omega.$$

Then taking the divergence of the above equation, P is solution of the following problem

$$\begin{aligned} \Delta P &= \operatorname{div} \boldsymbol{f} + \operatorname{div}((\mathbf{curl}\, \boldsymbol{b}) \times \boldsymbol{d} - (\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{u}) \quad \text{in} \quad \Omega, \\ P &= P_0 \quad \text{on } \Gamma_0 \quad \text{and} \quad P = P_0 + c_i \quad \text{on} \quad \Gamma_i. \end{aligned} \tag{22}$$

Since $\mathbf{curl}\, \boldsymbol{w} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega)$ and $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega) \hookrightarrow \boldsymbol{L}^{p^*}(\Omega)$, then $(\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{u} \in \boldsymbol{L}^r(\Omega)$. Besides, $\mathbf{curl}\, \boldsymbol{b} \in \boldsymbol{L}^p(\Omega)$ and $\boldsymbol{d} \in \boldsymbol{W}^{1,\frac{3}{2}}(\Omega) \hookrightarrow \boldsymbol{L}^3(\Omega)$. So $(\mathbf{curl}\, \boldsymbol{b}) \times \boldsymbol{d} \in \boldsymbol{L}^r(\Omega)$. Hence, we obtain that $\Delta P \in W^{-1\,r}(\Omega)$. Since P_0 belongs to $W^{1-1/r,r}(\Gamma)$, we deduce that the solution P of (22) belongs to $W^{1,r}(\Omega)$. Moreover, it satisfies the estimate

$$\|P\|_{W^{1,r}(\Omega)} \leqslant \left\|\operatorname{div} \boldsymbol{f}\right\|_{W^{-1,r}(\Omega)} + \|\operatorname{div}((\mathbf{curl}\, \boldsymbol{b}) \times \boldsymbol{d} - (\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{u})\|_{W^{-1,r}(\Omega)} + \|P_0\|_{W^{1-1/r,r}(\Gamma)}.$$

Applying the characterization of $[\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'$ given in (1), we obtain

$$\left\|\operatorname{div} \boldsymbol{f}\right\|_{W^{-1,r'}(\Omega)} \leqslant \left\|\boldsymbol{f}\right\|_{\left[\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)\right]'}. \tag{23}$$

Next, we have

$$\begin{aligned} \|\operatorname{div}((\mathbf{curl}\boldsymbol{b}) \times \boldsymbol{d})\|_{W^{-1,r}(\Omega)} &\leqslant \|(\mathbf{curl}\, \boldsymbol{b}) \times \boldsymbol{d}\|_{\boldsymbol{L}^r(\Omega)} \leq \|\mathbf{curl}\, \boldsymbol{b}\|_{\boldsymbol{L}^p(\Omega)} \|\boldsymbol{d}\|_{\boldsymbol{L}^3(\Omega)} \\ &\leq C_d \|\boldsymbol{b}\|_{\boldsymbol{W}^{1,p}(\Omega)} \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)}, \end{aligned} \tag{24}$$

where C_d is the constant related to the Sobolev embedding $\boldsymbol{W}^{1,\frac{3}{2}}(\Omega) \hookrightarrow \boldsymbol{L}^3(\Omega)$. Finally, we have

$$\begin{aligned} \|\operatorname{div}((\mathbf{curl}\boldsymbol{w}) \times \boldsymbol{u})\|_{W^{-1,r}(\Omega)} &\leqslant \|(\mathbf{curl}\boldsymbol{w}) \times \boldsymbol{u}\|_{\boldsymbol{L}^r(\Omega)} \leqslant \|\mathbf{curl}\boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} \|\boldsymbol{u}\|_{\boldsymbol{L}^{p^*}(\Omega)} \\ &\leq C \|\mathbf{curl}\boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} \|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}, \end{aligned} \tag{25}$$

where we have used the Sobolev embedding $\boldsymbol{W}^{1,p}(\Omega) \hookrightarrow \boldsymbol{L}^{p^*}(\Omega)$. Using estimates (23), (24), (25) combining with the estimate (15), we obtain the estimate (16) for the pressure. With the same arguments used in [8, Theorem 3.2], we can prove that the constants c_i, $1 \leq i \leq I$ satisfy the relation (18). □

5. Existence and uniqueness results of (MHD) for $\frac{3}{2} < p < 2$

The next Theorem 8 tells us that it is possible to extend the regularity of the solution of the nonlinear (MHD) problem for $\frac{3}{2} < p < 2$ in $\boldsymbol{W}^{1,p}(\Omega)$. For this, we apply Banach's fixed-point theorem over the linearized problem (11).

Theorem 8 (regularity $W^{1,p}$ with $\frac{3}{2} < p < 2$). *Assume that $\frac{3}{2} < p < 2$ and r with $\frac{1}{r} = \frac{1}{p} + \frac{1}{3}$. Let us consider*

$$\boldsymbol{f}, \boldsymbol{g} \in \left[\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)\right]' \quad \text{and} \quad P_0 \in W^{1-\frac{1}{r},r}(\Gamma)$$

satisfying the compatibility conditions (8)*,* (5)*.*

(i) *There exists a constant δ_1 such that, if*

$$\|\boldsymbol{f}\|_{\left[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)\right]'} + \|\boldsymbol{g}\|_{\left[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)\right]'} + \|P_0\|_{W^{1-\frac{1}{r},r}(\Gamma)} \leq \delta_1$$

Then, the (MHD) problem has at least a solution $(\boldsymbol{u}, \boldsymbol{b}, P, \boldsymbol{\alpha}) \in \boldsymbol{W}^{1,p}(\Omega) \times \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^I$. Moreover, we have the following estimates:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\boldsymbol{b}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leqslant C_1 \left(\|\boldsymbol{f}\|_{\left[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)\right]'} + \|\boldsymbol{g}\|_{\left[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)\right]'} + \|P_0\|_{W^{1-\frac{1}{r},r}(\Gamma)} \right), \tag{26}$$

$$\|P\|_{W^{1,r}(\Omega)} \leqslant C_1 \left(1 + C^* \eta\right) \left(\|\boldsymbol{f}\|_{\left[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)\right]'} + \|\boldsymbol{g}\|_{\left[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)\right]'} + \|P_0\|_{W^{1-\frac{1}{r},r}(\Gamma)} \right), \tag{27}$$

where $\delta_1 = (2C^2C^*)^{-1}$, $C_1 = C(1 + C^*\eta)$ with $C > 0$, $C^* > 0$ are the constants given in (29)*,* (31) *respectively and η defined by* (32)*. Furthermore, we have for all $1 \leqslant i \leqslant I$*

$$\alpha_i = \langle \boldsymbol{f}, \nabla q_i^N \rangle_\Omega - \int_\Omega (\mathbf{curl}\, \boldsymbol{u}) \times \boldsymbol{u} \cdot \nabla q_i^N \, d\boldsymbol{x} + \int_\Omega (\mathbf{curl}\, \boldsymbol{b}) \times \boldsymbol{b} \cdot \nabla q_i^N \, d\boldsymbol{x} - \int_\Gamma P_0 \nabla q_i^N \cdot \boldsymbol{n} \, ds$$

(ii) *Moreover, if the data satisfy that*

$$\|\boldsymbol{f}\|_{\left[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)\right]'} + \|\boldsymbol{g}\|_{\left[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)\right]'} + \|P_0\|_{W^{1-\frac{1}{r},r}(\Gamma)} \leq \delta_2,$$

for some $\delta_2 \in [0, \delta_1]$, then the weak solution of (MHD) problem is unique.

Proof. Let us define the space $\boldsymbol{Z}^p(\Omega) = \boldsymbol{W}_\sigma^{1,p}(\Omega) \times \boldsymbol{W}_\sigma^{1,p}(\Omega)$. For given $(\boldsymbol{w}, \boldsymbol{d}) \in \boldsymbol{B}_\eta \times \boldsymbol{B}_\eta$, define the operator T by $T(\boldsymbol{w}, \boldsymbol{d}) = (\boldsymbol{u}, \boldsymbol{b})$ with $(\boldsymbol{u}, \boldsymbol{b})$ is the component of the solution $(\boldsymbol{u}, \boldsymbol{b}, P, \boldsymbol{c})$ of (11) given by Theorem 7 and the neighbourhood $\boldsymbol{B}_\eta$ is defined by

$$\boldsymbol{B}_\eta = \left\{(\boldsymbol{w}, \boldsymbol{d}) \in \boldsymbol{Z}^p(\Omega),\ \|(\boldsymbol{w}, \boldsymbol{d})\|_{\boldsymbol{Z}^p(\Omega)} \leq \eta\right\}, \quad \eta > 0$$

which is equipped with the norm

$$\|(\boldsymbol{w}, \boldsymbol{d})\|_{\boldsymbol{Z}^p(\Omega)} = \|\boldsymbol{w}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,p}(\Omega)}.$$

We have to prove that T is a contraction from $\boldsymbol{B}_\eta$ to itself, i.e., let $(\boldsymbol{w}_1, \boldsymbol{d}_1), (\boldsymbol{w}_2, \boldsymbol{d}_2) \in \boldsymbol{B}_\eta$, we show that there exists $\theta \in (0, 1)$ such that:

$$\|T(\boldsymbol{w}_1, \boldsymbol{d}_1) - T(\boldsymbol{w}_2, \boldsymbol{d}_2)\|_{\boldsymbol{Z}^p(\Omega)} = \|(\boldsymbol{u}_1, \boldsymbol{b}_1) - (\boldsymbol{u}_2, \boldsymbol{b}_2)\|_{\boldsymbol{Z}^p(\Omega)} \leqslant \theta \|(\boldsymbol{w}_1, \boldsymbol{d}_1) - (\boldsymbol{w}_2, \boldsymbol{d}_2)\|_{\boldsymbol{Z}^p(\Omega)} \tag{28}$$

Since each $(\boldsymbol{u}_i, \boldsymbol{b}_i)$, $i = 1,2$ is a solution of (11) with $h = 0$, thanks to Theorem 7, we have the estimates

$$\|(\boldsymbol{u}_i, \boldsymbol{b}_i)\|_{\boldsymbol{Z}^p(\Omega)} \leq C \left(1 + \|\mathbf{curl}\, \boldsymbol{w}_i\|_{\boldsymbol{L}^{3/2}(\Omega)} + \|\boldsymbol{d}_i\|_{\boldsymbol{W}^{1,3/2}(\Omega)}\right) \gamma_1 \tag{29}$$

where

$$\gamma_1 = \|\boldsymbol{f}\|_{\left[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)\right]'} + \|\boldsymbol{g}\|_{\left[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)\right]'} + \|P_0\|_{W^{1-\frac{1}{r},r}(\Gamma)}.$$

Next, the differences $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$, $\boldsymbol{b} = \boldsymbol{b}_1 - \boldsymbol{b}_2$, $P = P_1 - P_2$ and $\boldsymbol{c}^1 - \boldsymbol{c}^2$ satisfies

$$\begin{cases} -\Delta \boldsymbol{u} + (\mathbf{curl}\, \boldsymbol{w}_1) \times \boldsymbol{u} + \nabla P - \mathbf{curl} \boldsymbol{b} \times \boldsymbol{d}_1 = \boldsymbol{f}_2 & \text{and } \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega \\ \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{b} - \mathbf{curl}(\boldsymbol{u} \times \boldsymbol{d}_1) = \boldsymbol{g}_2 & \text{and } \operatorname{div} \boldsymbol{b} = 0 \quad \text{in } \Omega \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} & \text{and } \boldsymbol{b} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma \\ P = 0 \text{ on } \Gamma_0 & \text{and } P = c_j \quad \text{on } \Gamma_j \\ \langle \boldsymbol{u} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_j} = 0 & \text{and } \langle \boldsymbol{b} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_j} = 0 \ \forall\ 1 \leqslant j \leqslant I \end{cases} \tag{30}$$

with $\boldsymbol{f}_2 = -(\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{u}_2 + (\mathbf{curl}\, \boldsymbol{b}_2) \times \boldsymbol{d}$ and $\boldsymbol{g}_2 = \mathbf{curl}(\boldsymbol{u}_2 \times \boldsymbol{d})$. Using (29), we obtain

$$\|(\boldsymbol{u}, \boldsymbol{b})\|_{\boldsymbol{Z}^p(\Omega)} \leq C^2 C^* \left(1 + C^* \eta\right)^2 \gamma_1 \|(\boldsymbol{w},\, \boldsymbol{d})\|_{\boldsymbol{Z}^p(\Omega)}, \tag{31}$$

where $C^* = C_w + C_d$ with $C_w > 0$ and $C_d > 0$ are such that, $\|\mathbf{curl}\, \boldsymbol{w}\|_{\boldsymbol{L}^{3/2}(\Omega)} \leq C_w \|\boldsymbol{w}\|_{\boldsymbol{W}^{1,p}(\Omega)}$ and $\|\boldsymbol{d}\|_{\boldsymbol{L}^3(\Omega)} \leq C_d \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,p}(\Omega)}$. Therefore, we can obtain estimate (28) if we choose, for example

$$\eta = (C^*)^{-1} \left(\left(2C^2 C^* \gamma_1\right)^{-1/2} - 1 \right) \quad \text{and} \quad \gamma_1 < \left(2C^2 C^*\right)^{-1}. \tag{32}$$

We conclude that T has a fixed point $(\boldsymbol{u}^*, \boldsymbol{b}^*) \in \boldsymbol{Z}^p(\Omega)$ satisfying:

$$\left\| \left(\boldsymbol{u}^*, \boldsymbol{b}^*\right) \right\|_{\boldsymbol{Z}^p(\Omega)} \leq C \left(1 + C^* \eta\right) \gamma_1, \tag{33}$$

which gives the estimate (26). The pressure estimate (27) and the uniqueness can be deduced as in [6, Theorem 19] and [7, Theorem 3.1]. □

Remark 9.

(i) Since the pressure is decoupled from the system, we can improve the regularity given in the previous results by choosing a convenient boundary condition P_0.

(ii) To obtain strong solutions for the (MHD) problem, we can consider the case where $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$ and $\boldsymbol{g}$ less regular in $\boldsymbol{L}^q(\Omega)$.

References

[1] G. V. Alekseev, “Mixed Boundary value problems for stationary magneto-hydrodynamic equations of a viscous heat-conducting fluid”, *J. Math. Fluid Mech.* **18** (2016), no. 3, p. 591-607.

[2] ———, “Solvability of an inhomogeneous boundary value problem for the stationary magnetohydrodynamic equations for a viscous incompressible fluid”, *Differ. Equ.* **52** (2016), no. 6, p. 739-748.

[3] G. V. Alekseev, R. V. Brizitskii, “Control problems for stationary magnetohydrodynamic equations of a viscous heat-conducting fluid under mixed boundary conditions”, *Comput. Math. Math. Physics.* **45** (2005), no. 12, p. 2049-2065.

[4] ———, “Solvability of the boundary value problem for stationary magnetohydrodynamic equations under mixed boundary conditions for the magnetic field”, *Appl. Math. Lett.* **32** (2014), p. 13-18.

[5] C. Amrouche, S. Boukassa, “Existence and regularity of solution for a model in magnetohydrodynamics”, *Nonlinear Anal., Theory Methods Appl.* **190** (2020), article ID 111602 (20 pages).

[6] C. Amrouche, M. Á. Rodriguez-Bellido, “Stationary Stokes, Oseen and Navier–Stokes equations with singular data”, *Arch. Ration. Mech. Anal.* **199** (2011), no. 2, p. 597-651.

[7] ———, “The Oseen and Navier–Stokes equations in a non-solenoidal framework”, *Math. Methods Appl. Sci.* **39** (2016), no. 17, p. 5066-5090.

[8] C. Amrouche, N. E. H. Seloula, “L^p-theory for the Navier–Stokes equations with pressure boundary conditions”, *Discrete Contin. Dyn. Syst.* **6** (2013), no. 5, p. 1113-1137.

[9] ———, “L^p-theory for vector potentials and Sobolev’s inequalities for vector fields: application to the Stokes equations with pressure boundary conditions”, *Math. Models Methods Appl. Sci.* **23** (2013), no. 1, p. 37-92.

[10] R. V. Brizitskii, D. A. Tereshko, “On the solvability of boundary value problems for the stationary magnetohydrodynamic equations with inhomogeneous mixed boundary conditions”, *Differ. Equ.* **43** (2007), no. 2, p. 246-258.

[11] C. Conca, C. Parés Madroñal, O. Pironneau, M. Thiriet, “Navier–Stokes equations with imposed pressure and velocity fluxes”, *Int. J. Numer. Methods Fluids* **20** (1995), no. 4, p. 267-287.

[12] J. Deng, Z. Tao, T. Zhang, “Iterative penalty finite element methods for the steady incompressible magnetohydrodynamic problem”, *Comput. Appl. Math.* **36** (2017), no. 4, p. 1637-1657.

[13] C. Greif, D. Li, D. Schötzau, X. Wei, “A mixed finite element method with exactly divergence-free velocities for incompressible magnetohydrodynamics”, *Comput. Methods Appl. Mech. Eng.* **199** (2010), p. 2840-2855.

[14] R. Hiptmair, L. Li, S. Mao, W. Zheng, "A fully divergence-free finite element method for magnetohydrodynamic equations", *Math. Models Methods Appl. Sci.* **28** (2018), no. 4, p. 659-695.

[15] W. Qiu, K. Shi, "A mixed DG method and an HDG method for incompressible magnetohydrodynamics", *IMA J. Numer. Anal.* **40** (2020), no. 2, p. 1356-1389.

[16] D. Schötzau, "Mixed finite element methods for stationary incompressible magneto-hydrodynamics", *Numer. Math.* **96** (2004), no. 4, p. 771-800.

[17] N. E. H. Seloula, "Mathematical analysis and numerical approximation of the Stokes and Navier–Stokes equations with non standard boundary conditions", PhD Thesis, Université de Pau et des Pays de l'Adour, (France), 2010.

[18] Y. Zeng, Z. Zhang, "Existence, regularity and uniqueness of weak solutions with bounded magnetic fields to the steady Hall-MHD system", *Calc. Var. Partial Differ. Equ.* **59** (2020), no. 2, article ID 84 (16 pages).