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Game theory and modeling / *Théorie des jeux et modélisation*

# Game-Theoretic and Inhibition-Based models for crowd motion

*Principes de théorie des jeux et comportement  
d'inhibition dans les modèles de foules*

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**Abstract.** We propose a new microscopic crowd motion model based on Game-Theoretic principles, from which we derive an Inhibition-Based model for evacuation situations. Each individual is supposed to have a desired velocity that they adapt to the behavior of neighbors that influence them. Possible adapted velocities are defined as instantaneous Nash equilibria: each individual does their best with respect to a personal objective (desired velocity), considering the behavior of the neighbors that influence them (to avoid overlapping). We address theoretical and modeling issues in various situations, in particular when each individual is influenced by all their neighbors, and in the case where the influence relations are structured in a hierarchical way. The second particular case is used to define the Inhibition-Based model.

**Résumé.** Nous proposons un nouveau modèle microscopique de mouvement de foule basé sur la théorie des jeux, à partir duquel nous dérivons une version particulière dédiée aux situations d'évacuation, basée sur un principe d'inhibition. Nous supposons que chaque individu a une certaine vitesse souhaitée qu'il adapte en fonction du comportement des voisins qui l'influencent. Les vitesses adaptées possibles sont définies comme des équilibres de Nash instantanés: chaque individu fait de son mieux par rapport à un objectif personnel (vitesse souhaitée), en tenant compte du comportement des voisins qui l'influencent (pour éviter un chevauchement avec eux). Nous abordons des questions relatives à la modélisation ainsi que les aspects théoriques du problème dans diverses situations, en particulier dans le cas où chaque individu est influencé par tous ses voisins, et dans le cas où les relations d'influence entre les individus ont une structure hiérarchique. Le second cas particulier fait l'objet du modèle basé sur un principe d'inhibition.

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## 1. Modelling

In [5, 6], a crowd motion model of the granular type was proposed, based on identifying individuals with rigid disks. It relies on the following principles: each individual has a desired, “selfish”, velocity (the velocity they would like to have if they were alone), and the actual velocity field is defined as the projection of the field of desired velocities on the set of velocity fields that are globally admissible (which do not lead to overlapping between discs). In this granular model, individuals are considered active and asocial: individuals tend to behave as if they were alone, and interactions do not correspond to individuals’ decisions, they are rather of mechanical nature (actual contact between grains). This model applies to highly congested situations where individuals passively undergo forces exerted by their neighbors (including some which they do not see). In real-life situations, even highly crowded ones, people tend to avoid hard physical contact by adapting their instantaneous velocity to the ones of their close neighbors. We propose here to model this very behavior by modifying the desired velocities before the projection step. We suppose that each individual is influenced by some others (practically those who are in their cone of vision). The adapted desired velocities that are likely to occur are such that each individual chooses the velocity that is the closest to their desired one, accounting for the behavior of others that influence them. Since the constraints on each velocity depends on velocities chosen by neighbors, the problem is very similar to finding Nash equilibria [7] in competitive games.

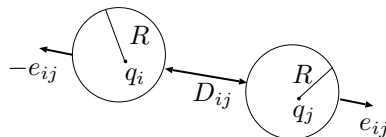
As we shall see, this approach does not properly define a single velocity field, it rather defines a set of velocity fields compatible with those requirements. The core of the approach therefore relies in the definition of the set of adapted desired velocities (set  $\Lambda$  defined by Equation (2) below). Defining this set does not provide a proper evolution model since, as we shall see, it might be empty in some situations and, when it is not, it generally contains more than one element. We establish in Section 2 properties of this set, in two particular situations: when the influence graph is complete (each individual is potentially influenced by all the others), and when it does not contain cycles. In the latter situation,  $\Lambda$  is reduced to a single element, which yields, along with the projection step, a proper evolution model which we shall call Inhibition-Based model (IB model). The relevance of this approach is supported by a comparison between the IB model and the purely granular (selfish) model in an evacuation situation: we check that the civilized behavior of individuals leads to a faster evacuation, which is known as the *Faster is Slower effect*.

### Mathematical formulation

Consider  $N$  individuals represented by disks of centers  $q_1, \dots, q_N \in \mathbb{R}^2$  and common radius  $R$ . The configuration of all individuals is denoted by  $q = (q_1, \dots, q_N) \in \mathbb{R}^{2N}$ . We denote by  $U_i \in \mathbb{R}^2$  the desired velocity of individual  $i$  and define the set of feasible configurations by:

$$K = \{q \in \mathbb{R}^{2N}, D_{ij}(q) \geq 0, \quad \forall i \neq j\}$$

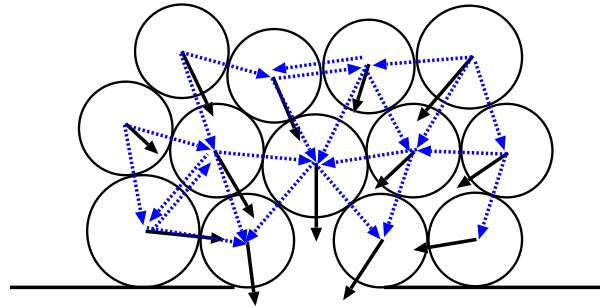
where  $D_{ij}(q) = |q_i - q_j| - 2R$  is the distance between individuals  $i$  and  $j$ .



**Figure 1.** Notation

To alleviate notation, we consider only non-overlapping constraints between individuals, keeping in mind that contacts between individuals and walls of the domain can be handled in the same manner.

Each individual is influenced by some others (not necessarily all of them) and we denote by  $I_i$  the set of pedestrians that influence  $i$ . We represent the influence relations between individuals by a directed graph built as follows: the nodes of the graph are the individuals, and an oriented edge  $i \rightarrow j$  exists if and only if  $j \in I_i$ .



**Figure 2.** Configuration of individuals with desired velocities (black vectors) and influence graph (blue dotted vectors)

For the granular model proposed in [5, 6], the actual velocity field is defined as the euclidean projection of the desired velocity field  $U = (U_1, \dots, U_N)$  on the set of globally admissible velocity fields defined by:

$$C(q) = \{v = (v_1, \dots, v_N) \in \mathbb{R}^{2N}, \forall j \neq i, \quad D_{ij}(q) = 0 \Rightarrow e_{ij}(q) \cdot (v_i - v_j) \leq 0\} \tag{1}$$

where  $e_{ij}(q) = (q_j - q_i) / |q_j - q_i|$ .

The Nash approach that we propose consists in considering that the desired velocity undergoes an *adaptation step*, which accounts for the fact that individuals tend to avoid overlapping with the people at distance zero from them, and *which they see* (i.e. which are in their influence set). Assuming that this first step leads to a proper adapted velocity, the latter may lead to a violation of the non overlapping constraints. Therefore this first step is supplemented by a *projection step*, which consists in projecting the adapted desired velocity on the set of feasible velocities  $C(q)$ . From the modeling standpoint, the first step accounts for decisional processes (individual adaptation to avoid collisions), whereas the second step has a mechanical nature: it accounts for unanticipated collisions between individuals. The overall approach writes:

(1) (Adaptation step) The first step writes

$$\left\{ \begin{array}{l} \text{Find } \tilde{U} \in \Lambda, \text{ where } \Lambda \text{ is defined by} \\ \Lambda = \left\{ v \in \mathbb{R}^{2N}, \quad v_i = \underset{w \in C_i(q, v_{-i})}{\operatorname{argmin}} \frac{1}{2} |w - U_i|^2, \quad \forall i = 1, \dots, N \right\} \end{array} \right. \tag{2}$$

where

$$C_i(q, v_{-i}) = \{w \in \mathbb{R}^2, \quad \forall j \in I_i, \quad D_{ij}(q) = 0 \Rightarrow e_{ij}(q) \cdot (w - v_j) \leq 0\} \tag{3}$$

with the usual notation  $v_{-i} = (v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_N)$ .

(2) (Projection step) The second step consists in defining the actual velocity field as the projection of  $\tilde{U}$  on the set of globally admissible velocity fields  $C(q)$  defined by (1):

$$u = P_{C(q)} \tilde{U}.$$

This approach raises delicate issues in terms of existence and uniqueness of adapted desired velocities since Nash equilibria are not unique in general and even existence is not always guaranteed. Notably, the classical theory about existence of Nash equilibria for generalized games (see for example [2]) does not apply for this problem due to the particular form of the minimization functional. In the next section, we consider two particular cases of influence graphs: the case of a complete graph (each individual is influenced by all the others) and the case of directed acyclic graph (hierarchical interactions between individuals). We prove that, in the first case, the set  $\Lambda$  is non-empty, and not reduced to a singleton in general. In the second case, existence of a unique adapted velocity field can be established, which properly defines an evolution process.

## 2. Theoretical issues

We address here the well-posedness of Problems (2)-(3) in two situations where existence can be proven. In the first one, all interactions are accounted for so that the influence graph is complete: each individual is influenced by all their neighbors. Solutions can then be constructed as classical solutions of a granular problem with arbitrary masses, and uniqueness does not hold in general. In the second situation, where the graph is assumed to be acyclic (hierarchical case), we shall prove existence and uniqueness of a solution.

Let us start by formulating the optimization problems that characterize individual velocities in a saddle point manner.

**Proposition 1.** *The collection of minimization Problems (2)-(3) is equivalent to the collection of saddle-point formulations: for each  $i = 1, \dots, N$ , there exist nonnegative Lagrange multipliers  $(\lambda_{ij})_{j \in I_i^c}$  such that*

$$\left\{ \begin{array}{l} \tilde{U}_i + \sum_{j \in I_i^c} \lambda_{ij} e_{ij} = U_i, \\ e_{ij} \cdot (\tilde{U}_i - \tilde{U}_j) \leq 0, \quad \forall j \in I_i^c, \\ \sum_{j \in I_i^c} \lambda_{ij} e_{ij} \cdot (\tilde{U}_i - \tilde{U}_j) = 0, \end{array} \right. \quad (4)$$

where  $I_i^c \subset I_i$  is the set of individuals  $j$  that influence  $i$ , and that are in contact with  $i$ , i.e. such that  $D_{ij} = 0$ .

**Proof.** The functional is quadratic and the constraints are affine, thus automatically qualified. Therefore, for each  $i = 1, \dots, N$ ,  $\tilde{U}_i$  is a solution of Problems (2)-(3) if and only if there exists nonnegative Lagrange multipliers  $(\lambda_{ij})_{j \in I_i^c}$  such that  $(\tilde{U}_i, \lambda)$  is a solution of the saddle point formulation (4) (by Kuhn–Tucker theorem, see [1] for more details).  $\square$

**Remark 2.** Each  $\lambda_{ij}$  quantifies the correction that  $i$  makes on their own velocity to preserve the constraint pertaining to their neighbor  $j$ . In the granular approach proposed in [5, 6], a similar Lagrange multiplier  $\lambda_{ij}$  was involved to account for the non-overlapping constraint between  $i$  and  $j$ , more precisely to quantify the interaction force between  $i$  and  $j$ . The fact that, in this granular setting,  $\lambda_{ij}$  is common to  $i$  and  $j$ , expresses the mechanical character of the interaction (the Law of Action–Reaction holds). The situation here is different: it may occur that  $\lambda_{ij} \neq \lambda_{ji}$ , which breaks the symmetry of the interaction, and underlines the fact that each  $\lambda_{ij}$  results from a personal decision made by the individual  $i$ .

*Complete influence graph situation*

In this case, each pedestrian takes into consideration the actions of all the others when choosing their own action. We state the existence of an adapted desired velocity field in the following proposition, which gives a constructive process to build an infinite number of equilibria. This process is based on mechanical principles, so that the law of action and reaction automatically holds: it restricts this approach to case where the influence graph is complete. Note also (see Proposition 6 below) that, in general, this process will *not* make it possible to build all Nash equilibria.

**Proposition 3.** *We assume that the influence graph is complete. We consider a collection of strictly positive masses  $m_1, \dots, m_N$  respectively attributed to individuals  $q_1, q_2, \dots, q_N$ . We shall denote by  $M = (m_1, \dots, m_N)$  the corresponding vector.*

*The problem*

$$\min_{v \in C(q)} \frac{1}{2} \sum_{i=1}^N m_i |v_i - U_i|^2, \tag{5}$$

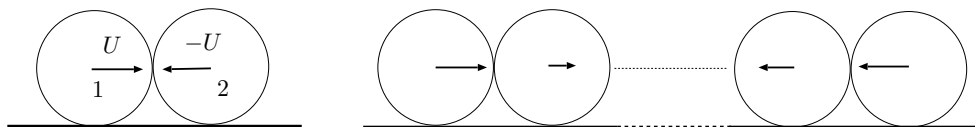
where  $C(q)$  is defined by (1), has a unique solution. This solution is a particular solution of Problems (2)-(3).

**Proof.** We proceed using the saddle point formulations of both Problems (2) and (5). Problem (5) is equivalent to its saddle point formulation, we denote by  $(u^M, \lambda^M)$ ,  $\lambda^M \geq 0$ , its saddle-point, so we have that, for all  $i = 1, \dots, N$ ,

$$\begin{cases} m_i u_i^M + \sum_{j \neq i, D_{ij}=0} \lambda_{ij}^M e_{ij} &= m_i U_i, \\ e_{ij} \cdot (u_i^M - u_j^M) \leq 0, \forall j \neq i, D_{ij} = 0, \\ \sum_{j \neq i, D_{ij}=0} \lambda_{ij}^M e_{ij} \cdot (u_i^M - u_j^M) &= 0. \end{cases} \tag{6}$$

Hence, setting  $\lambda_{ij} = \lambda_{ij}^M / m_i$ , the couple  $(u_i^M, (\lambda_{ij})_{j \neq i, D_{ij}=0})$  satisfies the saddle point formulation (4) of Problems (2)-(3). □

In general, varying  $M$  allows one to define infinitely many Nash equilibria. Let us consider for instance the interaction between two agents in contact (see Figure 3, left).



**Figure 3.** Two-agent and many-agent conflicts

Assume that the agent 1 on the left (resp. the agent 2 on the right) has a desired velocity  $+U$  (resp.  $-U$ ), with  $U > 0$ . Consider the mass vector  $M_\alpha = (1, \alpha)$ , with  $\alpha > 0$ . The projection of  $(U, -U)$  on the set of admissible velocities, for the norm associated to  $M_\alpha$ , corresponds to a common velocity  $u$  for 1 and 2, with

$$u = \frac{1 - \alpha}{1 + \alpha} U.$$

For  $\alpha$  varying between 0 and  $+\infty$ , we thus obtain a continuum of Nash equilibria in the form  $(u, u)$ , with  $u \in (-U, U)$ . The two limit cases are obtained by having  $\alpha$  go to 0 and  $+\infty$ .

Notice that, in this two-agent situation, different mass vectors (apart from scaling) lead to different equilibria. This one-to-one character may be ruled out in general. Consider for instance an array of discs (see Figure 3, right), where the desired velocities decrease from left to right. In this setting, for any mass vector  $M$ , the projection onto the set of feasible velocities is such that all velocities are the same, as stated by the following lemma.

**Lemma 4.** *We consider a cluster of  $N$  discs in a row, in the one-dimensional setting. The desired velocities  $U_1, U_2, \dots, U_N$  are assumed to be non-increasing, i.e.  $U_{i+1} \leq U_i$  for  $i = 1, \dots, N - 1$ . For any mass vector  $M$ , the projection of the desired velocity field on the set of admissible velocities, for the norm associated to  $M$ , has the form  $(u, u, \dots, u)$ .*

**Proof.** Let  $u = (u_1, \dots, u_N)$  be the projection of  $U = (U_1, \dots, U_N)$  on  $C(q)$ , for the norm associated to  $M$  (Problem (5)). Since  $u$  is admissible, it holds that  $u_{i+1} \geq u_i$  for every  $i$ . Let us prove that equality holds for every  $i$ . If  $u_{i+1} > u_i$  for some  $i$ , then either  $u_{i+1} > U_{i+1}$ , but then the distance can be reduced by changing  $u_{i+1}$  in  $u_{i+1} - \varepsilon$ , or  $u_{i+1} \leq U_{i+1}$ , but then  $u_i < U_{i+1} \leq U_i$ , and the cost can be reduced by changing  $u_i$  onto  $u_i + \varepsilon$ . As a consequence, it holds that  $u_{i+1} = u_i$  for every  $i$ , i.e. the solution writes  $(u, u, \dots, u)$ .  $\square$

Now consider a cluster of  $2N+1$  discs, indexed by  $i = -N, -N+1, \dots, 0, 1, \dots, N$ , and a collection of desired velocities that is nondecreasing from left to right, and *odd* with respect to the central disc, i.e.  $U_{-i} = -U_i$  for  $i = 0, 1, \dots, N$ . We furthermore assume that masses are symmetric with respect to the central discs, i.e.  $m_{-i} = m_i$  for  $i = 1, 2, \dots, N$ . By Lemma 4, the solution to Problem (5) is such that all discs have the same velocity, and by symmetry this common velocity is 0. This property holds for *any* symmetric mass distribution, which shows that the correspondence

$$\text{Mass vector } \bar{M} \longrightarrow \text{Solution to (5)}$$

is not injective in general, beyond the obvious scale degeneracy.

We shall see that some Nash equilibria cannot be recovered as limits of such mechanical equilibria. Let us first establish the closed character of  $\Lambda$ .

**Proposition 5.** *For the case of a complete influence graph, the set  $\Lambda$  of all Nash equilibria is closed in  $\mathbb{R}^{2N}$ .*

**Proof.** Let  $(\tilde{U}^n)_n$  be a sequence in  $\Lambda$  which converges to  $\tilde{U}$ . We denote by  $(\lambda_{ij}^n)_{j \neq i, D_{ij}=0}$  the nonnegative Lagrange multipliers associated to  $\tilde{U}_i^n$ , for all  $i = 1, \dots, N$  and  $n \in \mathbb{N}$ :

$$\left\{ \begin{array}{l} \tilde{U}_i^n + \sum_{j \neq i, D_{ij}=0} \lambda_{ij}^n e_{ij} = U_i \\ e_{ij} \cdot (\tilde{U}_i^n - \tilde{U}_j^n) \leq 0, \quad \forall j \neq i, D_{ij} = 0, \\ \sum_{j \neq i, D_{ij}=0} \lambda_{ij}^n e_{ij} \cdot (\tilde{U}_i^n - \tilde{U}_j^n) = 0 \end{array} \right. \quad (7)$$

Let  $J_i$  be the set:

$$J_i = \left\{ j \neq i, \left( \lambda_{ij}^n \right) \text{ has an infinite number of non-zero terms} \right\}.$$

Beyond some rank, the sequence  $\sum_{j \in J_i} \lambda_{ij}^n e_{ij}$  is equal to  $\sum_{j \neq i, D_{ij}=0} \lambda_{ij}^n e_{ij} = U_i - \tilde{U}_i^n$  and is then convergent. Since the set of nonnegative linear combinations of  $(e_{ij})_{j \in J_i}$  is closed, the limit can be written in the form  $\sum_{j \in J_i} \lambda_{ij} e_{ij}$  for some  $\lambda_{ij} \geq 0$ , for all  $j \in J_i$ . We set  $\lambda_{ij} = 0$  for all  $j \notin J_i$ , so we can write:

$$\tilde{U}_i + \sum_{j \neq i, D_{ij}=0} \lambda_{ij} e_{ij} = U_i.$$

We pass to the limit in the second equation of (7) to get:

$$e_{ij} \cdot (\tilde{U}_i - \tilde{U}_j) \leq 0$$

for all  $j \neq i$  and  $D_{ij} = 0$ . So it remains to be proved that the complementary condition

$$\sum_{j \neq i, D_{ij}=0} \lambda_{ij} e_{ij} \cdot (\tilde{U}_i - \tilde{U}_j) = 0$$

holds. For  $j \in J_i$ , there exists a sub-sequence still denoted by  $(\lambda_{ij}^n)_{j \neq i, D_{ij}=0}$  such that  $\lambda_{ij}^n$  is strictly positive starting from a given rank. The complementarity condition

$$\lambda_{ij}^n e_{ij} \cdot (\tilde{U}_i^n - \tilde{U}_j^n) = 0$$

is satisfied for all  $n \in \mathbb{N}$ , so  $e_{ij} \cdot (\tilde{U}_i^n - \tilde{U}_j^n) = 0$  for a subsequence of  $n$  going to infinity. Passing to the limit in the last equality we get

$$e_{ij} \cdot (\tilde{U}_i - \tilde{U}_j) = 0$$

for all  $j \in J_i$ . Then, the following holds for all  $i = 1, \dots, N$ :

$$\left\{ \begin{array}{l} \tilde{U}_i + \sum_{j \neq i, D_{ij}=0} \lambda_{ij} e_{ij} = U_i \\ e_{ij} \cdot (\tilde{U}_i - \tilde{U}_j) \leq 0, \quad \forall j \neq i, D_{ij} = 0, \\ \sum_{j \neq i, D_{ij}=0} \lambda_{ij} e_{ij} \cdot (\tilde{U}_i - \tilde{U}_j) = 0, \end{array} \right.$$

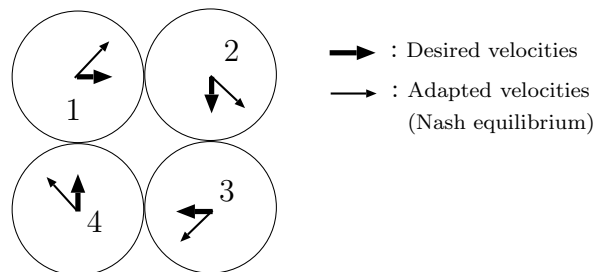
which means that  $(\tilde{U}, (\lambda_{ij})_{j \neq i, D_{ij}=0})$  is a solution of the saddle point formulation of Problems (2)-(3), and thus  $\tilde{U}$  belongs to  $\Lambda$ . □

We denote by  $\Lambda_g$  ( $g$  for “granular”) the set of all those velocity fields which can be obtained as a solution of (5), where  $M$  is a vector associated to masses  $m_1, \dots, m_N > 0$ . We have already shown that  $\Lambda_g \subset \Lambda$ , and the previous proposition extends the inclusion to the closure:  $\overline{\Lambda}_g \subset \Lambda$ . A natural question arises: does it hold that  $\overline{\Lambda}_g = \Lambda$ ? This question is also important from the modeling standpoint: do all Nash equilibria correspond to a global trade-off, the actual outcome of which would only depend on some sort of underlying hierarchy (encoded by the different masses)?

The answer is yes in dimension one and it can be proved by straightforward computations. We show in the following proposition that it is not true in dimension two, which means that some equilibria are genuinely of the Game-Theoretic type, i.e. they cannot be recovered by mechanical principles.

**Proposition 6.** *The inclusion  $\overline{\Lambda}_g \subset \Lambda$  is strict in dimension two.*

**Proof.** We consider the 4-disc configuration represented on Figure 4.



**Figure 4.** Four individuals forming a cycle



The desired velocities of 1, 2, 3 and 4 (bold arrows on the figure) are, respectively,  $(1, 0)$ ,  $(0, -1)$ ,  $(-1, 0)$ , and  $(0, 1)$ . Let us show that the collection of velocities  $(1, 1)$ ,  $(1, -1)$ ,  $(-1, -1)$ ,  $(-1, 1)$  (tiny arrows on the Figure 4) corresponds to a Nash equilibrium. Let us show first that the disc 2 realizes its optimum (i.e. minimal distance to desired velocity  $(0, -1)$ ), given the constraints exerted by discs 1 and 3. Let  $(u, v)$  be the velocity of disc 2, the corresponding constraints are  $u \geq 1$  and  $v \leq -1$ . The problem for 2 consists in minimizing  $u^2 + |v + 1|^2$  under those constraints, which yields  $(u, v) = (1, -1)$ . The very same approach can be carried out for each of the three other discs, which shows that the collection of velocities  $(1, 1)$ ,  $(1, -1)$ ,  $(-1, -1)$ ,  $(-1, 1)$  is indeed a Nash equilibrium.

We aim now at proving that this equilibrium is *not* the limit of equilibria associated to mass vectors (see Proposition 3). Consider the velocities that would result from a projection of the desired velocity field on the set of feasible velocities, for some non degenerated mass vector  $M$  (see Equation (5)). Since  $m_1$  and  $m_2$  are finite and positive, the horizontal velocity of 1 is necessarily reduced, which is not the case here (individual 1 fully imposes their horizontal velocity to individual 2). Focusing on the  $1 \leftrightarrow 2$  interaction, the considered adapted velocity field can be obtained only by having  $m_1/m_2$  go to infinity. Similarly, considering the remaining interactions  $2 \leftrightarrow 3$ ,  $3 \leftrightarrow 4$ , and  $4 \leftrightarrow 1$ , we obtain that  $m_2/m_3 \rightarrow +\infty$ ,  $m_3/m_4 \rightarrow +\infty$  and  $m_4/m_1 \rightarrow +\infty$ . This is impossible since, by cyclicity, the product of these four ratios is 1.  $\square$

### *Hierarchical influence graph*

This particular case of hierarchical influence graph is characterized by extreme asymmetric interactions between individuals: two individuals cannot influence each other mutually, neither directly nor indirectly. We state in the following proposition the existence and uniqueness of adapted desired velocity field belonging to  $\Lambda$ .

**Proposition 7.** *We suppose that the influence graph is directed and acyclic, then Problems (2)-(3) have a unique solution.*

**Proof.** The proof is based on a construction procedure that enables us to explicitly determine a unique solution of Problems (2)-(3).

We consider the following partition of nodes:

- $E_0$  is the subset of individuals that have no leaders. For any  $i \in E_0$ , we have  $\tilde{U}_i(q) = U_i$ .
- $E_1$  is the subset of individuals whose leaders are all in  $E_0$ . For any  $i \in E_1$ ,  $\tilde{U}_i(q)$  is uniquely determined as the solution to the minimization Problems (2)-(3), from the velocities of individuals in  $E_0$  that have already been determined.
- $E_k$ , for  $k = 2, 3, \dots$ , is the subset of individuals whose leaders are in  $E_0 \cup E_1 \cup \dots \cup E_{k-1}$ , with at least one leader in  $E_{k-1}$ . Like previously, for any  $i \in E_k$ ,  $\tilde{U}_i(q)$  is determined as the solution to Problems (2)-(3), from the velocities of individuals in  $E_0 \cup E_1 \cup \dots \cup E_{k-1}$ .

Since the set of individuals is finite, all individuals are handled after a finite number of steps, and this approach determines a solution to Problems (2)-(3) in a unique way.  $\square$

### **3. Inhibition-Based model**

We now consider the practical case of the evacuation of a room through a single exit.

In such a situation, individuals point toward the direction of the exit door, focusing on the direction of desired velocity and disregarding neighbors which are not located in front of them. We show in this section that, under some condition on the desired velocities and on the cone of vision, the influence relations between individuals become hierarchical, which makes the problem fully resolvable by Proposition 7.

As a main assumption, we consider that the desired velocity of an agent depends on their position only. More precisely, we consider that the desired velocity of agent  $i$  located at  $q_i$  is defined as  $U_i = U(q_i)$ , where  $x \mapsto U(x)$  is a global velocity field, shared by all agents.

We suppose that each individual is influenced by the neighbors which they can see, i.e. which lie in their cone of vision. The cone of vision of each individual is considered to be centered around the direction of their desired velocity with a fixed angle  $\alpha < \pi/2$ . The influence set of each individual contains all others whose positions are in their cone of vision (see Figure 5).

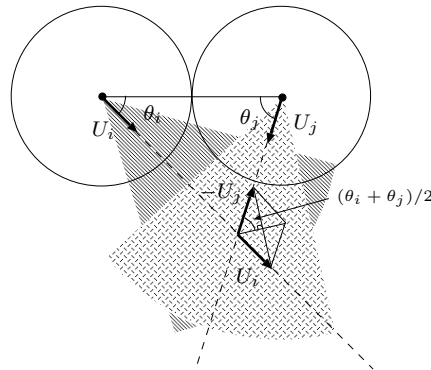
In the following lemma we show that, providing a certain condition on the global velocity field and the angle  $\alpha$  is met, the influence graph is hierarchical.

**Lemma 8.** *We suppose that the desired velocity field  $U$  and the angle of vision  $\alpha$  satisfy the following inequality:*

$$\|\nabla U\|_2 < \frac{\cos \alpha}{R}. \tag{8}$$

*The induced influence graph is then acyclic.*

**Proof.** We introduce  $\theta_i = \widehat{(U_i, e_{ij})}$  and  $\theta_j = \widehat{(U_j, e_{ji})}$  (see Figure 5). We ensure that two individuals  $i$  and  $j$  do not see each other mutually if  $\max(\theta_i, \theta_j) > \alpha$ .



**Figure 5.** Notation

To satisfy this constraint, it is sufficient to have

$$\cos\left(\frac{\theta_i + \theta_j}{2}\right) < \cos \alpha.$$

Moreover, straightforward computations (see Figure 5) yield

$$\cos\left(\frac{\theta_i + \theta_j}{2}\right) = \frac{\|U_i - U_j\|}{2} \leq R \|\nabla U\|_2.$$

By prescribing the last term to be less than  $\cos \alpha$ , we obtain the following condition on the angle of vision  $\alpha$  and the desired velocity field  $U$ :

$$\|\nabla U\|_2 < \frac{\cos \alpha}{R}, \tag{9}$$

which ends the proof of Lemma 8. □

By making this assumption, we ensure that the graph of influence induced by the cones of vision does not contain cycles, so that Problems (2)-(3) can be used to explicitly determine an inhibited velocity field.

As detailed in the previous section, each individual has two different types of interactions with the others:

- Interactions with individuals in their cone of vision: these interactions are based on a decision process and handled in the first step of the model (anticipation of possible collisions based on visual information).
- Interactions with the rest of individuals: these interactions are handled in a mechanical way (management of possible collisions between individuals that do not see each other).

Since the graph is acyclic we consider that individuals are enumerated according to the topological sorting algorithm (see proof of Proposition 7), so that the adapted desired velocity field for the IB model is the unique solution of Problems (2)-(3).

In other words, all adapted desired velocities are determined in a frontal way, starting from the most influential individuals (who do what they want) to the less influential ones (who do what they can). The next step consists in computing the actual velocity field as the projection of the adapted desired velocity field on the set of globally admissible velocity fields.

For this special case, characterized by an acyclic influence graph based on cones of vision around the desired direction, the decision taken by each individual (adaptation of the desired velocity) *reduces* their desired velocity in the direction where they want to go, which motivates the denomination *Inhibition-Based* (IB) model. This is asserted by the following proposition.

**Proposition 9.** *Let  $U$  be a desired velocity field and suppose that Condition (8) is satisfied. Then, denoting by  $\tilde{U}$  the unique element of  $\Lambda$ , the following holds:*

$$U_i \cdot \tilde{U}_i \leq \|U_i\|^2, \quad \forall i = 1, \dots, N.$$

**Proof.** The first equation of the saddle point formulation (4) implies that:

$$\tilde{U}_i \cdot U_i + \sum_{j \in I_i^c} \lambda_{ij} e_{ij} \cdot U_i = \|U_i\|^2, \quad \forall i = 1, \dots, N$$

which ends the proof since  $\lambda_{ij} e_{ij} \cdot U_i \geq 0$  for all  $j \in I_i^c$ . □

### Time discretization

We describe here a time discretization strategy to approximate solutions of the IB model. At each time step, we first compute the inhibited velocity field based on the current hierarchy. This field is then projected on the cone of feasible velocities, to handle the residual mechanical collisions which have not been prevented by the first step.

Let  $t_0 = 0$  be the initial time,  $\tau > 0$  a time step and  $t^n = n\tau$ . We suppose that Condition (8) is satisfied for every  $t \in [0, T]$ . Consider a given initial configuration  $q_0 = q(t_0) \in K$ . At each time step, we start by re-indexing the individuals according to the topological sorting algorithm, so that any individual  $i$  is influenced by individuals with an index  $j > i$ . We keep the same notation for readability reasons. We update the individuals' positions as follows:  $q^{n+1} = q^n + \tau u^{n+1}$  where  $u^{n+1}$  is the actual velocity computed in two steps, both based on a first order expansion of the non-overlapping constraint (as described in [5]).

The first step corresponds to individual adaptation (decision taking phase). We start with the highest index: individual  $N$  picks the velocity  $\tilde{u}_N^n$  that approaches best their desired one  $U_N$ , subject to constraints with their neighbors. When  $i$ 's turn comes, all velocities  $\tilde{u}_{i+1}^n, \dots, \tilde{u}_N^n$  have already been computed. For all  $j \in I_i$ , if  $i$  takes the velocity  $w$  during  $\tau$ , the first order expansion of  $D_{ij}$  writes

$$D_{ij}(q^n) + \tau e_{ij}(q^n) \cdot (\tilde{u}_j^n - w),$$

that is an affine expression which depends on velocities that have already been computed, thanks to the hierarchical ordering. We simply prescribe that the previous expression is non-negative, i.e. we prescribe

$$D_{ij}(q^n) + \tau e_{ij}(q^n) \cdot (\tilde{u}_j^n - \tilde{u}_i^n) \geq 0 \quad \forall j \in I_i.$$

The second step (global preservation of non-overlapping constraints) consists in projecting the adapted velocity  $\tilde{u}^n$  on the set of admissible velocities that ensure the non-overlapping of individuals at each time step. These velocities should satisfy, for all  $i \neq j$ ,

$$D_{ij}(q^n) + \tau e_{ij}(q^n) \cdot (u_j^n - u_i^n) \geq 0,$$

that is again the first order expansion of  $D_{ij}(q^n + \tau u^n) \geq 0$ .

To sum-up, the algorithm reads as follows:

- (1) (Adaptation step): We solve the following minimization problems in the following order  $i = N, N - 1, \dots, 1$ :

$$\tilde{u}_i^{n+1} = \operatorname{argmin}_{w \in C_i^+(q^n, \tilde{u}_{-i}^n)} \frac{1}{2} |w - U_i(q_i^n)|^2$$

where

$$C_i^+(q^n, \tilde{u}_{-i}^n) = \left\{ w \in \mathbb{R}^d, \forall j \in I_i(q^n), D_{ij}(q^n) + \tau e_{ij}(q^n) \cdot (\tilde{u}_j^n - w) \geq 0 \right\}.$$

Note that, because of the hierarchical indexing, all indices  $j$  correspond to individuals that have already decided their velocity  $\tilde{u}_j^n$ .

- (2) (Projection step): The vector of inhibited velocities  $\tilde{u}^{n+1}$  is projected on the set of globally admissible velocities (with respect to the non-overlapping constraint):

$$u^{n+1} = \operatorname{argmin}_{v \in C^\tau(q^n)} \frac{1}{2} |v - \tilde{u}^n|^2$$

$$C^\tau(q^n) = \left\{ v \in \mathbb{R}^{dN}, \forall j \neq i, D_{ij}(q^n) + \tau e_{ij}(q^n) \cdot (v_j - v_i) \geq 0 \right\}.$$

The minimization problems in the first step are local, they involve a very few degrees of freedom, and can be solved instantaneously. The problem in the second step is global, thus possibly more expensive, but it is a simple quadratic minimization problem with affine constraints, it can be e.g. solved by an Uzawa algorithm.

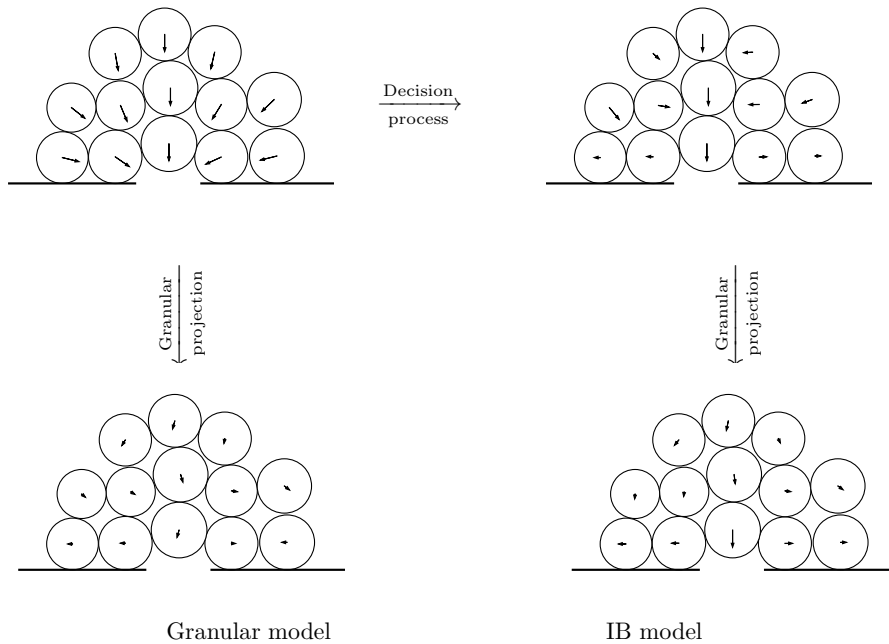
We illustrate in the following example the difference between the IB model and the purely granular one in evacuation situations.

**Example 10.** We consider some individuals trying to evacuate a room. The angle of the cone of vision is set at the value  $\pi/3$ . We represent their desired velocities, their adapted ones and their actual velocities according to the granular model and the IB one in Figure 6. When applying the granular projection directly to the desired velocity field, individuals get clogged and a jam is created upstream the door. However, the IB model gives the priority to the individual in front of the door to pass first and no jams then occur. This example illustrates the so-called Faster is Slower effect, as shall be detailed in the next section.

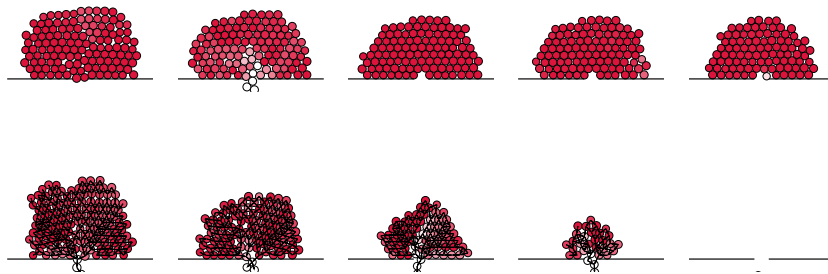
#### 4. Faster is slower effect

The Faster is Slower effect was described by Helbing et al. [4] as one of the characteristic features of escape panic. When pedestrians are in a rush, they tend to increase their velocity and show maladaptive pushing behavior that leads to a reduction of the flow through the exit. This effect has been proved experimentally in the experiences described in [3] where a group of individuals are asked to evacuate a room twice with low and high competitiveness level. The evacuation with low-competitiveness level gets to its end faster than the case of high competitiveness level.

Since individuals have the tendency to go slower for the IB model compared to the granular one, we propose to compare the behavior of individuals for both models in evacuation situations. For this purpose, we run some numerical simulations for the same initial configuration and



**Figure 6.** Example of computation of actual velocities according to the granular model (bottom left) and the IB one (bottom right).



**Figure 7.** Faster is Slower effect: Evacuation of 150 pedestrians with the Granular model (top) and IB model (bottom)

compare the numerical results. Some snapshots of an evacuation simulation are displayed in Figure 7. The influence graph is represented by black vectors for the IB model. For both cases individuals are colored according to their frustration level (red for high frustration level) computed as follows:

$$f_i = 1 - \frac{u_i \cdot U_i}{|U_i|^2}, \quad \forall i = 1, \dots, N.$$

The evacuation gets to its end faster for the IB model compared to the granular one where jams systematically occur during the evacuation. We also run some periodic evacuation simulations (evacuated individuals are re-injected at a random position at the back of the room) for both models and compute the mean of time lapses between consecutive egresses and the flow rate. The results are displayed in Table 1 with a 95% confidence level, and they clearly highlight the Faster is Slower effect, or equivalently, the Slower is Faster effect.

**Table 1.** Different evacuation situations with their respective mean of time lapses and flow rate.

Model	Time lapses (mean)	Flow rate
Granular	$0.41 \pm 0.02$ s	$2.42 \pm 0.1$ pers/s
IB	$0.31 \pm 0.004$ s	$3.18 \pm 0.04$ pers/s

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