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The subword complexity of polynomial subsequences of the Thue–Morse sequence

La complexité de facteurs des sous-suites polynomiales de la suite de Thue–Morse

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Abstract. Let $\mathbf{t} = (t(n))_{n\geqslant 0}$ be the Thue–Morse sequence in 0, 1. J.-P. Allouche and J. Shallit asked in 2003 whether the subword complexity of the subsequence $(t(n^2))_{n\geqslant 0}$ attains the maximal value. This problem was solved positively by Y. Moshe in 2007. Indeed Y. Moshe had shown that for all $H \in \mathbb{Q}[T]$ with $H(\mathbb{N}) \subseteq \mathbb{N}$ and $\deg H = 2$, all the subsequences $(t(H(n)))_{n\geqslant 0}$ attain the maximal subword complexity. Then he asked whether the same result holds for $\deg H \geqslant 3$. In this work, we shall give a positive answer to the above problem.

Résumé. Soit $\mathbf{t} = (t(n))_{n\geqslant 0}$ la suite de Thue–Morse en 0, 1. J.-P. Allouche et J. Shallit demandaient en 2003 si la complexité de facteurs de la sous-suite $(t(n^2))_{n\geqslant 0}$ atteint la maximale. Le problème était résolu positivement par Y. Moshe en 2007. En fait, Y. Moshe avait démontré que pour tout $H \in \mathbb{Q}[T]$ avec $H(\mathbb{N}) \subseteq \mathbb{N}$ et $\deg H = 2$, toutes les sous-suites $(t(H(n)))_{n\geqslant 0}$ atteignent la complexité maximale. Ensuite il demandait si le résultat est aussi valable pour $\deg H \geqslant 3$. Dans ce travail, nous allons donner une réponse positive au problème précédent.

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Soient Σ un alphabet fini et $\boldsymbol{u} = (u(n))_{n\geqslant 0}$ une suite sur Σ. Pour tout entier $m \geqslant 1$, désignons par $P_{\boldsymbol{u}}(m)$ le nombre de facteurs différents de longueur m dans $\boldsymbol{u}$, et appelons la fonction $P_{\boldsymbol{u}}$ la complexité de facteurs de $\boldsymbol{u}$. Ainsi $P_{\boldsymbol{u}}(m) \leqslant |\Sigma|^m$, où $|\Sigma|$ désigne le nombre d'éléments dans Σ.

Soit $\boldsymbol{t} = (t(n))_{n\geqslant 0}$ la suite de Thue–Morse en 0, 1. Elle est définie par $t(n) = s_2(n) (\bmod 2)$, pour tout entier $n \geqslant 0$, où $s_2(n)$ désigne le nombre de 1's dans la représentation binaire de n. La complexité de facteurs de $\boldsymbol{t}$ est compliquée mais déjà connue (voir S. Brlek [4], A. de Luca

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et S. Varricchio [8], et S. V. Avgustinovich [3]). J.-P. Allouche et J. Shallit demandaient dans [2] si $P_{\boldsymbol{u}}(m) = 2^m$ pour tout $m \geqslant 1$, où $\boldsymbol{u} = (t(n^2))_{n\geqslant 0}$. Le problème était résolu positivement par Y. Moshe [9]. Plus généralement, il a obtenu la même conclusion pour toutes les sous-suites $(t(H(n)))_{n\geqslant 0}$, avec $H \in \mathbb{Q}[T]$ tel que $H(\mathbb{N}) \subseteq \mathbb{N}$ et $\deg H = 2$, et il demandait ensuite si le résultat persiste encore pour $\deg H \geqslant 3$.

Dans la suite, nous allons donner une réponse positive au problème précédent de Y. Moshe.

Theorem. *Soit $H \in \mathbb{Q}[T]$ tel que $H(\mathbb{N}) \subseteq \mathbb{N}$ et $\deg H \geqslant 2$. Alors $P_{\boldsymbol{u}}(m) = 2^m$ pour tout entier $m \geqslant 1$, où $\boldsymbol{u} = (u(n))_{n\geqslant 0} = (t(H(n)))_{n\geqslant 0}$.*

1. Introduction

Let Σ be a finite alphabet and $\boldsymbol{u} = (u(n))_{n\geqslant 0}$ a sequence over Σ. For all integers $m \geqslant 1$, let $P_{\boldsymbol{u}}(m)$ be the number of different subwords in $\boldsymbol{u}$ with length m, and we call the function $P_{\boldsymbol{u}}$ the subword complexity of $\boldsymbol{u}$. So $P_{\boldsymbol{u}}(m) \leqslant |\Sigma|^m$, where $|\Sigma|$ denotes the number of elements in Σ. Note that automatic sequences have relatively low complexity $O(m)$, and random sequences have high complexity (see e.g. [2]).

Let $\boldsymbol{t} = (t(n))_{n\geqslant 0}$ be the Thue–Morse sequence in 0, 1, i.e., $t(n) = s_2(n) \pmod 2$, where $s_2(n)$ is the number of 1's in the binary representation of n. The subword complexity of $\boldsymbol{t}$ is complicated but already known (see S. Brlek [4], A. de Luca and S. Varricchio [8], and S. V. Avgustinovich [3]). It is well known that $\boldsymbol{t}$ is 2-automatic, and J.-P. Allouche showed in [1] that $(t(H(n))_{n\geqslant 0}$ is not 2-automatic, if $H \in \mathbb{Q}[T]$ with $H(\mathbb{N}) \subseteq \mathbb{N}$ and $\deg H \geqslant 2$. So $(t(n^2))_{n\geqslant 0}$ is not 2-automatic, and then J.-P. Allouche and J. Shallit asked in [2] whether $P_{\boldsymbol{u}}(m) = 2^m$ for all integers $m \geqslant 1$, where $\boldsymbol{u} = (t(n^2))_{n\geqslant 0}$. This problem was solved positively by Y. Moshe [9]. More generally, he has obtained the same conclusion for all $H \in \mathbb{Q}[T]$ with $H(\mathbb{N}) \subseteq \mathbb{N}$ and $\deg H = 2$, and then asked whether it holds also for $\deg H \geqslant 3$. Below we shall give a positive answer to this problem. For related works, see for example [5–7, 10, 11] and references therein.

Theorem 1. *Let $H \in \mathbb{Q}[T]$ such that $H(\mathbb{N}) \subseteq \mathbb{N}$ and $\deg H \geqslant 2$. Let $\boldsymbol{u} = (u(n))_{n\geqslant 0} = (t(H(n)))_{n\geqslant 0}$. Then $P_{\boldsymbol{u}}(m) = 2^m$ for all integers $m \geqslant 1$.*

Below let v_2 be the 2-adic valuation, and put $\mu(n) = 2^{v_2(n)}$ for all integers $n \geqslant 1$. For $\boldsymbol{e} = (e_j)_{1\leqslant j\leqslant n}$, $\boldsymbol{f} = (f_j)_{1\leqslant j\leqslant n} \in \mathbb{N}^n$, we say $\boldsymbol{e} < \boldsymbol{f}$ if $\exists\, k \in \mathbb{N} (1 \leqslant k \leqslant n)$ such that $e_j = f_j$ $(k < j \leqslant n)$ and $e_k < f_k$, and call it the colexicographic order. Finally put

$$|\boldsymbol{e}| := \sum_{k=1}^{n} e_k, \quad \text{and} \quad C_{\boldsymbol{e}} := \binom{|\boldsymbol{e}|}{e_1, e_2, \cdots, e_n} = |\boldsymbol{e}|! \Big/ \prod_{k=1}^{n} e_k!.$$

2. Some preliminary lemmas

Lemma 2. *Let $\ell \geqslant 1$ be an integer, and $\boldsymbol{e} = (e_j)_{1\leqslant j\leqslant 2^\ell} \in \mathbb{N}^{2^\ell}$ with $1 \leqslant |\boldsymbol{e}| < 2^\ell$. Let $J_{\boldsymbol{e}}$ be the set of $\boldsymbol{f} \in \mathbb{N}^{2^\ell}$ such that $|\boldsymbol{f}| = |\boldsymbol{e}|$ and $C_{\boldsymbol{f}} = C_{\boldsymbol{e}}$. Then $\operatorname{Card}(J_{\boldsymbol{e}})$ is even.*

Lemma 3. *Let $d \geqslant 0$, $\ell \geqslant 1$ be integers such that $d \leqslant 2^\ell - 1$. Let Y_d be the set of*

$$\boldsymbol{e} = \left(e_j\right)_{2\leqslant j\leqslant 2^\ell} \in \mathbb{N}^{2^\ell - 1} \quad \textit{with} \quad \sum_{j=2}^{2^\ell} e_j = d,$$

and $0 \leqslant e_j \leqslant 2 (2 \leqslant j \leqslant 2^\ell)$. Then $y_d := \operatorname{Card}(Y_d)$ is odd if and only if $d \equiv 0, 1 \pmod 3$.

Lemma 4. *Let $d \geqslant 0$, $\ell \geqslant 2$ be integers such that $d \leqslant 2^\ell - 2$. Let G_d be the set of*

$$\boldsymbol{e} = (e_j)_{3\leqslant j\leqslant 2^\ell} \in \mathbb{N}^{2^\ell-2} \quad \textit{with} \quad \sum_{j=3}^{2^\ell} e_j = d,$$

and $0 \leqslant e_j \leqslant 2 (3 \leqslant j \leqslant 2^\ell)$. Then $g_d := \mathrm{Card}(G_d)$ is odd if and only if $d \equiv 0, 2 \pmod 6$.

Lemma 5. *Let $d \geqslant 2$, $m \geqslant 1$ be integers, $H_j \in \mathbb{Z}[T]$ with $H_j(\mathbb{N}) \subseteq \mathbb{N}$ and*

$$H_j = \sum_{k=0}^{d} a_k^{(j)} T^k \left(1 \leqslant j \leqslant m\right).$$

Then $\exists\, N \geqslant 1$ such that for all $n > N$, $\exists\, A_n \geqslant 1$ such that $t(H_j(n+A_n)) = t(H_j(n))$ $(1 \leqslant j \leqslant m)$.

Lemma 6. *Let $d \geqslant 2$, $m \geqslant 1$ be integers, $H_j \in \mathbb{Z}[T]$ with $H_j(\mathbb{N}) \subseteq \mathbb{N}$ and*

$$H_j = \sum_{k=0}^{d} a_k^{(j)} T^k \left(1 \leqslant j \leqslant m\right)$$

such that $a_d^{(j)} = a_d > 0$ and $a_{d-1}^{(1)} < a_{d-1}^{(2)} < \cdots < a_{d-1}^{(m)}$. Then $\exists\, N \geqslant 1$ such that for all integers $n > N$, $\exists\, A_n \geqslant 1$ such that $t(H_j(n+A_n)) = t(H_j(n)) (1 \leqslant j < m)$ and $t(H_m(n+A_n)) = t(H_m(n)) + 1$.

Proof of Theorem 1. Write $H = \sum_{k=0}^{d} a_k T^k$ with $a_k \in \mathbb{Q}$, and $a_d \neq 0$. Take $Q \geqslant 2$ an integer such that $Qa_k \in \mathbb{Z}$ for $0 \leqslant k \leqslant d$. For $j \geqslant 1$, put $H_j(T) = H(QT + j)$. Then $H_j \in \mathbb{Z}[T]$, and we need to show that for all $b_j \in \mathbb{Z}/2\mathbb{Z}$ $(1 \leqslant j \leqslant m)$, we can find infinitely many integers n such that $t(H_j(n)) = b_j (1 \leqslant j \leqslant m)$.

By induction on m. If $m = 1$, by Lemma 6, $\exists\, N_1 \geqslant 1$ such that for all integers $n > N_1$, we can find $A_n \geqslant 1$ such that $t(H_1(n+A_n)) = t(H_1(n)) + 1$, hence $t(H_1(n)) = b_1$ or $t(H_1(n+A_n)) = b_1$.

Now assume that the result holds for $m-1$ with $m \geqslant 2$. Then there are infinitely many integers n such that $t(H_j(n)) = b_j (1 \leqslant j < m)$. If $b_m = t(H_m(n))$, then the desired result holds. Otherwise $b_m = t(H_m(n)) + 1$, and by Lemma 6, $\exists\, N_m \geqslant 1$ such that for all integers $n > N_m$, $\exists\, A_n \geqslant 1$ such that $t(H_j(n+A_n)) = t(H_j(n)) (1 \leqslant j < m)$ and $t(H_m(n+A_n)) = t(H_m(n)) + 1$. So the desired result holds. □

3. Proofs of lemmas

Proof of Lemma 2. Let S_{2^ℓ} be the symmetric group on 2^ℓ letters. For $\sigma \in S_{2^\ell}$ and $\boldsymbol{f} = (f_j)_{1\leqslant j\leqslant 2^\ell} \in J_{\boldsymbol{e}}$, put $\boldsymbol{f}_\sigma = (f_{\sigma(j)})_{1\leqslant j\leqslant 2^\ell} \in J_{\boldsymbol{e}}$, $J_{\boldsymbol{e}}(\boldsymbol{f}) = \{\boldsymbol{g} \in J_{\boldsymbol{e}} : \boldsymbol{g}_\sigma = \boldsymbol{f}\}$, and it suffices to show that $\mathrm{Card}(J_{\boldsymbol{e}}(\boldsymbol{f}))$ is even. Assume that $\boldsymbol{f}$ has r different values, each with multiplicity $s_i (1 \leqslant i \leqslant r)$. Then

$$\sum_{1\leqslant j\leqslant r} s_j = 2^\ell, \quad \text{and} \quad \mathrm{Card}(J_{\boldsymbol{e}}(\boldsymbol{f})) = \binom{2^\ell}{s_1, s_2, \cdots, s_r}.$$

Note that $s_i < 2^\ell (1 \leqslant i \leqslant r)$, otherwise $f_j = f_1$ $(1 \leqslant j \leqslant 2^\ell)$, and then $2^\ell > |\boldsymbol{f}| = 2^\ell f_1$. Finally we obtain

$$v_2\left(\mathrm{Card}\left(J_{\boldsymbol{e}}\left(\boldsymbol{f}\right)\right)\right) = \sum_{j=1}^{\ell}\left(\left\lfloor \frac{2^\ell}{2^j}\right\rfloor - \sum_{k=1}^{r}\left\lfloor \frac{s_k}{2^j}\right\rfloor\right) \geqslant \left\lfloor \frac{2^\ell}{2^\ell}\right\rfloor - \sum_{k=1}^{r}\left\lfloor \frac{s_k}{2^\ell}\right\rfloor = 1. \qquad \square$$

Proof of Lemma 3. Write $(1+x+x^2)^{2^\ell-1} = \sum_{i=0}^{2(2^\ell-1)} b_i x^i$, and set $\beta_k = \sum_{i=0}^{k} b_i$ $(1 \leqslant k < 2^\ell)$. Then

$$\left(1-x^3\right)^{2^\ell-1} = (1-x)^{2^\ell-1}\left(1+x+x^2\right)^{2^\ell-1} = (1-x)^{2^\ell-1}\sum_{i=0}^{2(2^\ell-1)} b_i x^i,$$

so

$$\sum_{i=0}^{2^\ell-1} x^{3i} \equiv \left(1-x^3\right)^{2^\ell-1} \equiv \frac{1-x^{2^\ell}}{1-x}\left(\sum_{i=0}^{2(2^\ell-1)} b_i x^i\right) \equiv \left(\sum_{i=0}^{2^\ell-1} x^i\right)\left(\sum_{i=0}^{2(2^\ell-1)} b_i x^i\right) \pmod 2.$$

The part of degree $< 2^\ell$ is $\sum_{k=0}^{2^\ell-1} \beta_k x^k$. Thus $\beta_k \equiv 1 \pmod 2$ iff $3 \mid k$. So $y_d = b_d = \beta_d - \beta_{d-1} \equiv 1 \pmod 2$ iff $d \equiv 0, 1 \pmod 3$. □

Proof of Lemma 4. Write $(1+x+x^2)^{2^\ell-2} = \sum_{i=0}^{2(2^\ell-2)} c_i x^i$. Then we have $c_d = g_d$. Note that

$$\left(1+x+x^2\right)\sum_{i=0}^{2(2^\ell-2)} c_i x^i = \left(1+x+x^2\right)^{2^\ell-1} = \sum_{i=0}^{2(2^\ell-1)} b_i x^i,$$

hence $c_0 = b_0 = 1$, $c_0 + c_1 = b_1$, $c_i + c_{i-1} + c_{i-2} = b_i$, for $2 \leqslant i \leqslant 2^\ell - 2$. By Lemma 3 and by induction on i, we obtain c_i is odd if and only if $i \equiv 0, 2 \pmod 6$. □

Proof of Lemma 5. Since $H_j(\mathbb{N}) \subseteq \mathbb{N}$ $(1 \leqslant j \leqslant m)$, we can find an integer $N \geqslant 1$ large enough such that for all integers $n > N$, we have $\sum_{k=i}^{d} \binom{k}{i} a_k^{(j)} n^{k-i} > 0$ $(0 \leqslant i \leqslant d)$. Take $M > d! \cdot \sum_{k=i}^{d} \binom{k}{i} a_k^{(j)} n^d$, possibly depending on n. Let ℓ, b be integers such that $2^\ell > d$, $b > n$, and b is odd. Take integers z_k such that $2^{z_1} > Mb^d$, and $2^{z_k} > 2^{dz_{k-1}} Mb^d$ $(2 \leqslant k \leqslant 2^\ell)$. Put $x_k = 2^{z_k} (1 \leqslant k \leqslant 2^\ell)$, and $A := A_n := b\sum_{k=1}^{2^\ell} x_k$.

Put $D(d,\ell) = \{\boldsymbol{e} = (e_k)_{1 \leqslant k \leqslant 2^\ell} \in \mathbb{N}^{2^\ell} : |\boldsymbol{e}| \leqslant d\}$. For $1 \leqslant j \leqslant m$ and $\boldsymbol{e} \in D(d,\ell)$, define

$$\alpha_j(\boldsymbol{e}) = b^{|\boldsymbol{e}|} C_{\boldsymbol{e}} \left(\sum_{k=|\boldsymbol{e}|}^{d} \binom{k}{|\boldsymbol{e}|} a_k^{(j)} n^{k-|\boldsymbol{e}|}\right) \prod_{i=1}^{2^\ell} x_i^{e_i} > 0. \tag{1}$$

Then $H_j(n) = \alpha_j(\mathbf{0})$ with $\mathbf{0} = (0, \ldots, 0)$, and by multinomial expansion, we obtain further

$$H_j(n+A) = \sum_{k=0}^{d} a_k^{(j)} (n+A)^k = \sum_{k=0}^{d} a_k^{(j)} \sum_{i=0}^{k} \binom{k}{i} n^{k-i} b^i \left(\sum_{l=1}^{2^\ell} x_l\right)^i = \sum_{\boldsymbol{e} \in D(d,\ell)} \alpha_j(\boldsymbol{e}), \tag{2}$$

where the last summation proceeds by the colexicographic order of $\boldsymbol{e}$, which begins with $\mathbf{0}$ and ends with $(0, \ldots, 0, d)$. Note that $\alpha_j(\boldsymbol{e}) < \mu(\alpha_j(\boldsymbol{f}))$ $(1 \leqslant j \leqslant m)$, for all $\boldsymbol{f} \in D(d,\ell)$ with $\boldsymbol{e} < \boldsymbol{f}$. Indeed, if we write $\boldsymbol{f} = (f_k)_{1 \leqslant k \leqslant 2^\ell}$ and let k_0 be the largest index k such that $e_k < f_k$, then $e_j = f_j$ $(k_0 < j \leqslant 2^\ell)$. Let $x_0 = 1$ if necessary, then we have

$$\begin{aligned}\mu\left(\alpha_j(\boldsymbol{f})\right) &\geqslant \prod_{k=1}^{2^\ell} x_k^{f_k} = \prod_{k=1}^{2^\ell} x_k^{f_k-e_k} \prod_{k=1}^{2^\ell} x_k^{e_k} = x_{k_0}^{f_{k_0}-e_{k_0}} \prod_{k<k_0} x_k^{f_k-e_k} \prod_{k=1}^{2^\ell} x_k^{e_k} \geqslant x_{k_0} \prod_{k<k_0} x_k^{-e_k} \prod_{k=1}^{2^\ell} x_k^{e_k} \\ &\geqslant x_{k_0} x_{k_0-1}^{-\sum_{k<k_0} e_k} \prod_{k=1}^{2^\ell} x_k^{e_k} \geqslant x_{k_0} x_{k_0-1}^{-d} \prod_{k=1}^{2^\ell} x_k^{e_k} > Mb^d \prod_{k=1}^{2^\ell} x_k^{e_k} \geqslant \alpha_j(\boldsymbol{e}).\end{aligned}$$

Hence the summation $\alpha_j(\boldsymbol{e}) + \alpha_j(\boldsymbol{f})$ has no carry under binary expansion, since $\alpha_j(\boldsymbol{e}) < \mu(\alpha_j(\boldsymbol{f}))$. By induction on $\boldsymbol{f}$ with its colexicographic order, we conclude that the binary expansion of $\sum_{\boldsymbol{e}<\boldsymbol{f}} \alpha_j(\boldsymbol{e})$ is a word of length $\leqslant v_2(\alpha_j(\boldsymbol{f}))$, thus the summation $\sum_{\boldsymbol{e}<\boldsymbol{f}} \alpha_j(\boldsymbol{e}) + \alpha_j(\boldsymbol{f})$ has no carry and yields a word of equal length with that of $\alpha_j(\boldsymbol{f})$. So does the summation in the formula (2), hence

$$t\left(H_j(n+A)\right) = \sum_{\boldsymbol{e} \in D(d,\ell)} t\left(\alpha_j(\boldsymbol{e})\right) = t\left(H_j(n)\right) + \sum_{i=1}^{d} \sum_{\boldsymbol{e} \in D(d,\ell), |\boldsymbol{e}|=i} t\left(\alpha_j(\boldsymbol{e})\right) = t\left(H_j(n)\right),$$

since by Lemma 2, the coefficient $C_{\boldsymbol{e}}$ $(\boldsymbol{e} \neq \mathbf{0})$ appears even times in the multinomial expansion, and $t(\alpha_j(\boldsymbol{e})) = t(\alpha_j(\boldsymbol{f}))$ if $C_{\boldsymbol{e}} = C_{\boldsymbol{f}}$ and $|\boldsymbol{e}| = |\boldsymbol{f}|$. □

Proof of Lemma 6. Let n, M be integers such that

$$da_d n + a_{d-1}^{(1)} > \frac{9}{10}\left(da_d n + a_{d-1}^{(m)}\right) > da_d, \sum_{k=i}^{d} \binom{k}{i} a_k^{(j)} n^{k-i} > 0,$$

and

$$M > d! \cdot \sum_{k=i}^{d} \binom{k}{i} a_k^{(j)} n^d \left(0 \leqslant i \leqslant d, \quad \text{and} \quad 1 \leqslant j \leqslant m\right).$$

Put $r = v_2(\frac{a_d d!}{\mu(a_d d!)} + 1) \geqslant 1$, and $q = v_2(\frac{a_d d!}{3\mu(a_d d!)} + 1)$ (if $d > 2$). Then $q = 1$ if $r > 1$ and $d > 2$. Put

$$B_j = (d-1)! \cdot \left(da_d n + a_{d-1}^{(j)}\right) / \mu(a_d d!) > 1 \left(1 \leqslant j \leqslant m\right),$$

and $B_0 = \frac{9}{10} B_m$. Then

$$B_m \geqslant B_j \geqslant B_1 > B_0, \quad \text{for} \quad da_d n + a_{d-1}^{(1)} > \frac{9}{10}\left(da_d n + a_{d-1}^{(m)}\right).$$

Below we shall choose appropriate $b, z_1 \in \mathbb{N}$ by distinguishing different cases.

Case 1. $d \equiv 0, 2 \pmod 6$ and $r \equiv 1 \pmod 2$. Then choose $b, z_1 \in \mathbb{N}$ such that $b > 16M$, $b \equiv 1 \pmod{2^{r+1}}$, and $2B_m b^{d-1} > 2^{z_1} > 2B_{m-1} b^{d-1}$, since for the integer u_0 large enough, we have

$$\begin{aligned}&\bigcup_{u \geqslant u_0} \left(\log_2\left(2B_{m-1}\left(2^{r+1}u+1\right)^{d-1}\right), \quad \log_2\left(2B_m\left(2^{r+1}u+1\right)^{d-1}\right)\right)\\ &\quad = \left(\log_2\left(2B_{m-1}\left(2^{r+1}u_0+1\right)^{d-1}\right), +\infty\right),\end{aligned}$$

since

$$\lim_{u\to+\infty}\left(\log_2\left(2B_m\left(2^{r+1}u+1\right)^{d-1}\right) - \log_2\left(2B_{m-1}\left(2^{r+1}(u+1)+1\right)^{d-1}\right)\right) = \log\frac{B_m}{B_{m-1}} > 0.$$

Case 2. $d \equiv 3, 5 \pmod 6$. Choose $b > 16M$, $b \equiv \frac{a_d d!}{\mu(a_d d!)} \pmod 4$, and $2B_m b^{d-1} > 2^{z_1} > 2B_{m-1} b^{d-1}$.

Case 3. $d \equiv 1 \pmod 3$, $q \equiv 1 \pmod 2$; or $d \geqslant 3$, $d \equiv 2 \pmod 6$, $r \equiv 0 \pmod 2$. Then choose $b > 16M$, $b \equiv 1 \pmod{2^{q+r+1}}$, and $\frac{4}{3} B_m b^{d-1} > 2^{z_1} > \frac{4}{3} B_{m-1} b^{d-1}$.

Case 4. $d = 2$, $r \equiv 0 \pmod 2$; or $d \equiv 1 \pmod 3$, $q \equiv 0 \pmod 2$ (thus $r = 1$). Then choose $b > 16M$, $b \equiv 1 \pmod{2^{q+r+1}}$, and $B_m b^{d-1} > 2^{z_1} > B_{m-1} b^{d-1}$.

Case 5. $d \equiv 0 \pmod 6$, $r \equiv 0 \pmod 2$. Take $b > 16M$, $b \equiv 1 \pmod{2^{q+r+1}}$, and $\frac{1}{2} B_m b^{d-1} > 2^{z_1} > \frac{1}{2} B_{m-1} b^{d-1}$.

Now fix $\ell \geqslant 1$ an integer such that $2^\ell > d$, and choose successively integers z_k $(2 \leqslant k \leqslant 2^\ell)$ such that $2^{z_k} > 2^{dz_{k-1}+2} M b^d$. Put $x_k = 2^{z_k}$ $(1 \leqslant k \leqslant 2^\ell)$, $A := A_n := b \sum_{k=1}^{2^\ell} x_k$. Then

$$x_k > 4Mb^d x_{k-1}^d \left(2 \leqslant k \leqslant 2^\ell\right),$$

and

$$x_1 = 2^{z_1} > \frac{1}{2} B_{m-1} b^{d-1} \geqslant \frac{1}{2} B_1 b^{d-1} > \frac{9}{20} B_m b^{d-1} > \frac{1}{4} B_m b^{d-1} \geqslant \frac{1}{4} B_j b^{d-1} \left(1 \leqslant j \leqslant m\right).$$

For all $\boldsymbol{e} = (e_j)_{1 \leqslant j \leqslant 2^\ell}, \boldsymbol{f} = (f_j)_{1 \leqslant j \leqslant 2^\ell} \in D(d, \ell)$ with $\boldsymbol{e} < \boldsymbol{f}$, we shall show below $\alpha_j(\boldsymbol{e}) < \alpha_j(\boldsymbol{f})$, and compare $\alpha_j(\boldsymbol{e})$ and $\mu(\alpha_j(\boldsymbol{f}))$ $(1 \leqslant j \leqslant m)$, where $\alpha_j(\boldsymbol{e})$ is defined as in the formula (1).

Define $\boldsymbol{e}' = (e_j)_{2 \leqslant j \leqslant 2^\ell}, \boldsymbol{f}' = (f_j)_{2 \leqslant j \leqslant 2^\ell}$. For $1 \leqslant j \leqslant m$, we distinguish different cases below.

Case a. $\boldsymbol{e}' < \boldsymbol{f}'$. Then $\mu(\alpha_j(\boldsymbol{f})) > 4\alpha_j(\boldsymbol{e})$. Indeed if k_0 is the largest index k such that $e_k < f_k$, then $e_j = f_j$ ($k_0 < j \leqslant 2^\ell$), and by the construction of M, b, and x_k's, we obtain

$$\begin{aligned}\alpha_j(\boldsymbol{f}) \geqslant \mu(\alpha_j(\boldsymbol{f})) &\geqslant \prod_{k=1}^{2^\ell} x_k^{f_k} = x_{k_0}^{f_{k_0}-e_{k_0}} \prod_{k<k_0} x_k^{f_k-e_k} \prod_{k=1}^{2^\ell} x_k^{e_k} \geqslant x_{k_0} \prod_{k<k_0} x_k^{-e_k} \prod_{k=1}^{2^\ell} x_k^{e_k} \\ &\geqslant x_{k_0} x_{k_0-1}^{-\sum_{k<k_0} e_k} \prod_{k=1}^{2^\ell} x_k^{e_k} \geqslant x_{k_0} x_{k_0-1}^{-d} \prod_{k=1}^{2^\ell} x_k^{e_k} > 4Mb^d \prod_{k=1}^{2^\ell} x_k^{e_k} \geqslant 4\alpha_j(\boldsymbol{e}).\end{aligned}$$

Case b. $\boldsymbol{e}' = \boldsymbol{f}'$, and $|\boldsymbol{e}| < d-1$. Then $\alpha_j(\boldsymbol{f}) \geqslant \mu(\alpha_j(\boldsymbol{f})) > 4\alpha_j(\boldsymbol{e})$. In fact, we have $B_m > 1$, and

$$\alpha_j(\boldsymbol{f}) \geqslant \mu(\alpha_j(\boldsymbol{f})) \geqslant x_1 \prod_{k=1}^{2^\ell} x_k^{e_k} > \frac{1}{4} B_m b^{d-1} \prod_{k=1}^{2^\ell} x_k^{e_k} \geqslant \frac{b}{4} B_m b^{|\boldsymbol{e}|} \prod_{k=1}^{2^\ell} x_k^{e_k} > 4MB_m b^{|\boldsymbol{e}|} \prod_{k=1}^{2^\ell} x_k^{e_k} > 4\alpha_j(\boldsymbol{e}).$$

Case c. $\boldsymbol{e}' = \boldsymbol{f}'$, and $|\boldsymbol{e}| = d-1$. Then $f_1 - e_1 = 1$, $|\boldsymbol{f}| = d$, and $\boldsymbol{f}$ is the successor of $\boldsymbol{e}$ in $D(d,\ell)$ (i.e., there does not exist $\boldsymbol{g} \in D(d,\ell)$ such that $\boldsymbol{e} < \boldsymbol{g} < \boldsymbol{f}$). Then by definition, we have $\frac{\alpha_j(\boldsymbol{f})}{\alpha_j(\boldsymbol{e})} > \frac{bx_1}{M} > 1$, and

$$\frac{\alpha_j(\boldsymbol{e})}{\mu(\alpha_j(\boldsymbol{f}))} = \frac{B_j b^{d-1}}{x_1} \frac{\mu((e_1+1)!)}{e_1!} \prod_{k=2}^{2^\ell} \frac{\mu(e_k!)}{e_k!}. \tag{3}$$

From above, we deduce that if $\alpha_j(\boldsymbol{e}) + \alpha_j(\boldsymbol{f})$ has a carry, then the pair $(\boldsymbol{e},\boldsymbol{f})$ belongs to the Case c.

Case 1. $d \equiv 0,2 \pmod 6$, and $r \equiv 1 \pmod 2$. Then

$$2B_m b^{d-1} > x_1 > 2B_{m-1} b^{d-1}, \frac{1}{2} \times \frac{10}{9} > \frac{B_m b^{d-1}}{x_1} > \frac{1}{2} \quad \text{and} \quad \frac{1}{2} > \frac{B_j b^{d-1}}{x_1} \ (1 \leqslant j < m),$$

thus for all $\boldsymbol{e} = (e_j)_{1\leqslant j\leqslant 2^\ell}, \boldsymbol{f} = (f_j)_{1\leqslant j\leqslant 2^\ell} \in D(d,\ell)$ with $\boldsymbol{e} < \boldsymbol{f}$, we have

$$\alpha_j(\boldsymbol{e}) < \mu(\alpha_j(\boldsymbol{f})) \ (1 \leqslant j < m), \quad \text{thus } t(H_j(n+A)) = t(H_j(n)) \ (1 \leqslant j < m),$$

just as for Lemma 5. Now $\frac{\alpha_m(\boldsymbol{e})}{\mu(\alpha_m(\boldsymbol{f}))} > 1$ iff $e_1 = 1$ and $e_k \in \{0,1,2\}$ ($2 \leqslant k \leqslant 2^\ell$), and then

$$1 < \frac{\alpha_m(\boldsymbol{e})}{\mu(\alpha_m(\boldsymbol{f}))} < 2, \frac{\alpha_m(\boldsymbol{f})}{\mu(\alpha_m(\boldsymbol{f}))} = \frac{b^d a_d d!}{\mu(a_d d!)} \equiv 2^r - 1 \pmod{2^{r+1}},$$

and we fall in the Case c. So the summation $\alpha_m(\boldsymbol{e}) + \alpha_m(\boldsymbol{f})$ takes the form

$$***0\underbrace{11\cdots1}_{r}\underbrace{00\cdots0}_{s}+1\underbrace{**\cdots*}_{s},$$

and carries exactly r times. But $\boldsymbol{f}$ is the successor of $\boldsymbol{e}$ in $D(d,\ell)$, so for all $\boldsymbol{g} \in D(d,\ell) \setminus \{\boldsymbol{e},\boldsymbol{f}\}$, either $\boldsymbol{g} < \boldsymbol{e}$ or $\boldsymbol{g} > \boldsymbol{f}$. In the first case, we have $\alpha_m(\boldsymbol{g}) < \mu(\alpha_m(\boldsymbol{e}))$ by the Cases a and b above. In the second case, we have $\alpha_m(\boldsymbol{e}) < \alpha_m(\boldsymbol{f})$ and $\boldsymbol{g}' > \boldsymbol{f}'$, thus $4\alpha_m(\boldsymbol{f}) < \mu(\alpha_m(\boldsymbol{g}))$ by the Case a, then $\alpha_m(\boldsymbol{e}) + \alpha_m(\boldsymbol{f}) < 2\alpha_m(\boldsymbol{f}) < \mu(\alpha_m(\boldsymbol{g}))$. As for the proof of Lemma 5, by induction on $\boldsymbol{e}$ with its colexicographic order, we obtain that the binary expansion of $\sum_{\boldsymbol{g}\in D(d,\ell),\boldsymbol{g}<\boldsymbol{e}} \alpha_m(\boldsymbol{g})$ (resp. $\sum_{\boldsymbol{g}\in D(d,\ell),\boldsymbol{g}\leqslant\boldsymbol{f}} \alpha_m(\boldsymbol{g})$) is a word of length $\leqslant v_2(\alpha_m(\boldsymbol{e}))$ (resp. $\leqslant v_2(\alpha_m(\boldsymbol{h}))$, for $\boldsymbol{h} \in D(d,\ell)$ with $\boldsymbol{f} < \boldsymbol{h}$). So the carries of $\alpha_m(\boldsymbol{e}) + \alpha_m(\boldsymbol{f})$ affect none of the other terms in the summation $\sum_{\boldsymbol{g}\in D(d,\ell)} \alpha_m(\boldsymbol{g})$. By Lemma 3, we get $t(H_m(n+A)) = t(H_m(n)) + ry_{d-2} = t(H_m(n)) + 1$, for r is odd, and $d-2 \equiv 0,1 \pmod 3$. □

Case 2. $d \equiv 3,5 \pmod 6$. Then for all $\boldsymbol{e},\boldsymbol{f} \in D(d,\ell)$ with $\boldsymbol{e} < \boldsymbol{f}$, we have $\alpha_j(\boldsymbol{e}) < \mu(\alpha_j(\boldsymbol{f}))$ $(1 \leqslant j < m)$, thus $t(H_j(n+A)) = t(H_j(n))$ $(1 \leqslant j < m)$. But $\frac{\alpha_m(\boldsymbol{e})}{\mu(\alpha_m(\boldsymbol{f}))} > 1$ iff $e_1 = 1$ and $e_k \in \{0,1,2\}$ ($2 \leqslant k \leqslant 2^\ell$), then we are in the Case c, and

$$1 < \frac{\alpha_m(\boldsymbol{e})}{\mu(\alpha_m(\boldsymbol{f}))} < 2, \frac{\alpha_m(\boldsymbol{f})}{\mu(\alpha_m(\boldsymbol{f}))} = \frac{b^d a_d d!}{\mu(a_d d!)} \equiv 1 \pmod 4,$$

so $\alpha_m(\boldsymbol{e})+\alpha_m(\boldsymbol{f})$ has one carry. As for the Case 1, by Lemma 3 and the fact that $d-2\equiv 0,1 \pmod 3$, we obtain

$$t\big(H_m(n+A)\big)=t\big(H_m(n)\big)+y_{d-2}=t\big(H_m(n)\big)+1.$$

□

Case 3. $d\equiv 1 \pmod 3$, $q\equiv 1 \pmod 2$; or $d\geqslant 3$, $d\equiv 2 \pmod 6$, $r\equiv 0 \pmod 2$. Then $\frac{4}{3}B_m b^{d-1}>x_1>\frac{4}{3}B_{m-1}b^{d-1}$, and we only consider the Case c. If $e_k\geqslant 3$ for some integer $k\geqslant 2$, then by the formula (3), we obtain

$$\frac{\alpha_j(\boldsymbol{e})}{\mu\big(\alpha_j(\boldsymbol{f})\big)}<\frac{3}{4}\times\frac{10}{9}\times 2\times\frac{1}{3}<1\ \big(1\leqslant j\leqslant m\big).$$

Below we assume $e_k\in\{0,1,2\}$ $(2\leqslant k\leqslant 2^{\ell})$.

If $e_1=0$ or 2, then

$$\frac{\alpha_j(\boldsymbol{e})}{\mu\big(\alpha_j(\boldsymbol{f})\big)}<1\ \big(1\leqslant j\leqslant m\big).$$

If $e_1=1$, then for $1\leqslant j\leqslant m$, we have $1<\frac{\alpha_j(\boldsymbol{e})}{\mu(\alpha_j(\boldsymbol{f}))}<2$, and $\frac{\alpha_j(\boldsymbol{f})}{\mu(\alpha_j(\boldsymbol{f}))}=\frac{b^d a_d d!}{\mu(a_d d!)}\equiv 2^r-1 \pmod{2^{r+1}}$, thus the summation $\alpha_j(\boldsymbol{e})+\alpha_j(\boldsymbol{f})$ has the form

$$***0\underbrace{11\cdots1}_{r}\underbrace{00\cdots0}_{s}+1\underbrace{**\ldots*}_{s},$$

hence it yields r carries (the number of such pairs is y_{d-2}).

If $e_1=3$, then

$$\frac{\alpha_j(\boldsymbol{e})}{\mu\big(\alpha_j(\boldsymbol{f})\big)}<1\ \big(1\leqslant j<m\big),1<\frac{\alpha_m(\boldsymbol{e})}{\mu\big(\alpha_m(\boldsymbol{f})\big)}<2,\frac{\alpha_m(\boldsymbol{f})}{\mu\big(\alpha_m(\boldsymbol{f})\big)}=\frac{b^d a_d d!}{3\mu(a_d d!)}\equiv 2^q-1\big(\mathrm{mod}\,2^{q+1}\big).$$

So the summation $\alpha_m(\boldsymbol{e})+\alpha_m(\boldsymbol{f})$ has the form

$$***0\underbrace{11\cdots1}_{q}\underbrace{00\cdots0}_{s}+1\underbrace{**\cdots*}_{s},$$

and gives q carries. Note that the number of such pairs is $y_{d-1-e_1}=y_{d-4}$, hence as for the Case 1, we obtain, by Lemma 3,

$$t\big(H_j(n+A)\big)=t\big(H_j(n)\big)+ry_{d-2}=t\big(H_j(n)\big)\big(1\leqslant j<m\big),$$
$$t\big(H_m(n+A)\big)=t\big(H_m(n)\big)+ry_{d-2}+qy_{d-4}=t\big(H_m(n)\big)+1.$$

□

Case 4. $d=2$, $r\equiv 0 \pmod 2$; or $d\equiv 1 \pmod 3$, $q\equiv 0 \pmod 2$ (thus $r=1$). So $B_m b^{d-1}>x_1>B_{m-1}b^{d-1}$. As above, we only consider the case c, and proceed similarly. If $e_k\geqslant 3$ for some integer $k\geqslant 2$, then

$$\frac{\alpha_j(\boldsymbol{e})}{\mu\big(\alpha_j(\boldsymbol{f})\big)}<\frac{10}{9}\times 2\times\frac{1}{3}<1\ \big(1\leqslant j\leqslant m\big).$$

Below we suppose that $e_k\in\{0,1,2\}(2\leqslant k\leqslant 2^{\ell})$. If $e_1=0,2$, then

$$\frac{\alpha_j(\boldsymbol{e})}{\mu\big(\alpha_j(\boldsymbol{f})\big)}<1\ \big(1\leqslant j<m\big),$$

and $1<\frac{\alpha_m(\boldsymbol{e})}{\mu(\alpha_m(\boldsymbol{f}))}<2$. For $e_1=0$, the summation $\alpha_m(\boldsymbol{e})+\alpha_m(\boldsymbol{f})$ has r carries (the number of such pairs is y_{d-1}); for $e_1=2$, the summation $\alpha_m(\boldsymbol{e})+\alpha_m(\boldsymbol{f})$ has q carries (the number of such pairs is y_{d-3}).

If $e_1=1$, then $1<\frac{\alpha_j(\boldsymbol{e})}{\mu(\alpha_j(\boldsymbol{f}))}<2$ $(1\leqslant j<m)$, and $2<\frac{\alpha_m(\boldsymbol{e})}{\mu(\alpha_m(\boldsymbol{f}))}<3$. For $1\leqslant j<m$, the summation $\alpha_j(\boldsymbol{e})+\alpha_j(\boldsymbol{f})$ has r carries (there are y_{d-2} such pairs), while the summation $\alpha_m(\boldsymbol{e})+\alpha_m(\boldsymbol{f})$ has $r-1$ carries (there are y_{d-2} such pairs).

If $e_1 = 3$, then $1 < \frac{\alpha_j(\boldsymbol{e})}{\mu(\alpha_j(\boldsymbol{f}))} < 2$ $(1 \leqslant j \leqslant m)$, and the summation $\alpha_j(\boldsymbol{e}) + \alpha_j(\boldsymbol{f})$ has q carries (there are y_{d-4} such pairs). If $e_1 \geqslant 4$, then

$$\frac{\alpha_j(\boldsymbol{e})}{\mu\big(\alpha_j(\boldsymbol{f})\big)} < 1 \ \big(1 \leqslant j \leqslant m\big).$$

As for the Case 1, we obtain, by Lemma 3,

$$t\big(H_j(n+A)\big) = t\big(H_j(n)\big) + r y_{d-2} + q y_{d-4} = t\big(H_j(n)\big) \ \big(1 \leqslant j < m\big),$$
$$t\big(H_m(n+A)\big) = t\big(H_m(n)\big) + r y_{d-1} + q y_{d-3} + (r-1) y_{d-2} + q y_{d-4} = t\big(H_m(n)\big) + 1. \qquad \square$$

Case 5. $d \equiv 0 \pmod 6$, and $r \equiv 0 \pmod 2$. Then $r > 1$, and $q = 1$, So $\frac{1}{2} B_m b^{d-1} > x_1 > \frac{1}{2} B_{m-1} b^{d-1}$. As above, we only consider the Case c. If $e_k > 4$ for some $k \geqslant 2$ or $\exists\, k_1, k_2 \geqslant 2$ with $k_1 \neq k_2$ such that $e_{k_1}, e_{k_2} \not\in \{0,1,2\}$, then

$$\frac{\alpha_j(\boldsymbol{e})}{\mu\big(\alpha_j(\boldsymbol{f})\big)} < 1 \ \big(1 \leqslant j \leqslant m\big).$$

Now suppose $e_k = 3$ or 4 for some unique integer $k \geqslant 2$. If $e_1 \neq 1$, then $\frac{\alpha_j(\boldsymbol{e})}{\mu(\alpha_j(\boldsymbol{f}))} < \frac{4}{3} \times \frac{1}{3} \times 2 \times \frac{10}{9} < 1$ $(1 \leqslant j \leqslant m)$. If $e_1 = 1$, then

$$1 < \frac{\alpha_j(\boldsymbol{e})}{\mu\big(\alpha_j(\boldsymbol{f})\big)} < 2, \quad \text{and} \quad \frac{\alpha_j(\boldsymbol{f})}{\mu\big(\alpha_j(\boldsymbol{f})\big)} = \frac{b^d a_d d!}{3\mu(a_d d!)} \equiv 2^q - 1 \ \big(\mathrm{mod}\ 2^{q+1}\big) \equiv 1 \ (\mathrm{mod}\ 4),$$

for $q = 1$. So $\alpha_j(\boldsymbol{e}) + \alpha_j(\boldsymbol{f})$ has one carry. One can check that there are $(2^\ell - 1)(g_{d-5} + g_{d-6})$ such pairs: $(2^\ell - 1) g_{d-5}$ pairs for $e_k = 3$, and $(2^\ell - 1) g_{d-6}$ pairs for $e_k = 4$.

Below we assume $e_k \in \{0,1,2\}$ $(2 \leqslant k \leqslant 2^\ell)$. If $e_1 \geqslant 4$, then we have $\frac{\alpha_j(\boldsymbol{e})}{\mu(\alpha_j(\boldsymbol{f}))} < 1$ $(1 \leqslant j \leqslant m)$.

If $e_1 = 3$, then for $1 \leqslant j \leqslant m$, we get

$$2 < \frac{8}{3} \times \frac{9}{10} < \frac{\alpha_j(\boldsymbol{e})}{\mu\big(\alpha_j(\boldsymbol{f})\big)} < \frac{8}{3} \times \frac{10}{9} < 3, \frac{\alpha_j(\boldsymbol{f})}{\mu\big(\alpha_j(\boldsymbol{f})\big)} = \frac{b^d a_d d!}{3\mu\big(a_d d!\big)} \equiv 1 \pmod 4,$$

thus the summation $\alpha_j(\boldsymbol{e}) + \alpha_j(\boldsymbol{f})$ has the form

$$***01\underbrace{00\cdots0}_{s}+10\underbrace{**\cdots*}_{s},$$

hence no carry.

If $e_1 = 0, 2$, then $1 < \frac{\alpha_j(\boldsymbol{e})}{\mu(\alpha_j(\boldsymbol{f}))} < 2 < \frac{\alpha_m(\boldsymbol{e})}{\mu(\alpha_m(\boldsymbol{f}))} < 3$ $(1 \leqslant j < m)$. If $e_1 = 0$, then $\alpha_j(\boldsymbol{e}) + \alpha_j(\boldsymbol{f})$ $(1 \leqslant j < m)$ has r carries, while $\alpha_m(\boldsymbol{e}) + \alpha_m(\boldsymbol{f})$ has $r-1$ carries, each of them has y_{d-1} such pairs. If $e_1 = 2$, then

$$\frac{\alpha_j(\boldsymbol{f})}{\mu\big(\alpha_j(\boldsymbol{f})\big)} = \frac{b^d a_d d!}{3\mu(a_d d!)} \equiv 1 \pmod 4 \ \big(1 \leqslant j \leqslant m\big),$$

hence $\alpha_j(\boldsymbol{e}) + \alpha_j(\boldsymbol{f})$ $(1 \leqslant j < m)$ has one carry, and there are y_{d-3} such pairs, while $\alpha_m(\boldsymbol{e}) + \alpha_m(\boldsymbol{f})$ does not have any carry.

If $e_1 = 1$, then

$$3 < \frac{\alpha_j(\boldsymbol{e})}{\mu\big(\alpha_j(\boldsymbol{f})\big)} < 4 < \frac{\alpha_m(\boldsymbol{e})}{\mu\big(\alpha_m(\boldsymbol{f})\big)} < 5 \ \big(1 \leqslant j < m\big), \frac{\alpha_j(\boldsymbol{f})}{\mu\big(\alpha_j(\boldsymbol{f})\big)} \equiv 2^r - 1 \ \big(\mathrm{mod}\ 2^{r+1}\big) \ \big(1 \leqslant j \leqslant m\big).$$

Thus $\alpha_j(\boldsymbol{e}) + \alpha_j(\boldsymbol{f})$ $(1 \leqslant j < m)$ has r carries, while $\alpha_m(\boldsymbol{e}) + \alpha_m(\boldsymbol{f})$ has $r-2$ carries, each of them has y_{d-2} such pairs.

Finally, proceeding as for the Case 1, we obtain, by Lemma 3 and Lemma 4,

$$t\big(H_j(n+A)\big) = t\big(H_j(n)\big) + \big(2^\ell - 1\big)\big(g_{d-5} + g_{d-6}\big) + r y_{d-1} + y_{d-3} + r y_{d-2} = t\big(H_j(n)\big) \ \big(1 \leqslant j < m\big),$$
$$t\big(H_m(n+A)\big) = t\big(H_m(n)\big) + \big(2^\ell - 1\big)\big(g_{d-5} + g_{d-6}\big) + (r-1) y_{d-1} + (r-2) y_{d-2} = t\big(H_m(n)\big) + 1. \qquad \square$$

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References

[1] J.-P. Allouche, "Somme des chiffres et transcendance", *Bull. Soc. Math. Fr.* **110** (1982), p. 279-285.

[2] J.-P. Allouche, J. Shallit, *Automatic sequences. Theory, applications, generalizations*, Cambridge University Press, 2003.

[3] S. V. Avgustinovich, "The number of different subwords of given length in the Morse–Hedlund sequence", *Sibirsk. Zh. Issled. Oper.* **1** (1994), no. 2, p. 3-7.

[4] S. Brlek, "Enumeration of factors in the Thue-Morse word", *Discrete Appl. Math.* **24** (1989), no. 1-3, p. 83-96.

[5] C. Dartyge, G. Tenenbaum, "Congruences de sommes de chiffres de valeurs polynomiales", *Bull. Lond. Math. Soc.* **38** (2006), no. 1, p. 61-69.

[6] M. Drmota, C. Mauduit, J. Rivat, "The sum-of-digits function of polynomial sequences", *J. Lond. Math. Soc.* **84** (2011), no. 1, p. 81-102.

[7] A. O. Gel'fond, "Sur les nombres qui ont des propriétés additives et multiplicatives données", *Acta Arith.* **13** (1967), p. 259-265.

[8] A. de Luca, S. Varricchio, "Some combinatorial properties of the Thue-Morse sequence and a problem in semi-groups", *Theor. Comput. Sci.* **63** (1989), no. 3, p. 333-348.

[9] Y. Moshe, "On the subword complexity of Thue-Morse polynomial extractions", *Theor. Comput. Sci.* **389** (2007), no. 1-2, p. 318-329.

[10] C. Müllner, L. Spiegelhofer, "Normality of the Thue-Morse sequence along Piatetski-Shapiro sequences. II", *Isr. J. Math.* **220** (2017), no. 2, p. 691-738.

[11] T. Stoll, "The sum of digits of polynomial values in arithmetic progressions", *Funct. Approximatio, Comment. Math.* **47** (2012), no. 2, p. 233-239.