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Partial differential equations / *Équations aux dérivées partielles*

Logarithmic estimates for mean-field models in dimension two and the Schrödinger–Poisson system

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Abstract. In dimension two, we investigate a free energy and the ground state energy of the Schrödinger–Poisson system coupled with a logarithmic nonlinearity in terms of underlying functional inequalities which take into account the scaling invariances of the problem. Such a system can be considered as a nonlinear Schrödinger equation with a cubic but nonlocal Poisson nonlinearity, and a local logarithmic nonlinearity. Both cases of repulsive and attractive forces are considered. We also assume that there is an external potential with minimal growth at infinity, which turns out to have a logarithmic growth. Our estimates rely on new logarithmic interpolation inequalities which combine logarithmic Hardy–Littlewood–Sobolev and logarithmic Sobolev inequalities. The two-dimensional model appears as a limit case of more classical problems in higher dimensions.

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1. The Schrödinger–Poisson system with a local logarithmic nonlinearity

The standard Schrödinger–Poisson (SP) system is a nonlinear Schrödinger equation with cubic but nonlocal nonlinearity. As for the nonlinear Schrödinger (NLS) equation with a local nonlinearity, scaling properties play a crucial role in the analysis of the solutions and depend on the dimension d of the Euclidean space. The fact that the nonlinearity in (SP) involves the Poisson convolution kernel makes existence results easier to study than for (NLS) because of the compactness properties induced by the convolution, but adds difficulties due to the non-locality of the mean field potential. We consider primarily the case $d = 2$.

Our purpose is to focus on the underlying functional inequalities and study the interaction of the Poisson term with other terms in the energy (external potential, local nonlinearities) with similar scaling properties: we shall consider quantities which are all critical for (SP) in the two-dimensional case. This is quite interesting from the mathematical point of view, as it is a threshold case for (SP) systems and involves a non sign-defined logarithmic kernel. The $d = 2$ case complements the results of [17, 18] in the limit regime involving logarithmic local nonlinearities. For related questions for $d = 3$, we refer to [17] and references therein. In higher dimensions, the problem is sub-critical if $d \le 5$ and critical for $d = 6$: see Section 3.2.

The (SP) system is used in quantum mechanics to represent a large number of particles by a single complex valued wave function. The local nonlinear term arises from local effects or thermodynamical considerations while the non-local Poisson potential accounts for long range forces which are either of repulsive nature (charged particles) or attractive (in case of gravitational and related models). Most models in the physics literature are justified only on an empirical basis as thermodynamical limits but are difficult to establish rigorously. This issue is anyway out of the scope of this paper.

The Schrödinger equation with a *logarithmic nonlinearity* is a remarkable model in physics, with interesting mathematical properties: see [5, 12, 34]. The equation has soliton-like solutions of Gaussian shape (called *Gaussons* in [5]). We shall refer to [14–16, 20] for some additional contributions in mathematics. *Schrödinger–Poisson* systems are commonly used in charged particles transport and particularly in semiconductor physics, in the repulsive case. In this direction, a classical reference for mathematical properties is [13] and we also quote [4, 36] for examples of applications. The mean-field attractive case (Newton equation) reflects gravitational forces instead of electrostatic forces. It is not studied as much as the repulsive case and it is mathematically more difficult: see for instance [36, Section 4]. As a side remark, we may notice that stationary solutions of (SP) share many properties with stationary solutions of two-dimensional models of chemotaxis, and the same functional inequalities are involved: see [24]. We can however handle the two cases, attractive and repulsive, in a common framework. We primarily focus on variational results, in relation with some interesting functional inequalities and their scaling properties.

For any function $u \in \mathrm{H}^1(\mathbb{R}^2)$, let us consider the *Schrödinger energy*

$$\mathscr{E}[u] := \int_{\mathbb{R}^2} |\nabla u|^2 \,\mathrm{d}x + \alpha \int_{\mathbb{R}^2} V\,|u|^2 \,\mathrm{d}x + 2\pi\,\beta \int_{\mathbb{R}^2} W\,|u|^2 \,\mathrm{d}x + \gamma \int_{\mathbb{R}^2} |u|^2 \log|u|^2 \,\mathrm{d}x \tag{1}$$

where α, β, γ are real parameters and the *self-consistent potential* W is obtained as a solution of the *Poisson equation*

$$-\Delta W = |u|^2 .$$

The solution W is defined only up to an additive constant: we make the specific choice $W = (-\Delta)^{-1}|u|^2$ given by the Green kernel as follows. Let us recall that on $\mathbb{R}^2$ the standard Green function G_y associated with $(-\Delta)$, that is, the solution of $-\Delta_x G = \delta_y(x)$, is given by

$$G(x,y) = -\frac{1}{2\pi} \log|x-y| \quad \forall\,(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2 .$$

Our choice amounts to take $W(x) = \int_{\mathbb{R}^2} |u(y)|^2\, G(x,y)\,\mathrm{d}y$. As a consequence, we have

$$W(x) \sim -\frac{\|u\|_2^2}{2\pi} \log|x| \quad \text{as} \quad |x| \to +\infty ,$$

and also $x \cdot \nabla W(x) < 0$ for large values of $|x|$ if, for instance, u is compactly supported. The cases $\beta > 0$ and $\beta < 0$ correspond to two very different physical situations. The case $\beta < 0$ is the *attractive* case of a Newton–Poisson coupling for gravitational mean-field models. With $\beta > 0$, the model represents the two-dimensional case of *repulsive* electrostatic forces, i.e., a mean field version of a quantum Coulomb gas of interacting particles in dimension $d = 2$.

The function V is an *external potential*, and we shall assume that it has a *critical growth*. The parameter $\alpha \in \mathbb{R}$ is a coupling parameter, whose value has to be discussed depending on the other terms. Without much loss of generality, we can assume that

$$V(x) = 2\log\left(1+|x|^2\right) \quad \forall\, x \in \mathbb{R}^2\,. \tag{2}$$

Concerning the *local nonlinearity*, the case $\gamma < 0$ corresponds to a *focusing local nonlinearity* while $\gamma > 0$ is the case a *defocusing local nonlinearity*. It is standard to observe that any critical point of $\mathscr{E}$ under the mass constraint

$$\int_{\mathbb{R}^2} |u|^2\,\mathrm{d}x = M$$

determines a standing wave of the *nonlinear Schrödinger–Poisson* system

$$i\,\frac{\partial\psi}{\partial t} = \Delta\psi + \alpha\,V\,\psi + \beta\,W\,\psi + \gamma\,\log|\psi|^2\,\psi\,.$$

In this paper we shall focus on finding conditions on α, β, $\gamma \in \mathbb{R}$ insuring that the functional $\mathscr{E}$ is either bounded or unbounded from below on

$$\mathscr{H}_M := \left\{u \in \mathrm{H}^1(\mathbb{R}^2) : \|u\|_2^2 = M\right\}.$$

This paper is organized as follows. We establish in Section 2 several new functional inequalities which generalize the logarithmic Hardy–Littlewood–Sobolev inequality, with an application to a *free energy* functional in dimension two: see Theorem 3. Section 3 is devoted to the boundedness from below of the Schrödinger energy $\mathscr{E}$, with main results in Theorem 10.

2. New logarithmic inequalities and free energy estimates

2.1. *Generalized logarithmic Hardy–Littlewood–Sobolev inequalities*

The *logarithmic Hardy–Littlewood–Sobolev inequality*

$$\int_{\mathbb{R}^2} \rho\,\log\left(\frac{\rho}{M}\right)\mathrm{d}x + \frac{2}{M}\iint_{\mathbb{R}^2\times\mathbb{R}^2} \rho(x)\,\rho(y)\,\log|x-y|\,\mathrm{d}x\,\mathrm{d}y + M\,(1+\log\pi) \ge 0 \tag{3}$$

has been established in optimal form in [11] by E. Carlen and M. Loss, and in [1] by W. Beckner for any $\rho \in \mathrm{L}^1_+(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2}\rho\,\mathrm{d}x = M > 0$. Equality is achieved by $\rho = \rho_\star$ with

$$\rho_\star(x) := \frac{M}{\pi\left(1+|x|^2\right)^2} \quad \forall\, x \in \mathbb{R}^2\,, \tag{4}$$

and also by any function obtained from $\rho_\star$ by a multiplication by a positive constant (with the corresponding mass constraint), a scaling or a translation. Alternative proofs based on fast diffusion flows have been obtained in [10, 21, 23]. Also see [2, 7, 22, 35] for further related results and considerations on dual Onofri type inequalities and [28] for a rearrangement-free proof of (3) using reflection positivity. Inequality (3) provides us with a useful lower bound on the free energy in the case of an attractive Poisson equation corresponding to the Keller–Segel model: see [6, 25], or in the case of a mean-field Newton equation in gravitational models. In presence of the potential V given by (2), we have

$$\int_{\mathbb{R}^2} \rho\,\log\left(\frac{\rho}{M}\right)\mathrm{d}x + 2\,\tau\int_{\mathbb{R}^2}\log\left(1+|x|^2\right)\rho\,\mathrm{d}x + M\,(1-\tau+\log\pi)$$
$$\ge \frac{2}{M}\,(\tau-1)\iint_{\mathbb{R}^2\times\mathbb{R}^2}\rho(x)\,\rho(y)\,\log|x-y|\,\mathrm{d}x\,\mathrm{d}y \tag{5}$$

for any $\tau \geq 0$ and for any function $\rho \in \mathrm{L}^1_+(\mathbb{R}^2)$ with $M = \int_{\mathbb{R}^2} \rho \, dx > 0$, according to [24]. Compared to [24], the discrepancy in the coefficient of M in the last term of the r.h.s. in (5) is due to the normalization of V as defined by (2). Equality again holds if $\rho = \rho_\star$ given by (4). When $\tau = 0$, (5) is nothing else than (3) while the case $\tau = 1$ is easily recovered by Jensen's inequality. Notice that the sign of the coefficient in front of the convolution term in the r.h.s. of (5) becomes positive if $\tau > 1$.

Let us divide (5) by $\tau > 0$ and then take the limit as $\tau \to +\infty$. By doing this, we obtain a new inequality, which differs from (3) and is of interest by itself.

Lemma 1. *For any function $\rho \in \mathrm{L}^1_+(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} \rho \, dx = M$, we have*

$$2 \int_{\mathbb{R}^2} \log\left(1+|x|^2\right) \rho \, dx - M \geq \frac{2}{M} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \rho(x)\, \rho(y) \log|x-y| \, dx\, dy. \tag{6}$$

Moreover equality in (6) *is achieved if and only if $\rho = \rho_\star$.*

Proof. We give a direct proof of (6), which does not rely on (5). A preliminary observation is that (6) makes sense, i.e., that

$$\rho \longmapsto \int_{\mathbb{R}^2} \log\left(1+|x|^2\right) \rho \, dx - \frac{1}{M} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \rho(x)\, \rho(y) \log|x-y| \, dx\, dy$$

is bounded from below. We may indeed notice that, for any x, $y \in \mathbb{R}^d$,

$$|x-y|^2 = |x|^2 + |y|^2 - 2\, x \cdot y \leq |x|^2 + |y|^2 + \left(1 + |x|^2\, |y|^2\right) = \left(1+|x|^2\right)\left(1+|y|^2\right),$$

so that, after multiplying by $\rho(x)\, \rho(y)$ and integrating with respect to x and y, we obtain

$$\begin{aligned} 2 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \rho(x)\, \rho(y) \log|x-y| \, dx\, dy &\leq \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \rho(x)\, \rho(y) \left(\log\left(1+|x|^2\right) + \log\left(1+|y|^2\right)\right) dx\, dy \\ &\leq 2\, M \int_{\mathbb{R}^2} \log\left(1+|x|^2\right) \rho \, dx. \end{aligned}$$

As a consequence, the problem is reduced to proving that the largest constant C such that

$$2 \int_{\mathbb{R}^2} \log\left(1+|x|^2\right) \rho \, dx - C \geq \frac{2}{M} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \rho(x)\, \rho(y) \log|x-y| \, dx\, dy$$

is $C = M$.

At heuristic level, if we admit that $\rho_\star$ realizes the equality case, this equality can be established as follows. The potential V given by (2) is such that $\mu_\star = \frac{1}{\pi}\, e^{-V} = \frac{\rho_\star}{M}$ is a probability measure and we have

$$\Delta V = 8\pi\, \mu_\star\,.$$

One can also check that

$$(-\Delta)^{-1} \mu_\star := -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x-y|\, \mu_\star(y)\, dy = -\frac{V}{8\pi} = -\frac{1}{4\pi} \log\left(1+|x|^2\right)$$

which requires a careful analysis of the integration constants. Indeed, in radial coordinates, by solving the ordinary differential equation

$$\left(r\, V'\right)' = \frac{8\, r}{\left(1+r^2\right)}\,, \quad V'(0) = 0\,, \quad V(0) = V_0\,,$$

a couple of integrations shows that

$$V'(r) = \frac{1}{r}\left(\frac{4}{1+r^2} - 4\right) \quad \text{and} \quad V(r) - V_0 = \int_0^r \frac{4\, s}{1+s^2}\, ds = 2 \log\left(1+r^2\right),$$

so that $8\pi\, (-\Delta)^{-1} \mu_\star = -(V + V_0)$ with $V_0 = 0$. Alternatively, a direct proof is obtained by observing that

$$V_0 = 4 \int_{\mathbb{R}^2} \log|y|\, \mu_\star(y)\, dy = 8 \int_0^{+\infty} \frac{r \log r}{\left(1+r^2\right)^2}\, dr = 0\,,$$

where the last equality is a consequence of the change of variables $r \mapsto 1/r$. Taking into account the identity

$$\int_{\mathbb{R}^2} \log\big(1+|x|^2\big)\,\mu_\star(x)\,\mathrm{d}x = \int_0^{+\infty} \frac{2\,r\,\log\big(1+r^2\big)}{\big(1+r^2\big)^2}\,\mathrm{d}r = 1\,,$$

this is consistent with the fact that $\rho_\star = M\,\mu_\star$ corresponds to the equality case in (3), according to [11]. Altogether, we have $C = M$, meaning that (6) is an equality if $\rho = \rho_\star$.

After these preliminary considerations, which are provided only for a better understanding of the functional framework, let us give a proof. With no loss of generality, we may assume that $M = 1$ because of the 1-homogeneity of (6). Let us notice that

$$2\int_{\mathbb{R}^2} \log\big(1+|x|^2\big)\,\rho\,\mathrm{d}x - 1 - 2\iint_{\mathbb{R}^2\times\mathbb{R}^2} \rho(x)\,\rho(y)\,\log|x-y|\,\mathrm{d}x\,\mathrm{d}y$$
$$= -2\iint_{\mathbb{R}^2\times\mathbb{R}^2} \big(\rho(x)-\mu_\star(x)\big)\big(\rho(y)-\mu_\star(y)\big)\,\log|x-y|\,\mathrm{d}x\,\mathrm{d}y.$$

We recover that the equality case in (6) is achieved if $\rho = \mu_\star$. With $W = -(-\Delta)^{-1}(\rho-\mu_\star)$, we obtain

$$-2\iint_{\mathbb{R}^2\times\mathbb{R}^2} \big(\rho(x)-\mu_\star(x)\big)\big(\rho(y)-\mu_\star(y)\big)\,\log|x-y|\,\mathrm{d}x\,\mathrm{d}y$$
$$= 4\pi\int_{\mathbb{R}^2} (\rho-\mu_\star)\,(-\Delta)^{-1}(\rho-\mu_\star)\,\mathrm{d}x = -4\pi\int_{\mathbb{R}^2} (\Delta W)\,W\,\mathrm{d}x = 4\pi\int_{\mathbb{R}^2} |\nabla W|^2\,\mathrm{d}x \ge 0\,,$$

where the last equality is obtained by a simple integration by parts. This can be done only because $\int_{\mathbb{R}^2} (\rho-\mu_\star)\,\mathrm{d}x = 0$, a necessary and sufficient condition to guarantee that ∇W is square integrable (for a proof, one has to study the behavior of the solution of the Poisson equation as $|x| \to +\infty$). At this point it is clear that $\int_{\mathbb{R}^2} |\nabla W|^2\,\mathrm{d}x = 0$ if and only if $\rho = \mu_\star$. The general case with an arbitrary $M > 0$ is obtained by writing $\rho_\star = M\,\mu_\star$, which concludes the proof. □

The equality case in (6) is achieved among radial functions. It is classical that the l.h.s. is decreasing under symmetric decreasing rearrangements, while the r.h.s. is increasing. The strict rearrangement inequality for the logarithmic kernel is proved in [11, Lemma 2]. As a limit case of $\iint_{\mathbb{R}^2\times\mathbb{R}^2} \big(\rho(x)-\mu_\star(x)\big)\big(\rho(y)-\mu_\star(y)\big)\,|x-y|^\lambda\,\mathrm{d}x\,\mathrm{d}y$ when $\lambda \to 0_-$, according to [30, Theorem 4.3] (also see [33] for interesting consequences), this is indeed expected. Justifying the square integrability of ∇W has therefore to be done only among radial functions, which is elementary using, e.g., a compactly supported function ρ and a density argument.

Also notice that one can now recover (5) as a simple consequence of (3) and (6). Next, we turn our attention to an inequality which is a consequence of convexity and Jensen's inequality. Let

$$\mathsf{J}_\eta[\rho] := \int_{\mathbb{R}^2} \rho\,\log\left(\frac{\rho}{\|\rho\|_1}\right)\mathrm{d}x + \eta\int_{\mathbb{R}^2} \log\big(1+|x|^2\big)\,\rho\,\mathrm{d}x \quad \forall\,\rho \in \mathrm{L}^1_+(\mathbb{R}^2)\,.$$

Lemma 2. *Let $\eta > 0$, $M > 0$ and $\mathscr{X}_M := \{\rho \in \mathrm{L}^1_+(\mathbb{R}^2) : \|\rho\|_1 = M\}$.*

(i) *If $\eta > 1$, then J_η is bounded from below on $\mathscr{X}_M$ and*

$$\int_{\mathbb{R}^2} \rho\,\log\left(\frac{\rho}{M}\right)\mathrm{d}x + \eta\int_{\mathbb{R}^2} \log\big(1+|x|^2\big)\,\rho\,\mathrm{d}x \ge M\,\log\left(\frac{\eta-1}{\pi}\right) \quad \forall\,\rho \in \mathscr{X}_M\,. \tag{7}$$

For any $\eta > 1$, equality in (7) is achieved by $\rho = M\,\rho_\eta$, where

$$\rho_\eta(x) := \frac{\eta-1}{\pi\big(1+|x|^2\big)^\eta} \quad \forall\,x \in \mathbb{R}^2\,.$$

(ii) *If $\eta \in (0,1]$, then $\inf_{\mathscr{X}_M} \mathsf{J}_\eta = -\infty$.*

If $\eta = 2$, then $\rho_2 = \rho_\star$, while (7) amounts to $\mathsf{J}_\eta[\rho] \ge \mathsf{J}_\eta[M\,\rho_\eta]$ for any $\eta > 1$. If τ is restricted to the range $[0,1]$, we notice as in [24] that (5) is a simple convex combination, with coefficients $(1-\tau)$ and τ, of (3) and (7) written with $\eta = 2$.

Proof. A direct computation based on $\frac{\mathrm{d}}{\mathrm{d}r}\left(1+r^2\right)^{1-\eta}=-2\,(\eta-1)\,r\left(1+r^2\right)^{-\eta}$ shows that

$$\int_{\mathbb{R}^2}\rho_\eta\,\mathrm{d}x=2\,(\eta-1)\int_0^{+\infty}r\left(1+r^2\right)^{-\eta}\,dr=1$$

for all $\eta>1$ and

$$\mathsf{J}_\eta[\rho]=\int_{\mathbb{R}^2}\rho\,\log\Big(\frac{\rho}{M\,\rho_\eta}\Big)\,\mathrm{d}x+M\log\Big(\frac{\eta-1}{\pi}\Big)\quad\forall\,\rho\in\mathscr{X}_M\,.$$

Using that $u\mapsto u\log u-u+1$ is a convex function whose minimum is 0, we get

$$\int_{\mathbb{R}^2}\rho\,\log\Big(\frac{\rho}{M\,\rho_\eta}\Big)\,\mathrm{d}x=\int_{\mathbb{R}^2}\frac{\rho}{M\,\rho_\eta}\,\log\Big(\frac{\rho}{M\,\rho_\eta}\Big)\,M\,\rho_\eta\,\mathrm{d}x\ge\int_{\mathbb{R}^2}\Big(\frac{\rho}{M\,\rho_\eta}-1\Big)\,M\,\rho_\eta\,\mathrm{d}x=0$$

by taking $u=\rho/(M\,\rho_\eta)$ and then integrating against $M\,\rho_\eta\,\mathrm{d}x$. This proves (7) for any $\eta>1$, where equality holds as a consequence of $\mathsf{J}_\eta[M\,\rho_\eta]=M\log\big(\frac{\eta-1}{\pi}\big)$.

Let us consider the case $\eta\in(0,1]$ and take $\rho=M\,\rho_\zeta$ with $\zeta>1$ as a test function. With a few integrations by parts, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2}\log\big(1+|x|^2\big)\,\rho_\zeta(x)\,\mathrm{d}x&=2\,(\zeta-1)\int_0^{+\infty}r\,\log\big(1+r^2\big)\,\big(1+r^2\big)^{-\zeta}\,\mathrm{d}r\\&=-\int_0^{+\infty}\frac{\mathrm{d}}{\mathrm{d}r}\Big(\big(1+r^2\big)^{1-\zeta}\Big)\log\big(1+r^2\big)\,\mathrm{d}r=2\int_0^{+\infty}r\,\big(1+r^2\big)^{-\zeta}\,\mathrm{d}r=\frac{1}{\zeta-1}\,,\end{aligned}$$

$$\int_{\mathbb{R}^2}\rho_\zeta\,\log\rho_\zeta\,\mathrm{d}x=\log\Big(\frac{\zeta-1}{\pi}\Big)\int_{\mathbb{R}^2}\rho_\zeta\,\mathrm{d}x-\zeta\int_{\mathbb{R}^2}\log\big(1+|x|^2\big)\,\rho_\zeta(x)\,\mathrm{d}x=\log\Big(\frac{\zeta-1}{\pi}\Big)-\frac{\zeta}{\zeta-1}\,,$$

so that $\lim_{\zeta\to1_+}\mathsf{J}_\eta[M\,\rho_\zeta]=-\infty$ because

$$\frac{1}{M}\,\mathsf{J}_\eta[M\,\rho_\zeta]=\int_{\mathbb{R}^2}\rho_\zeta\,\log\rho_\zeta\,\mathrm{d}x+\eta\int_{\mathbb{R}^2}\log\big(1+|x|^2\big)\,\rho_\zeta(x)\,\mathrm{d}x=\log\Big(\frac{\zeta-1}{\pi}\Big)-\frac{\zeta-\eta}{\zeta-1}\,.\qquad\square$$

2.2. *Boundedness from below of the free energy functional*

Let us consider the *free energy* functional defined by

$$\mathscr{F}_{\mathsf{a},\mathsf{b}}[\rho]:=\int_{\mathbb{R}^2}\rho\,\log\Big(\frac{\rho}{M}\Big)\,\mathrm{d}x+\mathsf{a}\int_{\mathbb{R}^2}\log\big(1+|x|^2\big)\,\rho\,\mathrm{d}x-\frac{\mathsf{b}}{M}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\rho(x)\,\rho(y)\,\log|x-y|\,\mathrm{d}x\,\mathrm{d}y$$

for any $\rho\in\mathrm{L}^1_+(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2}\rho\,\mathrm{d}x=M$. We look for the range of the parameters a and b such that

$$\mathscr{F}_{\mathsf{a},\mathsf{b}}[\rho]\ge\mathscr{C}(\mathsf{a},\mathsf{b})\,M\quad\forall\,\rho\in\mathrm{L}^1_+(\mathbb{R}^2)\quad\text{such that}\quad\big\|\rho\big\|_1=M,\tag{8}$$

for some constant $\mathscr{C}(\mathsf{a},\mathsf{b})$. Inequality (5) with $\tau\ge0$ is obtained as the special case $\mathsf{a}=2\,\tau$ and $\mathsf{b}=2\,(\tau-1)$, with $\mathscr{C}(\mathsf{a},\mathsf{b})=M\,(\tau-1-\log\pi)$, according to [24]. As a consequence, we also know that (8) holds for some $\mathscr{C}(\mathsf{a},\mathsf{b})>-\infty$ if $\mathsf{a}\ge2\,\tau$ and $\mathsf{b}=2\,(\tau-1)$, that is, $0\le\mathsf{b}+2\le\mathsf{a}$. This range can be improved. For instance, if $\mathsf{b}=0$, it is clear from Lemma 2 that the threshold is at $\mathsf{a}=1$ and not $\mathsf{a}=2$. Our result (see Figure 1) is as follows.

Theorem 3. *Inequality* (8) *holds for some* $\mathscr{C}(\mathsf{a},\mathsf{b})>-\infty$ *if either* $\mathsf{a}=0$ *and* $\mathsf{b}=-2$*, or*

$$\mathsf{a}>0\,,\quad-2\le\mathsf{b}<\mathsf{a}-1\quad\textit{and}\quad\mathsf{b}\le2\,\mathsf{a}-2\,.$$

If either $\mathsf{a}<0$ *or* $\mathsf{b}<-2$ *or* $\mathsf{b}>\min\{\mathsf{a}-1,2\,\mathsf{a}-2\}$ *or* $(\mathsf{a},\mathsf{b})=(1,0)$*, then*

$$\inf_{\rho\in\mathscr{X}_1}\mathscr{F}_{\mathsf{a},\mathsf{b}}[\rho]=-\infty\,.$$

If $0\le\mathsf{a}<1$ *and* $\mathsf{b}=2\,\mathsf{a}-2$*, then*

$$\mathscr{C}(\mathsf{a},2\,\mathsf{a}-2)=-\log\Big(\frac{e\,\pi}{1-\mathsf{a}}\Big)\,.$$

Moreover, if $\mathsf{a}>0$ *there is no minimizer for* $\mathscr{C}(\mathsf{a},2\,\mathsf{a}-2)$.

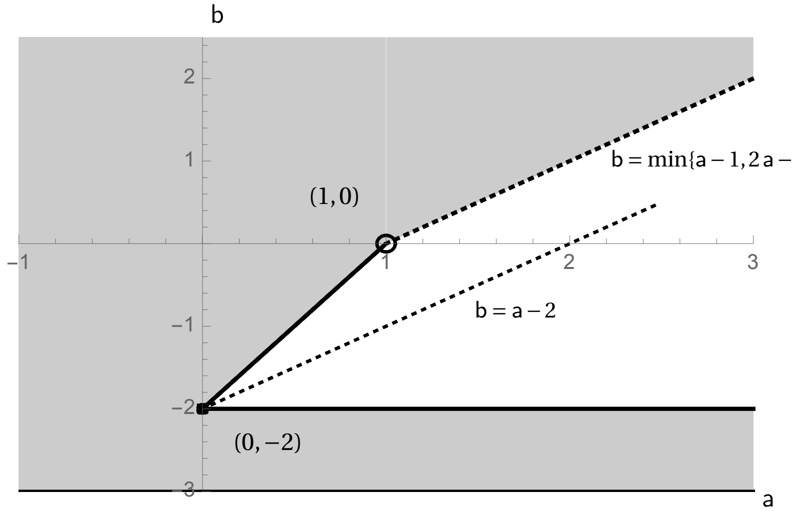

Figure 1. *White (resp. grey) area corresponds to the domain in which* (8) *holds for some finite constant* $\mathscr{C}(\mathsf{a},\mathsf{b})$ *(resp.* $\mathscr{C}(\mathsf{a},\mathsf{b})=-\infty$*). We also know that* $\mathscr{C}\big(\mathsf{a},2(\mathsf{a}-1)\big)=-\log(e\pi/(1-\mathsf{a}))$ *if* $0\le\mathsf{a}<1$ *and* $\mathscr{C}(1,0)=-\infty$*, while the boundedness from below of* $\mathscr{F}_{\mathsf{a},\mathsf{b}}$ *is not known in the threshold case* $\mathsf{b}=\mathsf{a}-1>0$*. On the dotted half-line* $\mathsf{b}=\mathsf{a}-2\ge-2$*, optimality is achieved by* $\rho_\star$ *and Inequality* (8) *corresponds to* (5) *with* $\mathsf{a}=2\tau$*,* $\mathsf{b}=2(\tau-1)$*, and* $\tau\ge0$*.*

The boundedness from below of $\mathscr{F}_{\mathsf{a},\mathsf{b}}$ is unknown only in the case $\mathsf{b}=\mathsf{a}-1>0$. If $\mathsf{b}=2\mathsf{a}-2<0$, we do not only show the semi-boundedness of $\mathscr{F}_{\mathsf{a},\mathsf{b}}$, but we actually compute the infimum $\mathscr{C}(\mathsf{a},\mathsf{b})$. The infimum is also known if $\mathsf{b}=\mathsf{a}-2\ge-2$ and in that case optimality is achieved by $\rho_\star$ according to (5). Note that for $\mathsf{a}=0$, the inequality $\mathscr{F}_{\mathsf{a},2(\mathsf{a}-1)}[\rho]\ge\mathscr{C}\big(\mathsf{a},2(\mathsf{a}-1)\big)\,M$ is the sharp logarithmic Hardy–Littlewood–Sobolev inequality (3) and as $\mathsf{a}\to1$ the infimum diverges to $-\infty$ consistently with the result of Lemma 2. For the convenience of the reader, we divide the proof of Theorem 3 in several intermediate results.

Lemma 4. *Inequality* (8) *holds for some* $\mathscr{C}(\mathsf{a},\mathsf{b})>-\infty$ *if either* $\mathsf{a}=0$ *and* $\mathsf{b}=-2$*, or*

$$\mathsf{a}>0\quad\textit{and}\quad-2\le\mathsf{b}<\min\{\mathsf{a}-1,2\mathsf{a}-2\}\,.$$

The proof for $-2\le\mathsf{b}<0$ and $\mathsf{a}>1-\mathsf{b}/2$ follows from the case $\mathsf{a}=1-\mathsf{b}/2$, which is treated in Lemmas 6 and 7 below, but we give the argument here nevertheless, since it is simpler.

Proof. The case $\mathsf{a}=0$ and $\mathsf{b}=-2$ corresponds to (3). The case $\mathsf{a}=\eta>1$ and $\mathsf{b}=0$ is (7).

If $\mathsf{b}<0$, the condition $\mathsf{b}<2\mathsf{a}-2$ arises by combining (3) and (7), respectively multiplied by $-\mathsf{b}/2$ and $1+\mathsf{b}/2$, with $\mathsf{a}=(1+\mathsf{b}/2)\,\eta$ for any $\eta>1$. In that case, (8) holds with

$$\mathscr{C}(\mathsf{a},\mathsf{b})=M(1+\log\pi)\,\frac{\mathsf{b}}{2}+M\log\left(\frac{\eta-1}{\pi}\right)\left(1+\frac{\mathsf{b}}{2}\right)=M(1+\log\pi)\,\frac{\mathsf{b}}{2}+M\log\left(\frac{2\mathsf{a}-2-\mathsf{b}}{\pi\,(\mathsf{b}+2)}\right)\frac{\mathsf{b}+2}{2}\,.$$

If $\mathsf{b}>0$, we sum (6) with a coefficient $\mathsf{b}/2$ and (7) with coefficient 1 and $\eta=\mathsf{a}-\mathsf{b}>1$. In that case, (8) holds with

$$\mathscr{C}(\mathsf{a},\mathsf{b})=M\,\frac{\mathsf{b}}{2}+M\log\left(\frac{\eta-1}{\pi}\right)=M\,\frac{\mathsf{b}}{2}+M\log\left(\frac{\mathsf{a}-\mathsf{b}-1}{\pi}\right).\qquad\square$$

With $M=1$, notice that

$$\mathscr{F}_{\mathsf{a},\mathsf{b}}[\rho]=\int_{\mathbb{R}^2}\rho\,\log\rho\,\mathrm{d}x+\mathsf{a}\int_{\mathbb{R}^2}\log\big(1+|x|^2\big)\,\rho\,\mathrm{d}x+2\pi\,\mathsf{b}\int_{\mathbb{R}^2}\rho\,(-\Delta)^{-1}\rho\,\mathrm{d}x\,.$$

Lemma 5. *If either* $\mathsf{a}<0$ *or* $\mathsf{b}<-2$ *or* $\mathsf{b}>\min\{\mathsf{a}-1,2\mathsf{a}-2\}$ *or* $(\mathsf{a},\mathsf{b})=(1,0)$*, then*

$$\inf_{\rho\in\mathscr{X}_1}\mathscr{F}_{\mathsf{a},\mathsf{b}}[\rho]=-\infty\,.$$

In Lemma 5, there is no loss of generality in assuming that $M = 1$. Under the assumptions on (a,b) of Lemma 5, Inequality (8) does not hold for some $\mathscr{C}(\mathsf{a},\mathsf{b}) > -\infty$. In that case, we shall simply write $\mathscr{C}(\mathsf{a},\mathsf{b}) = -\infty$. See Figure 1.

Proof. For an arbitrary $\rho \in \mathscr{X}_1$, i.e., $\rho \in \mathrm{L}^1_+(\mathbb{R}^2)$ such that $\|\rho\|_1 = 1$, let $\rho_{x_0}(x) := \rho(x - x_0)$. Since

$$\int_{\mathbb{R}^2} \log\left(1+|x|^2\right) \rho_{x_0}(x)\,\mathrm{d}x \sim 2\log|x_0| \int_{\mathbb{R}^2} \rho\,\mathrm{d}x \quad \text{as} \quad |x_0| \to +\infty$$

and all other integrals are unchanged, the conclusion is straightforward if $\mathsf{a} < 0$.

Assume now that $\rho \in \mathscr{X}_1$ is such that $\rho\log\rho$ and $\log\left(1+|x|^2\right)\rho$ are integrable, and let $\rho_\lambda(x) = \lambda^2\,\rho(\lambda\,x)$, for any $x \in \mathbb{R}^2$. We have

$$\int_{\mathbb{R}^2} \rho_\lambda \log\rho_\lambda\,\mathrm{d}x = \int_{\mathbb{R}^2} \rho\log\rho\,\mathrm{d}x + 2\log\lambda\,,$$

$$\int_{\mathbb{R}^2} \log\left(1+|x|^2\right)\rho_\lambda\,\mathrm{d}x = \int_{\mathbb{R}^2} \log\left(1+\lambda^{-2}|x|^2\right)\rho\,\mathrm{d}x\,,$$

$$\int_{\mathbb{R}^2} \rho_\lambda\,(-\Delta)^{-1}\rho_\lambda\,\mathrm{d}x = \int_{\mathbb{R}^2} \rho\,(-\Delta)^{-1}\rho\,\mathrm{d}x + \frac{\log\lambda}{2\pi}\,.$$

As $\lambda \to +\infty$, we obtain that $\mathscr{F}_{\mathsf{a},\mathsf{b}}[\rho_\lambda] \sim (\mathsf{b}+2)\log\lambda$, which proves our statement if $\mathsf{b} < -2$.

Assume additionally that $\rho(x) = 0$ if $|x| \not\in [1,2]$. Since on any compact set of $\mathbb{R}^2 \setminus \{0\}$, we have that $1+\lambda^{-2}|x|^2 \sim \lambda^{-2}|x|^2$ as $\lambda \to 0_+$ and deduce that

$$\int_{\mathbb{R}^2} \log\left(1+|x|^2\right)\rho_\lambda(x)\,\mathrm{d}x = \int_{\mathbb{R}^2} \log\left(1+\lambda^{-2}|x|^2\right)\rho(x)\,\mathrm{d}x \sim -2\log\lambda\,.$$

As $\lambda \to 0_+$, we obtain that $\mathscr{F}_{\mathsf{a},\mathsf{b}}[\rho_\lambda] \sim (\mathsf{b}+2-2\,\mathsf{a})\log\lambda$, which proves our statement if $\mathsf{b}+2-2\,\mathsf{a} > 0$.

Now, still assuming that $\rho(x) = 0$ if $|x| \not\in [1,2]$, let

$$\rho_{\varepsilon,\lambda}(x) = (1-\varepsilon)\,\rho(x) + \lambda^2\,\varepsilon\,\rho\,(\lambda\,x)$$

with parameters $(\varepsilon,\lambda) \in (0,1)^2$. Using that the supports of ρ and ρ_λ decouple if $\lambda < 1/2$, we have, for any given $\varepsilon \in (0,1)$, as $\lambda \to 0_+$

$$\int_{\mathbb{R}^2} \rho_{\varepsilon,\lambda}\log\rho_{\varepsilon,\lambda}\,\mathrm{d}x = \int_{\mathbb{R}^2} \rho\log\rho\,\mathrm{d}x + \varepsilon\log\varepsilon + (1-\varepsilon)\log(1-\varepsilon) + 2\,\varepsilon\log\lambda\,,$$

$$\int_{\mathbb{R}^2} \log\left(1+|x|^2\right)\rho_{\varepsilon,\lambda}\,\mathrm{d}x = (1-\varepsilon)\int_{\mathbb{R}^2} \log\left(1+|x|^2\right)\rho\,\mathrm{d}x + 2\,\varepsilon\int_{\mathbb{R}^2} \log|x|\,\rho\,\mathrm{d}x - 2\,\varepsilon\log\lambda + o\left(\log\lambda\right)\,,$$

$$\begin{aligned}\int_{\mathbb{R}^2} \rho_{\varepsilon,\lambda}\,(-\Delta)^{-1}\rho_{\varepsilon,\lambda}\,\mathrm{d}x = \left(\varepsilon^2+(1-\varepsilon)^2\right)\int_{\mathbb{R}^2} \rho\,(-\Delta)^{-1}\rho\,\mathrm{d}x + \varepsilon^2\,\frac{\log\lambda}{2\pi}&\\ - \frac{\varepsilon\,(1-\varepsilon)}{\pi}\int_{\mathbb{R}^2} \log|x|\,\rho(x)\,\mathrm{d}x + \frac{\varepsilon\,(1-\varepsilon)}{\pi}\log\lambda + o\left(\log\lambda\right)&.\end{aligned}$$

Thus,

$$\begin{aligned}\int_{\mathbb{R}^2} \rho_{\varepsilon,\lambda}\log\rho_{\varepsilon,\lambda}\,\mathrm{d}x + \mathsf{a}\int_{\mathbb{R}^2} \log\left(1+|x|^2\right)\rho_{\varepsilon,\lambda}\,\mathrm{d}x + 2\pi\,\mathsf{b}\int_{\mathbb{R}^2} \rho_{\varepsilon,\lambda}\,(-\Delta)^{-1}\rho_{\varepsilon,\lambda}\,\mathrm{d}x&\\ \sim 2\,\varepsilon\left(\left(1-\frac{\varepsilon}{2}\right)\mathsf{b}+1-\mathsf{a}\right)\log\lambda \quad \text{as} \quad \lambda \to 0_+\,.&\end{aligned}$$

This again proves our statement if $\mathsf{b}+1-\mathsf{a} > 0$, because $(1-\varepsilon/2)\,\mathsf{b}+1-\mathsf{a}$ can be made positive for $\varepsilon > 0$, small enough. □

The proof of Theorem 3 in the case $\mathsf{b} = 2\,(\mathsf{a}-1) \in [-2,0)$ is based on two ingredients exposed in Lemma 6 and Lemma 7. The first ingredient relates the minimization of $\mathscr{F}_{\mathsf{a},2\,(\mathsf{a}-1)}$ to a simpler, scale-invariant minimization problem. Let

$$\mathscr{G}_{\mathsf{a}}[\rho] := \int_{\mathbb{R}^2} \rho\log\rho\,\mathrm{d}x + 2\,\mathsf{a}\int_{\mathbb{R}^2} \log|x|\,\rho\,\mathrm{d}x + 2\,(\mathsf{a}-1)\iint_{\mathbb{R}^2\times\mathbb{R}^2} \rho(x)\log\frac{1}{|x-y|}\,\rho(y)\,\mathrm{d}x\,\mathrm{d}y$$

and

$$\mathscr{K}(\mathsf{a}) := \inf\left\{\mathscr{G}_{\mathsf{a}}[\rho]\,:\,\rho\ge 0\,,\ \int_{\mathbb{R}^2}\rho\,\mathrm{d}x = 1\right\}.$$

Lemma 6. *Let* $0\le\mathsf{a}<1$. *Then*

$$\mathscr{C}\big(\mathsf{a},2\,(\mathsf{a}-1)\big) = \mathscr{K}(\mathsf{a})\,.$$

Moreover, if $\mathsf{a}>0$ *there is no minimizer for* $\mathscr{C}\big(\mathsf{a},2\,(\mathsf{a}-1)\big)$.

Proof. Since $\log\big(1+|x|^2\big) > 2\log|x|$, we immediately obtain $\mathscr{F}_{\mathsf{a},2\,(\mathsf{a}-1)}[\rho] > \mathscr{G}_{\mathsf{a}}[\rho]$ whenever $\rho\not\equiv 0$ (and the functionals are finite). This implies that $\mathscr{C}\big(\mathsf{a},2\,(\mathsf{a}-1)\big)\ge\mathscr{K}(\mathsf{a})$ and that, if $\mathsf{a}>0$ and if $\mathscr{F}_{\mathsf{a},2\,(\mathsf{a}-1)}$ has a minimizer, then $\mathscr{C}\big(\mathsf{a},2\,(\mathsf{a}-1)\big) > \mathscr{K}(\mathsf{a})$.

We show now the opposite inequality $\mathscr{C}\big(\mathsf{a},2\,(\mathsf{a}-1)\big)\le\mathscr{K}(\mathsf{a})$, which will complete the proof. Let $\sigma\ge 0$ with $\int_{\mathbb{R}^2}\sigma\,\mathrm{d}x = 1$ and with compact support not containing the origin. Consider $\rho_\lambda(x) = \lambda^{-2}\sigma(x/\lambda)$ with $\lambda\gg 1$. Then, as in the proof of Lemma 5,

$$\mathscr{F}_{\mathsf{a},\mathsf{b}}[\rho_\lambda] = (-2+2\,\mathsf{a}-\mathsf{b})\log\lambda + \int_{\mathbb{R}^2}\sigma\,\log\sigma\,\mathrm{d}x + \mathsf{a}\int_{\mathbb{R}^2}\log\big(\lambda^{-2}+|x|^2\big)\,\sigma(x)\,\mathrm{d}x \\ + \mathsf{b}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\sigma(x)\,\log\frac{1}{|x-y|}\,\sigma(y)\,\mathrm{d}x\,\mathrm{d}y\,.$$

If $\mathsf{b} = 2\,(\mathsf{a}-1)$, then the coefficient of $\log\lambda$ vanishes and we obtain

$$\mathscr{C}\big(\mathsf{a},2\,(\mathsf{a}-1)\big)\le\liminf_{\lambda\to\infty}\mathscr{F}_{\mathsf{a},2\,(\mathsf{a}-1)}[\rho_\lambda] = \mathscr{G}_{\mathsf{a}}[\sigma]\,.$$

Taking the infimum over all σ (and removing the support assumptions by an approximation argument), we obtain $\mathscr{C}\big(\mathsf{a},2\,(\mathsf{a}-1)\big)\le\mathscr{K}(\mathsf{a})$, as claimed. □

Lemma 7. *Let* $0\le\mathsf{a}<1$. *Then*

$$\mathscr{K}(\mathsf{a}) = -\log\left(\frac{e\,\pi}{1-\mathsf{a}}\right).$$

For $\mathsf{a}>0$ *the infimum* $\mathscr{K}(\mathsf{a})$ *is achieved if and only if, for some* $\lambda>0$,

$$\rho(x) = \frac{1-\mathsf{a}}{\pi}\,\frac{\lambda^2}{|x|^{2\,\mathsf{a}}\big(\lambda^2+|x|^{2\,(1-\mathsf{a})}\big)^2}\,.$$

The idea of the proof is to apply a change of variables and to reduce the result to the case $\mathsf{a}=0$.

Proof. By symmetric decreasing rearrangement it suffices to bound $\mathscr{G}_{\mathsf{a}}[\rho]$ from below for radial decreasing ρ. In fact, in the following we only use that ρ is radial, and we use this in order to apply Newton's theorem. We set $\widetilde{\rho}(x) := |x|^{2\,\mathsf{a}}\,\rho(x)$ and then we define a radial function τ on $\mathbb{R}^2$ by $\tau(z) = \widetilde{\rho}\big(|z|^{1/(1-\mathsf{a})}\big)$ (with an obvious abuse of notation for the radial function $\widetilde{\rho}$). We have

$$\int_{\mathbb{R}^2}\tau(z)\,\mathrm{d}z = 2\pi\int_0^\infty\widetilde{\rho}\big(r^{1/(1-\mathsf{a})}\big)\,r\,\mathrm{d}r = 2\pi\,(1-\mathsf{a})\int_0^\infty\widetilde{\rho}(s)\,s^{1-2\,\mathsf{a}}\,\mathrm{d}s \\ = 2\pi\,(1-\mathsf{a})\int_0^\infty\rho(s)\,s\,\mathrm{d}s = (1-\mathsf{a})\int_{\mathbb{R}^2}\rho(x)\,\mathrm{d}x = 1-\mathsf{a}\,.$$

Moreover, by a similar computation,

$$\int_{\mathbb{R}^2}\rho\,\log\rho\,\mathrm{d}x + 2\,\mathsf{a}\int_{\mathbb{R}^2}\log|x|\,\rho\,\mathrm{d}x = \int_{\mathbb{R}^2}\widetilde{\rho}\,\log\widetilde{\rho}\,|x|^{-2\,\mathsf{a}}\,\mathrm{d}x = 2\pi\int_0^\infty\widetilde{\rho}(s)\,\log\widetilde{\rho}(s)\,s^{1-2\,\mathsf{a}}\,\mathrm{d}s \\ = \frac{2\pi}{1-\mathsf{a}}\int_0^\infty\tau(r)\,\log\tau(r)\,r\,\mathrm{d}r = \frac{1}{1-\mathsf{a}}\int_{\mathbb{R}^2}\tau(z)\,\log\tau(z)\,\mathrm{d}z\,.$$

Finally, by Newton's theorem,

$$\begin{aligned}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\rho(x)\log\frac{1}{|x-y|}\,\rho(y)\,\mathrm{d}x\,\mathrm{d}y &= \iint_{\mathbb{R}^2\times\mathbb{R}^2}\rho(x)\min\left\{\log\frac{1}{|x|},\log\frac{1}{|y|}\right\}\rho(y)\,\mathrm{d}x\,\mathrm{d}y\\ &= \iint_{\mathbb{R}^2\times\mathbb{R}^2}\widetilde{\rho}(x)\min\left\{\log\frac{1}{|x|},\log\frac{1}{|y|}\right\}\widetilde{\rho}(y)\,\frac{\mathrm{d}x}{|x|^{2\,\mathsf{a}}}\,\frac{\mathrm{d}y}{|y|^{2\,\mathsf{a}}}\\ &= (2\pi)^2\int_0^\infty\int_0^\infty\widetilde{\rho}(s)\min\left\{\log\frac{1}{s},\log\frac{1}{s'}\right\}\widetilde{\rho}(s')\,\frac{\mathrm{d}s}{s^{2\,\mathsf{a}-1}}\,\frac{\mathrm{d}s'}{(s')^{2\,\mathsf{a}-1}}\\ &= \frac{(2\pi)^2}{(1-\mathsf{a})^3}\int_0^\infty\int_0^\infty\tau(r)\min\left\{\log\frac{1}{r},\log\frac{1}{r'}\right\}\tau(r')\,r\,\mathrm{d}r\,r'\mathrm{d}r'\\ &= \frac{1}{(1-\mathsf{a})^3}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\tau(z)\,\log\frac{1}{|z-w|}\,\tau(w)\,\mathrm{d}z\,\mathrm{d}w\,.\end{aligned}$$

To summarize, we have

$$\mathscr{G}_{\mathsf{a}}[\rho]=\frac{1}{1-\mathsf{a}}\int_{\mathbb{R}^2}\tau(z)\,\log\tau(z)\,\mathrm{d}z-\frac{2}{(1-\mathsf{a})^2}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\tau(z)\,\log\frac{1}{|z-w|}\,\tau(w)\,\mathrm{d}z\,\mathrm{d}w\,.$$

By the logarithmic Hardy–Littlewood–Sobolev inequality (3), taking the normalization of τ into account, we deduce that

$$\mathscr{G}_{\mathsf{a}}[\rho]\ge-\log\left(\frac{e\pi}{1-\mathsf{a}}\right)$$

with equality if and only if, for some $\lambda>0$,

$$\tau(z)=\frac{1-\mathsf{a}}{\pi}\,\frac{\lambda^2}{(\lambda^2+|z|^2)^2}\,.$$

Translating this in terms of ρ, we obtain the claim of the lemma. □

2.3. *Additional remarks on the free energy and some open questions*

In Lemma 4, Inequality (8) holds for some finite constant $\mathscr{C}(\mathsf{a},\mathsf{b})$ if $(\mathsf{a},\mathsf{b})=(0,-2)$. We also know from Lemma 2 that $\lim_{\mathsf{a}\to1_+}\mathscr{C}(\mathsf{a},0)=-\infty$. If $\mathsf{b}=\mathsf{a}-1>0$, it is so far open to decide whether (8) holds for some $\mathscr{C}(\mathsf{a},\mathsf{b})>-\infty$. See Figure 1.

The free energy $\mathscr{F}_{\mathsf{a},\mathsf{b}}[\rho]$ is a natural Lyapunov functional for the drift-diffusion equation

$$\frac{\partial\rho}{\partial t}=\Delta\rho+\nabla\cdot\Big(\rho\left(\mathsf{a}\,\nabla V+4\pi\,\tfrac{\mathsf{b}}{M}\,\nabla W\right)\Big),\quad W=(-\Delta)^{-1}\rho\,. \tag{9}$$

Indeed we can write that $\Delta\rho=\nabla\cdot\left(\rho\,\nabla\log\rho\right)$ so that, for any smooth and sufficiently decreasing function ρ solving (9), we obtain using an integration by parts that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{F}_{\mathsf{a},\mathsf{b}}[\rho(t,\cdot)]=-\int_{\mathbb{R}^2}\rho\left|\nabla\log\rho+\mathsf{a}\,\nabla V+4\pi\,\tfrac{\mathsf{b}}{M}\,\nabla W\right|^2\mathrm{d}x\,.$$

Concerning the long time behavior of the solution of (9), we expect that $\mathscr{F}_{\mathsf{a},\mathsf{b}}[\rho(t,\cdot)]$ converges to $\mathscr{C}(\mathsf{a},\mathsf{b})$ as $t\to+\infty$ by analogy, e.g., with the Keller–Segel system (see [6, Section 4]), but it is an open question to deduce global decay rates of $\mathscr{F}_{\mathsf{a},\mathsf{b}}[\rho(t,\cdot)]$, for instance in a restricted class of solutions of (9), or even asymptotic decay rates as in [8]. Another issue is to understand the counterpart on $\mathbb{S}^2$ of the results on $\mathbb{R}^2$ using the inverse stereographic projection, as in [11,21,23].

For any $M>0$, the boundedness from below of

$$\mathscr{F}^{\mathsf{c}}_{\mathsf{a},\mathsf{b}}[\rho]:=\mathsf{a}\int_{\mathbb{R}^2}\log\left(1+|x|^2\right)\rho\,\mathrm{d}x-\frac{\mathsf{b}}{M}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\rho(x)\,\rho(y)\,\log|x-y|\,\mathrm{d}x\,\mathrm{d}y+\mathsf{c}\int_{\mathbb{R}^2}\rho\,\log\left(\frac{\rho}{M}\right)\mathrm{d}x$$

on the set $\mathscr{X}_M$ arises for any $\mathsf{c}>0$ as a straightforward consequence of Lemma 4 under the obvious condition $-2\,\mathsf{c}\le\mathsf{b}<\min\{\mathsf{a}-\mathsf{c},2\,\mathsf{a}-2\,\mathsf{c}\}$, by homogeneity. The case $\mathsf{c}=0$ is covered by Lemma 1. It is therefore a natural question to inquire what happens if $\mathsf{c}<0$.

Proposition 8. *For any* $(\mathsf{a},\mathsf{b})\in\mathbb{R}^2$ *and* $M>0$*, with the above notations, if* $\mathsf{c}<0$*, then*

$$\inf_{\rho\in\mathscr{X}_M}\mathscr{F}^{\mathsf{c}}_{\mathsf{a},\mathsf{b}}[\rho]=-\infty\,.$$

Proof. The key point of the proof is that $\rho\mapsto\mathsf{c}\int_{\mathbb{R}^2}\rho\,\log\rho\,\mathrm{d}x$ with $\mathsf{c}<0$ is a concave functional. Let $\rho\in\mathscr{X}_1$ be a function supported in the unit ball. For any $\varepsilon\in(0,1/4)$ and $n\in\mathbb{N}\setminus\{0\}$, let

$$R_{\varepsilon,n}(x):=\frac{1}{n^2}\sum_{k,\ell=1}^{n}\varepsilon^{-2}\,\rho\left(\varepsilon^{-1}\left(x-(k,\ell)\right)\right)\quad\forall\,x\in\mathbb{R}^2\,.$$

In order to investigate the limits $\varepsilon\to0_+$ and $n\to+\infty$, we compute

$$\begin{aligned}&\int_{\mathbb{R}^2}R_{\varepsilon,n}\,\log R_{\varepsilon,n}\,\mathrm{d}x=-\log\left(n^2\,\varepsilon^2\right)+\int_{\mathbb{R}^2}\rho\,\log\rho\,\mathrm{d}x=-2\,\log(n\,\varepsilon)+O(1)\,,\\&\int_{\mathbb{R}^2}\log\left(1+|x|^2\right)R_{\varepsilon,n}\,\mathrm{d}x\lesssim\log\left(1+2\,n^2\right)=2\,\log n\left(1+o(1)\right),\\&\left|\iint_{\mathbb{R}^2\times\mathbb{R}^2}R_{\varepsilon,n}(x)\,R_{\varepsilon,n}(y)\,\log|x-y|\,\mathrm{d}x\,\mathrm{d}y\right|\lesssim\frac{|\log\varepsilon|}{n^2}+\frac{\log\left(1+2\,n^2\right)}{2\,n^2}=\frac{|\log(\varepsilon/n)|}{n^2}\left(1+o(1)\right).\end{aligned}$$

With the choice $\varepsilon=n^{-A}$ for some $A>0$ large enough, we find that $\mathsf{c}\int_{\mathbb{R}^2}R_{\varepsilon,n}\,\log R_{\varepsilon,n}\,\mathrm{d}x\sim(A-1)\,|\mathsf{c}|\,\log n\to-\infty$ as $n\to+\infty$ and this term dominates the other ones. This concludes the proof. □

3. Logarithmic interpolation inequalities and Schrödinger energy estimates

We are now going to study the Schrödinger energy $\mathscr{E}$ defined by (1). As we shall see, the kinetic energy $\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x$ completely changes the picture and considering $\mathsf{c}<0$ makes sense.

3.1. *A new logarithmic interpolation inequality*

Here we combine logarithmic Hardy–Littlewood–Sobolev inequalities with the logarithmic Sobolev inequality to produce a new logarithmic interpolation inequality. This new inequality is more directly connected with the Schrödinger–Poisson system (SP).

In dimension $d=2$, with the Gaussian measure defined as $\mathrm{d}\mu=\mu(x)\,\mathrm{d}x$ where $\mu(x)=(2\,\pi)^{-1}\,\exp(-|x|^2/2)$, the *Gaussian logarithmic Sobolev inequality* reads

$$\int_{\mathbb{R}^2}|\nabla v|^2\,\mathrm{d}\mu\ge\frac{1}{2}\int_{\mathbb{R}^2}|v|^2\,\log|v|^2\,\mathrm{d}\mu\tag{10}$$

for any function $v\in\mathrm{H}^1(\mathbb{R}^2,\mathrm{d}\mu)$ such that $\int_{\mathbb{R}^2}|v|^2\,\mathrm{d}\mu=1$, and there is equality if and only if $v\equiv1$ (see [9, Theorem 4]). With $u=v\,\sqrt{\mu}$, it is a classical fact that Inequality (10) is equivalent to the standard *Euclidean logarithmic Sobolev inequality* established in [29] (also see [27] for an earlier related result) which can be written in dimension $d=2$ as

$$\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x\ge\frac{1}{2}\int_{\mathbb{R}^2}|u|^2\,\log\left(\frac{|u|^2}{\|u\|_2^2}\right)\mathrm{d}x+\frac{1}{2}\,\log\left(2\,\pi\,e^2\right)\|u\|_2^2\tag{11}$$

for any function $u\in\mathrm{H}^1(\mathbb{R}^2,\mathrm{d}x)$. This inequality is not invariant under scaling. By applying (11) to the scaled function $u_\lambda(x)=\lambda\,u(\lambda\,x)$, we obtain

$$\lambda^2\,\|\nabla u\|_2^2-\log\lambda\,\|u\|_2^2\ge\frac{1}{2}\int_{\mathbb{R}^2}|u|^2\,\log\left(\frac{|u|^2}{\|u\|_2^2}\right)\mathrm{d}x+\frac{1}{2}\,\log\left(2\,\pi\,e^2\right)\|u\|_2^2\tag{12}$$

for any $\lambda>0$. The scaling parameter λ can be optimized in order to obtain the *Euclidean logarithmic Sobolev inequality in scale invariant form*

$$\|u\|_2^2\,\log\left(\frac{1}{\pi\,e}\,\frac{\|\nabla u\|_2^2}{\|u\|_2^2}\right)\ge\int_{\mathbb{R}^2}|u|^2\,\log\left(\frac{|u|^2}{\|u\|_2^2}\right)\mathrm{d}x\tag{13}$$

for any function $u \in \mathrm{H}^1(\mathbb{R}^2, \mathrm{d}x)$, that can be found in [40, Theorem 2], [37, Inequality (2.3)], [19, Appendix B] or [9, Inequality (26)]. See [26, 38] for further references and consequences. Of course, (11) can be deduced from (13), so that (10), (11) and (13) are equivalent, and none of these inequalities is limited to $d = 2$, but constants in (11) and (13) have to be adapted to the dimension if $d \neq 2$.

It is possible to combine (3) and (11) with $\rho = |u|^2$ into

$$\int_{\mathbb{R}^2} |\nabla u|^2 \,\mathrm{d}x \geq \frac{2\pi}{\|u\|_2^2} \int_{\mathbb{R}^2} |u|^2 (-\Delta)^{-1} |u|^2 \,\mathrm{d}x + \frac{1}{2} \log(2e)\, \|u\|_2^2 \tag{14}$$

where

$$2\pi \int_{\mathbb{R}^2} |u|^2 (-\Delta)^{-1} |u|^2 \,\mathrm{d}x = -\iint_{\mathbb{R}^2 \times \mathbb{R}^2} |u(x)|^2 \log|x-y|\, |u(y)|^2 \,\mathrm{d}y.$$

By applying (14) to the scaled function $u_\lambda(x) = \lambda\, u(\lambda\, x)$, we obtain that

$$\lambda^2 \int_{\mathbb{R}^2} |\nabla u|^2 \,\mathrm{d}x - \|u\|_2^2 \log\lambda \geq \frac{2\pi}{\|u\|_2^2} \int_{\mathbb{R}^2} |u|^2 (-\Delta)^{-1} |u|^2 \,\mathrm{d}x + \frac{1}{2} \log(2e)\, \|u\|_2^2 \tag{15}$$

for any $\lambda > 0$. By optimizing on λ, we obtain the following scale invariant inequality.

Proposition 9. *For any function $u \in \mathrm{H}^1(\mathbb{R}^2)$, we have*

$$2\pi \int_{\mathbb{R}^2} |u|^2 (-\Delta)^{-1} |u|^2 \,\mathrm{d}x \leq \|u\|_2^4 \log\left(\frac{\|\nabla u\|_2}{\|u\|_2}\right). \tag{16}$$

Since (3) and (11) admit incompatible optimal functions, respectively the function $\rho = \rho_\star$ given by (4) and the Gaussian function $u(x) = (2\pi)^{-1/2} \sqrt{M}\, e^{-|x|^2/4} = \sqrt{M\,\mu(x)}$, up to multiplications by a constant, scalings and translations, equality is not achieved in (16) by a function $u \in \mathrm{H}^1(\mathbb{R}^2)$.

3.2. *Interpolations inequalities in higher dimensions*

For comparison, let us briefly consider the case of higher dimensions, that is, the case of the Euclidean space $\mathbb{R}^d$ with $d \geq 3$. We can refer for instance to [3] for more detailed considerations on scalings in absence of an external potential. The Gagliardo–Nirenberg inequality

$$\mathscr{C}_{\mathrm{GN}}\, \|\nabla u\|_2^{\vartheta}\, \|u\|_2^{1-\vartheta} \geq \|u\|_p \quad \forall\, u \in \mathrm{H}^1(\mathbb{R}^d) \tag{17}$$

holds with $\theta = d\,\frac{p-2}{2p}$ for any $p \in (2, 2^*]$, where $2^* = \frac{2d}{d-2}$ is the critical Sobolev exponent. Optimality is attained by the so-called *Lommel functions*, which are radial functions according to, e.g., [39], and are defined by the Euler–Lagrange but have no explicit formulation in terms of the usual special functions: see [31,32]. This can be combined with the *critical Hardy–Littlewood–Sobolev inequality*,

$$\frac{1}{(d-2)\,|\mathbb{S}^{d-1}|} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\rho(x)\,\rho(y)}{|x-y|^{d-2}} \,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^d} \rho\, (-\Delta)^{-1} \rho \,\mathrm{d}x \leq \mathscr{C}_{\mathrm{HLS}} \left(\int_{\mathbb{R}^d} |\rho|^{\frac{2d}{d+2}} \,\mathrm{d}x\right)^{1+\frac{2}{d}} \tag{18}$$

for any function $\rho \in \mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)$, to establish for $\rho = |u|^2$ that

$$\mathscr{C}_d \int_{\mathbb{R}^d} |u|^2 (-\Delta)^{-1} |u|^2 \,\mathrm{d}x \leq \|\nabla u\|_2^{d-2}\, \|u\|_2^{6-d} \quad \forall\, u \in \mathrm{H}^1(\mathbb{R}^d), \tag{19}$$

under the condition that $\frac{4d}{d+2} \leq \frac{2d}{d-2}$, that is, for

$$3 \leq d \leq 6.$$

Let us notice that the inequality is *critical* if $d = 6$ in the sense that $\int_{\mathbb{R}^6} |u|^2 (-\Delta)^{-1} |u|^2 \,\mathrm{d}x$ and $\left(\int_{\mathbb{R}^6} |\nabla u|^2 \,\mathrm{d}x\right)^2$ have the same homogeneity and scaling invariance, which is a standard source of loss of compactness along an arbitrary minimizing sequence satisfying a given $\|u\|_2$ constraint. From (17) and (18), we find out that

$$\mathscr{C}_d \ge \mathscr{C}_{\mathrm{GN}}^{-4}\,\mathscr{C}_{\mathrm{HLS}}^{-1}\,.$$

The above estimate is strict because optimal functions do not coincide in (17) and (18) if $3 \le d \le 5$. In dimension $d = 6$, we have that $\mathscr{C}_6 = \mathscr{C}_{\mathrm{GN}}^{-4}\,\mathscr{C}_{\mathrm{HLS}}^{-1}$ is sharp, with equality in (19) achieved by the Aubin–Talenti function $x \mapsto (1+|x|^2)^{-2}$.

3.3. *Bounds on the Schrödinger energy*

Let $\gamma_+ := \max\{\gamma, 0\}$ and consider $\mathscr{E}$ as in (1).

Theorem 10. *Let α, β, γ be real parameters and assume that $M > 0$. Then*

(i) *$\mathscr{E}$ is not bounded from below on $\mathscr{H}_M$ if one of the following conditions is satisfied:*
  (a) *$\alpha < 0$,*
  (b) *$\alpha \ge 0$ and $M\beta > \min\{2\alpha - \gamma, 4\alpha - 2\gamma\}$.*

(ii) *$\mathscr{E}$ is bounded from below on $\mathscr{H}_M$ if either $\alpha = 0$, $\beta \le 0$ and $M\beta + 2\gamma \le 0$, or $\alpha > 0$ and one of the following conditions is satisfied:*
  (a) *$\gamma \le 0$ and $M\beta \le 2\alpha$,*
  (b) *$\gamma > 0$, $M\beta \le 4\alpha - 2\gamma$ and $M\beta < 2\alpha - \gamma$.*

Two cases covered by Theorem 10 are shown in Figure 2.

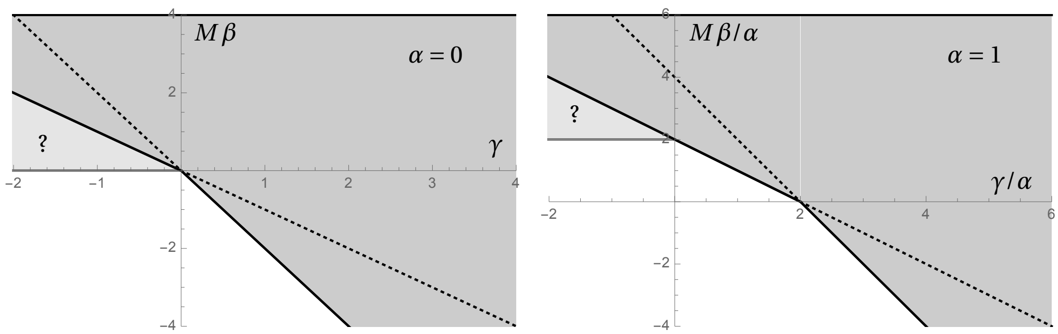

Figure 2. White (resp. dark grey) area corresponds to the domain in which $\mathscr{E}$ is bounded (resp. unbounded) from below with $\alpha = 0$ on the left and $\alpha = 1$ on the right. Whether $\mathscr{E}$ is bounded in the light grey domain or not is open so far.

Proof. Let us start by the proof of (i), i.e., the cases for which $\inf\{\mathscr{E}[u] : u \in \mathscr{H}_M\} = -\infty$. Case (a) corresponds to $\alpha < 0$ and can be dealt with using translations as in the proof of Lemma 5: $\lim_{|x_0|\to+\infty} \mathscr{E}[u(\cdot - x_0)] = -\infty$. Next let $u_\lambda(x) := \lambda\, u(\lambda\, x)$ and notice that

$$\int_{\mathbb{R}^2} |\nabla u_\lambda|^2\,\mathrm{d}x = \lambda^2 \int_{\mathbb{R}^2} |\nabla u|^2\,\mathrm{d}x = o(\log\lambda) \quad \text{as} \quad \lambda \to 0_+\,,$$

so that, with $\rho_\lambda = |u_\lambda|^2$,

$$\mathscr{E}[u_\lambda] \sim 2\alpha \int_{\mathbb{R}^2} \log\left(1+|x|^2\right)\rho_\lambda\,\mathrm{d}x + 2\pi\beta \int_{\mathbb{R}^2} \rho_\lambda\,(-\Delta)^{-1}\rho_\lambda\,\mathrm{d}x + \gamma \int_{\mathbb{R}^2} \rho_\lambda \log\rho_\lambda\,\mathrm{d}x\,.$$

By arguing as in Lemma 5, we obtain that $\lim_{\lambda\to 0_+} \mathscr{E}[u_\lambda] = -\infty$ in case (b).

Concerning (ii), the boundedness from below of $\mathscr{E}$ is as follows. From (12) and (15), we learn that

$$\int_{\mathbb{R}^2} |\nabla u|^2 \,\mathrm{d}x \ge \frac{1}{2\,\lambda_1^2} \int_{\mathbb{R}^2} |u|^2 \log\left(\frac{|u|^2}{M}\right) \mathrm{d}x + \frac{\log\left(2\,\pi\, e^2\,\lambda_1^2\right)}{2\,\lambda_1^2}\, M \tag{20}$$

and

$$\int_{\mathbb{R}^2} |\nabla u|^2 \,\mathrm{d}x \ge \frac{2\,\pi}{M\,\lambda_2^2} \int_{\mathbb{R}^2} |u|^2\,(-\Delta)^{-1}|u|^2 \,\mathrm{d}x + \frac{\log\left(2\,e\,\lambda_2^2\right)}{2\,\lambda_2^2}\, M \tag{21}$$

with $\|u\|_2^2 = M$. Here λ_1 and λ_2 are two arbitrary positive parameters. Let us distinguish various cases:

(1) If $\alpha = 0$, $\beta \le 0$ and $\gamma \le 0$, the boundedness from below of $\mathscr{E}$ is a direct consequence of (20) and (21). The case $\alpha = 0$, $\beta < 0$ and $\gamma > 0$ can be reduced to the case $\alpha = 0$ and $\gamma = 0$ using (3) if $M\,\beta + 2\,\gamma \le 0$.

(2) If either $\alpha > 0$, $\beta \le 0$ and $\gamma \le 0$, or $\alpha > 0$, $\beta > 0$, $\gamma \le 0$ and $M\,\beta + 2\,\gamma \le 0$, we conclude as above.

(3) If $\alpha > 0$, $\beta > 0$ and $\gamma \le 0$, the boundedness from below is a direct consequence of Lemma 1 if $M\,\beta - 2\,\alpha \le 0$.

(4) If $\alpha > 0$, $\gamma > 0$ and $M\,\beta + 2\,\gamma \ge 0$, we notice that $\mathscr{E}[u] \ge \gamma\,\mathscr{F}_{\mathsf{a},\mathsf{b}}\left[|u|^2\right]$ with $\mathsf{a} = 2\,\alpha/\gamma$ and $\mathsf{b} = M\,\beta/\gamma$. The result of Lemma 4 applies and the condition $\mathsf{b} < \min\{\mathsf{a}-1, 2\,\mathsf{a}-2\}$ can be rewritten as $M\,\beta < \min\left\{2\,\alpha-\gamma, 4\,\alpha-2\,\gamma\right\}$. The case $M\,\beta = 4\,\alpha-2\,\gamma$ corresponds to $\mathsf{b} = 2\,\mathsf{a}-2$ and it is covered by Lemmas 6 and 7.

(5) If $\alpha > 0$, $\gamma > 0$ and $M\,\beta + 2\,\gamma < 0$, we conclude by observing that

$$\mathscr{E}[u] \ge \gamma\,\mathscr{F}_{\mathsf{a},-2}\left[\,|u|^2\,\right] + \int_{\mathbb{R}^2} |\nabla u|^2 \,\mathrm{d}x + \frac{2\pi}{M}\,(M\,\beta + 2\,\gamma) \int_{\mathbb{R}^2} |u|^2\,(-\Delta)^{-1}|u|^2 \,\mathrm{d}x\,,$$

where, because $M\,\beta + 2\,\gamma < 0$, the sum of the last two terms is bounded from below in view (21) and where Lemma 4 guarantees that $\mathscr{F}_{\mathsf{a},-2}\left[\,|u|^2\,\right]$ is bounded from below. □

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