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Fast reaction limit and forward-backward diffusion: A Radon–Nikodym approach

Jakub Skrzeczkowski[*, a]

[a] Faculty of Mathematics, Informatics and Mechanics, University of Warsaw, Stefana Banacha 2, 02-097 Warsaw, Poland

E-mail: jakub.skrzeczkowski@student.uw.edu.pl

Abstract. We consider two singular limits: a fast reaction limit with a non-monotone nonlinearity and a regularization of the forward-backward diffusion equation. We derive pointwise identities satisfied by the Young measure generated by these problems. As a result, we obtain an explicit formula for the Young measure even without the non-degeneracy assumption used in the previous works. The main new idea is an application of the Radon–Nikodym theorem to decompose the Young measure.

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1. Introduction and main results

1.1. *Presentation of the problem*

In this paper, we are interested in the limiting behavior (as $\varepsilon \to 0$) of the following problems: for the reaction-diffusion system

$$\partial_t u^\varepsilon = \frac{v^\varepsilon - F\left(u^\varepsilon\right)}{\varepsilon}, \qquad u^\varepsilon(0,x) = u_0(x), \qquad \frac{\partial}{\partial \mathbf{n}} u^\varepsilon(t,x) = 0 \ \text{ for } \ x \in \partial\Omega \tag{1}$$

$$\partial_t v^\varepsilon = \Delta v^\varepsilon + \frac{F\left(u^\varepsilon\right) - v^\varepsilon}{\varepsilon}, \qquad v^\varepsilon(0,x) = v_0(x), \qquad \frac{\partial}{\partial \mathbf{n}} v^\varepsilon(t,x) = 0 \ \text{ for } \ x \in \partial\Omega \tag{2}$$

and for the regularization of the forward-backward parabolic equation $\partial_t u = \Delta F(u)$

$$\partial_t u^\varepsilon = \Delta v^\varepsilon, \qquad u^\varepsilon(0,x) = u_0(x), \tag{3}$$

$$v^\varepsilon = F\left(u^\varepsilon\right) + \varepsilon\, \partial_t u^\varepsilon, \qquad \frac{\partial}{\partial \mathbf{n}} v^\varepsilon(t,x) = 0 \ \text{ for } \ x \in \partial\Omega \tag{4}$$

[*] Corresponding author.

Both problems are posed on some bounded and smooth domain $\Omega \subset \mathbb{R}^d$. Moreover, they admit unique, global-in-time classical solutions cf. Lemmas 10 and 17.
The initial conditions u_0, v_0 and the nonlinearity F satisfy the following.

Assumption 1 (Initial data for (1)–(2)**).** *Functions $u_0(x)$, $v_0(x)$ satisfy*

(1) *(nonnegativity) $u_0, v_0 \geq 0$.*
(2) *(regularity) $u_0, v_0 \in C^{2+\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$.*
(3) *(boundary condition) u_0, v_0 satisfy the Neumann boundary condition.*

Assumption 2 (Initial data for (3)–(4)**).** *Function $u_0(x)$ belongs to $L^\infty(\Omega)$ and satisfy $u_0(x) \geq 0$ for a.e. $x \in \Omega$.*

Assumption 3 (Reaction function F**).** *We assume that the function $F(u)$ satisfies:*

(1) *(nonnegativity) $F(0) = 0$ and $F \geq 0$.*
(2) *(piecewise monotonicity) There are $\alpha_- < \alpha_+ < \beta_- < \beta_+$ such that $F(\beta_-) = F(\alpha_-)$, $F(\alpha_+) = F(\beta_+)$, F is strictly increasing on $(-\infty, \alpha_+) \cup (\beta_-, \infty)$ and strictly decreasing on (α_+, β_-) (see Figure 1). Moreover, $\lim_{u\to\infty} F(u) = \infty$.*
(3) *(regularity) F is Lipschitz continuous. Moreover, it is continuously differentiable on each of the intervals $(-\infty, \alpha_+)$, (α_+, β_-) and (β_-, ∞).*

In what follows, it will be crucial to introduce a notation related to the inverses of function F.

Notation 4. *Let $S_1(\lambda) \leq S_2(\lambda) \leq S_3(\lambda)$ be the solutions of equation $F(S_i(\lambda)) = \lambda$ (see Figure 1). These are inverses of F satisfying*

$$S_1 : \left(-\infty, f_+\right] \to (-\infty, \alpha_+], \qquad S_2 : \left(f_-, f_+\right) \to \left(\alpha_+, \beta_-\right), \qquad S_3 : \left[f_-, \infty\right) \to \left[\beta_-, \infty\right).$$

Their role is to focus our analysis on parts of the plot of F where monotonicity of F does not change. By a small abuse of notation, we extend functions S_i by a constant value to the whole of $\mathbb{R}$. We usually write

$$\begin{aligned} I_1 &= (-\infty, \alpha_+], & I_2 &= \left(\alpha_+, \beta_-\right), & I_3 &= \left[\beta_-, \infty\right), \\ J_1 &= \left(-\infty, f_+\right], & J_2 &= \left(f_-, f_+\right), & J_3 &= \left[f_-, \infty\right). \end{aligned}$$

for images of functions S_1, S_2, S_3 and for their domains.

System (1)–(2) is an interesting toy model for studying oscillations in reaction-diffusion systems as they are known to occur in their steady states [28]. For monotone F the problem is fairly classical and has been studied for a great variety of reaction-diffusion systems, also with more than two components [5, 6, 14, 29] or reaction-diffusion equation coupled with an ODE [21]. In the limit $\varepsilon \to 0$, one obtains widely studied cross-diffusion systems [9, 10, 15, 16, 22, 24] where the gradient of one quantity induces a flux of another one. A slightly different yet connected type of problem deals with the fast-reaction limit for irreversible reactions which leads to free boundary problems [11, 17, 20]. Finally, for non-monotone F as in this paper, the only available result was established very recently in [33] (see below). We also refer to the recent stability analysis of problems of type (1)–(2) [12, 13, 25].
System (3)–(4) was extensively studied by Plotnikov [34, 35] who identified the limits as $\varepsilon \to 0$ in terms of Young measures (see below) and by Novick–Cohen and Pego who studied its asymptotics with $\varepsilon > 0$ fixed [31]. The regularization term in (3)–(4) was also generalized in [3, 4, 40]. Recently, so-called nonstandard analysis was used to study the limit problem in the space of grid functions [7, 8].
It is known [33, 35] that both systems exhibit the following surprising phenomenon: as $\varepsilon \to 0$, $F(u^\varepsilon) \to v$ and $v^\varepsilon \to v$ converge strongly without any known a priori estimates allowing to conclude so. As a consequence, u^ε converges weakly to

$$u(t,x) = \lambda_1(t,x)\, S_1\big(v(t,x)\big) + \lambda_2(t,x)\, S_2\big(v(t,x)\big) + \lambda_3(t,x)\, S_3\big(v(t,x)\big)$$

where $\sum_{i=1}^{3}\lambda_i(t,x)=1$. More precisely, if $\{\mu_{t,x}\}_{t,x}$ is the Young measure generated by $\{u^\varepsilon\}_{\varepsilon>0}$, we have

$$\mu_{t,x}=\lambda_1(t,x)\,\delta_{S_1(v(t,x))}+\lambda_2(t,x)\,\delta_{S_2(v(t,x))}+\lambda_3(t,x)\,\delta_{S_3(v(t,x))}$$

which represents oscillations between phases $S_1(v(t,x))$, $S_2(v(t,x))$ and $S_3(v(t,x))$. The proof exploits a family of energies as well as analysis of related Young measures in the spirit of Murat and Tartar's work on conservation laws and compensated compactness [30, 41]. The numerical simulations suggest that the middle state, referred to as an unstable phase, is not present [19] which motivates research on two-phase solutions to such problems [23,26,39,42] with a result of nonuniqueness when the unstable phase is present [43].

So far, the main assumption on F that allows to deduce the strong convergence is the so-called non-degeneracy condition: for (1)–(2) it reads

$$\text{for all intervals } R\subset\left(f_-,f_+\right):\quad \sum_{i=1}^{3} a_i\,\left(S_i'(r)+1\right)=0\quad \text{for } r\in R \implies a_1+a_2+a_3=0 \tag{5}$$

while for (3)–(4) it reads

$$\text{for all intervals } R\subset\left(f_-,f_+\right):\quad \sum_{i=1}^{3} a_i\,S_i'(r)=0\qquad \text{for } r\in R \implies a_1+a_2+a_3=0. \tag{6}$$

While it is fairly classical for this type of problems [1, 31, 35], it is hard to be verified for a given nonlinearity F. Moreover, the non-degeneracy condition excludes piecewise affine functions used in more explicit computations as in [26].

1.2. *Main results and outline of the paper*

In this paper, we take a slightly different approach to study the strong convergence. Although we use a family of energy identities to characterize the Young measure as Plotnikov [35], we aim at pointwise identities to obtain an optimal amount of information from these energy identities, in particular we deduce new results. To achieve this, we use the Radon–Nikodym Theorem as explained below.

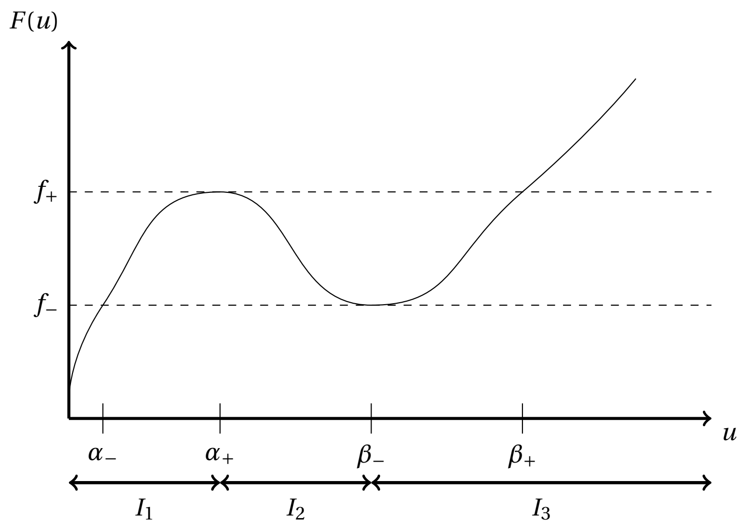

Figure 1. Plot of a typical function F. It is strictly increasing in the intervals $I_1:=(-\infty,\alpha_+]$, $I_3:=[\beta_-,\infty)$ and strictly decreasing in $I_2:=(\alpha_+,\beta_-)$. For $r\in[f_-,f_+]$, the function F is not invertible and equation $F(u)=r$ has three roots $u=S_1(r)\le S_2(r)\le S_3(r)$.

Let $\{\mu_{t,x}\}_{t,x}$ be the Young measure generated by sequence $\{u^{\varepsilon}\}_{\varepsilon\in(0,1)}$ solving either (1)–(2) or (3)–(4), i.e. for any bounded function $G:\mathbb{R}\to\mathbb{R}$ we have (up to a subsequence and for a.e. $(t,x)\in(0,T)\times\Omega$)

$$G\big(u^{\varepsilon}\big)\xrightarrow{*}\int_{\mathbb{R}}G(\lambda)\,\mathrm{d}\mu_{t,x}(\lambda),$$

see Appendix A.3 if necessary. To analyze the amount of $\mu_{t,x}$ on intervals I_1, I_2 and I_3, see Figure 1, we introduce restrictions

$$\mu^{(1)}_{t,x}:=\mu_{t,x}\mathbb{1}_{I_1},\qquad \mu^{(2)}_{t,x}:=\mu_{t,x}\mathbb{1}_{I_2},\qquad \mu^{(3)}_{t,x}:=\mu_{t,x}\mathbb{1}_{I_3}.$$

The reason we introduce these measures is that in the sequel, we will gain information only about measure $F^{\#}\mu_{t,x}$, i.e. a push-forward (image) of $\mu_{t,x}$ along F defined as

$$F^{\#}\mu_{t,x}=\mu_{t,x}\Big(F^{-1}(A)\Big),\qquad A\subset\mathbb{R}^{+}.$$

Observe that for all $i=1,2,3$, measures $F^{\#}\mu^{(i)}_{t,x}$ are absolutely continuous with respect to $F^{\#}\mu_{t,x}$. Therefore, the Radon–Nikodym theorem implies that there exist densities $g^{(1)}(\lambda)$, $g^{(2)}(\lambda)$ and $g^{(3)}(\lambda)$ such that

$$F^{\#}\mu^{(i)}_{t,x}(A)=\int_A g^{(i)}(\lambda)\,\mathrm{d}F^{\#}\mu_{t,x}(\lambda),\qquad i=1,2,3. \tag{7}$$

We also note that for all $A\subset\mathbb{R}^{+}$

$$\sum_{i=1}^{3}F^{\#}\mu^{(i)}_{t,x}(A)=\sum_{i=1}^{3}\mu_{t,x}\Big(F^{-1}(A)\cap I_i\Big)=\mu_{t,x}\Big(F^{-1}(A)\Big)=F^{\#}\mu_{t,x}(A). \tag{8}$$

In particular, from (7) and (8) we deduce that for $F^{\#}\mu_{t,x}$-a.e. λ we have

$$\sum_{i=1}^{3}g_i(\lambda)=1. \tag{9}$$

The main result of this paper reads:

Theorem 5.

(A) *Let $\{\mu_{t,x}\}_{t,x}$ be the Young measure generated by sequence $\{u^{\varepsilon}\}_{\varepsilon\in(0,1)}$ solving* (1)–(2). *Then, for almost all λ_0 (with respect to $F^{\#}\mu_{t,x}$) and all $\tau_0\neq f_-,f_+$ we have*

$$\sum_{i=1}^{3}\big(S_i'(\tau_0)+1\big)\Big[\mathbb{1}_{\lambda_0>\tau_0}\,g_i(\lambda_0)-F^{\#}\mu^{(i)}_{t,x}(\tau_0,\infty)\Big]+\big(S_1'(\tau_0)-S_2'(\tau_0)\big)\Big(F^{\#}\mu^{(1)}_{t,x}(\mathbb{R}^{+})-g_1(\lambda_0)\Big)=0.$$

where S_i are the inverses of F as in Notation 4 and g_i are the Radon–Nikodym densities as in (7). *Moreover, for $\lambda_0\neq f_-,f_+$ we have*

$$\big(1-F^{\#}\mu_{t,x}\{\lambda_0\}\big)\sum_{i=1}^{3}\big(S_i'(\lambda_0)+1\big)\,g_i(\lambda_0)=0. \tag{10}$$

(B) *Let $\{\mu_{t,x}\}_{t,x}$ be the Young measure generated by sequence $\{u^{\varepsilon}\}_{\varepsilon\in(0,1)}$ solving* (3)–(4). *Then, for almost all λ_0 (with respect to $F^{\#}\mu_{t,x}$) and all $\tau_0\neq f_-,f_+$ we have*

$$\sum_{i=1}^{3}S_i'(\tau_0)\Big[\mathbb{1}_{\lambda_0>\tau_0}\,g_i(\lambda_0)-F^{\#}\mu^{(i)}_{t,x}(\tau_0,\infty)\Big]+\big(S_1'(\tau_0)-S_2'(\tau_0)\big)\Big(F^{\#}\mu^{(1)}_{t,x}(\mathbb{R}^{+})-g_1(\lambda_0)\Big)=0.$$

where S_i are the inverses of F as in Notation 4 and g_i are the Radon–Nikodym densities as in (7). *Moreover, for $\lambda_0\neq f_-,f_+$ we have*

$$\big(1-F^{\#}\mu_{t,x}\{\lambda_0\}\big)\sum_{i=1}^{3}S_i'(\lambda_0)\,g_i(\lambda_0)=0. \tag{11}$$

As $F^{\#}\mu_{t,x}$ turns out to be the Young measure generated by $\{v^{\varepsilon}\}_{\varepsilon>0}$ cf. Corollary 11, strong convergence $v^{\varepsilon} \to v$ can be deduced if one proves that $F^{\#}\mu_{t,x}$ is the Dirac measure cf. Lemma 23 (A). Equation (10) shows that the latter follows if one finds λ_0 in the support such that the sum $\sum_{i=1}^{3}(S_i'(\lambda_0)+1)\,g_i(\lambda_0)$ does not vanish (some additional care is needed when $\lambda_0 = f_-, f_+$, cf. Lemma 15).

We remark that similar forms of the entropy equality as in Theorem 5 are well-known however they have not been formulated as in our paper. In particular, they are usually stated without explicitly identified coefficients standing next to $(S_i'(\tau_0)+1)$.
First, we show that the form presented in Theorem 5 can be used to recover the result of Plotnikov [35] and of Perthame and Skrzeczkowski [33].

Theorem 6. *Suppose that non-degeneracy condition* (5)–(6) *is satisfied. Then, $v^{\varepsilon} \to v$ strongly in $L^2((0,T)\times\Omega)$. Moreover, there are nonnegative numbers $\lambda_1(t,x)$, $\lambda_2(t,x)$, $\lambda_3(t,x)$ such that $\sum_{i=1}^{3}\lambda_i(t,x) = 1$ and*

$$\mu_{t,x} = \lambda_1(t,x)\,\delta_{S_1(v(t,x))} + \lambda_2(t,x)\,\delta_{S_2(v(t,x))} + \lambda_3(t,x)\,\delta_{S_3(v(t,x))}.$$

Now, we move to the new results that easily follow from Theorem 5. The first one asserts that if one knows a priori that the Young measure $\{\mu_{t,x}\}_{t,x}$ is not supported in the interval I_2 where F is decreasing, the strong convergence occurs. The fact concerning the support of $\{\mu_{t,x}\}_{t,x}$ was observed in the numerical simulations [19] and so, the next theorem may serve as a tool to prove strong convergence without the non-degeneracy condition.

Theorem 7. *Suppose that:*

- *there exists $\tau_0 \in (f_-, f_+)$ such that $S_1'(\tau_0) - S_3'(\tau_0) \neq 0$,*
- *Young measure $\{\mu_{t,x}\}_{t,x}$ is not supported in the interval I_2 (see Figure 1).*

Then, $v^{\varepsilon} \to v$ strongly in $L^2((0,T)\times\Omega)$. Moreover, there are nonnegative numbers $\lambda_1(t,x)$, $\lambda_3(t,x)$ such that $\lambda_1(t,x)+\lambda_3(t,x) = 1$ and

$$\mu_{t,x} = \lambda_1(t,x)\,\delta_{S_1(v(t,x))} + \lambda_3(t,x)\,\delta_{S_3(v(t,x))}.$$

The next result shows that the systems (1)–(2) and (3)–(4) are not exactly the same in view of the strong convergence. Indeed, for the first one, we can establish a simple condition on F implying strong convergence of $v^{\varepsilon} \to v$ that does not exclude piecewise affine functions as in the case of non-degeneracy condition (5).

Theorem 8. *Let $\{\mu_{t,x}\}_{t,x}$ be the Young measure generated by sequence $\{u^{\varepsilon}\}_{\varepsilon\in(0,1)}$ solving* (1)–(2)*. Suppose that:*

- *there exists $\tau_0 \in (f_-, f_+)$ such that $S_1'(\tau_0) - S_3'(\tau_0) \neq 0$,*
- *$S_2'(\lambda)+1 > 0$ for all $\lambda \in (f_-, f_+)$.*

Then, $v^{\varepsilon} \to v$ strongly in $L^2((0,T)\times\Omega)$. Moreover, there are nonnegative numbers $\lambda_1(t,x)$, $\lambda_2(t,x)$, $\lambda_3(t,x)$ such that $\sum_{i=1}^{3}\lambda_i(t,x) = 1$ and

$$\mu_{t,x} = \lambda_1(t,x)\,\delta_{S_1(v(t,x))} + \lambda_2(t,x)\,\delta_{S_2(v(t,x))} + \lambda_3(t,x)\,\delta_{S_3(v(t,x))}.$$

As an example, the following function F satisfies assumptions of Theorem 8:

$$F(\lambda) = \begin{cases} 2\lambda & \text{if } \lambda \in [0,1], \\ 3-2\lambda & \text{if } \lambda \in \left[1, \frac{5}{4}\right], \\ 4\lambda - \frac{9}{2} & \text{if } \lambda \in \left[\frac{5}{4}, \infty\right). \end{cases}$$

Then, $S_1'(\lambda) = \frac{1}{2}$, $S_2'(\lambda) = -\frac{1}{2}$ and $S_3'(\lambda) = \frac{1}{4}$ so that $S_1'(\lambda) - S_3'(\lambda) = \frac{1}{4} \neq 0$ and $S_2'(\lambda)+1 = \frac{1}{2} > 0$. Note that F does not satisfy non-degeneracy condition (5) that was used in the previous paper on the fast reaction limit with non-monotone reaction function [33].

The proofs of Theorem 7 and 8 are based on equation (10), namely one uses $g_1(\lambda_0)+g_2(\lambda_0)+g_3(\lambda_0)=1$ to show that for $\lambda_0 \in \operatorname{supp} F$ we have $F^{\#}\mu_{t,x}\{\lambda_0\}=1$. Note that (10) is not valid for $\lambda_0 = f_-, f_+$ so some additional care is needed if the support of measure $F^{\#}\mu_{t,x}$ accumulates only in these points. This is studied in Lemma 15 and it requires an additional assumption that $S_1'(\tau)-S_3'(\tau)$ does not vanish at least for one value of τ, see also Remark 16.

The structure of the paper is as follows. In Section 2 we review (well-known) properties of the fast-reaction system (1)–(2). Then, in Section 3 we use the compensated compactness approach to prove Theorem 5. In Section 4 we prove Theorems 6, 7 and 8 while in Section 5 we show how to easily adapt proofs of Theorems 5–7 to the case of system (3)–(4). Finally, Appendix A provides the necessary background on Young measures, supports of measures and compensated compactness results.

2. Properties of the fast-reaction system (1)–(2)

We begin by recalling the energy equality and the well-posedness result from [33]. As parts (4)–(6) of Lemma 10 were not stated in [33] in the form we need here, we prove these results below.

Lemma 9 (energy equality). *Given a smooth test function $\phi:\mathbb{R}\to\mathbb{R}$, we define*

$$\Psi(\lambda) := \int_0^\lambda \phi\big(F(\tau)\big)\,\mathrm{d}\tau, \qquad \Phi(\lambda) := \int_0^\lambda \phi(\tau)\,\mathrm{d}\tau. \tag{12}$$

Then, if $(u^\varepsilon, v^\varepsilon)$ solve (1)–(2), it holds

$$\partial_t \Psi\left(u^\varepsilon\right) + \partial_t \Phi\left(v^\varepsilon\right) = \Delta\Phi\left(v^\varepsilon\right) - \phi'\left(v^\varepsilon\right)\left|\nabla v^\varepsilon\right|^2 - \frac{\left(v^\varepsilon - F\left(u^\varepsilon\right)\right)\Big(\phi\left(v^\varepsilon\right)-\phi\left(F\left(u^\varepsilon\right)\right)\Big)}{\varepsilon}. \tag{13}$$

Proof. Multiplying equation for u^ε in (1)–(2) with $\phi(F(u^\varepsilon))$ and equation for v^ε in (1)–(2) with $\phi(v^\varepsilon)$ we obtain

$$\begin{aligned}
\partial_t \Psi\left(u^\varepsilon\right) &= \frac{v^\varepsilon - F\left(u^\varepsilon\right)}{\varepsilon}\,\phi\left(F\left(u^\varepsilon\right)\right),\\
\partial_t \Phi\left(v^\varepsilon\right) &= \Delta\Phi\left(v^\varepsilon\right) - \phi'\left(v^\varepsilon\right)\left|\nabla v^\varepsilon\right|^2 + \frac{F\left(u^\varepsilon\right)-v^\varepsilon}{\varepsilon}\,\phi\left(v^\varepsilon\right).
\end{aligned}$$

Summing up these equations we deduce (13). □

Lemma 10. *There exists the unique classical solution $u^\varepsilon, v^\varepsilon : [0,\infty)\times\Omega \to \mathbb{R}$ of (1)–(2) which is nonnegative and has regularity*

$$u^\varepsilon \in C^{\alpha, 1+\alpha/2}\Big([0,\infty)\times\overline{\Omega}\Big), \qquad v^\varepsilon \in C^{2+\alpha, 1+\alpha/2}\Big([0,\infty)\times\overline{\Omega}\Big).$$

Moreover, we have

1. $0 \le u^\varepsilon \le M$, $0 \le v^\varepsilon \le M$ *with* $M = \max(\|F(u_0)\|_\infty, \|u_0\|_\infty, \|v_0\|_\infty, f_+, \beta_+)$,
2. $\{\nabla v^\varepsilon\}_{\varepsilon\in(0,1)}$ *is uniformly bounded in* $L^2((0,\infty)\times\Omega)$,
3. $\{\frac{F(u^\varepsilon)-v^\varepsilon}{\sqrt{\varepsilon}}\}_{\varepsilon\in(0,1)}$ *and* $\{\sqrt{\varepsilon}\,\Delta v^\varepsilon\}_{\varepsilon\in(0,1)}$ *are uniformly bounded in* $L^2((0,\infty)\times\Omega)$,
4. $\{\partial_t u^\varepsilon + \partial_t v^\varepsilon\}_{\varepsilon\in(0,1)}$ *is uniformly bounded in* $L^2(0,T;H^{-1}(\Omega))$,
5. *for all smooth* $\varphi:\mathbb{R}\to\mathbb{R}$, $\{\nabla\varphi(v^\varepsilon)\}_{\varepsilon\in(0,1)}$ *is uniformly bounded in* $L^2((0,\infty)\times\Omega)$,
6. *for all smooth* $\phi:\mathbb{R}\to\mathbb{R}$, $\{\partial_t\Psi(u^\varepsilon)+\partial_t\Phi(v^\varepsilon)\}_{\varepsilon\in(0,1)}$ *is uniformly bounded in* $(C(0,T;H^k(\Omega)))^*$ *for sufficiently large* $k\in\mathbb{N}$.

Proof. Existence and uniqueness of the global solution as well as points (1)–(3) were proven in [33, Theorem 3.1] so we only sketch the argument. First, local well-posedness and nonnegativity follows from the classical theory [37]. To extend these results to an arbitrary interval of time, we need to prove a priori estimates as in (1). To this end, we note that thanks to (13), the nonnegative map

$$t \mapsto \int_{\Omega} \left[\Psi\left(u^{\varepsilon}(t,x)\right) + \Phi\left(v^{\varepsilon}(t,x)\right) \right] \mathrm{d}x$$

is nonincreasing whenever $\phi' \geq 0$. Choosing ϕ vanishing on $(0,M)$ and stricly increasing for (M,∞) we obtain (1) and the global well-posedness. Then, (2) and (3) follows from (13) with $\phi(v) = v$. Furthemore, (4) follows from the equality $\partial_t u^{\varepsilon} + \partial_t v^{\varepsilon} = \Delta v^{\varepsilon}$ and property (2) while (5) follows from the chain rule for Sobolev functions, boundedness of v^{ε} from (1) and (2). Finally, to see (6) we choose $k \geq d$ so that $H^k(\Omega)$ embedds continuously into $L^{\infty}(\Omega)$. Let $\varphi \in C(0,T;H^k(\Omega))$. Note that there is a constant C such that

$$\|\varphi\|_{\infty} \leq C \|\varphi\|_{C(0,T;H^k(\Omega))}, \qquad \|\varphi\|_{L^2(0,T;H^1(\Omega))} \leq C \|\varphi\|_{C(0,T;H^k(\Omega))}. \tag{14}$$

Thanks to (13) we have

$$\begin{aligned}\int_{(0,T)\times\Omega} \left(\partial_t \Psi\left(u^{\varepsilon}\right) + \partial_t \Phi\left(v^{\varepsilon}\right)\right) \varphi \,\mathrm{d}t\,\mathrm{d}x - \int_{(0,T)\times\Omega} \nabla\Phi\left(v^{\varepsilon}\right)\cdot\nabla\varphi \,\mathrm{d}t\,\mathrm{d}x \\ = -\int_{(0,T)\times\Omega} \phi'\left(v^{\varepsilon}\right)\left|\nabla v^{\varepsilon}\right|^2 \varphi \,\mathrm{d}t\,\mathrm{d}x - \int_{(0,T)\times\Omega} \frac{\left(v^{\varepsilon} - F\left(u^{\varepsilon}\right)\right)\left(\phi\left(v^{\varepsilon}\right) - \phi\left(F\left(u^{\varepsilon}\right)\right)\right)}{\varepsilon} \varphi \,\mathrm{d}t\,\mathrm{d}x.\end{aligned}$$

As $|\phi'(v^{\varepsilon})| \leq C$ and $|\phi(v^{\varepsilon}) - \phi(F(u^{\varepsilon})| \leq C|v^{\varepsilon} - F(u^{\varepsilon})|$ we use bounds (14) together with points (2) and (3) to deduce for some possibly larger constant C (independent of ε)

$$\left|\int_{(0,T)\times\Omega} \left(\partial_t \Psi\left(u^{\varepsilon}\right) + \partial_t \Phi\left(v^{\varepsilon}\right)\right) \varphi \,\mathrm{d}t\,\mathrm{d}x\right| \leq C \|\varphi\|_{C(0,T;H^k(\Omega))}. \qquad \square$$

Corollary 11. *Let $\{\mu_{t,x}\}_{t,x}$ and $\{\nu_{t,x}\}_{t,x}$ be the Young measures generated by sequences $\{u^{\varepsilon}\}_{\varepsilon>0}$ and $\{v^{\varepsilon}\}_{\varepsilon>0}$ respectively. Combining Lemma 10* (3) *and Lemma 23 (B, C) we obtain that $F^{\#}\mu_{t,x} = \nu_{t,x}$.*

3. Proof of Theorem 5 for fast-reaction system (1)–(2)

We begin by formulating the entropy equality.

Lemma 12 (Entropy equality). *Let Ψ and Φ be defined with* (12), *$\{\mu_{t,x}\}$ be the Young measure generated by sequence $\{u^{\varepsilon}\}_{\varepsilon>0}$ solving* (1)–(2) *and g_i be the densities given by* (7). *Then, for almost all λ_0 (with respect to $F^{\#}\mu_{t,x}$) we have*

$$\sum_{i=1}^{3} \left(\Psi\left(S_i(\lambda_0)\right) + \Phi(\lambda_0)\right) g_i(\lambda_0) = \sum_{i=1}^{3} \int_{\mathbb{R}^+} \left(\Psi\left(S_i(\lambda)\right) + \Phi(\lambda)\right) g_i(\lambda)\,\mathrm{d}F^{\#}\mu_{t,x}(\lambda), \tag{15}$$

where S_i are the inverses of F as in Notation 4.

Proof. Thanks to Lemma 10 (6), for all smooth $\phi : \mathbb{R} \to \mathbb{R}$, $\{\partial_t\Psi(u^{\varepsilon}) + \partial_t\Phi(v^{\varepsilon})\}_{\varepsilon\in(0,1)}$ is uniformly bounded in $(C(0,T;H^k(\Omega)))^*$. Similarly, for all smooth $\varphi : \mathbb{R} \to \mathbb{R}$, $\{\nabla\varphi(v^{\varepsilon})\}_{\varepsilon\in(0,1)}$ is uniformly bounded in $L^2((0,\infty)\times\Omega)$. Hence, Lemma 19 implies

$$\operatorname*{w^*-lim}_{\varepsilon\to 0} \left(\Psi\left(u^{\varepsilon}\right) + \Phi\left(v^{\varepsilon}\right)\right)\varphi\left(v^{\varepsilon}\right) = \operatorname*{w^*-lim}_{\varepsilon\to 0} \left(\Psi\left(u^{\varepsilon}\right) + \Phi\left(v^{\varepsilon}\right)\right) \operatorname*{w^*-lim}_{\varepsilon\to 0} \varphi\left(v^{\varepsilon}\right).$$

As $v^{\varepsilon} - F(u^{\varepsilon}) \to 0$ cf. Lemma 10 (3), we may replace v^{ε} with $F(u^{\varepsilon})$ in the identity above to obtain

$$\operatorname*{w^*-lim}_{\varepsilon\to 0} \left(\Psi\left(u^{\varepsilon}\right) + \Phi\left(F\left(u^{\varepsilon}\right)\right)\right)\varphi\left(F\left(u^{\varepsilon}\right)\right) = \operatorname*{w^*-lim}_{\varepsilon\to 0} \left(\Psi\left(u^{\varepsilon}\right) + \Phi\left(F\left(u^{\varepsilon}\right)\right)\right) \operatorname*{w^*-lim}_{\varepsilon\to 0} \varphi\left(F\left(u^{\varepsilon}\right)\right).$$

In the language of Young measures, this identity reads

$$\int_{\mathbb{R}^+} \left(\Psi(\lambda) + \Phi(F(\lambda))\right)\varphi(F(\lambda))\,\mathrm{d}\mu_{t,x}(\lambda) = \int_{\mathbb{R}^+} \left(\Psi(\lambda) + \Phi(F(\lambda))\right)\mathrm{d}\mu_{t,x}(\lambda) \int_{\mathbb{R}^+} \varphi(F(\lambda))\,\mathrm{d}\mu_{t,x}(\lambda).$$

We observe that $\lambda = \sum_{i=1}^{3} S_i(F(\lambda))\, \mathbb{1}_{\lambda\in I_i}$. Hence, we may use the concept of push-forward measure to write

$$\sum_{i=1}^{3}\int_{\mathbb{R}^+}\big(\Psi\big(S_i(\lambda)\big)+\Phi(\lambda)\big)\,\varphi(\lambda)\,\mathrm{d}F^{\#}\mu_{t,x}^{(i)}(\lambda)$$
$$=\sum_{i=1}^{3}\int_{\mathbb{R}^+}\big(\Psi\big(S_i(\lambda)\big)+\Phi(\lambda)\big)\,\mathrm{d}F^{\#}\mu_{t,x}^{(i)}(\lambda)\int_{\mathbb{R}^+}\varphi(\lambda)\,\mathrm{d}F^{\#}\mu_{t,x}(\lambda).$$

Using (7) with densities $g_1(\lambda)$, $g_2(\lambda)$ and $g_3(\lambda)$ we obtain

$$\sum_{i=1}^{3}\int_{\mathbb{R}^+}\big(\Psi\big(S_i(\lambda)\big)+\Phi(\lambda)\big)\,\varphi(\lambda)\,g_i(\lambda)\,\mathrm{d}F^{\#}\mu_{t,x}(\lambda)$$
$$=\sum_{i=1}^{3}\int_{\mathbb{R}^+}\big(\Psi\big(S_i(\lambda)\big)+\Phi(\lambda)\big)\,g_i(\lambda)\,\mathrm{d}F^{\#}\mu_{t,x}(\lambda)\int_{\mathbb{R}^+}\varphi(\lambda)\,\mathrm{d}F^{\#}\mu_{t,x}(\lambda).$$

Hence, when λ_0 belongs to the support of measure $F^{\#}\mu_{t,x}$, we obtain

$$\sum_{i=1}^{3}\big(\Psi\big(S_i(\lambda_0)\big)+\Phi(\lambda_0)\big)\,g_i(\lambda_0)=\sum_{i=1}^{3}\int_{\mathbb{R}^+}\big(\Psi\big(S_i(\lambda)\big)+\Phi(\lambda)\big)\,g_i(\lambda)\,\mathrm{d}F^{\#}\mu_{t,x}(\lambda).$$

□

To analyze the entropy inequality, we need to deal with integrals of the form $\int_0^{S_i(\lambda)}\phi(F(\tau))\,\mathrm{d}\tau$. This is the content of the next lemma.

Lemma 13. *We have*

$$\Psi\big(S_i(\lambda_0)\big)=\int_0^{S_i(\lambda_0)}\phi(F(\tau))\,\mathrm{d}\tau=\int_0^{\lambda_0}\phi(\tau)\,S_i'(\tau)\,\mathrm{d}\tau+C_i(\phi)$$

where $C_1(\phi)=0$ *and* $C_2(\phi)=C_3(\phi)=\int_0^{f_+}\phi(\tau)\,(S_1'(\tau)-S_2'(\tau))\,\mathrm{d}\tau$.

Proof. For $i=1$ we note that F is invertible on $(0,S_1(\lambda))$ so that a simple change of variables implies

$$\Psi\big(S_1(\lambda_0)\big)=\int_0^{S_1(\lambda_0)}\phi(F(\tau))\,\mathrm{d}\tau=\int_0^{\lambda_0}\phi(\tau)\,S_1'(\tau)\,\mathrm{d}\tau.$$

For $i=2$ we first split the integral for two intervals $(0,\alpha_+)$, (α_+,λ_0) cf. Notation 4. On each of them, F is invertible so we can apply a change of variables again:

$$\begin{aligned}\Psi\big(S_2(\lambda_0)\big)&=\int_0^{\alpha_+}\phi(F(\tau))\,\mathrm{d}\tau+\int_{\alpha_+}^{S_2(\lambda_0)}\phi(F(\tau))\,\mathrm{d}\tau\\&=\int_0^{f_+}\phi(\tau)\,S_1'(\tau)\,\mathrm{d}\tau-\int_{\lambda_0}^{f_+}\phi(\tau)\,S_2'(\tau)\,\mathrm{d}\tau=C_2(\phi)+\int_0^{\lambda_0}\phi(\tau)\,S_2'(\tau)\,\mathrm{d}\tau.\end{aligned}$$

For $i=3$ we split the integral for three intervals and apply a change of variables again:

$$\begin{aligned}\Psi\big(S_3(\lambda_0)\big)&=\int_0^{\alpha_+}\phi(F(\tau))\,\mathrm{d}\tau+\int_{\alpha_+}^{\beta_-}\phi(F(\tau))\,\mathrm{d}\tau+\int_{\beta_-}^{S_3(\lambda_0)}\phi(F(\tau))\,\mathrm{d}\tau\\&=\int_0^{f_+}\phi(\tau)\,S_1'(\tau)\,\mathrm{d}\tau-\int_{f_-}^{f_+}\phi(\tau)\,S_2'(\tau)\,\mathrm{d}\tau+\int_{f_-}^{\lambda_0}\phi(\tau)\,S_3'(\tau)\,\mathrm{d}\tau.\end{aligned}$$

As $S_2'(\tau)=0$ and $S_3'(\tau)=0$ for $\tau\in(0,f_-)$, the proof is concluded. □

Lemma 14. *Consider function*

$$\mathscr{F}(\tau_0)=\sum_{i=1}^{3}\big(S_i'(\tau_0)+1\big)\,F^{\#}\mu_{t,x}^{(i)}\big((\tau_0,\infty)\big)+\big(S_1'(\tau_0)-S_2'(\tau_0)\big)\left(1-F^{\#}\mu_{t,x}^{(1)}\big(\mathbb{R}^+\big)\right).$$

Then, for almost all λ_0 (with respect to $F^{\#}\mu_{t,x}$) and $\tau_0 \neq f_-, f_+$ we have

$$\mathbb{1}_{\lambda_0>\tau_0} \sum_{i=1}^{3} \left(S_i'(\tau_0)+1\right) g_i(\lambda_0) + \left(S_1'(\tau_0) - S_2'(\tau_0)\right)\left(1-g_1(\lambda_0)\right) = \mathscr{F}(\tau_0).$$

Proof. We consider $\phi(\tau) = \phi^{\delta}(\tau) = \frac{1}{\delta}\mathbb{1}_{[\tau_0,\tau_0+\delta]}$ and send $\delta \to 0$ so that $\Phi(\lambda_0) = \int_0^{\lambda_0} \phi^{\delta}(\tau)\,\mathrm{d}\tau \to \mathbb{1}_{\lambda>\tau_0}$. Moreover, $\int_0^{\lambda_0} \phi^{\delta}(\tau)\, S_i'(\tau)\,\mathrm{d}\tau \to S_i'(\tau_0)\,\mathbb{1}_{\lambda_0>\tau_0}$. Therefore, from Lemmas 12 and 13 we deduce

$$\sum_{i=1}^{3}\left(\mathbb{1}_{\lambda_0>\tau_0}\left(S_i'(\tau_0)+1\right) + \left(S_1'(\tau_0)-S_2'(\tau_0)\right)\mathbb{1}_{i=2,3}\right) g_i(\lambda_0) =$$
$$= \sum_{i=1}^{3}\int_{\mathbb{R}^+}\left(\mathbb{1}_{\lambda>\tau_0}\left(S_i'(\tau_0)+1\right) + \left(S_1'(\tau_0)-S_2'(\tau_0)\right)\mathbb{1}_{i=2,3}\right) g_i(\lambda)\,\mathrm{d}F^{\#}\mu_{t,x}(\lambda).$$

Using identities from (8) and (9)

$$1-g_1(\lambda_0) = g_2(\lambda_0)+g_3(\lambda_0), \qquad 1-F^{\#}\mu^{(1)}_{t,x}\left(\mathbb{R}^+\right) = F^{\#}\mu^{(2)}_{t,x}\left(\mathbb{R}^+\right) + F^{\#}\mu^{(3)}_{t,x}\left(\mathbb{R}^+\right),$$

we conclude the proof. □

Proof of Theorem 5. The first part of Theorem 5 is proved in Lemma 14. To see the second one, fix $\lambda_0 \neq f_-, f_+$. For $\tau_0 := \eta > \lambda_0$ we obtain

$$\sum_{i=1}^{3}\left(S_i'(\eta)+1\right) F^{\#}\mu^{(i)}_{t,x}\left((\eta,\infty)\right) + \left(S_1'(\eta)-S_2'(\eta)\right)\left(F^{\#}\mu^{(1)}_{t,x}\left(\mathbb{R}^+\right) - g_1(\lambda_0)\right) = 0$$

while for $\tau_0 := \xi < \lambda_0$ we deduce

$$\sum_{i=1}^{3}\left(S_i'(\xi)+1\right)\left(g_i(\lambda_0) - F^{\#}\mu^{(i)}_{t,x}\left((\xi,\infty)\right)\right) + \left(S_1'(\xi)-S_2'(\xi)\right)\left(F^{\#}\mu^{(1)}_{t,x}\left(\mathbb{R}^+\right) - g_1(\lambda_0)\right) = 0.$$

Sending $\xi,\eta \to \lambda_0$ and using continuity of $\lambda \mapsto S_i'(\lambda)$ at $\lambda \neq f_-, f_+$ we obtain

$$\sum_{i=1}^{3}\left(S_i'(\lambda_0)+1\right) g_i(\lambda_0) = \sum_{i=1}^{3}\left(S_i'(\lambda_0)+1\right) F^{\#}\mu^{(i)}_{t,x}\{\lambda_0\}.$$

Finally, we note that for almost all λ_0 (with respect to $F^{\#}\mu_{t,x}$) $F^{\#}\mu^{(i)}_{t,x}\{\lambda_0\} = g_i(\lambda_0)\,F^{\#}\mu_{t,x}\{\lambda_0\}$ and this concludes the proof. □

4. Proofs of Theorems 6, 7 and 8 for fast-reaction system (1)–(2)

Proof of Theorem 6. If $\operatorname{supp} F^{\#}\mu_{t,x} \cap (0, f_-)$ is nonempty, we take any $\lambda_0 \in \operatorname{supp} F^{\#}\mu_{t,x} \cap (0, f_-)$. Note that $S_2'(\lambda_0) = S_3'(\lambda_0) = 0$. Moreover, (10) in Theorem 5 implies

$$\left(1 - F^{\#}\mu_{t,x}\{\lambda_0\}\right)\left(S_1'(\lambda_0)+1\right) g_1(\lambda_0) = 0.$$

For almost all $\lambda_0 \in (0, f_-)$ we have $g_1(\lambda_0) = 1$ so we conclude $F^{\#}\mu_{t,x}\{\lambda_0\} = 1$. A similar argument works in the case $\lambda_0 \in (f_+, \infty)$.

Now, let $\lambda_0 \in [f_-, f_+] \cap \operatorname{supp} F^{\#}\mu_{t,x}$. If $\operatorname{supp} F^{\#}\mu_{t,x} = \{\lambda_0\}$, we conclude $F^{\#}\mu_{t,x} = \delta_{\lambda_0}$. Otherwise, there are $\lambda_1, \lambda_2 \in \operatorname{supp} F^{\#}\mu_{t,x}$ such that $f_- \le \lambda_1 < \lambda_2 \le f_+$. For any $\tau_0 \in (\lambda_1, \lambda_2)$ we use Theorem 5 with $\lambda_0 = \lambda_1, \lambda_2$ to obtain two equations:

$$\sum_{i=1}^{3}\left(S_i'(\tau_0)+1\right)\left[g_i(\lambda_2) - F^{\#}\mu^{(i)}_{t,x}(\tau_0,\infty)\right] + \left(S_1'(\tau_0)-S_2'(\tau_0)\right)\left(F^{\#}\mu^{(1)}_{t,x}\left(\mathbb{R}^+\right) - g_1(\lambda_2)\right) = 0,$$
$$-\sum_{i=1}^{3}\left(S_i'(\tau_0)+1\right) F^{\#}\mu^{(i)}_{t,x}(\tau_0,\infty) + \left(S_1'(\tau_0)-S_2'(\tau_0)\right)\left(F^{\#}\mu^{(1)}_{t,x}\left(\mathbb{R}^+\right) - g_1(\lambda_1)\right) = 0.$$

Hence, $\sum_{i=1}^3 (S_i'(\tau_0)+1)\, g_i(\lambda_2) + (S_1'(\tau_0) - S_2'(\tau_0))\,(g_1(\lambda_1) - g_1(\lambda_2)) = 0$. But then, non-degeneracy condition (5) implies that $\sum_{i=1}^3 g_i(\lambda_2) = 0 \neq 1$ raising contradiction.

It follows that $F^\# \mu_{t,x}$ is the Dirac measure. From Corollary 11 we deduce that the Young measure $\{\nu_{t,x}\}_{t,x}$ generated by $\{v^\varepsilon\}_{\varepsilon>0}$ is also the Dirac measure so $v^\varepsilon \to v$ strongly and $\nu_{t,x} = \delta_{v(t,x)}$, cf. Lemma 23. The representation formula for $\mu_{t,x}$ follows from $F^\# \mu_{t,x} = \delta_{v(t,x)}$. □

Before proceeding to the proofs of Theorems 7 and 8, we will state a simple lemma concerning the case when $F^\# \mu_{t,x}$ is supported only at f_- and f_+. This needs some care as functions S_1', S_2' and S_3' are not continuous at these points.

Lemma 15 (Accumulation at the interface). *Suppose that there exists $\tau_0 \in (f_-, f_+)$ such that $S_1'(\tau_0) - S_3'(\tau_0) \neq 0$. Assume that* $\operatorname{supp} F^\# \mu_{t,x} \subset \{f_-, f_+\}$. *Then, $F^\# \mu_{t,x} = \delta_{f_-}$ or $F^\# \mu_{t,x} = \delta_{f_+}$.*

Proof. Aiming at contradiction, we assume that $F^\# \mu_{t,x}\{f_+\} > 0$ and $F^\# \mu_{t,x}\{f_-\} > 0$. Note that $F^{-1}(f_+) \notin I_2$ so that

$$0 = \mu^{(2)}_{t,x}\Big(F^{-1}\left(f_+\right) \cap I_2\Big) = F^\# \mu^{(2)}_{t,x}\left\{f_+\right\} = g_2\left(f_+\right) F^\# \mu_{t,x}\left\{f_+\right\}.$$

It follows that $g_2(f_+) = 0$ and similarly $g_2(f_-) = 0$. Applying Theorem 5 with $\tau_0 \in (f_-, f_+)$ and $\lambda_0 \in \{f_-, f_+\}$ we obtain

$$\sum_{i=1}^3 \left(S_i'(\tau_0)+1\right)\left[\mathbb{1}_{\lambda_0>\tau_0}\, g_i(\lambda_0) - F^\# \mu^{(i)}_{t,x}(\tau_0,\infty)\right] + \left(S_1'(\tau_0) - S_2'(\tau_0)\right)\left(F^\# \mu^{(1)}_{t,x}(\mathbb{R}^+) - g_1(\lambda_0)\right) = 0.$$

As $\tau_0 \in (f_-, f_+)$, we have

$$F^\# \mu^{(i)}_{t,x}(\tau_0,\infty) = F^\# \mu^{(i)}_{t,x}\left\{f_+\right\} = g_i\left(f_+\right) F^\# \mu_{t,x}\left\{f_+\right\}.$$

But this implies

$$\left(\mathbb{1}_{\lambda_0>\tau_0} - F^\# \mu_{t,x}\left\{f_+\right\}\right) \sum_{i=1,3} \left(S_i'(\tau_0)+1\right) g_i(\lambda_0) + \left(S_1'(\tau_0) - S_2'(\tau_0)\right)\left(F^\# \mu^{(1)}_{t,x}(\mathbb{R}^+) - g_1(\lambda_0)\right) = 0.$$

Considering $\lambda_0 = f_+, f_-$ and using $1 - F^\# \mu_{t,x}\{f_+\} = F^\# \mu_{t,x}\{f_-\}$ we obtain two equations:

$$F^\# \mu_{t,x}\left\{f_-\right\} \sum_{i=1,3} \left(S_i'(\tau_0)+1\right) g_i\left(f_+\right) + \left(S_1'(\tau_0) - S_2'(\tau_0)\right)\left(F^\# \mu^{(1)}_{t,x}(\mathbb{R}^+) - g_1(f_+)\right) = 0, \tag{16}$$

$$-F^\# \mu_{t,x}\left\{f_+\right\} \sum_{i=1,3} \left(S_i'(\tau_0)+1\right) g_i\left(f_-\right) + \left(S_1'(\tau_0) - S_2'(\tau_0)\right)\left(F^\# \mu^{(1)}_{t,x}(\mathbb{R}^+) - g_1(f_-)\right) = 0. \tag{17}$$

Using $1 - F^\# \mu_{t,x}\{f_+\} = F^\# \mu_{t,x}\{f_-\}$ once again we obtain

$$\begin{aligned} F^\# \mu^{(1)}_{t,x}(\mathbb{R}^+) - g_1\left(f_+\right) &= g_1\left(f_+\right) F^\# \mu_{t,x}\left\{f_+\right\} + g_1\left(f_-\right) F^\# \mu_{t,x}\left\{f_-\right\} - g_1\left(f_+\right) \\ &= \left(g_1\left(f_-\right) - g_1\left(f_+\right)\right) F^\# \mu_{t,x}\left\{f_-\right\} \end{aligned}$$

and similarly for $F^\# \mu^{(1)}_{t,x}(\mathbb{R}^+) - g_1(f_-)$. As we assume that $F^\# \mu_{t,x}\{f_-\}, F^\# \mu_{t,x}\{f_+\} > 0$, we may simplify (16)–(17) to obtain

$$\sum_{i=1,3} \left(S_i'(\tau_0)+1\right) g_i\left(f_+\right) + \left(S_1'(\tau_0) - S_2'(\tau_0)\right)\left(g_1\left(f_-\right) - g_1\left(f_+\right)\right) = 0, \tag{18}$$

$$-\sum_{i=1,3} \left(S_i'(\tau_0)+1\right) g_i\left(f_-\right) + \left(S_1'(\tau_0) - S_2'(\tau_0)\right)\left(g_1\left(f_+\right) - g_1\left(f_-\right)\right) = 0. \tag{19}$$

We observe further that $g_1(\lambda_0) + g_3(\lambda_0) = 1$, cf. (9), so that

$$\sum_{i=1,3} \left(S_i'(\tau_0)+1\right) g_i(\lambda_0) = \left(S_1'(\tau_0) - S_3'(\tau_0)\right) g_1(\lambda_0) + \left(S_3'(\tau_0)+1\right).$$

Hence, we may further simplify (18)–(19) to get

$$\left(S_1'(\tau_0) - S_3'(\tau_0)\right) g_1\left(f_+\right) + \left(S_3'(\tau_0)+1\right) + \left(S_1'(\tau_0) - S_2'(\tau_0)\right)\left(g_1\left(f_-\right) - g_1\left(f_+\right)\right) = 0, \tag{20}$$

$$-\left(S_1'(\tau_0) - S_3'(\tau_0)\right) g_1\left(f_-\right) - \left(S_3'(\tau_0)+1\right) + \left(S_1'(\tau_0) - S_2'(\tau_0)\right)\left(g_1\left(f_+\right) - g_1\left(f_-\right)\right) = 0. \tag{21}$$

By assumption, there is $\tau_0 \in (f_-, f_+)$ such that $S_1'(\tau_0) - S_3'(\tau_0) \neq 0$. Using (20)–(21) for such τ_0 we see that $g_1(f_+) = g_1(f_-)$. But then, coming back to (18)–(19), we deduce that

$$\sum_{i=1,3} \left(S_i'(\tau_0) + 1\right) g_i\left(f_+\right) = 0, \qquad \sum_{i=1,3} \left(S_i'(\tau_0) + 1\right) g_i\left(f_-\right) = 0.$$

As S_1, S_3 are increasing, this implies $g_1(f_-) = g_3(f_-) = g_1(f_+) = g_3(f_+) = 0$ raising contradiction with $g_1(f_-) + g_3(f_-) = 1$ and $g_1(f_+) + g_3(f_+) = 1$. □

Remark 16. Without the assumption that there is $\tau_0 \in (f_-, f_+)$ such that $S_1'(\tau_0) - S_3'(\tau_0) \neq 0$ we observe that (20)–(21) degenerate to the same equation:

$$g_1\left(f_+\right) - g_1\left(f_-\right) = \frac{1 + S_3'(\tau_0)}{S_1'(\tau_0) - S_2'(\tau_0)}$$

valid for all $\tau_0 \in (f_-, f_+)$. Hence, it the function $\tau_0 \mapsto \frac{1+S_3'(\tau_0)}{S_1'(\tau_0) - S_2'(\tau_0)}$ is not constant, we may also obtain contradiction. Nevertheless, we believe that the assumption on $S_1'(\tau_0) - S_3'(\tau_0)$ is easier to formulate.

Proof of Theorem 7. As in the proof of Theorem 6, we may assume that $\operatorname{supp} F^{\#}\mu_{t,x} \subset [f_-, f_+]$ (this did not use the non-degeneracy condition!). By assumption of the theorem, for any set $A \subset \mathbb{R}^+$

$$0 = \mu_{t,x}\Big(F^{-1}(A) \cap I_2\Big) = F^{\#}\mu^{(2)}_{t,x}(A) = \int_A g_2(\lambda)\, \mathrm{d}F^{\#}\mu_{t,x}(\lambda)$$

so $g_2(\lambda) = 0$ for almost all λ. Hence, when $\lambda_0 \in \operatorname{supp} F^{\#}\mu_{t,x} \cap (f_-, f_+)$, the sum

$$\sum_{i=1}^{3} \left(S_i'(\lambda_0) + 1\right) g_i(\lambda_0) \geq \min\left(S_1'(\lambda_0) + 1, S_3'(\lambda_0) + 1\right) > 0$$

because $g_1(\lambda_0) + g_3(\lambda_0) = 1$ and S_1, S_3 are strictly increasing. It follows from Theorem 5 that $F^{\#}\mu_{t,x}\{\lambda_0\} = 1$, i.e. $F^{\#}\mu_{t,x} = \delta_{\lambda_0}$. Finally, if there is no such $\lambda_0 \in \operatorname{supp} F^{\#}\mu_{t,x} \cap (f_-, f_+)$, we apply Lemma 15.

It follows that $F^{\#}\mu_{t,x}$ is the Dirac measure so that we can conclude as in Theorem 6. □

Proof of Theorem 8. Mimicking the proof of Theorem 7, we let $\lambda_0 \in \operatorname{supp} F^{\#}\mu_{t,x} \cap (f_-, f_+)$ and we observe that the sum

$$\sum_{i=1}^{3} \left(S_i'(\lambda_0) + 1\right) g_i(\lambda_0) \geq \min(1, \delta(\lambda_0)) \sum_{i=1}^{3} g_i(\lambda_0) = \min(1, \delta(\lambda_0)) > 0$$

where $\delta(\lambda_0)$ is such that $S_2'(\lambda_0) + 1 > \delta(\lambda_0) > 0$. We conclude as in the proof of Theorem 7. □

5. Proof of Theorems 5–7 to the forward-backward diffusion system (3)–(4)

We first formulate a basic well-posedness result for (3)–(4). This comes mostly from [31, 35] but the compactness estimates are simplified.

Lemma 17. *Let $u_0 \in L^\infty(\Omega)$. Then, there exists the unique solution $u^\varepsilon : [0,\infty) \times \Omega \to \mathbb{R}$ of (3)–(4) which is nonnegative and has regularity $C^1([0,T]; L^2(\Omega)) \cap L^\infty(\Omega)$. Moreover, we have*

(1) *for $M = \max(\|F(u_0)\|_\infty, f_+)$ we have $0 \leq u^\varepsilon \leq M$,*
(2) *$\{\nabla v^\varepsilon\}_{\varepsilon \in (0,1)}$ is uniformly bounded in $L^2((0,T) \times \Omega)$,*
(3)
$$\left\{\frac{v^\varepsilon - F(u^\varepsilon)}{\sqrt{\varepsilon}}\right\}_{\varepsilon \in (0,1)} = \left\{\sqrt{\varepsilon}\, u^\varepsilon_t\right\}_{\varepsilon \in (0,1)}$$
are uniformly bounded in $L^2((0,T) \times \Omega)$,
(4) *for all smooth $\varphi : \mathbb{R} \to \mathbb{R}$, $\{\nabla \varphi(v^\varepsilon)\}_{\varepsilon \in (0,1)}$ is uniformly bounded in $L^2((0,T) \times \Omega)$,*

(5) *for all smooth* $\phi : \mathbb{R} \to \mathbb{R}$, $\{\partial_t \Psi(u^\varepsilon)\}_{\varepsilon\in(0,1)}$ *is uniformly bounded in* $(C(0,T;H^k(\Omega)))^*$ *for sufficiently large* $k \in \mathbb{N}$.

Proof. We observe that the equation is equivalent to the following ODE:

$$\partial_t u^\varepsilon = (I-\varepsilon\Delta)^{-1}\,\Delta F\big(u^\varepsilon\big).$$

As long as $\varepsilon > 0$, the (RHS) is Lipschitz continuous, say on $L^2(\Omega)$, so the local well-posedness follows. To obtain global well-posedness, we consider functions Ψ, Φ defined in (12). We have

$$\begin{aligned}\partial_t \Psi\left(u^\varepsilon\right) = \phi\left(F\left(u^\varepsilon\right)\right)\, u_t^\varepsilon &= \left(\phi\left(F\left(u^\varepsilon\right)\right) - \phi\left(v^\varepsilon\right)\right)\, u_t^\varepsilon + \phi\left(v^\varepsilon\right)\Delta v^\varepsilon \\ &= \left(\phi\left(F\left(u^\varepsilon\right)\right) - \phi\left(v^\varepsilon\right)\right)\, u_t^\varepsilon + \Delta\Phi\left(v^\varepsilon\right) - \phi'\left(v^\varepsilon\right)\left|\nabla v^\varepsilon\right|^2.\end{aligned} \tag{22}$$

If ϕ is nondecreasing, we have

$$\left(\phi\left(F\left(u^\varepsilon\right)\right) - \phi\left(v^\varepsilon\right)\right) u_t^\varepsilon = \left(\phi\left(F\left(u^\varepsilon\right)\right) - \phi\left(v^\varepsilon\right)\right)\frac{v^\varepsilon - F\big(u^\varepsilon\big)}{\varepsilon} \le 0$$

so after integration in space, the (RHS) of (22) is nonnegative. Hence, $\partial_t \int_\Omega \Psi(u^\varepsilon) \le 0$. Choosing $\phi = 0$ for $[0,M]$ and $\phi'(x) > 0$ for $x \notin [0,M]$ we prove (1) and conclude the proof of global well-posedness. To see (2) and (3) we take $\phi(x) = x$ and integrate (22) in time and space. Part (4) easily follows from the chain rule and (2). Finally, (5) follows from (22) and exactly the same computations as in Lemma 10. □

Now, we formulate an analog of Lemma 12.

Lemma 18 (Entropy equality). *Let* Ψ *be defined with* (12), $\{\mu_{t,x}\}$ *be the Young measure generated by sequence* $\{u^\varepsilon\}_{\varepsilon>0}$ *solving* (3)–(4) *and* g_i *be the densities given by* (7). *Then, for almost all* λ_0 *(with respect to* $F^\#\mu_{t,x}$*) we have*

$$\sum_{i=1}^{3}\Psi\big(S_i(\lambda_0)\big)\,g_i(\lambda_0) = \sum_{i=1}^{3}\int_{\mathbb{R}^+}\Psi\big(S_i(\lambda)\big)\,g_i(\lambda)\,\mathrm{d}F^\#\mu_{t,x}(\lambda), \tag{23}$$

where S_i *are the inverses of* F *as in Notation 4.*

Proof. Thanks to Lemma 17 (5), for all smooth $\phi : \mathbb{R} \to \mathbb{R}$, $\{\partial_t\Psi(u^\varepsilon)\}_{\varepsilon\in(0,1)}$ is uniformly bounded in $(C(0,T;H^k(\Omega)))^*$. Similarly, for all smooth and bounded $\varphi : \mathbb{R}\to\mathbb{R}$, $\{\nabla\varphi(v^\varepsilon)\}_{\varepsilon\in(0,1)}$ is uniformly bounded in $L^2((0,\infty)\times\Omega)$. Hence, Lemma 19 implies

$$\mathrm{w}^*\text{-}\lim_{\varepsilon\to 0}\Psi\left(u^\varepsilon\right)\varphi\left(v^\varepsilon\right) = \mathrm{w}^*\text{-}\lim_{\varepsilon\to 0}\Psi\left(u^\varepsilon\right)\,\mathrm{w}^*\text{-}\lim_{\varepsilon\to 0}\varphi\left(v^\varepsilon\right).$$

As $v^\varepsilon - F(u^\varepsilon) = \varepsilon\, u_t^\varepsilon \to 0$ cf. Lemma 17 (3), we may replace v^ε with $F(u^\varepsilon)$ in the identity above to obtain

$$\mathrm{w}^*\text{-}\lim_{\varepsilon\to 0}\Psi\left(u^\varepsilon\right)\varphi\left(F\left(u^\varepsilon\right)\right) = \mathrm{w}^*\text{-}\lim_{\varepsilon\to 0}\Psi\left(u^\varepsilon\right)\,\mathrm{w}^*\text{-}\lim_{\varepsilon\to 0}\varphi\left(F\left(u^\varepsilon\right)\right).$$

In the language of Young measures, this identity reads

$$\int_{\mathbb{R}^+}\Psi(\lambda)\,\varphi(F(\lambda))\,\mathrm{d}\mu_{t,x}(\lambda) = \int_{\mathbb{R}^+}\Psi(\lambda)\,\mathrm{d}\mu_{t,x}(\lambda)\int_{\mathbb{R}^+}\varphi(F(\lambda))\,\mathrm{d}\mu_{t,x}(\lambda).$$

We observe that $\lambda = \sum_{i=1}^3 S_i(F(\lambda))\,\mathbb{1}_{\lambda\in I_i}$. Hence, we may use push-forward measure to write

$$\sum_{i=1}^{3}\int_{\mathbb{R}^+}\Psi\big(S_i(\lambda)\big)\,\varphi(\lambda)\,\mathrm{d}F^\#\mu_{t,x}^{(i)}(\lambda) = \sum_{i=1}^{3}\int_{\mathbb{R}^+}\Psi\big(S_i(\lambda)\big)\,\mathrm{d}F^\#\mu_{t,x}^{(i)}(\lambda)\int_{\mathbb{R}^+}\varphi(\lambda)\,\mathrm{d}F^\#\mu_{t,x}(\lambda).$$

Using densities $g_1(\lambda)$, $g_2(\lambda)$ and $g_3(\lambda)$ we obtain

$$\begin{aligned}&\sum_{i=1}^{3}\int_{\mathbb{R}^+}\Psi\big(S_i(\lambda)\big)\,\varphi(\lambda)\,g_i(\lambda)\,\mathrm{d}F^\#\mu_{t,x}(\lambda) \\ &\qquad = \sum_{i=1}^{3}\int_{\mathbb{R}^+}\Psi\big(S_i(\lambda)\big)\,g_i(\lambda)\,\mathrm{d}F^\#\mu_{t,x}(\lambda)\int_{\mathbb{R}^+}\varphi(\lambda)\,\mathrm{d}F^\#\mu_{t,x}(\lambda).\end{aligned}$$

Hence, if λ_0 belongs to the support of the measure $F^{\#}\mu_{t,x}$, we obtain (23). □

Proof of Theorems 5–7. Comparing formulations of Lemmas 12 and 18 we see that it is sufficient to modify proofs in Sections 3-4 by replacing $S_1'+1$, $S_2'+1$ and $S_3'+1$ with S_1', S_2' and S_3' respectively. □

Note that Theorem 8 is only true for fast-reaction limit (1)–(2) because its proof exploits presence of term $S_2'+1$ in the entropy formulations.

Appendix A. Useful notions and results

A.1. *Compensated compactness lemma*

We formulate the lemma used in the proof of Theorem 5, more precisely in Lemma 12. For the proof see [27, Proposition 1].

Lemma 19. *Let $\Omega\subset\mathbb{R}^n$ be a bounded domain. Suppose that $\{a_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^2(0,T;H^1(\Omega))$ and $\{b_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^2(0,T;L^2(\Omega))$. Moreover, assume that the sequence of distributional time derivatives $\{\partial_t b_n\}_{n\in\mathbb{N}}$ is uniformly bounded in the dual space $C(0,T;H^m(\Omega))^*$ for some $m\in\mathbb{N}$. Then, if $a_n\rightharpoonup a$ and $b_n\rightharpoonup b$ we have $a_n b_n\to a b$ in the sense of distributions.*

In our case, the considered sequences are also in $L^\infty((0,T)\times\Omega)$ so the resulting convergence is true in the weak* sense.

A.2. *Support of a measure*

We recall definition of the support of measure on $\mathbb{R}^n$ [38, Definition 1.14]. For this, let $B(x,r)$ denote a ball of radius $r>0$ centered at $x\in\mathbb{R}^n$.

Definition 20. *Let μ be a nonnegative measure on $\mathbb{R}^n$. We say that $x\in\operatorname{supp}\mu$ if and only if $\mu(B(x,r))>0$ for all $r>0$.*

Remark 21. When a given property (like an equation) is satisfied for almost every x (with respect to μ) one may worry that it is not true for the particularly chosen value of x. This is not the problem if one takes $x\in\operatorname{supp}\mu$ because in each neighbourhood of x there is $y\in\operatorname{supp}\mu$ such that the considered property has to be satisfied as the measure of each neighbourhood is nonzero.

A.3. *Young measures*

Finally, we recall the theory of Young measures introduced by Young [44, 45] and recalled in the seminal paper of Ball [2]. Reader interested in a modern presentation may consult [18], [32, Chapter 6] or [36, Chapter 4]. For simplicity, we formulate it for sequences of functions $\{u_n\}_{n\in\mathbb{N}}$ uniformly bounded in $L^p(\Omega)$ with some $1\le p\le\infty$ and $\Omega\subset\mathbb{R}^n$ being a bounded domain. We start by recalling the most important result that we cite from [32, Theorem 6.2]:

Theorem 22 (Fundamental Theorem of Young Measures). *Let $\Omega\subset\mathbb{R}^n$ be a a bounded domain and let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence bounded in $L^p(\Omega)$ with $1\le p\le\infty$. Then, there exists a subsequence (not relabeled) and a weakly-$*$ measurable family of probability measures $\{\mu_x\}_{x\in\Omega}$ such that for all bounded and smooth $G:\mathbb{R}\to\mathbb{R}$, we have*

$$G\big(u_n(x)\big)\overset{*}{\rightharpoonup}\int_{\mathbb{R}} G(\lambda)\,\mathrm{d}\mu_x(\lambda)\qquad \text{in } L^\infty(\Omega). \tag{24}$$

We say that the sequence $\{u_n\}_{n\in\mathbb{N}}$ generates the family of Young measures $\{\mu_x\}_{x\in\Omega}$.

Now we list properties of Young measures used in the paper.

Lemma 23. *Under the notation of Theorem 22, the following hold true.*

(A) *We have $u_n \to u$ a.e. (up to a subsequence) if and only if $\mu_{t,x} = \delta_{u(t,x)}$.*
(B) *If $\{w_n\}_{n\in\mathbb{N}}$ is another bounded sequence such that $u_n - w_n \to 0$ a.e. then Young measures generated by $\{u_n\}_{n\in\mathbb{N}}$ and $\{w_n\}_{n\in\mathbb{N}}$ coincide.*
(C) *If $F : \mathbb{R} \to \mathbb{R}$, sequence $\{F(u_n)\}_{n\in\mathbb{N}}$ generates Young measure $F^{\#}\mu_{t,x}$ (i.e. push-forward $\mu_{t,x} \circ F^{-1}$).*

Sketch of the proof. For (A) we consider $G(u) = u$ and $G(u) = u^2$ to deduce that $u_n \to u$ in $L^2(\Omega)$ so that (up to a subsequence) u_n converges a.e. The opposite direction is clear because $G(u_n(x)) \to G(u(x))$ a.e. For (B) we note that for all bounded and smooth G, weak limits of $G(u_n(x))$ and $G(w_n(x))$ coincide. For (C), it is sufficient to write

$$G\big(F(u_n)\big) = \int_{\mathbb{R}} G(F(\lambda))\,\mathrm{d}\mu_{t,x}(\lambda) = \int_{\mathbb{R}} G(\lambda)\,\mathrm{d}\Big(\mu_{t,x} \circ F^{-1}\Big)(\lambda).$$

□

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