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Toric mirror symmetry revisited

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Abstract. The Cox construction presents a toric variety as a quotient of affine space by a torus. The category of coherent sheaves on the corresponding stack thus has an evident description as invariants in a quotient of the category of modules over a polynomial ring. Here we give the mirror to this description, and in particular, a clean new proof of mirror symmetry for smooth toric stacks.

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1. Introduction

Let $\mathbb{G}_m^n \circlearrowright \mathbb{A}^n$ be the standard action, $G \subset \mathbb{G}_m^n$ a subgroup, and $Z \subset \mathbb{A}^n$ a closed invariant subset. The coherent sheaf category on the (stack) quotient $Y = (\mathbb{A}^n \setminus Z)/G$ can be calculated as:

$$\mathrm{Coh}(Y) = \left(\mathrm{Coh}\left(\mathbb{A}^n\right) / \mathrm{Coh}(Z)\right)^G \tag{1}$$

The purpose of this note is to translate Equation (1) through homological mirror symmetry for monomial Z, (the case relevant to the construction of toric varieties [4]), and in particular give a new proof of homological mirror symmetry for smooth toric stacks.

Some brief historical remarks. The mirror to coherent sheaves on a toric variety is a partially wrapped Fukaya category of a torus dual to $\mathbb{T} = \mathbb{G}_m^n/G$. When Y is smooth, the wrapping can be described in terms of a "superpotential" function on the torus [18]; more generally, the stop for the wrapping is a certain singular Legendrian introduced in [5, 6]. There are several approaches to this mirror symmetry: the microlocal sheaf theoretic approach of [3, 5, 6, 27] is completed, first by [21], and also by logically independent methods in [30] and [28]; meanwhile [1, 2, 16, 17] provides a direct Floer theoretic approach. (Comparing sheaf and Floer results requires the foundational [13–15] together with the Lagrangian skeleton calculations of [12, 31].)

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Here we give yet another proof, logically independent of all previous ones, and perhaps more conceptual – we only need hands-on calculations of categories for mirror symmetry for $\mathbb{A}^1$. In particular, we will discover, rather than posit, the conic Lagrangian of [5,6]. In addition we will see clearly on the A-side the significance of asking for a simplicial fan. We work with microlocal sheaf theory; one can pass to Fukaya categories by the comparison result [15]. It would also be straightforward to translate our methods to work directly with Fukaya categories.

Finally, let us mention that toric mirror symmetry often serves as a useful building block for various further constructions [9–12,24].

Conventions

We always work with the stable ∞-categories given by localizing categories of complexes along quasi-isomorphisms; we always use the "homotopy" or "derived" versions of functors as appropriate for this context. Throughout we work over some fixed commutative ring k. We write Coh to mean IndCoh (or QCoh as we only work with smooth spaces); the usual notion is recovered by taking compact objects. For foundations see [8,22,23].

We use the microlocal sheaf theory as developed originally in [20], though we freely make recourse to the advances in homological algebra since that time. For M a manifold, $\Lambda \subset T^*M$ a conic Lagrangian, we write $\mathrm{Sh}_\Lambda(M)$ for the stable ∞-category of (unbounded complexes of) sheaves of k-modules with microsupport in Λ.

2. Mirror symmetry for $\mathbb{A}^1$

Here we record all the calculations we will need in this article; essentially we need to check by hand equivariant mirror symmetry for $\mathbb{A}^1$. The proofs of the following lemmas are elementary and sketched or omitted.

Lemma 1. $\mathrm{Coh}(\mathbb{G}_m) \cong \mathrm{Sh}_{T^*_{S^1}S^1}(S^1)$.

Proof. Both are computed by modules over $k[t,t^{-1}]$, this ring being identified as the endomorphisms of the structure sheaf on $\mathbb{G}_m$ or as the stalk functor at any point of S^1. □

Remark. This isomorphism identifies the tensor product monoidal structure on one side with the convolution monoidal structure on the other.

We identify $S^1 = \mathbb{R}/\mathbb{Z}$ and write $T_0^- S^1$ for the negative cotangent fiber at zero. We abbreviate

$$\multimap := T^*_{S^1} S^1 \cup T_0^- S^1$$

Lemma 2. $\mathrm{Coh}(\mathbb{A}^1) \cong \mathrm{Sh}_{\multimap}(S^1)$.

Proof. Both are computed by modules over $k[t]$; identified variously as the endomorphisms of the structure sheaf, the stalk functor at a nonzero point. □

Remark. There are some choices in this isomorphism; in particular, as the categories in question are isomorphic to their opposites, we could also have taken an isomorphism $\mathrm{Coh}(\mathbb{A}^1) \cong \mathrm{Sh}_{\multimap}(S^1)^{op}$ as is done in some references.

Corollary 3. $\mathrm{Coh}(\mathbb{A}^n) \cong \mathrm{Sh}_{\multimap^n}((S^1)^n)$.

Proof. Künneth. □

In fact we need the following functoriality. The inclusion $i : 0 \in \mathbb{A}^1$ corresponds to a pullback functor $i^* : \mathrm{Coh}(\mathbb{A}^1) \to \mathrm{Mod}(k)$. Recall that at a smooth Lagrangian point of some $\Lambda \subset T^*M$, there is a functor of 'microstalk' $\mu : \mathrm{Sh}_\Lambda(M) \to \mathrm{Mod}(k)$. More precisely, there are many such functors differing by autoequivalences of $\mathrm{Mod}(k)$; one can fix a canonical choice after specifying data related to Maslov indices [20, Chapter 7.5]. Fix any such choice $\mu : \mathrm{Sh}_{\multimap}(S^1) \to \mathrm{Mod}(k)$ for the microstalk at $T^-_0 S^1$.

Lemma 4. *The isomorphism of Lemma 2 can be chosen to intertwine i^* and μ.*

For $\chi \in \mathbb{G}_m$, we write $\mathscr{L}_\chi$ for the local system on S^1 with monodromy χ. We have a $\mathbb{G}_m$ action on $\mathbb{A}^1$, inducing an action on $\mathrm{Coh}(\mathbb{A}^1)$ by pushforward. We have a $\mathbb{G}_m$ action on any category of sheaves with prescribed microsupport over S^1 where χ acts by $\otimes \mathscr{L}_\chi$.

Lemma 5. *The isomorphism of Lemma 2 intertwines the $\mathbb{G}_m$ actions.*

3. Mirror to deletions

Let $Z \subset \mathbb{A}^n$ be some union of coordinate strata. We wish to produce the mirror of $\mathrm{Coh}(\mathbb{A}^n \setminus Z)$. Recall that if $V \subset X$ is a closed embedding of schemes, $\mathrm{Coh}(X \setminus V)$ is the quotient of $\mathrm{Coh}(X)$ by the image of $\mathrm{Coh}(V)$ under the (not fully faithful) morphism of pushforward under inclusion.

The space $\multimap^n$ is the union of coordinate-wise nonpositive conormals to all strata in the product stratification arising from $S^1 = \mathbb{R}/\mathbb{Z} = 0 \cup (0,1)$, or equivalently the closure of the union of the strictly negative conormals. Here, by the (strictly) negative conormal to a stratum, we mean that one takes codirections which are (strictly) negative along every component corresponding to a 0 rather than a $(0,1)$.

There is an evident bijection between the above strata in $(S^1)^n$ and the coordinate strata in $\mathbb{A}^n$.

Definition 6. *Λ_Z is the union of the nonpositive conormals to strata in $(S^1)^n$ whose corresponding strata in $\mathbb{A}^n$ are not contained in Z.*

Example 7. Consider the case of the diagonal $\mathbb{G}_m$ action on $\mathbb{A}^n$. Then $Z = \{0\}$, and $Y = \mathbb{P}^{n-1}$. The Lagrangian Λ_Z is the closure of the union of negative conormals to coordinate strata of $(S^1)^n$ except for the point stratum 0^n. In this case, Λ_Z is obtained from $\multimap^n$ by deleting the strictly negative conormals to the stratum $0^n \in (S^1)^n$.

Proposition 8. *There is an isomorphism* $\mathrm{Coh}(\mathbb{A}^n \setminus Z) \cong \mathrm{Sh}_{\Lambda_Z}((S^1)^n)$.

Proof. We must show that $\mathrm{Sh}_{\Lambda_Z}((S^1)^n)$ is the quotient of $\mathrm{Sh}_{\Lambda^n}((S^1)^n)$ by the images of categories mirror to the pushforwards of $\mathrm{Coh}(L)$ for $L \subset Z$ a coordinate subspace. By Lemma 4, the proof of Lemma 2, Künneth, and the above discussion, this means we should take the quotient by the microstalks along the strictly negative conormals to strata of $(S^1)^n$ which correspond to strata of Z. Taking the quotient by microstalks is the same as deleting the component from the microsupport. The result of deleting these components from $\multimap^n$ differs from Λ_Z only by subcritical isotropics, which do not affect the microlocal theory of sheaves [20, Theorem 6.5.4]. $\square$

Remark. In the above argument, we took the quotient of $\mathrm{Coh}(\mathbb{A}^n)$ by all coherent sheaf categories from strata in Z. This is the same as taking the quotient by the coherent sheaf categories from top dimensional strata; but on the mirror side it is not completely obvious that deleting only conormals to the corresponding largest dimensional strata results in a category equivalent to that determined by Λ_Z. To see this is true directly, one must remember e.g. that the microsupport of a sheaf can never be a manifold with boundary, or a manifold which is locally closed but not closed.

4. Mirror to quotients

There is an evident equivalence between sheaves on S^1 and periodic sheaves on $\mathbb{R}$:

$$\operatorname{Sh}\left(S^1\right) = \operatorname{Sh}(\mathbb{R})^{\mathbb{Z}} \tag{2}$$

On the other hand, $\operatorname{Sh}(S^1)$ carries a $\mathbb{G}_m$ action, where $\lambda \in \mathbb{G}_m$ acts as tensor product with the rank one local system of monodromy λ. Passing to the universal cover $\mathbb{R} \to S^1$ as trivializes this action, leading one to expect

$$\operatorname{Sh}\left(S^1\right)^{\mathbb{G}_m} \stackrel{?}{=} \operatorname{Sh}(\mathbb{R}) \tag{3}$$

More generally, one has the following categorical incarnation of Pontrjagin duality:

Lemma 9. *Let $G \subset \mathbb{G}_m^n$ be any subgroup and let $L = \operatorname{Hom}(G, \mathbb{G}_m)$. Then there are inverse equivalences between: the category of presentable dg categories with a G action, and the category of presentable dg categories with an L action, which take $\mathscr{C} \mapsto \mathscr{C}^G$ and $\mathscr{D} \mapsto \mathscr{D}^L$.*

The following argument was explained to me by Justin Hilburn and Nick Rozenblyum.

Proof. An action of L, i.e. a map $L \to \operatorname{End}(\mathscr{D})$, is the same as an action of the monoidal category of L-graded k-modules. This in turn is equivalent to the monoidal category of G-representations, i.e. the category $(\operatorname{QCoh}(BG), \otimes)$. Now we will use the fact that BG is "1-affine" [7].[1]

In general a stack X is said to be "1-affine" when the 'global sections' map

$$\Gamma : \text{sheaves of categories on } X \to \text{categories with an action of } \mathscr{O}_X$$

is an equivalence. It is moreover the case [7, Section 10.3.2] that 'sheaves of categories on BG' are naturally identified with "categories with an action of $(\operatorname{QCoh}(G), \star)$", where $\star$ is the convolution product coming from the multiplication on G. 1-affineness translates to the assertion that adjunction between "equivariantization" and "de-equivariantization" is an equivalence:

$$\begin{aligned} \left(\operatorname{QCoh}(G), \star\right) - \text{modules} &\leftrightarrow \left(\operatorname{QCoh}(BG), \otimes\right) - \text{modules} \\ \mathscr{C} &\mapsto \mathscr{C}^G = \operatorname{Hom}_{\operatorname{QCoh}(G)}(\operatorname{Mod}(k), \mathscr{C}) \\ \operatorname{Mod}(k) \otimes_{\operatorname{QCoh}(BG)} \mathscr{D} = \mathscr{D}_{BG} &\hookleftarrow \mathscr{D} \end{aligned}$$

Above, the actions on $\operatorname{Mod}(k)$ are trivial.

It remains to check $\mathscr{D}_L \cong \mathscr{D}^L$. We explain for $L = \mathbb{Z}$, the general case is similar. A $\mathbb{Z}$-action is encoded by the automorphism F by which $1 \in \mathbb{Z}$ acts; the co/invariants are given by the co/equalizer of the diagram $\mathscr{D} \rightrightarrows \mathscr{D}$ comparing the maps $id_{\mathscr{D}}$ and F. Our categories are presentable; taking adjoints interchanges F with F^{-1} and invariants with co-invariants. A second argument: $\operatorname{QCoh}(B\mathbb{G}_m)$ is rigid, so $\operatorname{Mod}(k)$ is self-dual as a $\operatorname{QCoh}(B\mathbb{G}_m)$-module [8, Vol. 1, Chapter 1, Proposition 9.4.4], hence invariants and co-invariants agree. ☐

We leave the reader to check that when $\mathscr{D} = \operatorname{Sh}(\mathbb{R})$ and the $L = \mathbb{Z}$ action is pullback by translation, then the corresponding $\mathbb{G}_m$ action is the one described above, hence establishing the validity of Equation 3. Analogous statements then follow for $\mathscr{D} = \operatorname{Sh}(\mathbb{R}^n/M)$ and $L = \mathbb{Z}^n/M$ for any subgroup $M \subset \mathbb{Z}^n$, and moreover for full subcategories defined by a microsupport condition preserved by the L action.

We return to our setting: $G \subset \mathbb{G}_m^n$ acts on affine space $\mathbb{A}^n$. We have the sequence of groups

$$1 \to G \to \mathbb{G}_m^n \to \mathbb{T} \to 1 \tag{4}$$

[1] In [7] it is assumed that the ground ring k is a field of characteristic zero. However the role of this hypothesis is only to ensure semisimplicity of the category of G representations, so we do not need for $\mathbb{G}_m^n$. Subgroups of $\mathbb{G}_m^n$ are then also 1-affine by [7, Corollary 3.2.7]

and the corresponding sequence of characters

$$0 \to M \to \mathbb{Z}^n \to \mathrm{Hom}(G, \mathbb{G}_m) \to 0 \tag{5}$$

In this sequence we identify $\mathbb{Z}^n = \pi_1((\mathbb{R}/\mathbb{Z})^n)$. We write $c_M : \mathbb{R}^n/M \to \mathbb{R}^n/\mathbb{Z}^n = (S^1)^n$ for the cover corresponding to the subgroup $M \subset \mathbb{Z}^n$. We write $\Lambda_{Z,M}$ for the preimage of Λ_Z under the induced cover of cotangent bundles.

Proposition 10. $\mathrm{Sh}_{\Lambda_{Z,M}}(\mathbb{R}^n/M) \cong \mathrm{Coh}(\mathbb{A}^n \setminus Z)^G$.

Proof. By descent and Proposition 8,

$$\mathrm{Sh}_{\Lambda_{Z,M}}\left(\mathbb{R}^n/M\right)^{\mathbb{Z}^n/M} = \mathrm{Sh}_{\Lambda_Z}\left(\mathbb{R}^n/\mathbb{Z}^n\right) = \mathrm{Coh}\left(\mathbb{A}^n \setminus Z\right)$$

Now we take G invariants apply Lemma 9 on the left hand side. The fact that the G actions in question are the expected ones follows from Lemma 5. □

Remark. Various choices (e.g. the orientation of S^1) may lead to the G or L actions being conjugated by an automorphism of G or L. This does not affect the categories of invariants.

Remark. The interaction of mirror symmetry with equivariance and covers has been known since at least early 2000's, see e.g. [26]; in the present context it is at least implicit in the relationship between [5] and [27]. The work [28] argues for a version of Lemma 9 without appeal to [7], and uses it to deduce the "nonequivariant" equivalence conjectured in [27] from the "equivariant" results of [5]. The recent work [19,32] also takes a point of view similar to the present article.

5. The Cox construction and the FLTZ skeleton

Proposition 10 gives the mirror construction to Equation (1). However, it does not provide an entirely satisfying mirror for the the toric stack $Y = (\mathbb{A}^n \setminus Z)/G$. In particular, this stack is $n - g$ dimensional, where $g = \dim G$. One would expect a $2(n-g)$ dimensional mirror; whereas $T^*(\mathbb{R}^n/M)$ is $2n$ dimensional.

We will write $A = \mathbb{R}^n/M$ and $M_{\mathbb{R}} = M \otimes \mathbb{R}$. Consider the sequence

$$0 \to M_{\mathbb{R}}/M \to A = \mathbb{R}^n/M \xrightarrow{\pi} \mathbb{R}^n/M_{\mathbb{R}} \to 0$$

So A is a $(n-g)$-torus bundle over the g-dimensional vector space $\mathbb{R}^n/M_{\mathbb{R}}$. For $\gamma \in \mathbb{R}^n/M_{\mathbb{R}}$, we will write A_γ for the corresponding $(n-g)$-torus fiber.

Consider the induced map $p : T^*A \to \mathbb{R}^n/M_{\mathbb{R}}$. We denote the symplectic reduction maps as $r : p^{-1}(\gamma) \to T^*A_\gamma$. For $\Lambda \subset T^*A$, we write

$$\Lambda_\gamma := r\left(\Lambda \cap p^{-1}(\gamma)\right) \subset T^*A_\gamma$$

We say $\Lambda \subset T^*A$ is π-noncharacteristic if Λ contains no nonzero covectors pulled back from the base; i.e. no nonzero covectors to the fibers. By [20, Proposition 5.4.13.i], we have:

Lemma 11. *For π-noncharacteristic Λ, restriction at γ gives a map* $\mathrm{Sh}_\Lambda(A) \to \mathrm{Sh}_{\Lambda_\gamma}(A_\gamma)$.

Remark. In fact the π-noncharacteristic hypothesis is unnecessary. In general one should take *the closure* of the projection of Λ to the relative cotangent bundle $T^*\pi$, and then restrict to the fiber T^*A_γ (see e.g. [25, Lemma 3.13]). Because of the linear nature of the locus $\Lambda_{Z,M}$, the projection is already closed. We will not use this fact here, but it is relevant for studying non-simplicial fans.

The torus A_γ has dimension $n-g$ as desired. It remains to describe $\Lambda_{Z,M,\gamma}$ explicitly, study when $\Lambda_{Z,M}$ is π-noncharacteristic, and finally determine whether the above map is an isomorphism.

We now restrict ourselves to the setting where Z is zero locus of the Cox irrelevant ideal of a toric variety or stack. Let us recall from [4] the combinatorial data relevant to this situation:

Definition 12. *By fan data, we mean a lattice N, a rationally surjective lattice map $f:\mathbb{Z}^n \to N$ and a set $\widetilde{\Sigma}$ of closed coordinate strata of $\mathbb{R}^n_{\geq 0}$, such that*

- *$\widetilde{\Sigma}$ is closed under taking coordinate sub-strata*
- *$\widetilde{\Sigma}$ includes all coordinate rays*
- *The images under f of the (relative) interiors of coordinate strata in $\widetilde{\Sigma}$ are disjoint*

The image $\sigma = f(\widetilde{\sigma})$ of a stratum is a cone in $N\otimes\mathbb{R}$. It is more typical to introduce directly these cones, the set of which is denoted Σ and called the fan. The fan is said to be simplicial if these cones are simplices; equivalently, if the maps $\widetilde{\sigma}\to f(\widetilde{\sigma})$ are bijections. Our definition of toric data ensures that $\widetilde{\Sigma}$ includes the zero stratum, so no coordinate ray is collapsed to a point.

Remark. Taking f rationally surjective ensures the toric variety is a quotient of $\mathbb{A}^n$. More generally it would be a quotient of some $\mathbb{A}^n\times\mathbb{G}_m^k$, giving some extra circle factors on the mirror.

We write $M = \mathrm{Hom}(N,\mathbb{Z})$ and thus obtain an injective map $M \xrightarrow{f} \mathbb{Z}^n$, determining the sequence (5) above, which in turn determines $G = \mathrm{Hom}(\mathbb{Z}^n/M,\mathbb{G}_m)\subset \mathrm{Hom}(\mathbb{Z}^n,\mathbb{G}_m) = \mathbb{G}_m^n$.

Fan data determines a locus $Z\subset\mathbb{A}^n$, as follows. For each $\widetilde{\sigma}\in\widetilde{\Sigma}$, the face $\widetilde{\sigma}\subset\mathbb{R}^n_{\geq 0}$ is characterized as the zero locus of some variables $x_{i_1},\dots,x_{i_n}$. Now for each such face *not in* $\widetilde{\Sigma}$, we include in Z the zero locus of the *complementary* variables.

Example 13. Let us consider $\mathbb{P}^2$. That is, consider $N=\mathbb{Z}^3/(1,1,1)\mathbb{Z}$. We take $\widetilde{\Sigma}$ to include all faces except the maximal face, $\mathbb{R}^3_{\geq 0}$. The open face is that on which no variables vanish; correspondingly $Z\subset\mathbb{A}^3$ consists only of the origin, the locus where all variables vanish.

Example 14. Let us consider $\mathbb{P}^2$ minus a vertex. Again $N=\mathbb{Z}^3/(1,1,1)\mathbb{Z}$, but now we take $\widetilde{\Sigma}$ to include all faces except the maximal face, $\mathbb{R}^3_{\geq 0}$, and the face given by $x_1=0$. Then we should include in $Z\subset\mathbb{A}^3$ the origin as before, and also the locus on which x_2 and x_3 vanish, namely the x_1-axis. That is, Z is the x_1-axis.

From [4, Theorem 2.1], we learn that with these definitions, the stack $(\mathbb{A}^n\setminus Z)/G$ has coarse moduli space given by the toric variety which would ordinarily be associated to the fan Σ, and that the stack is Deligne–Mumford iff the fan is simplicial.

Recall we obtain Λ_Z by beginning with $\multimap^n$ and deleting strata associated to coordinate subspaces in Z. The space $\multimap^n$ itself is a subset of $T^*(\mathbb{R}^n/\mathbb{Z}^n)$; it is natural to identify the cotangent fiber with the $\mathbb{R}^n$ which contains $\widetilde{\Sigma}$. Indeed, this $\mathbb{R}^n$ projects to the cotangent fiber of $(M\otimes\mathbb{R})/M$, which is naturally identified with $N\otimes\mathbb{R}$.

For $\widetilde{\sigma}\subset\mathbb{R}^n$, and any lattice $L\subset\mathbb{Z}^n\subset\mathbb{R}^n$, we write

$$\widetilde{\sigma}^\perp := \left\{p \,\middle|\, \langle p,\widetilde{\sigma}\rangle\in\mathbb{Z}\right\}\subset\mathbb{R}^n/L$$

It is a union of translates of images of linear subspaces of $\mathbb{R}^n$.

Combining the definitions of Z and in terms of $\widetilde{\Sigma}$ and of Λ_Z in terms of Z, we have:

$$\Lambda_Z = \bigcup_{\widetilde{\sigma}\in\widetilde{\Sigma}} \widetilde{\sigma}^\perp\times -\widetilde{\sigma}\subset(\mathbb{R}/\mathbb{Z})^n\times\mathbb{R}^n = T^*(\mathbb{R}/\mathbb{Z})^n$$

Our conventions for $\widetilde{\sigma}^\perp$ ensure that $\Lambda_{Z,M}$ is given by the same formula, now as a subset of T^*A. Finally, it follows from the definitions that:

$$\Lambda_{Z,M,\gamma} = \bigcup_{\widetilde{\sigma}\in\widetilde{\Sigma}} \left(\widetilde{\sigma}^\perp\cap A_\gamma\right)\times -f(\widetilde{\sigma})$$

In particular, $\Lambda_{Z,M,0}$ is precisely the conic Lagrangian proposed by [6].

Proposition 15. *The following are equivalent*

- Σ *is simplicial*
- $\Lambda_{Z,M}$ *is* π*-noncharacteristic*
- *Every* $\pi : \widetilde{\sigma}^{\perp} \to \mathbb{R}^n / M_{\mathbb{R}}$ *is a smooth submersion*

Proof. As noted above, Σ is simplicial if the map $\mathbb{R}^n \to N \otimes \mathbb{R}$ induces always bijections on $\widetilde{\sigma} \to f(\sigma)$. This map is the quotient by the cotangent directions to $\mathbb{R}^n / M_{\mathbb{R}}$, so it is a bijection iff said directions are never contained in the cones $\widetilde{\sigma}$; i.e. if $\Lambda_{Z,M}$ is π-noncharacteristic. As $\widetilde{\sigma}$ is an open subset of the conormal to $\widetilde{\sigma}^{\perp}$, the second and third conditions above are also equivalent. $\square$

Remark. When Σ is not simplicial, there will be some $\widetilde{\sigma}^{\perp}$ of lower dimension than $\mathbb{R}^n / M_{\mathbb{R}}$, and consequently the corresponding $\widetilde{\sigma}^{\perp} \cap A_{\gamma}$ will be generically, but not always, empty. Correspondingly, the $\mathrm{Sh}_{\Lambda_{Z,M,\gamma}}(A_{\gamma})$ will certainly depend on γ.

Recall (from [13,14] and references therein) that a Weinstein pair (Q, R) is a Weinstein manifold Q together with a Weinstein hypersurface $R \subset \partial_{\infty} Q$. The relative skeleton of such a pair is the union of the skeleton of Q and the cone of the skeleton on R. We understand the cotangent bundle of a noncompact manifold as an "open Liouville sector" in the sense of [14].

Proposition 16. *There is a Weinstein pair* $(T^* A, W)$ *with relative skeleton* $\Lambda_{Z,M}$*. When the fan* Σ *is simplicial, there is a Weinstein pair* $(T^* A_{\gamma}, W_{\gamma})$ *with relative skeleton* $\Lambda_{Z,M,\gamma}$*, and a Weinstein deformation equivalence* $(T^* A, W) \simeq (T^* A_{\gamma}, W_{\gamma}) \times T^*(\mathbb{R}^n / M_{\mathbb{R}})$.

Proof. We first construct W. The product structure of $\bigcirc\!\!\!-^{\,n}$ makes it clear that the corresponding singular Legendrian $\partial \bigcirc\!\!\!-^{\,n} \subset S^*(S^1)^n$ is the skeleton of a Weinstein domain D; see e.g. the discussion in [13, Section 6.1]. One has the following handle presentation. Begin with the negative conormals to the codimension-1 strata of $(S^1)^n$. This is a disjoint union of $(S^1)^{n-1}$ inside the cosphere bundle, which naturally thicken to their cotangent bundles. We view these $(S^1)^{n-1}$ as the Bott minima of some Morse function on D. The negative conormals to the codimension-2 strata of $(S^1)^n$ arise as the stable flow from a $(S^1)^{n-2}$-parameterized 1-handle attachment; etcetera. We define D_Z by attaching only handles with cores in Λ_Z. Then W is an appropriate cover of D_Z.

If the fan Σ is simplicial, Proposition 15 ensures that all this attaching data (a sequence of Legendrian manifolds in various contact levels) projects submersively to $\mathbb{R}^n / M_{\mathbb{R}}$, hence naturally giving a family of said attaching data. We take the fiber at γ to define W_{γ}. The base $\mathbb{R}^n / M_{\mathbb{R}}$ is contractible, so we may isotope the family of attaching data to a constant family. $\square$

Remark. In fact, in [12,31], the W_{γ} described above are identified with hypersurfaces in $(\mathbb{C}^*)^{n-g}$ with Newton polynomial corresponding to the fan data.

Theorem 17. *For* Σ *simplicial, the map* $\mathrm{Sh}_{\Lambda_{Z,M}}(A) \to \mathrm{Sh}_{\Lambda_{Z,M,\gamma}}(A_{\gamma})$ *is an equivalence.*

Proof. The point is that sheaf categories are invariant under deformation through Weinstein pairs. For cotangent bundles (which covers the present situation) this is [29]; for another argument and more general results see [25]. Alternatively, pass by [15] to Fukaya categories, where deformation invariance is obvious. Now the result follows from Proposition 16. $\square$

Remark. Note $M = \pi_1(A) = \pi_1(A_{\gamma})$; the equivalence of Theorem 17 respects the $\mathrm{Hom}(M, \mathbb{G}_m)$ action. Equivariantization as in Section 4 recovers the main result of [5].

Remark. Rather than one mirror skeleton $\Lambda_{Z,M,0}$ as in [5, 21, 27, 28, 30], we obtain a family of them parameterized by $\mathbb{R}^n / M_{\mathbb{R}}$, hence a local system of categories over this base. Points $\gamma \in \mathbb{R}^n / M_{\mathbb{R}}$ which differ by the image of the integer lattice $\mathbb{Z}^n / M$ give the same $\Lambda_{Z,M,\gamma}$, giving an action of $\mathbb{Z}^n / M$ on $\mathrm{Sh}_{\Lambda_{Z,M,\gamma}}(A_{\gamma})$. Per [4], $\mathbb{Z}^n / M$ is the group of toric divisors; and one can see from the considerations of this article that the corresponding action on coherent sheaves is tensor product by the corresponding line bundles. This is a version of the main result of [16].

Remark. A key result of [12] is that the microlocalization $\mu : \mathrm{Sh}_{\Lambda_{Z,M,\gamma}}(A_\gamma) \to \mu\mathrm{Sh}(\partial_\infty \Lambda_{Z,M,\gamma})$ is mirror to restriction of coherent sheaves from a toric variety to its toric boundary. One can deduce this from Lemma 4 by the methods of the present article.

Remark. One can study singular varieties by studying the mirror (in terms of $\Lambda_{Z,M,\gamma}$) of the effect of resolution of singularities on categories of coherent sheaves, as is done in [21]. It would be interesting to directly investigate the question: what is mirror to forming the coarse moduli space?

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