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Comptes Rendus

Mathématique

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Volume 358, issue 9-10 (2020), p. 1073-1083

Published online: 5 January 2021

https://doi.org/10.5802/crmath.128

Les Comptes Rendus. Mathématique *sont membres du*
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 1778-3569

Comptes Rendus
Mathématique
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https://doi.org/10.5802/crmath.128

Partial Differential Equation / *Équations aux dérivées partielles*

Large-amplitude internal fronts in two-fluid systems

Fronts internes de grandes amplitudes pour des systèmes à deux fluides

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Abstract. In this announcement, we report results on the existence of families of large-amplitude internal hydrodynamic bores. These are traveling front solutions of the full two-phase incompressible Euler equation in two dimensions. The fluids are bounded above and below by flat horizontal walls and acted upon by gravity. We obtain continuous curves of solutions to this system that bifurcate from the trivial solution where the interface is flat. Following these families to the their extreme, the internal interface either overturns, comes into contact with the upper wall, or develops a highly degenerate "double stagnation" point.

Our construction is made possible by a new abstract machinery for global continuation of monotone front-type solutions to elliptic equations posed on infinite cylinders. This theory is quite robust and, in particular, can treat fully nonlinear equations as well as quasilinear problems with transmission boundary conditions.

Résumé. Dans cette note, nous présentons des résultats d'existence d'ondes de Mascaret de grandes amplitudes. Cela correspond à des ondes progressives pour l'équation d'Euler incompressible à deux phases en deux dimensions d'espace. Les fluides sont délimités au-dessus et au-dessous par des parois horizontales et sont soumis à leurs gravités. Nous obtenons des courbes continues de solutions à ce système qui bifurquent de la solution triviale où l'interface est plate. A la limite, l'interface interne se renverse, entre en contact avec la paroi supérieure, ou développe un point de « double stagnation » très dégénéré.

Notre construction est rendue possible grâce à une nouvelle méthode abstraite pour la continuation globale des solutions de type front monotone aux équations elliptiques, posées sur des cylindres infinis. Cette théorie est assez robuste et, en particulier, peut traiter des équations entièrement non linéaires ainsi que des problèmes quasi-linéaires avec des conditions aux limites de transmission.

2020 Mathematics Subject Classification. 35B32, 76B15, 35J60, 35J66.

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Funding. The research of RMC is supported in part by the NSF through DMS-1907584. The research of SW is supported in part by the NSF through DMS-1812436. A portion of this work was completed during a Research-in-Teams Program generously supported by the Erwin Schrödinger Institute for Mathematics and Physics, University of Vienna.

Manuscript received 31st July 2020, revised 3rd October 2020, accepted 5th October 2020.

1. Introduction

The world's oceans are stratified in the sense that the fluid density increases with depth. While small in relative terms, this density variation can dramatically affect the dynamics and, in particular, allows for the formation of large scale *internal waves* that remain coherent over long distances. In many settings there are two regions with nearly constant density separated by a thin layer, called the *pycnocline*, where density gradients are large. This permits the system to be modeled as two constant density fluids with different densities, divided by a sharp interface along which waves can propagate. Unlike surface waves in a homogeneous density fluid[1], these internal waves can take the form of fronts or (smooth) *hydrodynamical bores*. These are steady solutions where the internal interface is asymptotically flat both upstream and downstream of the wave but with different heights.

Let us restrict attention to the simplest configuration where the two fluid layers are irrotational and bounded from above and below by rigid flat boundaries as shown in Figure 1. There is an extensive applied literature on this problem, mostly centered around linear or weakly nonlinear model equations which are valid only for small amplitudes [18], as well as a growing body of rigorous results. For bores in the full nonlinear equations, the first rigorous existence results date back to the work of Amick and Turner [4], confirming formal predictions based on the weakly nonlinear extended Korteweg–de Vries equation. Alternative proofs have subsequently been given using different methods by Mielke [28], Makarenko [25], and the authors [7].

In this announcement, we report the first construction of genuinely large-amplitude bores. One can no longer expect to base such an analysis on a well-chosen model equation, and instead we rely on a new abstract global bifurcation theory tailored to front-type solutions of elliptic equations in cylindrical domains [8].

1.1. *Formulation and existence theory*

The problem can be mathematically formulated as follows. The unknown interface $\mathscr{S} = \{(x, y) : y = \eta(x)\}$ separates two open fluid regions $\mathscr{D}_1$ and $\mathscr{D}_2$ as shown in Figure 1. Here the lower region $\mathscr{D}_1$ has constant density $\rho_1 > 0$, and is bounded below by a rigid barrier at height $y = -\lambda$. The upper region $\mathscr{D}_2$ is likewise bounded above by a rigid barrier at $y = 1-\lambda$ and has constant density $0 < \rho_2 < \rho_1$. Note that the total height of the channel is normalized to 1. Assuming incompressibility, the velocity field in each fluid is given by $(\partial_y\psi, -\partial_x\psi)$ for some *stream function* ψ satisfying

$$\Delta\psi = 0 \quad \text{in } \mathscr{D}_1 \cup \mathscr{D}_2 \tag{1a}$$

[1]The nonexistence of irrotational bores in constant density water was first established by Rayleigh [32]; see also Lamb [24, Chapter VIII, part 187]. In the rotational setting, it is a consequence of the analysis in [41, Section 3.2]

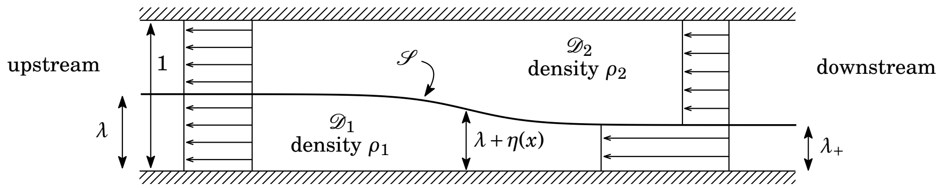

Figure 1. A bore in a two layer fluid.

together with the so-called kinematic boundary conditions

$$\begin{aligned} \psi &= 0 && \text{on } \mathscr{S}, \\ \psi &= \lambda && \text{on } y=-\lambda, \\ \psi &= \lambda-1 && \text{on } y=1-\lambda, \end{aligned} \tag{1b}$$

and the dynamic boundary condition

$$\frac{1}{2}[\![\rho|\nabla\psi|^2]\!] + \frac{[\![\rho]\!]}{F^2}y = \frac{[\![\rho]\!]}{2} \qquad \text{on } \mathscr{S}, \tag{1c}$$

where $F>0$ is a dimensionless parameter called the *Froude number* and $[\![\cdot]\!] = (\cdot)_2 - (\cdot)_1$ denotes the jump of a quantity across the interface $\mathscr{S}$. These are supplemented with the asymptotic conditions

$$\nabla\psi \to (0,-1), \quad \eta \to 0 \qquad \text{as } x \to -\infty, \tag{1d}$$

$$\eta \to \lambda_+ - \lambda \neq 0 \qquad \text{as } x \to +\infty \tag{1e}$$

as shown in Figure 1.

We are interested in classical solutions to (1) which enjoy the regularity

$$\psi \in C_{\mathrm{b}}^{2+\alpha}(\overline{\mathscr{D}_1}) \cap C_{\mathrm{b}}^{2+\alpha}(\overline{\mathscr{D}_2}) \cap C_{\mathrm{b}}^{0}(\overline{\mathscr{D}_1 \cup \mathscr{D}_2}), \quad \eta \in C_{\mathrm{b}}^{2+\alpha}(\mathbb{R}),$$

for a fixed $\alpha \in (0,1)$, and where the subscript "b" indicates uniform boundedness. As is well known in the literature on internal waves, this situation is only possible provided the Froude number F and constant λ_+ are given explicitly by

$$F^2 = \frac{\sqrt{\rho_1}-\sqrt{\rho_2}}{\sqrt{\rho_1}+\sqrt{\rho_2}}, \quad \lambda_+ = \frac{\sqrt{\rho_1}}{\sqrt{\rho_1}+\sqrt{\rho_2}}, \tag{2}$$

see for instance [23, Appendix A]. Note that we are working in units where the height of the channel is the length scale, and the upstream (relative) velocity is the velocity scale.

Our first theorem is the following global bifurcation result [8].

Theorem 1 (Large-amplitude bores). *Fix* $\alpha \in (0,1)$ *and densities* $0 < \rho_2 < \rho_1$. *There exist* C^0 *curves*

$$\mathscr{C}^\pm = \left\{(\psi(s),\eta(s),\lambda(s)) : \pm s \in (0,\infty)\right\}$$

of classical solutions to the internal wave problem (1)–(2) *with the following properties.*

(a) (Strict monotonicity) *Each solution on* $\mathscr{C}^\pm$ *is a strictly monotone bore:*

$$\begin{aligned} \pm\partial_x\eta(s) &< 0 && \textit{on} \quad \mathbb{R}, \\ \pm\partial_x\psi(s) &> 0 && \textit{in} \quad \mathscr{D}_1(s)\cup\mathscr{D}_2(s)\cup\mathscr{S}(s), \\ \partial_y\psi(s) &< 0 && \textit{in} \quad \overline{\mathscr{D}_1(s)\cup\mathscr{D}_2(s)}. \end{aligned}$$

(b) (Stagnation limit) *Following* $\mathscr{C}^\pm$, *we encounter waves that are arbitrarily close to having a horizontal stagnation point on the internal interface:*

$$\lim_{s\to\pm\infty} \sup_{\mathscr{S}(s)} \partial_y\psi_i(s) = 0, \quad \textit{for } i = 1 \textit{ or } 2. \tag{3}$$

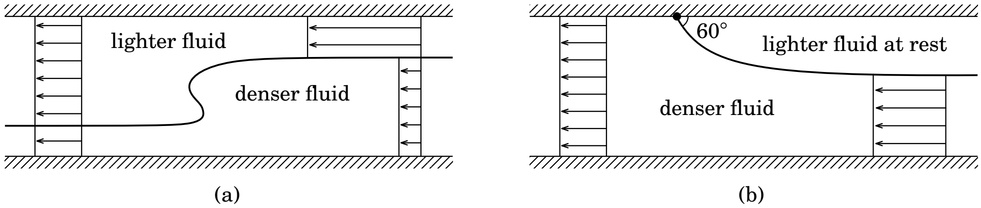

Figure 2. Limiting configurations of monotone bores.

(c) (Laminar origin) *Both* $\mathscr{C}^-$ *and* $\mathscr{C}^+$ *emanate from the same laminar solution in that*

$$\eta(s) \to 0, \quad \nabla\psi(s) \to (0,-1), \quad \lambda(s) \to \lambda_+ \pm \quad \text{as} \quad s \to 0\pm.$$

An overview of the proof of Theorem 1 is given at the end of Section 2.5.

1.2. *Overhanging water waves*

The next result characterizes the limiting form of the profile along $\mathscr{C}^\pm$.

Theorem 2 (Limiting interface).

(a) (Overturning or singularity) *In the limit along* $\mathscr{C}^-$*, either the interface* overturns *in that*

$$\limsup_{s\to-\infty} \|\partial_x \eta(s)\|_{L^\infty(\mathbb{R})} = \infty,$$

or it becomes singular *in that we can extract a translated subsequence*

$$\eta(s) \longrightarrow \eta^* \in \operatorname{Lip}(\mathbb{R}) \quad \text{in } C^\varepsilon_{\mathrm{loc}} \quad \text{for all } \varepsilon \in (0,1)$$

such that $\{y < \eta^*(x)\}$ *simultaneously fails to satisfy both an interior sphere and exterior sphere condition at a single point on its boundary.*

(b) (Overturning or contact) *Following* $\mathscr{C}^+$*, either the interface overturns or it comes into contact with the upper wall:*

$$\limsup_{s\to\infty} \lambda(s) = 1 \quad \text{or} \quad \limsup_{s\to\infty} \|\partial_x \eta(s)\|_{L^\infty(\mathbb{R})} = \infty.$$

We conjecture that the singularity alternative in Theorem 2 (a) can be eliminated with further analysis, and hence that overturning occurs. This would be consistent with numerical work of Dias and Vanden-Broeck [15], which suggests that the interfaces along $\mathscr{C}^-$ overturn while those along $\mathscr{C}^+$ come into contact with the upper wall as shown in Figure 2 to form a so-called "gravity current". In the latter case, von Kármán [21] asserted that the interface must meet the wall at exactly a 60° angle. This was based on a formal calculation in the vein of the classical work of Stokes [36] on extreme surface water waves. The computations in [15] corroborate von Kármán's conjecture, but rigorous verification remains an important open question.

The conclusion of Theorem 2 should be compared with earlier work on overturning or *overhanging* water waves. These are waves for which the interface $\mathscr{S}$ ceases to be a graph. Since we are speaking of traveling wave solutions, this configuration persists for all time, a quite strange phenomenon that has sparked intense mathematical interest.

There are three main physical effects to be considered: gravity, surface tension forces along the interface, and vorticity, either continuously distributed throughout the fluid or concentrated into a internal vortex sheet as in (1). When only gravity is present, overturning waves cannot exist [3, 35, 38]. When only surface tension forces or capillarity is present, on the other hand, there is an exact family of overhanging periodic solutions due to Crapper [14]. These waves have subsequently been perturbed by Akers, Ambrose, and Wright [1] and Córdoba, Enciso, and

Grubic [13] to obtain overhanging gravity-capillary waves where the (dimensionless) gravity is small. Very recently, another family of explicit overhanging waves has been discovered [16,19,20] where only vorticity is present.

In the above existence results, gravity is either neglected or treated as a small parameter. However, numerical work has shown that overhanging waves can also exist when gravity effects are $O(1)$, and in particular in the absence of surface tension forces [30,39]. Rigorous verification of these results remains an outstanding open problem, but significant progress has been made by Constantin, Varvaruca, and Strauss [12]. They used global bifurcation theory to construct a continuous curve of solutions, which in principle are allowed to overturn. Based on subsequent numerics [16, 17] it is conjectured that these curves indeed contain overhanging waves, but a rigorous proof has so far been elusive. We mention related work on periodic internal gravity waves [26,37] and internal gravity-capillary waves [2].

Theorem 2 (a) is tantalizingly close to a proof of overturning. The only other possibility is a highly degenerate type of singularity that has not been observed in numerics [15]. Moreover, there is hope that such singularities could be ruled out through a completely local analysis. By comparison, the global bifurcation results in [2,12,37] allow for a wide range of possibilities. The price we pay for this apparent advantage is twofold. First, we work with a reformulation of (1) that degenerates as an overturning wave is approached. This allows us to detect overturning more easily, but prevents us from continuing further to obtain truly overhanging waves. Second, we construct bores rather than periodic or solitary waves. This introduces serious difficulties related to the unboundedness of the fluid domain as well as the lack of symmetry for the solutions. Once these considerable obstacles have been overcome, however, we find ourselves with more concrete information about the solutions than would be available in the periodic or solitary wave cases.

2. Global bifurcation of monotone fronts

2.1. *Motivation from second-order ODEs*

As mentioned above, Theorem 1 is obtained through a much more general set of results on the global bifurcation of monotone fronts in elliptic PDE. Before presenting those ideas, let us briefly discuss the setting of second-order ODEs where it is easier to construct concrete examples.

In fact, an equation of this type is frequently used as a simplified model for the internal wave system (1). Under the assumption that the waves are long (in some appropriate sense) but not necessarily small amplitude, Miyata [29] and Choi–Camassa [10, 11] independently derived a time-dependent PDE related to the Serre–Green–Naghdi system. Referred to as the MCC equation, this model reduces to the extended Korteweg–de Vries equation mentioned in the introduction in the small-amplitude limit, but it is far more accurate for waves of moderate and even large amplitude. With our current notation, the MCC equation reads

$$\zeta_x^2 = \frac{3\zeta^2}{2F^2}\,\frac{(\lambda+\zeta)(1-\lambda-\zeta+F^2)\rho_2-(1-\lambda-\zeta)(\lambda+\zeta-F^2)\rho_1}{(1-\lambda)^2(\lambda+\zeta)\rho_2+\lambda^2(1-\lambda-\zeta)\rho_1} \tag{4}$$

in integrated form, where here we write ζ rather than η for the deflection of the interface to emphasize the distinction with the full system (1). In differentiated form, (4) can be written as

$$\ddot{\zeta}+V_z(\zeta,\lambda)=0 \tag{5}$$

for an explicit $V=V(z,\lambda)$ that is analytic in its arguments and where dot denotes derivative in x. Here we are viewing the densities ρ_1,ρ_2 as fixed and the Froude number F as given by (2), so that the upstream depth λ of the lower fluid layer is the only parameter. A bore now corresponds to a heteroclinic orbit of (5) connecting two distinct equilibria.

For general equations of the form (5) and a given pair of equilibria, it is relatively straightforward to formulate general conditions which guarantee the existence of heteroclinic orbits. For instance we have the following.

Proposition 3. *Consider the second-order ODE* (5). *Suppose that for a fixed parameter λ_0, there are two distinct rest points $Z_-(\lambda_0)$ and $Z_+(\lambda_0)$ that are* conjugate *in that*

$$V(Z_-(\lambda_0),\lambda_0) = V(Z_+(\lambda_0),\lambda_0). \tag{6}$$

Assume also that the potential satisfies a heteroclinic nondegeneracy condition

$$V(z,\lambda_0) < V(Z_\pm(\lambda_0),\lambda_0) \qquad \text{for } z \text{ between } Z_+(\lambda_0) \text{ and } Z_-(\lambda_0), \tag{7}$$

and spectral nondegeneracy condition

$$V_{zz}(Z_-(\lambda_0),\lambda_0),\ V_{zz}(Z_+(\lambda_0),\lambda_0) < 0. \tag{8}$$

Then there exists a solution (ζ_0,λ_0) to (5) *with $\zeta_0(x) \to Z_\pm(\lambda_0)$ as $x \to \pm\infty$.*

Stated simply, the problem of finding heteroclinic solutions to the ODE (5) amounts to verifying the existence of conjugate rest points of $V(\cdot,\lambda_0)$ satisfying a type of heteroclinic nondegeneracy condition (7) and a spectral nondegeneracy condition (8). We can moreover consider the case when there is a smooth family of conjugate rest points $Z_+(\lambda)$ and $Z_-(\lambda)$ that satisfy (6)–(8) for λ in a neighborhood of λ_0. It is not hard to see that there will then exist a local curve $\mathscr{K}_{\mathrm{loc}}$ of heteroclinic orbits bifurcating from (ζ_0,λ_0). Clearly, one can continue this curve at least as far as the above hypotheses are satisfied along it.

Applying Proposition 3 to the MCC model (4), we find that for any $\lambda \in (0,1)$ there is always a unique smooth heteroclinic orbit connecting the rest points $Z_-(\lambda) = 0$ and $Z_+(\lambda) = \lambda_+$, where here λ_+ is given by (2). Recall that for the full problem, numerical evidence [15] suggests that some bores are instead overturning. Such waves would violate the long-wave assumption made in the derivation of (4), and so this discrepancy is to be expected.

2.2. *Monotone fronts solutions to elliptic PDE*

Keeping in mind the above discussion, consider now the following (fully) nonlinear PDE:

$$\begin{cases} A(y,u,\nabla u, D^2u,\lambda) &= 0 \quad \text{in } \Omega, \\ B(y,u,\nabla u,\lambda) &= 0 \quad \text{on } \Gamma_1, \\ u &= 0 \quad \text{on } \Gamma_0, \end{cases} \tag{9}$$

where $\lambda \in \mathbb{R}$ is a parameter, and the domain $\Omega = \mathbb{R}\times\Omega'$ is an infinite cylinder with bounded base $\Omega' \subset \mathbb{R}^{d-1}$. For simplicity, assume that Ω is connected with a $C^{2+\alpha}$ boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$, for a fixed $\alpha \in (0,1)$ and such that $\Gamma_0 \cap \Gamma_1 = \emptyset$. Points in Ω will be denoted (x,y), where $x \in \mathbb{R}$ and $y \in \Omega'$.

We assume that A and B are real analytic in all of their arguments and that (9) is uniformly elliptic with a uniformly oblique boundary condition on Γ_1. Through the Dubreil–Jacotin transform, the internal waves problem (1) can be rewritten roughly in this form with upstream layer depth ratio as the parameter. In fact, the dynamic condition (1c) will lead to a nonlinear transmission problem, but this can be handled through a small modification.

Define a *front* to be a solution (u,λ) of (9) that enjoys the classical regularity $u \in C^{2+\alpha}_{\mathrm{b}}(\overline{\Omega})$ and has distinct point-wise limits as $x \to -\infty$ and $x \to +\infty$; thinking of water waves, we call these the upstream and downstream states, respectively. From the structure of the equation, one can prove that they are in fact x-independent solutions of (9). We call a front *monotone* provided $\partial_x u \le 0$ (or $\partial_x u \ge 0$) in Ω, and *strictly monotone* if $\partial_x u < 0$ (or $\partial_x u > 0$) in $\Omega \cup \Gamma_1$.

Fronts are the PDE analogues of heteroclinic solutions to the ODE (5) with the (unbounded) axial direction identified with the evolution variable. We may then ask: (i) under what conditions does (9) support (monotone) fronts, and (ii) do these fronts persist for non-perturbative parameter values. The first of these questions has been pursued by many authors. The most common approaches include monotonicity methods [5, 40] and center manifold reduction [22, 27], which has been applied to our system (1) in [4, 7, 28].

Our main abstract result addresses the second problem, namely the global continuation of a given curve $\mathscr{C}_{\mathrm{loc}}$ of perturbative strictly monotone fronts. In brief, it gives conditions analogous to those of Proposition 3 under which the local curve $\mathscr{C}_{\mathrm{loc}}$ can be extended to a larger curve $\mathscr{C}$ of strictly monotone fronts. These hypotheses are discussed in the next subsection. In Section 2.4, we give a sharp set of alternatives that characterize the limiting behavior as one follows the resulting global curve to its extreme. The statement of the global bifurcation theorem is found in Section 2.5.

2.3. *Hypotheses*

In what follows, we suppose that there exists a local curve $\mathscr{C}_{\mathrm{loc}}$ of strictly monotone front solutions to (9). To simplify the notation, it is useful to write (9) as the abstract operator equation

$$\mathscr{F}(u,\lambda)=0.$$

One can easily verify that $\mathscr{F}$ is real analytic as a mapping $C_{\mathrm{b}}^{2+\alpha}(\overline{\Omega})\times\mathbb{R}\to C_{\mathrm{b}}^{\alpha}(\overline{\Omega})\times C_{\mathrm{b}}^{1+\alpha}(\Gamma_1)$.

First, we note that the system (9) is invariant under translation in x, and so $\partial_x u$ lies in $\ker\mathscr{F}_u(u,\lambda)$ by an elliptic regularity argument. For simplicity, we assume that along the local curve the kernel is exactly one dimensional:

$$\ker\mathscr{F}_u(u,\lambda)=\operatorname{span}\{\partial_x u\}\quad\text{for all } (u,\lambda)\in\mathscr{C}_{\mathrm{loc}}. \tag{H1}$$

The next hypothesis corresponds to the spectral non-degeneracy condition (8) in the ODE setting. For a monotone front (u,λ), the Fréchet derivative $\mathscr{F}_u(u,\lambda)$ is a linear elliptic operator whose coefficients have well-defined limits as $x\to\pm\infty$. Restricting the domain to x-independent functions, this gives elliptic operators on Ω' that we call the *transversal linearized operators* at $x=+\infty$ and $x=-\infty$. One can show that these will have principal eigenvalues that we will denote by $\sigma_0^{\pm}(u,\lambda)$. Recall from elliptic theory, the principal eigenvalue is real and lies strictly to the right of the rest of the spectrum.

In analogy to the assumption (8) in Proposition 3, we focus on the situation where

$$\sigma_0^-(u,\lambda),\ \sigma_0^+(u,\lambda)<0\quad\text{for all } (u,\lambda)\in\mathscr{C}_{\mathrm{loc}}. \tag{H2}$$

Observe that (H2) is equivalent to the essential spectrum of the limiting linearized operators being properly contained in left complex half-plane $\mathbb{C}_-$.

The final hypothesis is made with an eye towards applications. Usually, one obtains $\mathscr{C}_{\mathrm{loc}}$ through a preliminary local bifurcation argument. A common scenario on unbounded domains is that $\mathscr{C}_{\mathrm{loc}}$ originates from an x-independent solution to (9) that is singular in the sense that the linearized operator there fails to be Fredholm. With that in mind, suppose that $\mathscr{C}_{\mathrm{loc}}$ admits the C^0 parameterization

$$\mathscr{C}_{\mathrm{loc}}=\{(u(\varepsilon),\lambda(\varepsilon)):0<\varepsilon<\varepsilon_0\}\subset\mathscr{F}^{-1}(0),$$

where

$$(u(\varepsilon),\lambda(\varepsilon))\to(u_0,\lambda_0)\quad\text{as }\varepsilon\to 0+,\qquad\text{and}\qquad \sigma_0^+(u_0,\lambda_0)=0\text{ or }\sigma_0^-(u_0,\lambda_0)=0. \tag{H4}$$

We label this condition (H4) rather than (H3) for consistency with [8].

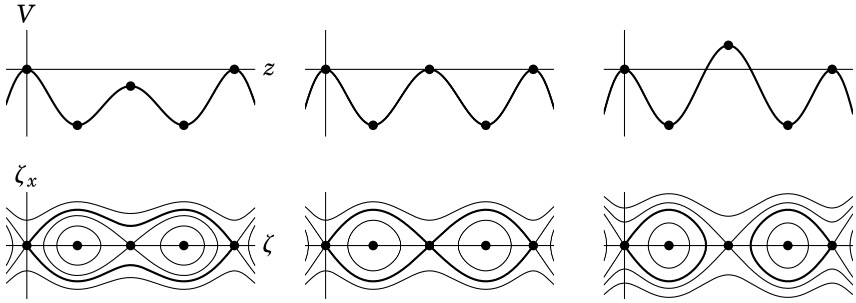

Figure 3. An ODE of the form (5) that experiences a heteroclinic degeneracy (A2).

2.4. *Alternatives*

Taking for granted that $\mathscr{C}_{\mathrm{loc}}$ can be extended, the next question is what we might encounter at the extreme of the resulting global curve. To form intuition for the PDE case, let us consider in tandem the simpler task of continuing the curve $\mathscr{K}_{\mathrm{loc}}$ of heteroclinic solutions to the ODE (5).

An obvious possibility is that the heteroclinic orbits persist for all parameter values (hence λ is unbounded along the curve) or that arbitrarily large fronts exist (that is, ζ is unbounded in norm). This alternative has a straightforward translation to the PDE setting: we say that a sequence of monotone fronts $\{(u_n, \lambda_n)\}$ experiences *blowup* provided that

$$\|u_n\|_{C^{2+\alpha}(\Omega)} + |\lambda_n| \longrightarrow \infty. \tag{A1}$$

Note that in applications, it is often necessary to formulate the theory for (u, λ) lying in an open subset of $C^{2+\alpha}(\overline{\Omega}) \times \mathbb{R}$ rather than the whole space. For example, our analysis of internal waves supposes the absence of horizontal stagnation, which corresponds to a pointwise inequality for a certain derivative of u. In that case, the definition of blowup will include the possibility that the sequence limits to the boundary of this set.

Another alternative for the ODE (5) is that the heteroclinic nondegeneracy condition (7) is violated in the limit. It could then happen that the heteroclinic orbit between the equilibria $Z_-(\lambda)$ and $Z_+(\lambda)$ breaks down and a new heteroclinics is born that connects one of them to an intermediate rest point as in Figure 3. For the PDE (9), the upstream and downstream states play the role of the equilibria in the original heteroclinic orbit, and the intermediate equilibrium would correspond to a distinct x-independent solution.

To formulate this more precisely, observe that by composing with a sequence of translations in the x-direction, we can shift the incipient intermediate state upstream or downstream so that the solutions locally — but not uniformly — converge to a new front. With that in mind, we say that a sequence of strictly monotone fronts $\{(u_n, \lambda_n)\}$ experiences a *heteroclinic degeneracy* if there is a sequence $x_n \to \pm\infty$ so that the three limits

$$\lim_{x\to\mp\infty}\lim_{n\to\infty} u_n(x+x_n,\cdot), \quad \lim_{n\to\infty}\lim_{x\to+\infty} u_n(x,\cdot), \quad \lim_{n\to\infty}\lim_{x\to-\infty} u_n(x,\cdot) \quad \text{exist and are distinct.} \tag{A2}$$

One can further assume that $(u_n(\cdot + x_n, \cdot), \lambda_n)$ converges in $C^2_{\mathrm{loc}}(\overline{\Omega})$ to a monotone front (u_*, λ_*).

Finally, it can happen that as we continue $\mathscr{K}_{\mathrm{loc}}$, the spectral non-degeneracy condition (8) fails. Figure 4 shows how this might occur in a specific example. For $\lambda < \lambda^*$, there is a monotone increasing heteroclinic orbit connecting the constant solutions $Z_- = 0$ and $Z_+ = 1$. This orbit persists for $\lambda = \lambda^*$, but the spectral condition (8) is violated at Z_-. The Jacobian matrix for the corresponding planar system will then cease to be invertible downstream, and the orbit no longer

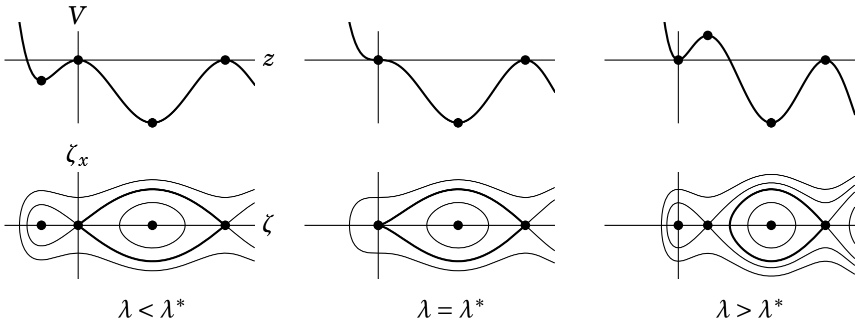

Figure 4. An ODE of the form (5) that experiences a spectral degeneracy (A3).

decays exponentially as $x \to -\infty$. For $\lambda > \lambda^*$, the heteroclinic orbit degenerates into a homoclinic orbit to $\zeta = 1$, while $\zeta = 0$ becomes a center.

The analogous scenario in the PDE setting should naturally involve the spectrum of the linearized problem at infinity. In particular, we say a sequence of strictly monotone fronts $\{(u_n, \lambda_n)\}$ experiences *spectral degeneracy* if

$$\sigma_0^-(u_n, \lambda_n) \to 0 \quad \text{or} \quad \sigma_0^+(u_n, \lambda_n) \to 0. \tag{A3}$$

Recalling (H2), we see that spectral degeneracy indicates resonance: the essential spectrum of the linearized problem upstream or downstream moves through the origin. Were this to occur, $\mathscr{F}$ will lose semi-Fredholmness and its zero-set may not be relatively compact. In connection to traveling waves in reaction-diffusion equations, (A3) corresponds to the onset of "essential instability" [33, 34].

2.5. *Statement of abstract results*

Having developed the necessary intuition, we are now prepared to present the main global bifurcation theorem.

Theorem 4 (Global bifurcation). *Consider the elliptic PDE* (9). *Let $\mathscr{C}_{\mathrm{loc}}$ be a curve of strictly monotone front solutions which bifurcates from a singular point as in* (H4) *and satisfies the kernel* (H1) *and spectral* (H2) *conditions. Then $\mathscr{C}_{\mathrm{loc}}$ is contained in a global C^0 curve*

$$\mathscr{C} := \{(u(s), \lambda(s)) : 0 < s < \infty\} \subset \mathscr{F}^{-1}(0)$$

of strictly monotone front solutions with the properties enumerated below.

(a) (Alternatives) *For any sequence $s_n \to +\infty$, along some subsequence, $(u(s_n), \lambda(s_n))$, the blowup* (A1)*, heteroclinic degeneracy* (A2)*, or spectral degeneracy* (A3) *alternative will occur.*

(b) (Analyticity) *At each point, $\mathscr{C}$ admits a local real-analytic reparameterization.*

(c) *For all s sufficiently large, $(u(s), \lambda(s)) \notin \mathscr{C}_{\mathrm{loc}}$. In particular, $\mathscr{C}$ is not a closed loop.*

It bears repeating that the above theorem applies to a broad class of problems as it makes no structural hypotheses on the system beyond analyticity of (A, B) and ellipticity. There is a substantive body of work on fronts for semi-linear PDEs arising in reaction-diffusion equations (see [40] and the references therein). To the best of our knowledge, however, Theorem 4 is the first systematic treatment that applies even to fully nonlinear problems. For example, in addition to the water wave applications discussed above, the general theory is used in a forthcoming paper to construct large nonlinear elastostatic fronts [9].

Theorem 4 is also distinctive in that it avoids making assumptions on the compactness properties of $\mathscr{F}$ beyond the local curve. Classical global bifurcation theory makes comparatively stringent requirements that are appropriate for elliptic PDEs set on bounded domains but not the present problem. For instance, Buffoni–Toland [6] ask that the zero-set $\mathscr{F}^{-1}(0)$ be locally compact and $\mathscr{F}_u(u,\lambda)$ be Fredholm index 0 for $(u,\lambda) \in \mathscr{F}^{-1}(0)$. The seminal work of Rabinowitz [31] assumes that $\mathscr{F}$ is locally proper and Fredholm index 0 throughout its domain (though it need not be analytic).

The basic philosophy inherent to our approach is that, on unbounded domains, it is more natural to think of the failure of these compactness properties as an *alternative*, and then seek to classify it in terms of qualitative features of the solutions. It is truly remarkable that the simple set of possibilities for the ODE (5), when properly interpreted, exhaustively categorize the limiting behavior for solution curves to the vastly more complicated PDE (9).

Let us conclude by briefly outlining how Theorem 4 is used to construct large-amplitude bores. Local curves $\mathscr{C}_{\mathrm{loc}}$ of small-amplitude monotone front solutions to the internal wave problem (1) were obtained in [4, 7, 25, 28]. In [7] this was done using a novel center manifold reduction method that is particularly well suited to verifying that the hypotheses (H1), (H2), and (H4) hold. Due to its variational structure, (1) possess several conserved quantities: the mass flux, energy, and flow force through any vertical cross-section of the fluid domain must be the same. The upstream and downstream states must therefore be *conjugate* in that the values of these three quantities will agree. For (1), this requirement is so restrictive that, in fact, at every λ, there is a unique downstream state that is conjugate to the fixed upstream state. This insight drastically simplifies the task of computing the spectrum of the transversal linearized operators at $x = \pm\infty$, and indeed, we are able to rule out spectral degeneracy (A3) entirely. It also disqualifies the heteroclinic degeneracy alternative, as the three limiting states in (A2) would be distinct and pairwise conjugate, which is impossible. Thus blowup (A1) occurs as we follow the global bore curve. Through elliptic regularity theory, we obtain uniform a priori bounds that show this must coincide with the stagnation limit (3).

References

[1] B. F. Akers, D. M. Ambrose, J. D. Wright, "Gravity perturbed Crapper waves", *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **470** (2014), no. 2161, article ID 20130526.

[2] D. M. Ambrose, W. A. Strauss, J. D. Wright, "Global bifurcation theory for periodic traveling interfacial gravity-capillary waves", *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* **33** (2016), no. 4, p. 1081-1101.

[3] C. J. Amick, "Bounds for water waves", *Arch. Ration. Mech. Anal.* **99** (1987), no. 2, p. 91-114.

[4] C. J. Amick, R. E. L. Turner, "Small internal waves in two-fluid systems", *Arch. Ration. Mech. Anal.* **108** (1989), no. 2, p. 111-139.

[5] H. Berestycki, L. Nirenberg, "Travelling fronts in cylinders", *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* **9** (1992), no. 5, p. 497-572.

[6] B. Buffoni, J. F. Toland, *Analytic theory of global bifurcation. An introduction*, Princeton Series in Applied Mathematics, Princeton University Press, 2003.

[7] R. M. Chen, S. Walsh, M. H. Wheeler, "Center manifolds without a phase space for quasilinear problems in elasticity, biology, and hydrodynamics", https://arxiv.org/abs/1907.04370, 2019.

[8] ———, "Global bifurcation for monotone fronts of elliptic equations", https://arxiv.org/abs/2005.00651, 2020.

[9] ———, "Global bifurcation of anti-plane shear fronts", Preprint,https://arxiv.org/abs/2008.09453, 2020.

[10] W. Choi, R. Camassa, "Long internal waves of finite amplitude", *Phys. Rev. Lett.* **77** (1996), no. 9, p. 1759-1762.

[11] ———, "Fully nonlinear internal waves in a two-fluid system", *J. Fluid Mech.* **396** (1999), p. 1-36.

[12] A. Constantin, W. Strauss, E. Vărvărucă, "Global bifurcation of steady gravity water waves with critical layers", *Acta Math.* **217** (2016), no. 2, p. 195-262.

[13] D. Córdoba, A. Enciso, N. Grubic, "On the existence of stationary splash singularities for the Euler equations", *Adv. Math.* **288** (2016), p. 922-941,.

[14] G. D. Crapper, "An exact solution for progressive capillary waves of arbitrary amplitude", *J. Fluid Mech.* **2** (1957), p. 532-540.

[15] F. Dias, J.-M. Vanden-Broeck, "On internal fronts", *J. Fluid Mech.* **479** (2003), p. 145-154.

[16] S. A. Dyachenko, V. M. Hur, "Stokes waves with constant vorticity: folds, gaps and fluid bubbles", *J. Fluid Mech.* **878** (2019), p. 502-521.

[17] ———, "Stokes waves with constant vorticity: I. Numerical computation", *Stud. Appl. Math.* **142** (2019), no. 2, p. 162-189.

[18] K. R. Helfrich, W. K. Melville, "Long nonlinear internal waves", in *Annual review of fluid mechanics*, Annual review of fluid mechanics, vol. 38, Annual Reviews, 2006, p. 395-425.

[19] V. M. Hur, J.-M. Vanden-Broeck, "A new application of Crapper's exact solution to waves in constant vorticity flows", *Eur. J. Mech. B Fluids* **83** (2020), p. 190-194.

[20] V. M. Hur, M. H. Wheeler, "Exact free surfaces in constant vorticity flows", *J. Fluid Mech.* **896** (2020), article ID R1.

[21] T. von Kármán, "The engineer grapples with non-linear problems", *Bull. Am. Math. Soc.* **46** (1940), p. 615-683.

[22] K. Kirchgässner, "Wave-solutions of reversible systems and applications", *J. Differ. Equations* **45** (1982), no. 1, p. 113-127.

[23] O. Laget, F. Dias, "Numerical computation of capillary-gravity interfacial solitary waves", *J. Fluid Mech.* **349** (1997), p. 221-251.

[24] H. Lamb, *Hydrodynamics*, 6th ed., Cambridge Mathematical Library, vol. 6, Cambridge University Press, 1993, with a foreword by Russel E. Caflisch.

[25] N. I. Makarenko, "Smooth bore in a two-layer fluid", in *Free boundary problems in continuum mechanics. International conference on free boundary problems in continuum mechanics, Novosibirsk, Russia, July 15-19, 1991*, ISNM. International Series of Numerical Mathematics, vol. 106, Birkhäuser, 1992, p. 195-204.

[26] D. V. Maklakov, "A note on the existence of pure gravity waves at the interface of two fluids", *Physica D* **401** (2020), no. 5, article ID 132157.

[27] A. Mielke, "Reduction of quasilinear elliptic equations in cylindrical domains with applications", *Math. Methods Appl. Sci.* **10** (1988), no. 1, p. 51-66.

[28] ———, "Homoclinic and heteroclinic solutions in two-phase flow", in *Proceedings of the IUTAM/ISIMM Symposium on Structure and Dynamics of Nonlinear Waves in Fluids (Hannover, 1994)*, Advanced Series in Nonlinear Dynamics, vol. 7, World Scientific, 1995, p. 353-362.

[29] M. Miyata, "Long internal waves of large amplitude", in *Nonlinear water waves*, International Union of Theoretical and Applied Mechanics, Springer, 1988, p. 399-406.

[30] D. I. Pullin, R. H. J. Grimshaw, "Finite-amplitude solitary waves at the interface between two homogeneous fluids", *Phys. Fluids* **31** (1988), no. 12, p. 3550-3559.

[31] P. H. Rabinowitz, "Some global results for nonlinear eigenvalue problems", *J. Funct. Anal.* **7** (1971), p. 487-513.

[32] L. Rayleigh, "On the theory of long waves and bores", *Proc. Roy. Soc. Lond., Ser. A* **90** (1914), no. 619, p. 324-328.

[33] B. Sandstede, A. Scheel, "Essential instability of pulses and bifurcations to modulated travelling waves", *Proc. R. Soc. Edinb., Sect. A, Math.* **129** (1999), no. 6, p. 1263-1290.

[34] ———, "Essential instabilities of fronts: bifurcation, and bifurcation failure", *Dyn. Syst.* **16** (2001), no. 1, p. 1-28.

[35] E. R. Spielvogel, "A variational principle for waves of infinite depth", *Arch. Ration. Mech. Anal.* **39** (1970), p. 189-205.

[36] G. G. Stokes, *On the theory of oscillatory waves*, Mathematical and Physical Papers, vol. 1, Cambridge University Press, 1880, 197-229 pages.

[37] S.-M. Sun, "Existence of large amplitude periodic waves in two-fluid flows of infinite depth", *SIAM J. Math. Anal.* **32** (2001), no. 5, p. 1014-1031,.

[38] J. F. Toland, "Stokes waves", *Topol. Methods Nonlinear Anal.* **7** (1996), no. 1, p. 1-48.

[39] R. E. L. Turner, J.-M. Vanden-Broeck, "Broadening of interfacial solitary waves", *Phys. Fluids* **31** (1988), no. 9, p. 2486-2490.

[40] A. I. Volpert, V. A. Volpert, V. A. Volpert, *Traveling wave solutions of parabolic systems*, Translations of Mathematical Monographs, vol. 140, American Mathematical Society, 1994, Translated from the Russian manuscript by James F. Heyda.

[41] M. H. Wheeler, "Solitary water waves of large amplitude generated by surface pressure", *Arch. Ration. Mech. Anal.* **218** (2015), no. 2, p. 1131-1187.