



INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus

Mathématique

Vladimir Pestov

An amenability-like property of finite energy path and loop groups

Volume 358, issue 11-12 (2020), p. 1139-1155

Published online: 25 January 2021

<https://doi.org/10.5802/crmath.134>

This article is licensed under the
CREATIVE COMMONS ATTRIBUTION 4.0 INTERNATIONAL LICENSE.
<http://creativecommons.org/licenses/by/4.0/>



Les Comptes Rendus. Mathématique sont membres du
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 1778-3569



Group Theory / Théorie des groupes

An amenability-like property of finite energy path and loop groups

Une propriété semblable à la moyennabilité des groupes de chemins et de lacets à énergie finie

Vladimir Pestov^{a, b}

^a Departamento de Matemática, Universidade Federal de Santa Catarina, Campus Universitário Trindade, CEP 88.040-900 Florianópolis-SC, Brasil

^b Département de mathématiques et de statistique, Université d'Ottawa, Complexe STEM, 150 Louis-Pasteur Pvt, Ottawa, Ontario K1N 6N5 Canada

E-mail: vladimir.pestov@uottawa.ca

Abstract. We show that the groups of finite energy loops and paths (that is, those of Sobolev class H^1) with values in a compact connected Lie group, as well as their central extensions, satisfy an amenability-like property: they admit a left-invariant mean on the space of bounded functions uniformly continuous with regard to a left-invariant metric. Every strongly continuous unitary representation π of such a group (which we call skew-amenable) has a conjugation-invariant state on $B(\mathcal{H}_\pi)$.

Résumé. Nous montrons que les groupes de lacets et de chemins à énergie finie (c.à.d. de classe H^1 de Sobolev) à valeurs dans un groupe de Lie compact et connexe, ainsi que leurs extensions centrales, satisfont une version de la moyennabilité: ils admettent une moyenne invariante à gauche sur l'espace de fonctions bornées uniformément continues par rapport à une métrique invariante à gauche. Chaque représentation unitaire continue, π , d'un tel groupe (que nous disons d'être "moyennable en biais") possède un état sur $B(\mathcal{H}_\pi)$ invariant sous conjugaison.

Funding. CNPq (bolsa Pesquisador Visitante, processo 310012/2016) and CAPES (bolsa Professor Visitante Estrangeiro Sénior, processo 88881.117018/2016-01).

Manuscript received 12th August 2020, revised and accepted 20th October 2020.

Version française abrégée

Motivation et résultats

On s'intéresse à la question suivante de Carey et Grundling ([5, p. 114]). Soit X une variété riemannienne compacte. Les groupes $C^k(X, SU(n))$, $C^\infty(X, SU(n))$, et $H^k(X, SU(n))$, munis des

topologies correspondantes, sont-ils moyennables? Cette question, à son tour, est orientée vers le problème de l'existence d'un état de plus basse énergie d'une théorie de jauge.

On rappelle qu'un groupe topologique G (pas forcément localement compact) est dit moyennable (au sens de Pierre de la Harpe [14]) si toute action continue de G sur un espace compact quelconque possède une mesure de probabilité borelienne régulière et invariante. Cela équivaut à l'existence d'une moyenne invariante à gauche sur l'espace $\text{RUCB}(G)$ des fonctions bornées uniformément continues à droite. Ici, nous suivons Bourbaki ([3, Ch. III, p. 20]) en disant qu'une fonction $f: G \rightarrow \mathbb{R}$ est uniformément continue à droite (respectivement, à gauche) si pour chaque $\varepsilon > 0$ il existe un voisinage V de l'unité de G tel que, quels que soient $x, y \in G$ avec $xy^{-1} \in V$ (resp., $x^{-1}y \in V$), on a $|f(x) - f(y)| < \varepsilon$.

À notre connaissance, la question de Carey et Grundling reste ouverte pour toutes les valeurs de $k \geq 0$ (on exige $k > \dim X/2$ pour que les groupes de Sobolev soient bien définis). La seule exception est le cas $k = 0$ dans la dimension un, c'est-à-dire, le cas des groupes de chemins et de lacets continus, dont la moyennabilité est une conséquence immédiate des résultats de Marie-Paule Malliavin et Paul Malliavin [16] (voir ci-après la section 2).

Dans cette Note, nous nous occupons des groupes de chemins et lacets d'une classe de régularité légèrement plus élevée, à savoir, de la classe de Sobolev H^1 (aussi connus comme des chemins et lacets à énergie finie). Bien que nous ne sommes pas en mesure de montrer la moyennabilité des groupes en question, nous établissons une variation de cette propriété étroitement liée et, de toute évidence, bien pertinante.

Disons qu'un groupe topologique G est *moyennable en biais*¹ s'il existe une moyenne invariante à gauche sur les fonctions bornées uniformément continues à gauche. Théorème 2.2.1 dans [12] entraîne que pour les groupes localement compacts, la moyennabilité en biais et la moyennabilité sont deux notions équivalentes. Ceci est inexact sans l'hypothèse de la compacité locale: le groupe unitaire $U(\ell^2)$ muni de la topologie forte est moyennable [14], mais pas moyennable en biais ([19, Ex. 3.6.3]). En revanche, nous ne savons pas si tout groupe moyennable en biais est moyennable.

Voici le résultat principal de la Note.

Théorème 1. *Soit K un groupe de Lie compact et connexe. Les groupes de chemins et de lacets à énergie finie à valeurs dans K , ainsi que leurs variantes basées et extensions centrales, sont moyennables en biais.*

Tout groupe topologique G moyennable en biais possède la propriété suivante. Soit π une représentation unitaire et fortement continue de G dans un espace hilbertien \mathcal{H}_π . Alors, l'algèbre d'opérateurs $B(\mathcal{H}_\pi)$ admet un état ϕ invariant par conjugaison: quels que soient $T \in B(\mathcal{H}_\pi)$ et $g \in G$, on a $\phi(\pi_g^* T \pi_g) = \phi(T)$. En d'autres mots, la représentation π est moyennable au sens de Bekka [2]. Pour la preuve, voir [11, Prop. 4.5]. On en déduit:

Corollaire 2. *Chaque représentation unitaire et continue, π , d'un groupe mentionné dans le théorème 1 possède un état invariant ϕ sur l'algèbre $B(\mathcal{H}_\pi)$.*

À notre avis, il serait profitable d'enquêter sur la moyennabilité en biais des groupes vus dans la question de Carey et Grundling au début de la Note.

Esquisse de la preuve

Soit K un groupe de Lie compact réel d'algèbre de Lie \mathfrak{k} . On réalise K et \mathfrak{k} par des matrices, et munit $M_n(\mathbb{R})$ de la norme de Hilbert-Schmidt, notée $\|\cdot\|_2$. Toute application $f: [0, 1] \rightarrow K$

¹Cette suggestion terminologique, visant l'expression anglaise *skew-amenable group*, appartient à Colas Bardavid.

absolument continue possède la *dérivée logarithmique à droite* définie presque partout par $\partial^{\log} f(t) = f'(t) \cdot f(t)^{-1}$. Si f vérifie la condition $\int_0^1 \|\partial^{\log} f(t)\|^2 dt < \infty$, on dit que f est un chemin à énergie finie. Cela est équivalent à la condition que f soit de classe de Sobolev H^1 . L'ensemble de tous les chemins à énergie finie à valeurs dans K , muni de la topologie H^1 , forme un groupe topologique par rapport à la multiplication ponctuelle (en effet, un groupe de Lie banachique, voir [1, §1.8]), noté $H^1(\mathbb{I}, K)$. (Ici et dans le reste de l'article, \mathbb{I} denote l'intervalle $[0, 1]$). De même, pour le groupe de lacets, $H^1(\mathbb{S}^1, K)$.

La restriction de la dérivée logarithmique à droite sur le sous-groupe $H_0^1(\mathbb{I}, K)$ des chemins basés (c.à.d., ceux-ci vérifiant $f(0) = e$) établit un homéomorphisme entre le groupe en question et l'espace hilbertien $L^2(\mathbb{I}, \mathfrak{k})$, l'application réciproque étant donnée par l'intégrale multiplicative [7],

$$t \mapsto \prod_0^t \exp f(s) ds. \quad (1)$$

La dérivée logarithmique étant un cocycle (voir l'équation (3)), elle établit un isomorphisme entre le groupe topologique $H_0^1(\mathbb{I}, K)$ et l'espace $L^2(\mathbb{I}, \mathfrak{k})$ muni de la loi multiplicative

$$f * g = f + \text{Ad}_{\Pi \exp f} g.$$

Ici, $\Pi \exp f$ désigne la fonction dans l'équation (1), et Ad est la représentation adjointe, appliquée ponctuellement (pour les groupes matriciels, il s'agit tout simplement de la conjugaison). On note, en particulier, que la structure uniforme additive de l'espace hilbertien $L^2(\mathbb{I}, \mathfrak{k})$ est la structure uniforme à gauche du groupe topologique $(L^2(\mathbb{I}, \mathfrak{k}), *)$, parce que elle est invariante par translations et rotations (c.à.d., opérateurs orthogonaux). Par conséquent, dans le but de montrer la moyennabilité en biais du groupe $H_0^1(\mathbb{I}, K)$ (où, de même, de sa copie isomorphe $(L^2(\mathbb{I}, \mathfrak{k}), *)$), il suffit de construire une moyenne invariante sur l'espace des fonctions bornées et uniformément continues sur $L^2(\mathbb{I}, \mathfrak{k})$ invariante par toutes les translations, ainsi que par les rotations sous la forme spéciale Ad_f , où $f \in H_0^1(\mathbb{I}, K)$.

Pour chaque entier positif, N , notons V_N le sous-espace euclidien de $L^2(\mathbb{I}, \mathfrak{k})$ de dimension finie $N \cdot \dim \mathfrak{k}$ formé par des fonctions simples prenant une valeur constante sur chaque interval $[i/N, (i+1)/N]$. Notons B_N la boule unité fermée dans l'espace V_N . Soit (R_N) une suite des réels positifs convergeant vers l'infini. Notons ν_N la restriction de la mesure de Lebesgue sur la boule $R_N B_N$, normalisée pour devenir une mesure de probabilité. (L'on peut aussi bien utiliser une mesure gaussienne centrée de variance R_N^2). Fixons un ultrafiltre libre \mathcal{U} sur \mathbb{N} , et définissons, pour chaque fonction F bornée et uniformément continue sur $L^2(\mathbb{I}, \mathfrak{k})$,

$$\phi(F) = \lim_{N \rightarrow \mathcal{U}} \int_{V_N} F(x) d\nu_N(x).$$

La fonctionnelle ϕ est linéaire, positive, et de norme 1. Elle est invariante par translations dès que $R_N = \omega(N^{1/2})$. Sous cette hypothèse, un raisonnement géométrique simple montre que la valeur de la mesure de la différence symétrique $B_N \Delta(f_N + B_N)$ converge vers zéro quand $N \rightarrow \infty$ pour toute suite (f_N) bornée dans $L^2(\mathbb{I}, \mathfrak{g})$ et telle que $f_N \in V_N$. Maintenant on use la densité de sous-espace vectoriel $\cup_N V_N$ dans $L^2(\mathbb{I}, \mathfrak{k})$.

La preuve de l'invariance par les transformations Ad_r , où $r \in H_0^1(\mathbb{I}, K)$, est légèrement plus compliquée. Elle a lieu sous l'hypothèse $R_N = o(N/\log N)$. Étant donné une fonction r ci-dessus, on l'approxime avec une fonction $\rho: \mathbb{I} \rightarrow K$ simple, prenant des valeurs constantes sur les intervalles $[i/N, (i+1)/N]$. La transformation ρ conserve la mesure ν_N , d'où il suffit d'estimer la norme de $(\text{Ad}_r - \text{Ad}_\rho)f$ pour un élément quelconque f de $R_N B_N$.

En utilisant la continuité 1/2-hölderienne de r , on obtient

$$\|\text{Ad}_r f - \text{Ad}_\rho f\|_2 \leq CN^{-1/2} \left\| \partial^{\log} r \right\| \max_{i=0}^{N-1} (N^{-1/2} \|f_i\|_2),$$

où $f_i \in \mathfrak{k}$ est la valeur constante de f sur l'intervalle i -ième. À l'aide de l'inégalité de concentration de Paul Lévy pour la boule euclidienne ([15, Prop. 2.9, p. 30]), on montre que la mesure ν_N de l'ensemble de toutes les fonctions f ayant la propriété $\max_{i=0}^{N-1} (N^{-1/2} \|f_i\|_2) < R_N N^{-1/2} \log N$ converge vers 1 quand $N \rightarrow \infty$. D'où on déduit que

$$\int_{L^2} [F(\text{Ad}_r f) - F(f)] d\nu_N(f) \rightarrow 0$$

pour toute fonction F bornée et uniformément continue sur $L^2(\mathbb{I}, \mathfrak{k})$. On vient donc d'établir que le groupe de chemins basés $H_0^1(\mathbb{I}, K)$ est moyennable en biais.

Pour transférer le résultat au groupe de lacets basés $H_0^1(\mathbb{S}^1, K)$, nous montrons que tout sous-groupe normal et co-compact d'un groupe moyennable en biais est moyennable en biais. Notons que en général, la moyennabilité en biais n'est pas transmise au sous-groupes topologiques fermés, comme le témoigne l'exemple dans [4].

Le résultat pour les groupes de chemins et de lacets en est déduit à l'aide du théorème suivant: le produit semi-direct d'un groupe compact agissant sur un groupe moyennable en biais est moyennable en biais. Le résultat est donc plus faible et l'argument, plus délicat que pour les groupes moyennables, dû au fait que l'action du groupe à gauche sur les fonctions uniformément continues à gauche est généralement discontinue. Finallement, un argument beaucoup plus simple établit le résultat pour les extensions centrales. Effectivement, nous ne savons pas si la classe des groupes moyennables en biais est fermée par les extensions les plus générales.

1. Motivation and statements of results

We are interested in the following open question.

Question 1 (Carey and Grundling [5, p. 114]). *Let X be a compact Riemannian manifold. Are the groups $C^k(X, SU(n))$, $C^\infty(X, SU(n))$, and $H^k(X, SU(n))$ with their natural group topologies amenable?*

Amenability of a topological group G is meant in the sense of de la Harpe [14]: every continuous action of G on a compact space admits an invariant regular Borel probability measure. Equivalently, the space $\text{RUCB}(G)$ of bounded right uniformly continuous functions on G admits a left-invariant mean. The right uniform continuity of a function f is meant in the sense of Bourbaki ([3, Ch. III, p. 20]): $|f(x) - f(vx)| < \varepsilon$ for every $\varepsilon > 0$ when v belongs to a sufficiently small neighbourhood of identity, V . In other words, the map sending each $g \in G$ to the function ${}^g f$, ${}^g f(x) = f(g^{-1}x)$, is continuous in the supremum norm.

The question was advertised by its authors since at least 2000, as a possible tool to prove the existence of invariant vacuum states in gauge field theories. More generally, it can be asked about the group of vertical automorphisms of a smooth principal K -bundle for any compact connected Lie group K . It can be shown that the C^∞ case is equivalent to the amenability of the C^k class groups for all $k \geq 0$, or of the class H^k groups for all $k > \dim X/2$, where X is the base. To our knowledge, the problem remains open for all values of k .

An exception is the case of continuous paths and loops (that is, $k = 0$ and $X = [0, 1]$ or \mathbb{S}^1). Here a positive answer follows from the work of Marie-Paule Malliavin e Paul Malliavin [16]. (See Section 2 below.)

In this Note we study groups of finite energy paths and loops (those of Sobolev class $H^1 = W_2^1$, strictly intermediate between C^0 and C^1). Here is the main result.

Theorem 2. *Let K be a compact connected Lie group. The topological groups of finite energy paths $H^1(\mathbb{I}, K)$, loops $H^1(\mathbb{S}^1, K)$, based paths $H_0^1(\mathbb{I}, K)$, and based loops $H_0^1(\mathbb{S}^1, K)$, as well as their central extensions, admit a left-invariant mean on the space of left uniformly continuous bounded functions.*

A function f is *left uniformly continuous* if for every $\varepsilon > 0$, we have $|f(x) - f(xv)| < \varepsilon$ when $v \in V$ and V is a sufficiently small neighbourhood of identity. In the case of a metrizable group, it is the same as uniform continuity with regard to a compatible left-invariant metric.

We adopt a suggestion by Martin Schneider to call a topological group G admitting a left-invariant mean on the space $\text{LUCB}(G)$ *skew-amenable*. We still do not know whether the groups of H^1 paths and loops are amenable. However, it appears that the skew-amenability of a topological group allows to construct invariant states on algebras of operators with more ease than amenability.

Given a strongly continuous unitary representation π of G in a Hilbert space \mathcal{H} , a state ϕ on the algebra $B(\mathcal{H})$ (that is, a positive linear functional with $\phi(1) = 1$) is invariant if $\phi(\pi_{g^*} T \pi_g) = \phi(T)$ for every bounded operator T and each $g \in G$. A unitary representation admitting an invariant state is called *amenable* in the sense of Bekka [2].

Proposition 3 (Giordano and Pestov [11, Prop. 4.5]). *Every strongly continuous unitary representation of a skew-amenable topological group G is amenable in the sense of Bekka.*

For a locally compact group G skew-amenability is equivalent to amenability: since the left regular representation of G is amenable by the above, the group is amenable by Theorem 2.2 in [2]. (For a stronger result, see [12, Thm. 2.2.1].) At the same time, the unitary group $U(\ell^2)$ with the strong operator topology is amenable [14] but not skew-amenable, by Proposition 3 above and the proof of Proposition 2 in [13] or Example 3.6.3 in [19]. We do not know if every skew-amenable group is amenable.

Corollary 4. *Let K be a compact connected Lie group. The groups of finite energy paths and loops with values in K , as well as their central extensions, admit an invariant state for every strongly continuous unitary representation.*

Note that in a toy example discussed in [5, §4], the group of gauge transformations ($C^0(\mathbb{I}, SU(n))$ with the relative weak topology, where $\mathbb{I} = [0, 1]$) is not only amenable, but skew-amenable as well, because it is a SIN group (the right and left uniform structures coincide). We suggest the following.

Question 5. *Let K be a compact Lie group, and P a principal smooth K -fibre bundle. Are the groups of vertical automorphisms of P of classes C^k , H^k , $k \geq 0$, and C^∞ with the corresponding topologies skew-amenable?*

Acknowledgements

The results were announced at the Second Brazilian Workshop in Geometry of Banach Spaces, Ubatuba, SP, Brazil, August 2018. The author is thankful to the Workshop Organizers for an invitation to the extremely pleasant event, and to F. Martin Schneider for stimulating conversations.

2. Continuous paths and loops: result of Malliavin and Malliavin

According to [16], on the group of C^0 -loops taking values in a compact connected Lie group K , integration with regard to the nonlinear Wiener measure w_t of parameter t is asymptotically left-invariant in the limit $t \rightarrow \infty$ under the action of the subgroup of C^1 -loops. In other words, if f is a bounded Borel function on the group $C^0(\mathbb{S}^1, K)$, then for every C^1 -loop g ,

$$\int f(g^{-1}x) dw_t(x) - \int f(x) dw_t(x) \rightarrow 0 \text{ as } t \rightarrow \infty.$$

(Apply Theorem 1.1 from [16], with $p = 2$, and the Cauchy inequality.) Let now \mathcal{U} be an ultrafilter on \mathbb{R} containing every interval $[t, +\infty)$. It follows that the ultralimit

$$\phi(f) = \lim_{t \rightarrow \mathcal{U}} \int f(x) d\omega_t(x)$$

is a C^1 -left invariant mean on the space of bounded Borel functions on the group of C^0 -loops. The definition of right uniform continuity easily implies that the restriction of ϕ to the bounded right uniformly continuous functions is already invariant under left translations by all C^0 -loops. So the group $C^0(\mathbb{S}^1, K)$ is amenable.

A similar argument holds for the path group, but is unnecessary, because the group $C^0(\mathbb{I}, K)$ is isomorphic to a topological quotient group of the loop group, by restricting the domain of each loop to a fixed copy of $\mathbb{I} \subseteq \mathbb{S}^1$. We have:

Corollary 6. *The groups of continuous paths and loops with values in a compact connected Lie group, with the C^0 -topology, are amenable.*

3. Finite energy paths and loops

Let G be a Lie group, and $t \in \mathbb{I}$ a point of smoothness of a function $f: \mathbb{I} \rightarrow G$. The *right logarithmic derivative* of f at t is an element of the Lie algebra \mathfrak{g} of G :

$$\partial^{\log} f(t) = f'(t) \cdot f(t)^{-1}. \quad (2)$$

Here we use a simplified notation for the image of $f'(t) \in T_{f(t)}G$ in $T_e(G) = \mathfrak{g}$ under the right translation by $f(t)^{-1}$. If G is a matrix group, it becomes a genuine product of matrices.

If t is a common point of smoothness for two maps $f, g: \mathbb{I} \rightarrow G$, a direct calculation verifies the cocycle condition:

$$\partial^{\log}(fg)(t) = \partial^{\log} f(t) + \text{Ad}_{f(t)} \partial^{\log} g(t). \quad (3)$$

The formula holds in a very general context of infinite-dimensional Lie groups G , see [17, p. 250]. For matrix groups, the adjoint representation is the conjugation: $\text{Ad}_g y = gyg^{-1}$.

An absolutely continuous mapping $f: \mathbb{I} \rightarrow G$ has the right logarithmic derivative defined almost everywhere. Such a mapping has *finite energy* if, for some (or any) norm on the Lie algebra \mathfrak{g} ,

$$\int_0^1 \left\| \partial^{\log} f(t) \right\|_{\mathfrak{g}}^2 dt < \infty.$$

Now assume $G = K$ to be a compact connected Lie group, realized as a Lie subgroup of $SO(n)$ inside of $M_n(\mathbb{R})$ with the Hilbert–Schmidt norm (which we denote $\|\cdot\|_2$). The Lie algebra \mathfrak{k} of K is realized as a Lie subalgebra of $\mathfrak{o}(n)$ inside $M_n(\mathbb{R})$. Since the right hand side in the formula (2) becomes a matrix multiplication, and the matrix $f(t)^{-1}$ is orthogonal, it preserves the norm. It follows that for K compact, the finite energy paths are just the mappings $f: \mathbb{I} \rightarrow K$ of Sobolev class $H^1 = W_2^{(1)}$.

The space $H^1(\mathbb{I}, M_n(\mathbb{R})) \cong H^1[0, 1] \otimes M_n(\mathbb{R}) \cong M_n(H^1[0, 1])$ with the Sobolev norm

$$\|f\|_{H^1} = \left(\|f\|_2^2 + \|f'\|_2^2 \right)^{1/2} \quad (4)$$

forms a unital Banach algebra under the pointwise multiplication (because so does $H^1[0, 1]$ [18, Cor. 9.7, p. 30]). Consequently, the collection $H^1(\mathbb{I}, K)$ forms a topological subgroup of the group of invertible elements $H^1(\mathbb{I}, M_n(\mathbb{R}))^\times$. (It is even a Banach–Lie group, see a more general result in [1, §1.8], but we will not use this.) In the same way, finite energy loops form a topological group, $H^1(\mathbb{S}^1, K)$, isomorphic to a closed cocompact subgroup of $H^1(\mathbb{I}, K)$.

On the subgroup $H_0^1(\mathbb{I}, K)$ of based paths ($f(0) = e_G$), the Maurer–Cartan cocycle map $\partial^{\log}: H_0^1(\mathbb{I}, K) \rightarrow L^2(\mathbb{I}, \mathfrak{k})$ is clearly injective. It is also surjective, with the inverse given by the product integral [7, 17]. If f is a step function taking constant values in \mathfrak{k} on each interval $[t_i, t_{i+1}]$, $i = 0, 1, \dots, n-1$, and if $t \in [t_j, t_{j+1}]$, the product integral of f between 0 and t is defined by

$$\prod_0^t \exp f(s) ds = \exp_K(t - t_j) f(t_j) \exp_K(t_j - t_{j-1}) f(t_{j-1}) \dots \exp_K t_1 f(0).$$

Afterwards, the product integral is extended over all L^1 -functions by continuity. In particular, it is well defined on the L^2 -functions. The function $\mathbb{I} \ni t \mapsto \prod_0^t \exp f(s) ds \in K$ is absolutely continuous and satisfies

$$\partial_t^{\log} \prod_0^t \exp f(s) ds = f(t), \quad t\text{-a.e.}$$

(See [7, Thm. 1.2, and §1.8, p. 55, (8.6)].)

Notice that for $f \in H_0^1(\mathbb{I}, K)$, $\|f'\|_2 = \|\partial^{\log} f\|_2$. It follows that on the group $H_0^1(\mathbb{I}, K)$, the metric induced by the Sobolev norm in Eq. (4) is equivalent to the metric induced from the space $L^2(\mathbb{I}, \mathfrak{k})$ through the cocycle map:

$$d(f, g) = \|\partial^{\log} f - \partial^{\log} g\|_2.$$

This metric d is left invariant, because the L^2 -norm is invariant under translations and orthogonal transformations.

One can now identify $H_0^1(\mathbb{I}, K)$ as a topological group with the space $L^2(\mathbb{I}, \mathfrak{k})$ equipped with the multiplication operation

$$f * g = f + \text{Ad}_{\Pi \exp f} g. \quad (5)$$

Here $\Pi \exp f$ is the short for the function $t \mapsto \prod_0^t \exp f(s) ds$, and the adjoint representation Ad is applied pointwise. In our case of K being a matrix group, $\text{Ad}_g x = gxg^{-1}$.

Under the topological group isomorphism

$$\partial^{\log}: H_0^1(\mathbb{I}, K) \rightarrow (L^2(\mathbb{I}, \mathfrak{k}), *), \quad (6)$$

the additive uniform structure on $L^2(\mathbb{I}, \mathfrak{k})$ corresponds to the left uniform structure on $H_0^1(\mathbb{I}, K)$. It follows that the group $H_0^1(\mathbb{I}, K)$ is separable and Weil-complete, that is, complete in the left (or, equivalently, right) uniformity. In particular, it is a Polish group. Since $H^1(\mathbb{I}, K)$ is an extension of the based path group by a compact subgroup, and the group $H^1(\mathbb{I}, K)$ contains $H^1(\mathbb{S}^1, K)$ and $H_0^1(\mathbb{S}^1, K)$ as closed subgroups, they are separable and Weil-complete too. At the same time, these are not SIN groups.

Proposition 7. *Let K be a non-abelian connected compact Lie group. On each of the groups $H_0^1(\mathbb{I}, K)$, $H^1(\mathbb{I}, K)$, $H_0^1(\mathbb{S}^1, K)$ and $H^1(\mathbb{S}^1, K)$ the left and the right uniform structures are different.*

Proof. Enough to prove the result for the former group (which is obtained from every other group by taking subgroups and/or quotients). Fix any element $y \in \mathfrak{k} \setminus Z(\mathfrak{k})$. Identify $H_0^1(\mathbb{I}, K)$ as a topological group with $(L^2(\mathbb{I}, \mathfrak{k}), *)$ as in Eqs. (5)–(6). Let π be the orthogonal projection of $L^2(\mathbb{I}, \mathfrak{k})$ onto the one-dimensional subspace spanned by the constant function y . This π is (left) uniformly continuous. Given $\varepsilon > 0$, there is $f \in L^2(\mathbb{I}, \mathfrak{k})$ with $\|f\|_2 < \varepsilon$ and such that $\text{Ad}_{\prod_0^t \exp f(s) ds} y \neq \pm y$ for almost all t . Since the adjoint action is by orthogonal transformations, it follows that $\|\pi(\text{Ad}_{\prod_0^t \exp f(s) ds} y - y)\|_2 > 0$. Now for a sufficiently large $R > 0$,

$$\|\pi(f * Ry) - Ry\|_2 \geq R \|\pi(\text{Ad}_{\prod_0^t \exp f(s) ds} y - y)\|_2 - \varepsilon$$

will be as large as desired. So, π is not right uniformly continuous. \square

4. Skew-amenableability of the group of based paths

Lemma 8. Let s_n be a sequence of positive reals with $s_n = o(n^{-1/2})$. Let B_n denote the unit ball in the n -dimensional Euclidean space, and λ_n the Lebesgue measure. Then $\lambda_n((B_n + s_n e_1) \setminus B_n) / \lambda_n(B_n) \rightarrow 0$ as $n \rightarrow \infty$.

Proof. For $i = n, n-1$ denote $\tilde{\lambda}_i$ the Lebesgue measure on the ball B_i normalized to one. Since $s_n^2 = o(1/n)$, it is possible to choose a sequence of positive reals α_n with

$$\alpha_n = o(1/n) \cap \omega(s_n^2).$$

The normalized $(n-1)$ -volume of the ball $(1 - \alpha_n)B_{n-1}$ equals $(1 - \alpha_n)^{n-1} = 1 - o(1)$. For every $x \in e_1^\perp$ with $\|x\| \leq 1 - \alpha_n$, the height $h_n(x)$ of the section of the ball B_n in the direction e_1 over the point x satisfies $h_n(x)^2 + (1 - \alpha_n)^2 = 1$, thus $h_n(x)^2 = 2\alpha_n - \alpha_n^2$ and $h_n(x) = \Omega(\alpha_n^{1/2})$. In particular, $h_n(x) = \omega(s_n)$, uniformly in x . Using the Fubini theorem,

$$\begin{aligned} \frac{\lambda_n((B_n + s_n e_1) \setminus B_n)}{\lambda_n(B_n)} &= \tilde{\lambda}_n((B_n + s_n e_1) \setminus B_n) \\ &\leq \int_{B_{n-1} \setminus (1 - \alpha_n)B_{n-1}} d\tilde{\lambda}_{n-1}(x) + \int_{(1 - \alpha_n)B_{n-1}} \frac{s_n}{h_n(x)} d\tilde{\lambda}_{n-1}(x) \\ &= o(1) + (1 - o(1))o(1) \\ &= o(1). \end{aligned}$$

□

We will freely use the topological group isomorphism in Eq. (6). For every N , denote P_N the uniform partition of the unit interval into subintervals of length $1/N$. Let V_N be the space of all functions in $L^2(\mathbb{I}, \mathfrak{k})$ constant a.e. on elements of P_N . The union

$$V_\infty = \bigcup_{N=1}^{\infty} V_N$$

forms a vector space which is dense in $L^2(\mathbb{I}, \mathfrak{k})$.

Let $R_N \uparrow \infty$ be a sequence of positive real numbers. Denote B_N the unit ball in the Nd -dimensional Euclidean space V_N , where $d = \dim \mathfrak{k}$. Let v_N be the Lebesgue measure on the ball $R_N B_N$ normalized to one.

Let \mathcal{U} be a non-principal ultrafilter on the natural numbers. Define a mean ϕ on the space of all bounded (left) uniformly continuous functions F on $L^2(\mathbb{I}, \mathfrak{k})$ by

$$\phi(F) = \lim_{N \rightarrow \mathcal{U}} \int F(x) dv_N(x).$$

We will show that, under suitable assumptions on the rate of growth of R_N , the mean ϕ is left-invariant with respect to the group operation $*$ from Eq. (5). It is enough to show separately that for every $g \in L^2(\mathbb{I}, \mathfrak{k})$ we have $\int (F(x+g) - F(x)) dv_N(x) \rightarrow 0$, and for every $r \in H_0^1(\mathbb{I}, K)$, $\int (F(\text{Ad}_r x) - F(x)) dv_N(x) \rightarrow 0$, as $N \rightarrow \infty$.

Lemma 9. Suppose

$$R_N = \omega(N^{1/2}).$$

Let F be a bounded Borel measurable function on $L^2(\mathbb{I}, \mathfrak{k})$, and let (g_N) be a norm-bounded sequence, where $g_N \in V_N$ for each N . Then

$$\int [F(g_N + f) - F(f)] dv_N(f) \rightarrow 0,$$

and the rate of convergence only depends on $\sup_N \|g_N\|$ and $\|F\|_\infty$.

Proof. If every element of the sequence $R_N B_N$ is seen as a Følner set in the additive group of the finite-dimensional Euclidean space V_N , then the Følner constants only depend on N as long as the elements $g_N \in V_N$ have norm $\leq M$. In other words, the measure of the symmetric difference

between the unit ball B_N and its shift by a vector of length $\leq M$ in whatever direction converges to zero as $N \rightarrow \infty$. This is assured by Lemma 8:

$$\begin{aligned} \int_{L^2} [F(g_N + f) - F(f)] d\nu_N(f) &= \int_{R_N B_N} F(g_N + f) d\nu_N(f) - \int_{R_N B_N} F(f) d\nu_N(f) \\ &= \int_{(R_N B_N + g_N) \cap R_N B_N} F(f) d\nu_N(f) + \int_{(R_N B_N + g_N) \cap R_N B_N} F(f) d\nu_N(f) \\ &\quad - \int_{(R_N B_N + g_N) \cap R_N B_N} F(f) d\nu_N(f) - \int_{R_N B_N \setminus (R_N B_N + g_N)} F(f) d\nu_N(f) \\ &\leq 2 \|F\|_\infty \nu_N(R_N B_N \setminus (g_N + R_N B_N)) \\ &= 2 \|F\|_\infty \tilde{\lambda}_n(B_N \setminus (\|g_N\| R_N^{-1} e_1 + B_N)) \\ &\leq 2 \|F\|_\infty \tilde{\lambda}_n(B_N \setminus (M R_N^{-1} e_1 + B_N)) \\ &\rightarrow 0, \end{aligned}$$

because $M R_N^{-1} = o(N^{-1/2})$ by assumption. \square

Recall that for an element $f \in L^2(\mathbb{I}, \mathfrak{k})$ and a function $r: [0, 1] \rightarrow K$, we define $\text{Ad}_r f$ pointwise a.e., that is, $(\text{Ad}_r f)(x) = \text{Ad}_{r(x)} f(x)$. Since we realize \mathfrak{k} and K inside of $M_n(\mathbb{R})$, the adjoint representation is just the conjugation: $\text{Ad}_g(x) = g x g^{-1}$. Denoting the uniform operator norm on matrices $\|\cdot\|_u$, and the Hilbert–Schmidt norm by $\|\cdot\|_2$, we have, for two unitary operators u, v and a matrix $A \in M_n$,

$$\begin{aligned} \|\text{Ad}_u A - \text{Ad}_v A\|_2 &= \|u A u^{-1} - v A v^{-1}\|_2 \\ &\leq \|u A u^{-1} - u A v^{-1}\|_2 + \|u A v^{-1} - v A v^{-1}\|_2 \\ &\leq 2 \|u - v\|_u \|A\|_2. \end{aligned}$$

Lemma 10. *Suppose*

$$R_N = o(N/\log N).$$

Then for every H^1 -function $r: [0, 1] \rightarrow K \subseteq M_n$ and every bounded uniformly continuous function F on $L^2((0, 1), \mathfrak{k})$,

$$\int_{L^2} [F(\text{Ad}_r f) - F(f)] d\nu_N(f) \rightarrow 0 \text{ as } N \rightarrow \infty,$$

and the convergence is uniform in r over every set bounded in the norm $\|\partial^{\log} r\|_2$, and also over any uniformly equicontinuous and uniformly bounded family of functions F .

Proof. Every H^1 -function $r: [0, 1] \rightarrow K \subseteq M_n$ is Hölder continuous with exponent $1/2$, or, more exactly, for $0 \leq s \leq t \leq 1$,

$$\begin{aligned} \|r(s) - r(t)\|_u &\leq C \left(\int_s^t \|r'(x)\|_2^2 dx \right)^{1/2} |s - t|^{1/2}, \\ &= C \left(\int_s^t \|r'(x) r(x)^{-1}\|_2^2 dx \right)^{1/2} |s - t|^{1/2}. \end{aligned}$$

For every $i = 0, 1, \dots, N - 1$, denote

$$c_i = \left(\int_{i/N}^{(i+1)/N} \|r'(x)\|_2^2 dx \right)^{1/2}.$$

Thus, $\sum_{i=0}^{N-1} c_i^2 = \|\partial^{\log} r\|^2$. Given $t \in [i/N, (i+1)/N]$,

$$\|r(i/N) - r(t)\|_u \leq C c_i N^{-1/2}.$$

Approximate r with a step function ρ taking the value $r(i/N)$ on the i -th interval of the partition. For t as above,

$$\|r(t) - \rho(t)\|_u \leq C c_i N^{-1/2}.$$

Let $f \in B_N$, the unit ball in the Euclidean space V_N . Denote f_i the constant value taken by f on the interval $[i/N, (i+1)/N]$. We have:

$$\begin{aligned} \|\text{Ad}_r f - \text{Ad}_\rho f\|_2^2 &= \int_0^1 \|(\text{Ad}_{r(x)} - \text{Ad}_{\rho(x)})f(x)\|_2^2 dx \\ &\leq \sum_{i=0}^{N-1} \int_{i/N}^{(i+1)/N} 4 \|r(x) - \rho(x)\|_u^2 \|f_i\|_2^2 dx \\ &\leq 4 \sum_{i=0}^{N-1} \int_{i/N}^{(i+1)/N} C^2 c_i^2 N^{-1} \|f_i\|_2^2 dx \\ &= C' N^{-1} \sum_{i=0}^{N-1} c_i^2 \int_{i/N}^{(i+1)/N} \|f_i\|_2^2 dx \\ &= C' N^{-1} \sum_{i=0}^{N-1} c_i^2 N^{-1} \|f_i\|_2^2 \\ &\leq C' N^{-1} \left\| \partial^{\log} r \right\|^2 \max_{i=0}^{N-1} (N^{-1} \|f_i\|_2^2). \end{aligned}$$

Thus,

$$\|\text{Ad}_r f - \text{Ad}_\rho f\|_2 \leq C'' N^{-1/2} \left\| \partial^{\log} r \right\| \max_{i=0}^{N-1} (N^{-1/2} \|f_i\|_2). \quad (7)$$

For every i , the function

$$B_N \ni f \mapsto N^{-1/2} f_i \in \mathfrak{k} \subseteq M_n$$

is 1-Lipschitz. Fix an orthonormal basis e_1, \dots, e_d in \mathfrak{k} . The function $B_N \ni f \mapsto \langle N^{-1/2} f_i, e_j \rangle \in \mathbb{R}$ is 1-Lipschitz, with zero as the median value. The ball B_N has dimension Nd . By the Lévy concentration inequality for the Euclidean ball ([15, Prop. 2.9, p. 30]), for every $\varepsilon > 0$

$$\tilde{\lambda}_{dN} \{f \in B_N : |\langle N^{-1/2} f_i, e_j \rangle| > \varepsilon\} < 2 \exp(-c\varepsilon^2 Nd),$$

where $\tilde{\lambda}_{dN}$ is the Lebesgue measure on B_N normalized to one. Consequently,

$$\begin{aligned} \tilde{\lambda}_{dN} \{f \in B_N : N^{-1/2} \|f_i\|_2 > \varepsilon\} &\leq \tilde{\lambda}_{dN} \{x \in B_N : \exists j = 1, \dots, d, |\langle N^{-1/2} f_i, e_j \rangle| > \varepsilon\} \\ &< 2d \exp(-c\varepsilon^2 Nd). \end{aligned}$$

Substituting $\varepsilon = \log N / \sqrt{N}$,

$$\tilde{\lambda}_{dN} \{x \in B_N : N^{-1/2} \|f_i\|_2 > \log N \cdot N^{-1/2}\} < 2d \exp(-cd \log^2 N),$$

and

$$\begin{aligned} \tilde{\lambda}_{dN} \left\{ x \in B_N : \max_{i=0}^{N-1} (N^{-1/2} \|f_i\|_2) > \log N \cdot N^{-1/2} \right\} &< 2Nd \exp(-cd \log^2 N) \\ &= 2Nd N^{-cd \log N} \\ &= 2d N^{1-cd \log N} \\ &= N^{-\omega(1)}. \end{aligned}$$

Denote

$$A_N = \{f \in B_N : \forall i, N^{-1/2} \|f_i\| \leq \log N \cdot N^{-1/2}\}$$

and $A_N^c = B_N \setminus A_N$.

Let F be a uniformly continuous bounded function on $L^2(\mathbb{I}, \mathfrak{g})$, and denote

$$\varepsilon_F(\delta) = \sup \{|F(f) - F(h)| : \|f - h\|_2 < \delta\}.$$

We have $\varepsilon_F(\delta) \rightarrow 0$ when $\delta \rightarrow 0$.

Since Ad_ρ is an orthogonal operator, it leaves invariant the balls B_N and $R_N B_N$ and the corresponding measures $\tilde{\lambda}_{dN}$ and ν_N . We have

$$\begin{aligned} & \int_{L^2} [F(\text{Ad}_r f) - F(f)] d\nu_N(f) \\ &= \int_{R_N B_N} [F(\text{Ad}_r f) - F(\text{Ad}_\rho f)] d\nu_N(f) + \int_{R_N B_N} [F(\text{Ad}_\rho f) - F(f)] d\nu_N(f), \end{aligned}$$

where the second integral vanishes. Let us estimate the first integral separately over $R_N A_N$ and $R_N A_N^c$. For the latter,

$$\begin{aligned} \left| \int_{R_N A_N^c} [F(\text{Ad}_r f) - F(f)] d\nu_N(f) \right| &\leq \int_{R_N A_N^c} |F(\text{Ad}_r f) - F(f)| d\nu_N(f) \\ &\leq 2 \|F\|_\infty \nu_N(R_N A_N^c) \\ &= 2 \|F\|_\infty \tilde{\lambda}_{dN}(A_N^c) \\ &< 4d \|F\|_\infty N^{1-cd\log N} \\ &\rightarrow 0, \end{aligned}$$

and the convergence is obviously uniform in r . For $R_N A_N$, we have, using Eq. (7),

$$\begin{aligned} \sup_{f \in R_N A_N} \|\text{Ad}_r f - \text{Ad}_\rho f\|_2 &\leq \sup_{f \in R_N A_N} N^{-1/2} \left\| \partial^{\log r} \right\|_2 \max_{i=0}^{N-1} (N^{-1/2} \|f_i\|_2) \\ &\leq N^{-1/2} \left\| \partial^{\log r} \right\|_2 \frac{\log N}{N^{1/2}} R_N \\ &= \left\| \partial^{\log r} \right\|_2 \frac{\log N}{N} R_N \\ &\rightarrow 0, \end{aligned}$$

because of the hypothesis $R_N = o(N/\log N)$. Now,

$$\begin{aligned} \left| \int_{R_N A_N} [F(\text{Ad}_r f) - F(\text{Ad}_\rho f)] d\nu_N(f) \right| &\leq \int_{R_N A_N} |F(\text{Ad}_r f) - F(\text{Ad}_\rho f)| d\nu_N(f) \\ &\leq \varepsilon_F \left(\sup_{f \in R_N A_N} \|\text{Ad}_r f - \text{Ad}_\rho f\| \right) \\ &\leq \varepsilon_F \left(\left\| \partial^{\log r} \right\|_2 \frac{\log N}{N} R_N \right) \\ &\rightarrow 0. \end{aligned}$$

The convergence is uniform in r on any set where the values of $\|\partial^{\log r}\|_2$ are bounded. It is also uniform for any uniformly bounded family of functions F having a common modulus of uniform continuity. \square

Lemma 11. Suppose $R_N = o(N/\log N) \cap \omega(N^{1/2})$. Let D be a compact subset of the group $H_0^1(\mathbb{I}, K)$. For every bounded left uniformly continuous function F on the group we have, uniformly in $g \in D$,

$$\int [F(g * f) - F(f)] d\nu_N(f) \rightarrow 0. \quad (8)$$

The convergence is uniform over any uniformly equicontinuous and uniformly bounded family of functions F .

Proof. Denote $M = \max_{f \in D} \|f\| + 1$. Given $\varepsilon > 0$, find $\delta > 0$ such that whenever $\|f - h\|_2 < \delta$, $|F(f) - F(h)| < \varepsilon$. Now find N so large that, for each $n \geq N$,

- D is contained in the δ -neighbourhood of V_n ,

- $\int [F(g + f) - F(f)] dv_n(f) < \varepsilon$ whenever $g' \in V_N$ and $\|g'\| \leq M$,
- for each $g \in D$, $\int [F(\text{Ad}_{\Pi \exp g} f) - F(f)] dv_n(f) < \varepsilon$.

Given $g \in D$ and $n \geq N$, find $g' \in V_n$ with $\|g - g'\| < \delta$. Now

$$\begin{aligned} \left| \int [F(g * f) - F(f)] dv_n \right| &= \left| \int [F(g + \text{Ad}_{\Pi \exp g} f) - F(f)] dv_n \right| \\ &\leq \left| \int [F(g + \text{Ad}_{\Pi \exp g} f) - F(g' + \text{Ad}_{\Pi \exp g} f)] dv_n \right| \\ &\quad + \left| \int [F(g' + \text{Ad}_{\Pi \exp g} f) - F(\text{Ad}_{\Pi \exp g} f)] dv_n \right| \\ &\quad + \left| \int [F(\text{Ad}_{\Pi \exp g} f) - F(f)] dv_n \right| \\ &< 3\varepsilon. \end{aligned}$$

□

As an immediate consequence, we obtain the skew-amenability for the group $H_0^1(\mathbb{I}, K)$ of based paths.

5. Co-compact normal subgroups

In this Section we will establish the skew-amenability of the group of based loops, as a direct consequence of the following result.

Theorem 12. *A normal co-compact subgroup of a skew-amenable metrizable group is skew-amenable.*

(The result is valid without an assumption of metrizability, but the proof is more involved.)

For a topological group G , denote $\mathcal{L}(G)$ the Gelfand space of the C^* -algebra $\text{LUCB}(G)$. This is the maximal (Samuel) compactification ([20, Ch. III]) of the left uniform space of G . The left action of G on the algebra, $f \mapsto {}^g f$, is by C^* -algebra automorphisms, and so it determines an action of G on the left by homeomorphisms on the space $\mathcal{L}(G)$, although this action will in general be discontinuous. By the Riesz representation theorem, G is skew-amenable if and only if $\mathcal{L}(G)$ supports a left-invariant regular Borel probability measure.

It is instructive to compare $\mathcal{L}(G)$ with the Samuel compactification of the right uniform space of G , known as the *greatest ambit*, $\mathcal{S}(G)$ (see e.g. [6, 19]). In this case, the left action of G on $\mathcal{S}(G)$ is continuous, and G is amenable if and only if $\mathcal{S}(G)$ admits a left-invariant regular Borel probability measure. It is easy to see, using the inversion map $g \mapsto g^{-1}$, that the *right* action on the “skew ambit” $\mathcal{L}(G)$ is continuous, and G is amenable if and only if $\mathcal{L}(G)$ admits a *right* invariant probability measure. Equivalently, G is skew-amenable if the greatest ambit admits a *right* invariant probability measure.

Now let H be a normal subgroup of a metrizable topological group G , such that the quotient group $K = G/H$ is compact. Fix a bounded left-invariant compatible metric d on G , and denote $\pi: G \rightarrow G/H$ the quotient homomorphism. It extends to a continuous G -equivariant map $\bar{\pi}: \mathcal{L}(G) \rightarrow G/H$. The formula $\bar{d}(xH, yH) = \inf\{d(xh_1, yh_2) : h_1, h_2 \in H\}$ defines a left-invariant compatible metric on G/H (this is where we are using the normality of H). Let μ be a left-invariant regular Borel probability measure on $\mathcal{L}(G)$. For $\delta > 0$, cover K with finitely many open δ -balls. They are all left translations of each other. The same is true of their inverse images under $\bar{\pi}$, all of which consequently have the same strictly positive measure. The open set $V_\delta = \bar{\pi}^{-1} B_\delta^\bar{d}(e)$ satisfies $HV_\delta = V_\delta$, so is invariant under the left translations by elements of H . Let μ_δ be the restriction of the measure μ to V_δ , normalized to one. The measure μ_δ is a regular Borel probability measure on $\mathcal{L}(G)$, invariant under left translations by elements of H .

Every bounded uniformly continuous function on a metric space is uniformly approximated with Lipschitz functions (see e.g. [10, Cor. 1]). Therefore, it is enough to define a left-invariant mean on the bounded Lipschitz functions on (H, d) , or, essentially the same, a left-invariant positive affine functional on the bounded 1-Lipschitz functions sending 0 to 0 and 1 to 1. Given a bounded 1-Lipschitz function $F: H \rightarrow \mathbb{R}$, extend it to a bounded 1-Lipschitz function $\tilde{F}: G \rightarrow \mathbb{R}$ in a usual way:

$$\tilde{F}(g) = \inf\{F(h) + d(g, h) : h \in H\}.$$

Further, uniquely extend \tilde{F} to a continuous function $\bar{F}: \mathcal{L}(G) \rightarrow \mathbb{R}$. Now choose an ultrafilter \mathcal{U} on \mathbb{R} containing all the intervals $(0, \delta)$, $\delta > 0$, and set

$$\phi(F) = \lim_{\delta \rightarrow \mathcal{U}} \int_{\mathcal{L}(G)} \bar{F}(x) d\mu_\delta(x).$$

Left-invariance of ϕ . For every $k \in H$ and $g \in G$,

$$\begin{aligned} {}^k \tilde{F}(g) &= \tilde{F}(k^{-1}g) \\ &= \inf\{F(h) + d(k^{-1}g, h) : h \in H\} \\ &= \inf\{F(k^{-1}h) + d(k^{-1}g, k^{-1}h) : h \in H\} \\ &= \inf\{{}^k F(h) + d(g, h) : h \in H\} \\ &= \widetilde{kF}(g). \end{aligned}$$

By continuity, one must have ${}^k \bar{F} = \widetilde{kF}$, and since the measures μ_δ are left-invariant, we conclude.

Other properties of ϕ . Given $x \in V_\delta \cap G$, there is $h \in H$ with $d(h, x) < \delta$. It means that the value at x of every function \tilde{F} , where F is 1-Lipschitz on H , is δ -approximated by the value of F at h . Consequently, for every constant function c on H , $\sup_{x \in V_\delta} |\bar{c}(x) - c| \leq \delta$, and since $\text{supp } \mu_\delta \subseteq V_\delta$,

$$\int |\bar{c}(x) - c| d\mu_\delta(x) \leq \delta.$$

For every positive 1-Lipschitz function $F \geq 0$, we deduce that $\tilde{F} \geq -\delta$ on V_δ , so

$$\int \bar{F}(x) d\mu_\delta(x) \geq -\delta.$$

For any 1-Lipschitz functions F, G and $t \in [0, 1]$, we have

$$\sup_{x \in V_\delta} |tF + (1-t)G(x) - t\tilde{F}(x) - (1-t)\tilde{G}(x)| \leq 3\delta,$$

and

$$\int |tF + (1-t)G(x) - t\tilde{F}(x) - (1-t)\tilde{G}(x)| d\mu_\delta(x) \leq 3\delta.$$

In the ultralimit, we obtain the desired properties of ϕ . □

6. Semidirect products with compact groups

The groups of paths $H^1(\mathbb{I}, K)$ and loops $H^1(\mathbb{S}^1, K)$ decompose into semidirect products of the normal subgroups, $H_0^1(\mathbb{I}, K)$ and $H_0^1(\mathbb{S}^1, K)$ respectively, with the compact subgroup of constant paths (= loops), isomorphic to K . Therefore, to establish their skew-amenability, it is enough to prove to following result.

Theorem 13. *The semidirect product of a compact group and a skew-amenable group is skew-amenable.*

Recall that a topological group G is an *inner* semidirect product of a subgroup K and a normal subgroup H if every element of G is uniquely written as a product of an element of K and an element of H , and the resulting map $K \times H \rightarrow G$ is a homeomorphism. In this case, for $x, y \in K$ and $g, h \in H$,

$$xgyh = xy \cdot y^{-1}gy \cdot h,$$

and the mapping $\varsigma: K \rightarrow \text{Aut } H$, $\varsigma_y g = y^{-1}gy$, is an antihomomorphism, that is, $\varsigma_{yz}g = z^{-1}y^{-1}gyz = \varsigma_z\varsigma_y g$. The group operation becomes

$$(x, g)(y, h) = (xy, \varsigma_y g \cdot h).$$

We will denote this realization of outer semidirect product $K \ltimes_\varsigma^1 H$.

Note however that the group G also admits a unique decomposition as $G = H \cdot K$: indeed, $g = xh = xhx^{-1} \cdot x$, and the mapping $(h, x) \mapsto hx = x \cdot x^{-1}hx \in G$ is also a homeomorphism. In this case,

$$gxhy = g \cdot xhx^{-1} \cdot xy,$$

and the map of K to $\text{Aut } H$ defined by $\tau_x h = xhx^{-1}$ is a homomorphism. The group operation on the product $K \times H$ becomes

$$(x, g)(y, h) = (xy, g\tau_x h).$$

We will denote this realization of semidirect product $K \ltimes_\tau^2 H$. Here is an isomorphism between the two:

$$K \ltimes_\tau^1 H \ni (x, h) = xh \mapsto xhx^{-1} \cdot x = (x, \tau_x h) \in K \ltimes_\tau^2 H. \quad (9)$$

Lemma 14. *Under our assumptions, the compactification $\mathcal{L}(K \ltimes_\tau^2 H)$ can be canonically identified with the product $K \times \mathcal{L}(G)$: the unique continuous map $\mathcal{L}(K \ltimes_\tau^2 H) \rightarrow K \times \mathcal{L}(G)$ extending the identity map on $K \ltimes_\tau^2 H$ is a homeomorphism.*

Proof. Let $f: K \ltimes^2 H \rightarrow \mathbb{R}$ be a bounded uniformly continuous function. For every $x \in K$, the function

$$f_x: G \ni g \mapsto f(x, g) \in \mathbb{R}$$

is bounded left uniformly continuous on G , and so extends uniquely to a continuous function $\bar{f}_x: \mathcal{L}(G) \rightarrow \mathbb{R}$. Coalescing the functions \bar{f}_x , $x \in K$, we get a bounded function

$$\bar{f}: K \times \mathcal{L}(G) \ni (x, g) \mapsto \bar{f}_x(g) \in \mathbb{R},$$

so it only remains to show it is continuous.

Let $(x_0, \xi_0) \in K \times \mathcal{L}(G)$, and let $\varepsilon > 0$. Since the right action by K on $\mathcal{L}(G)$ is continuous, there is a neighbourhood of identity, V , in K , with the property: for all $x \in V$ and all $(y, h) \in K \ltimes^2 H$,

$$f(yx, h) \stackrel{\varepsilon}{\approx} f(y, h)$$

(here $(y, h)(x, e) = (yx, h)$). Let $\zeta \in \mathcal{L}(G)$ be arbitrary. There exists a net of elements of G with $g_\alpha \rightarrow \zeta$. For every $x \in K$, $\bar{f}(x, g_\alpha) = \bar{f}_x(g_\alpha) \rightarrow \bar{f}_x(\zeta) = \bar{f}(x, \zeta)$. We conclude: for all $x \in V$ and all $(y, \zeta) \in K \ltimes \mathcal{L}(G)$,

$$\bar{f}(yx, \zeta) \stackrel{\varepsilon}{\approx} \bar{f}(y, \zeta). \quad (10)$$

Now define an open neighbourhood of ξ_0 in $\mathcal{L}(G)$ by

$$U = \left(\overline{f}_{x_0} \right)^{-1} (f(x_0, \xi_0) - \varepsilon, f(x_0, \xi_0) + \varepsilon). \quad (11)$$

The set $x_0 V \times U$ is an open neighbourhood of (x_0, ξ_0) in $K \times \mathcal{L}(G)$. Let $(x, \xi) \in x_0 V \times U$. For some $v \in V$, we have $x = x_0 v$, and

$$\begin{aligned} |\bar{f}(x, \xi) - \bar{f}(x_0, \xi_0)| &\leq |\bar{f}(x_0 v, \xi) - \bar{f}(x_0, \xi)| + |\bar{f}(x_0, \xi) - \bar{f}(x_0, \xi_0)| \\ &< 2\varepsilon, \end{aligned}$$

by Eqs. (10) and (11). □

Observe that the compactness of K in the lemma above cannot be dropped: for an infinite discrete group G , the “skew ambit” $\mathcal{L}(G)$ is just the Stone–Čech compactification, βG , and it is well known that $\beta(\mathbb{N} \times \mathbb{N}) \neq \beta\mathbb{N} \times \beta\mathbb{N}$ ([8, Exercice 3.6.D(b)]). In fact, the Stone–Čech compactification of the product of a compact group K with a discrete group D need not be equal to the product of K with βD either ([8, Exercises 3.6.D(a) and 3.2.H(b)]).

Every topological group automorphism ς_x , $x \in K$ of G extends to a C^* -algebra automorphism of $\text{LUCB}(G)$, and further to a homeomorphism of $\mathcal{L}(G)$. We will denote it with the same symbol, ς_x .

Lemma 15. *Under our assumptions, the compactification $\mathcal{L}(K \times_{\tau}^1 G)$ can be canonically identified with the product $K \times \mathcal{L}(G)$, with the action of the group $K \times_{\tau}^1 G$ given by*

$$(x, g)(y, \xi) = (xy, \varsigma_y g \cdot \xi), \quad (12)$$

for all $x, y \in K$, $g \in G$, and $\xi \in \mathcal{L}(G)$.

Proof. The group isomorphism in Eq. (9) preserves the left uniform structure, sends the subgroup K to itself, and preserves every fiber $\{x\} \times G$, from where the first statement follows.

To prove the formula, choose a net g_α of elements of G converging to ξ . For fixed $x \in K$ and $g \in G$, the left translations of $\mathcal{L}(K \times G) \cong K \times \mathcal{L}(G)$ by elements of the group $K \times G$ are continuous, and so are the transformations τ_x and the multiplication on the left on $\mathcal{L}(G)$, therefore

$$\begin{aligned} (x, g)(y, \xi) &= (x, g)(y, \lim_\alpha g_\alpha) \\ &= (x, g) \lim_\alpha (y, g_\alpha) \\ &= \lim_\alpha (x, g)(y, g_\alpha) \\ &= \lim_\alpha (xy, \varsigma_y g \cdot g_\alpha) \\ &= (xy, \lim_\alpha \varsigma_y g \cdot g_\alpha) \\ &= (xy, \varsigma_y g \lim_\alpha g_\alpha) \\ &= (xy, \varsigma_y g \xi). \end{aligned}$$

To get the equalities on lines 2 and 6, we have implicitly used the topological identification $\mathcal{L}(K \times_{\tau}^1 G) \cong K \times \mathcal{L}(G)$. \square

Now we are ready to prove Theorem 13. Fix a left-invariant regular Borel probability measure μ on $\mathcal{L}(G)$, and let ν be the normalized Haar measure on K . We claim that the product probability measure $\nu \otimes \mu$ on $K \times_{\varsigma}^1 \mathcal{L}(G)$ is left-invariant. Indeed, for every continuous function f on $K \times \mathcal{L}(G)$ and each $(\kappa', g) \in K \times_{\varsigma}^1 G$, we have, using the Fubini theorem ([9, Thm. 252B]):

$$\begin{aligned} \int^{(\kappa', g)} f(\kappa, s) d(\nu \otimes \mu)(\kappa, s) &= \int_K d\nu(\kappa) \int_{\mathcal{L}(G)} f((\kappa', g)^{-1} \cdot (\kappa, s)) d\mu(s) \\ &= \int_K d\nu(\kappa) \int_{\mathcal{L}(G)} f((\kappa'^{(-1)}, \tau_{\kappa'^{(-1)}} g^{-1}) \cdot (\kappa, s)) d\mu(s) \\ &= \int_K d\nu(\kappa) \int_{\mathcal{L}(G)} f(\kappa'^{(-1)} \kappa, \tau_{\kappa'^{(-1)} \kappa} g^{-1} \cdot s) d\mu(s) \\ &= \int_K d\nu(\kappa) \int_{\mathcal{L}(G)} f(\kappa'^{(-1)} \kappa, s) d\mu(s) \\ &= \int_K d\nu(\kappa) \left[\kappa' \mapsto \int_{\mathcal{L}(G)} f(\kappa, s) d\mu(s) \right] \\ &= \int_K d\nu(\kappa) \int_{\mathcal{L}(G)} f(\kappa, s) d\mu(s) \\ &= \int f(\kappa, s) d(\nu \otimes \mu)(\kappa, s). \end{aligned} \quad \square$$

7. Central extensions

Proposition 16. *A central extension of a skew-amenable topological group is skew-amenable.*

Proof. Let $e \rightarrow Z \rightarrow G \rightarrow G/Z \rightarrow e$, where Z is central and G/Z is skew-amenable. Select an invariant mean ϕ_Z on $\ell^\infty(Z)$ and a left-invariant mean, $\phi_{G/Z}$, on $\text{LUCB}(G/Z)$. Given a bounded left-uniformly continuous function f on G , define

$$\phi_G(f) = \phi_{G/Z} [gZ \mapsto \phi_Z [z \mapsto f(zg)|_Z]].$$

The intermediate function $gZ \mapsto \phi_Z [z \mapsto f(zg)|_Z]$ is well-defined (that is, constant on the Z -cosets), because for each $z' \in Z$,

$$\begin{aligned} \phi_Z [z \mapsto f(zgz')|_Z] &= \phi_Z [z \mapsto f(z'zg)|_Z] \\ &= \phi_Z [z \mapsto z'^{-1}f(zg)|_Z] \\ &= \phi_Z [z \mapsto f(zg)|_Z]. \end{aligned}$$

The intermediate function is also obviously bounded, and left uniformly continuous: if V is such that $g^{-1}h \in V$ implies $f(g) \approx^\varepsilon f(h)$, then for all $z \in Z$, also $f(zg) \approx^\varepsilon f(zh)$, and $\phi_Z [z \mapsto f(zg)] \approx^\varepsilon \phi_Z [z \mapsto f(zh)]$. Thus, the value $\phi_G(f)$ above is well-defined, and is clearly a mean on $\text{LUCB}(G)$. Remains to verify the left-invariance:

$$\begin{aligned} \phi_G({}^h f) &= \phi_{G/Z} [gZ \mapsto \phi_Z [z \mapsto {}^h f(zg)|_Z]] \\ &= \phi_{G/Z} [gZ \mapsto \phi_Z [z \mapsto f(h^{-1}zg)|_Z]] \\ &= \phi_{G/Z} [gZ \mapsto \phi_Z [z \mapsto f(zh^{-1}g)|_Z]] \\ &= \phi_{G/Z} ({}^{hZ} [gZ \mapsto \phi_Z [z \mapsto f(zg)|_Z]]) \\ &= \phi_{G/Z} [gZ \mapsto \phi_Z [z \mapsto f(zg)|_Z]] \\ &= \phi_G(f). \end{aligned}$$

□

The argument is of course standard, but uses the centrality of Z in an essential way. We do not know if Proposition 16 and Theorem 13 can be generalized to show that skew-amenability is closed under extensions.

In the non-locally compact case, skew-amenability appears to be more fragile than amenability, because of discontinuity of the left action and the resulting measurability problems. It could certainly help, to have a general criterion of skew-amenability as powerful as the criterion of amenability for all topological groups established recently by Schneider and Thom ([21, Thm. 3.2]). According to a private communication from Martin Schneider, such a result is possible and will be published soon.

References

- [1] S. Albeverio, R. Høegh-Krohn, J. A. Marion, D. H. Testard, B. S. Torrésani, *Noncommutative distributions. Unitary representation of gauge groups and algebras*, Pure and Applied Mathematics, vol. 175, Marcel Dekker, 1993.
- [2] M. E. B. Bekka, “Amenable unitary representations of locally compact groups”, *Invent. Math.* **100** (1990), no. 2, p. 383–401.
- [3] N. Bourbaki, *Topologie Générale I*, Hermann, 1971.
- [4] A. Carderi, A. Thom, “An exotic group as limit of finite special linear groups”, *Ann. Inst. Fourier* **68** (2018), no. 1, p. 257–273.
- [5] A. Carey, H. Grundling, “On the problem of the amenability of the gauge group”, *Lett. Math. Phys.* **68** (2004), no. 2, p. 113–120.
- [6] J. De Vries, *Elements of topological dynamics*, Mathematics and its Applications, vol. 257, Kluwer Academic Publishers, 1993.

- [7] J. D. Dollard, C. N. Friedman, *Product integration with applications to differential equations*, Encyclopedia of Mathematics and its Applications, vol. 10, Addison-Wesley Publishing Group, 1979.
- [8] R. Engelking, *General Topology*, Monografie Matematyczne, vol. 60, PWN - Polish Scientific Publishers, 1977.
- [9] D. H. Fremlin, *Measure theory. Vol. 2. Broad foundations*, Torres Fremlin, 2003, corrected second printing of the 2001 original.
- [10] M. I. Garrido, J. A. Jaramillo, “Lipschitz-type functions on metric spaces”, *J. Math. Anal. Appl.* **340** (2008), no. 1, p. 282-290.
- [11] T. Giordano, V. Pestov, “Some extremely amenable groups related to operator algebras and ergodic theory”, *J. Inst. Math. Jussieu* **6** (2007), no. 2, p. 279-315.
- [12] F. P. Greenleaf, *Invariant means on topological groups and their applications*, Van Nostrand Mathematical Studies, vol. 16, Van Nostrand Reinhold Co., 1969.
- [13] P. de la Harpe, “Moyennabilité de quelques groupes topologiques de dimension infinie”, *C. R. Math. Acad. Sci. Paris* **277** (1973), p. 1037-1040.
- [14] ———, “Moyennabilité du groupe unitaire et propriété P de Schwartz des algèbres de von Neumann”, in *Algèbres d’opérateurs (Sém., Les Plains-sur-Bex, 1978)*, Lecture Notes in Mathematics, vol. 725, Springer, 1979, p. 220-227.
- [15] M. Ledoux, *The concentration of measure phenomenon*, Mathematical Surveys and Monographs, vol. 89, American Mathematical Society, 2001.
- [16] M.-P. Malliavin, P. Malliavin, “Integration on loop group III. Asymptotic Peter-Weyl orthogonality”, *J. Funct. Anal.* **108** (1992), p. 13-46.
- [17] H. Omori, *Infinite-dimensional Lie groups*, Translations of Mathematical Monographs, vol. 158, American Mathematical Society, 1997.
- [18] R. S. Palais, *Foundations of global non-linear analysis*, Mathematics Lecture Note Series, W.A. Benjamin, Inc., 1968.
- [19] V. Pestov, *Dynamics of Infinite-Dimensional Groups: The Ramsey–Dvoretzky–Milman Phenomenon*, University Lecture Series, vol. 40, American Mathematical Society, 2006.
- [20] P. Samuel, “Ultrafilters and compactification of uniform spaces”, *Trans. Am. Math. Soc.* **64** (1948), p. 100-132.
- [21] F. M. Schneider, A. Thom, “On Følner sets in topological groups”, *Compos. Math.* **154** (2018), no. 7, p. 1333-1361.