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Meromorphic solutions of a first order differential equations with delays

Yu Chen[a] **and Tingbin Cao**[*, a]

[a] Department of Mathematics, Nanchang University, Nanchang city, Jiangxi 330031, P. R. China

E-mails: chenyu@email.ncu.edu.cn (Y. Chen), tbcao@ncu.edu.cn (T. B. Cao)

Abstract. The main purpose of this paper is to study meromorphic solutions of the first order differential equations with delays

$$w(z+1)-w(z-1)+a(z)\left(\frac{w'(z)}{w(z)}\right)^k = R(z,w(z))$$

and

$$w(z+1)+a(z)\left(\frac{w'(z)}{w(z)}\right)^k = R(z,w(z)),$$

where k is a positive integer, $a(z)$ is a rational function, $R(z,w)$ is rational in w with rational coefficients. Some necessary conditions on the degree of $R(z,w)$ are obtained for the equation to admit a transcendental meromorphic solution of minimal hypertype. These are extensions of some previous results due to Halburd, Korhonen, Liu and others. Some examples are given to support our conclusions.

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1. Introduction and Main Results

In recent years, the application of Nevanlinna theory and its difference analogues in the study of meromorphic solutions of complex delay differential equations has become an important topic. We assume that the readers are familiar with the basic definitions and notations of Nevanlinna theory such as the counting function $N(r,f)$, proximity function $m(r,f)$ and characteristic function $T(r,f)$ for a meromorphic function $f(z)$ on the complex plane (refer to see [1, 3] or other references therein).

By using the reduction of integrable delay differential equation, the Painlevé equation with formal continuum limit is obtained. For example, Quispel et al. [10] obtained the equation with logarithmic derivative as follow,

$$w(z)(w(z+1)-w(z-1))+aw'(z)=bw(z), \tag{1}$$

[*] Corresponding author.

with a and b are constants, as the symmetry reduction of the Kac–Van Moerbeke equation. They also observed that (1) possess the formal continuum limit in the first Painlevé equation.

In 2017, Halburd and Korhonen [2] considered the extended version of (1) and studied the first order delay differential difference equation

$$w(z+1)-w(z-1)+a(z)\frac{w'(z)}{w(z)}=R(z,w(z))=\frac{P(z,w(z))}{Q(z,w(z))}. \tag{2}$$

They obtained the following result:

Theorem 1 ([2]). *Assume $w(z)$ is a transcendental meromorphic solution of (2), where $a(z)$ is a rational function in z, $P(z,w(z))$ is a polynomial in w with rational coefficients in z, and $Q(z,w(z))$ is a polynomial in w having zeros that are nonzero rational functions in z but are not zeros of $P(z,w(z))$*[1]*. If the hyper-order of w is strictly less than 1, then*

$$\deg_w(P)=\deg_w(Q)+1\le 3 \quad or \quad \deg_w(R)\le 1,$$

where $\deg_w(P)=\deg_w(P(z,w))$ denotes the degree of P as a polynomial in w and $\deg_w(R)=\max\{\deg_w(P),\deg_w(Q)\}$ denotes the degree of R as a rational function of w.

If $\deg_w(R(z,w))=0$, then equation (2) becomes

$$w(z+1)-w(z-1)+a(z)\frac{w'(z)}{w(z)}=b(z), \tag{3}$$

where $a(z)$ and $b(z)$ are rational functions. Halburd and Korhonen singled out the equation (1) from the class (3) by introducing an additional condition that the meromorphic solution has sufficiently many simple zeros.

Theorem 2 ([2]). *Assume $w(z)$ is a transcendental meromorphic solution of (3), where $a(z)\not\equiv 0$ and $b(z)$ is rational. If the hyper-order of w is less than one and for any $\epsilon>0$,*

$$\overline{N}\left(r,\frac{1}{w}\right)\ge\left(\frac{3}{4}+\epsilon\right)T(r,w)+S(r,w), \tag{4}$$

then the coefficients $a(z)$ and $b(z)$ are both constants.

Recently, there arises some further studies on entire or meromorphic solutions of delay differential equations related to the results of Halburd and Korhonen, refer to see [4, 7, 11, 13, 14, 16] and therein. In which, Liu and Song [7] considered equation (2) dropping into only one delay.

Theorem 3 ([11]). *Let w be a transcendental meromorphic solution of*

$$c(z)w(z+1)+a(z)\frac{w'(z)}{w(z)}=R(z,w(z))=\frac{P(z,w(z))}{Q(z,w(z))}, \tag{5}$$

where $P(z,w)$ and $Q(z,w)$ also satisfy the conditions above Theorem 1. If the hyper-order of w is strictly less than one, then

$$\deg_w(P)=\deg_w(Q)+1=2 \quad or \quad \deg_w(R)\le 1.$$

It is interesting to ask: *How about the above theorems if the logarithmic derivative $\frac{w'(z)}{w(z)}$ is changed by $\left(\frac{w'(z)}{w(z)}\right)^k$ for arbitrary positive integer k?*

The main purpose of this paper is to deal with this topic, and obtain the following results which extend the results of Theorems 1–3. The hyperorder strictly less than one is improved to the case of minimal hypertype. Theorem 1 is just the special case of $k=1$ in the first theorem below.

[1] In fact, from the original proof of Theorem 1 one can see that the $Q(z,w(z))$ is either of $\deg_w(Q)=0$, or a polynomial in w having zeros that are nonzero rational functions in z but are not zeros of $P(z,w(z))$.

Theorem 4. *Let w be a transcendental meromorphic solution of*

$$w(z+1)-w(z-1)+a(z)\left(\frac{w'(z)}{w(z)}\right)^k = R(z,w(z)) = \frac{P(z,w(z))}{Q(z,w(z))}, \tag{6}$$

where k is a positive integer, $a(z)$ is a rational function, $P(z,w)$ is a polynomial in w with rational coefficients in z, and $Q(z,w)$ is of degree zero or a polynomial in w with roots that are nonzero rational functions of z and not roots of $P(z,w)$. If w is of minimal hypertype, that is,

$$\limsup_{r\to\infty}\frac{\log^+ T(r,f)}{r}=0,$$

then

(i) *if* $\deg_w(Q)\le 1$, *we have* $\deg_w(P)\le \deg_w(Q)+k$;
(ii) *if* $\deg_w(Q)\ge 2$, *we obtain*

$$\deg_w(Q)<\deg_w(P)\le \deg_w(Q)+k\le \min\{\deg_w(P)+k, 2k+2\}.$$

We will give two examples to show the conclusion (i) of Theorem 4.

Example 5. The meromorphic function $w(z)=\sec(\pi z)$ solves

$$w(z+1)-w(z-1)+a(z)\left(\frac{w'(z)}{w(z)}\right)^2 = \pi^2 a(z)(w^2(z)-1),$$

where $k=2$ and $a(z)(\not\equiv 0)$ is any rational function. We can see that $\deg_w(Q)=0$, and $\deg_w(P)=2=\deg_w(Q)+k$. This means that the equality of the conclusion (i) of Theorem 4 could be possible.

Example 6. The meromorphic function $w(z)=\sec^2(\pi z)$ solves

$$w(z+1)-w(z-1)+a(z)\left(\frac{w'(z)}{w(z)}\right)^4 = 16\pi^4 a(z)(w(z)-1)^2,$$

where $k=4$ and $a(z)(\not\equiv 0)$ is any rational function. We can see that $\deg_w(Q)=0$, and $\deg_w(P)=2<\deg_w(Q)+k$. This shows that strict inequality sign in the conclusion (i) of Theorem 4 is available.

Then, we give the following example to show the conclusion (ii) of Theorem 4.

Example 7. The meromorphic function $w(z)=\frac{1}{e^z+1}$ solves

$$w(z+1)-w(z-1)+a(z)\left(\frac{w'(z)}{w(z)}\right)^2 = \frac{P(z,w(z))}{Q(z,w(z))},$$

$$\begin{aligned}P(z,w(z)) &= (2-e-e^{-1})a(z)w^4(z)+(3e+3e^{-1}-6)a(z)w^3(z)\\ &\quad +((3-e-e^{-1})a(z)+2-e-e^{-1})w^2(z)+(e+e^{-1})w(z)+a(z),\\ Q(z,w(z)) &= (2-e-e^{-1})w^2(z)+(e+e^{-1}-2)w(z)+1,\end{aligned}$$

where $a(z)(\not\equiv 0)$ is an arbitrary rational function. We see that $k=2$, $\deg_w(Q)=2$ and $\deg_w(P)=\deg_w(Q)+k=4$.

We find an interesting phenomenon that Theorem 2 can not be extended to the arbitrary case of $k>1$, based on the next theorem.

Theorem 8. *Assume $w(z)$ is a transcendental meromorphic solution of*

$$w(z+1)-w(z-1)+a(z)\left(\frac{w'(z)}{w(z)}\right)^k = b(z), \tag{7}$$

where $a(z)\not\equiv 0$ and $b(z)$ is rational. If w is of minimal hypertype and for any $\epsilon>0$,

$$\overline{N}\left(r,\frac{1}{w}\right)\ge\left(\frac{3}{4}+\epsilon\right)T(r,w)+S(r,w), \tag{8}$$

then $k=1$, and thus this reduces into Theorem 2.

The following example shows that the assumption (8) in Theorem 8 is necessary.

Example 9. The entire function $w(z) = e^{2\pi i z}$ solves

$$w(z+1) - w(z-1) + a(z)\left(\frac{w'(z)}{w(z)}\right)^k = (2\pi i)^k a(z)$$

where k is an arbitrary positive integer and $a(z)(\not\equiv 0)$ is an arbitrary rational function.

We also consider the situation which the equation drops into only possessing one delay. Theorem 3 becomes the special case of $k = 1$ in the following result.

Theorem 10. *If the following equation*

$$w(z+1) + a(z)\left(\frac{w'(z)}{w(z)}\right)^k = R(z, w(z)) = \frac{P(z, w(z))}{Q(z, w(z))} \tag{9}$$

possesses a transcendental meromorphic solution w such that w is of minimal hypertype, where $a(z)$ is a rational function in z, $P(z, w(z))$ is a polynomial in w with rational coefficients in z, and $Q(z, w(z))$ is of $\deg_w(Q) = 0$ or a polynomial in w having zeros that are nonzero rational functions in z but are not zeros of $P(z, w(z))$, then

(i) *If* $\deg_w(Q) = 0$, *we have* $\deg_w(P) \le \deg_w(Q) + k$.
(ii) *If* $\deg_w(Q) \ge 1$, *we have*

$$\deg_w(Q) < \deg_w(P) \le \deg_w(Q) + k \le \min\{\deg_w(P) + k, 2k+1\}.$$

We give two examples to example the conclusion (i) of Theorem 10.

Example 11. The meromorphic function $w(z) = \csc(\pi z)$ solves

$$w(z+1) - w(z-1) + a(z)\left(\frac{w'(z)}{w(z)}\right)^4 = \pi^4 a(z)(w^2(z) - 1)^2,$$

where $a(z)(\not\equiv 0)$ is any rational function. It gives the equality of conclusion (i) of Theorem 10, since $k = 4$, $\deg_w(Q) = 0$ and $\deg_w(P) = 4 = \deg_w(Q) + k$.

Example 12. The meromorphic function $w(z) = \csc^2(\pi z)$ solves

$$w(z+1) - w(z-1) + a(z)\left(\frac{w'(z)}{w(z)}\right)^2 = 4\pi^2 a(z)(w(z) - 1),$$

where $a(z)(\not\equiv 0)$ is any rational function. It gives the inequality of conclusion (i) of Theorem 10, since $k = 2$, $\deg_w(Q) = 0$ and $\deg_w(P) = 1 < \deg_w(Q) + k$.

Next, we give an example to example the conclusion (ii) of Theorem 10.

Example 13. The meromorphic function $w(z) = \frac{1}{e^z - 1}$ solves the follow equation

$$w(z+1) + a(z)\left(\frac{w'(z)}{w(z)}\right)^2 = \frac{P(z, w(z))}{Q(z, w(z))},$$

$$\begin{aligned} P(z, w(z)) &= (e-1)a(z)w^3(z) + (3e-2)a(z)w^2(z) + (3e-1)a(z)w(z) + ea(z), \\ Q(z, w(z)) &= (e-1)w(z) + e, \end{aligned}$$

where $a(z)(\not\equiv 0)$ is an arbitrary rational function. We see that $k = 2$, $\deg_w(Q) = 1$ and $\deg_w(P) = \deg_w(Q) + k = 3$.

In addition, the following example shows that the assumption "$Q(z, w)$ has roots that are nonzero rational functions of z" in Theorem 10 is necessary.

Example 14. Let $w(z) = \mathscr{P}(z+z_0;\omega_1,\omega_2)$ $(z_0 \in \mathbb{C})$ with either ω_1 or ω_2 equals to one be a Weierstrass elliptic function satisfying the Weierstrass equation

$$(w^{'}(z))^2 = 4w^3(z) - c_2 w(z) - c_3,$$

where c_2 and c_3 are constants [9]. Then w solves the delay differential equation

$$w(z+1) - \frac{1}{4}\left(\frac{w'(z)}{w(z)}\right)^2 = \frac{c_2 w(z) + c_3}{4w^2(z)}.$$

We can see that in the above equation, $k=2$, $\deg_w(Q) = 2$ but $\deg_w(P) = 1 < \deg_w(Q)$.

Finally, it is very interesting that for one delay case, the assumption (4) in Theorem 2 (or say (8) in Theorem 8) is invalid whenever w is of minimal hypertype, according to the following result.

Theorem 15. *Assume $w(z)$ is a transcendental meromorphic solution of*

$$w(z+1) + a(z)\left(\frac{w'(z)}{w(z)}\right)^k = b(z), \tag{10}$$

where $a(z) \not\equiv 0$ and $b(z)$ is rational. If w is of minimal hypertype, then

$$N\left(r, \frac{1}{w}\right) = S(r,w), \quad \textit{and} \quad N(r,w) \neq S(r,w).$$

At the end of the first section, we give two examples to show that the conclusions as in Theorem 8 are not true for one delay case.

Example 16. The meromorphic function $w(z) = \frac{z}{e^{2\pi i z}+1}$ solves

$$w(z+1) - \frac{z}{2\pi i}\cdot\frac{w'(z)}{w(z)} = z - \frac{1}{2\pi i}.$$

Take $a(z) = \frac{z}{2\pi i}$ and $b(z) = z - \frac{1}{2\pi i}$. So we can not obtain the conclusion that $a(z)$ and $b(z)$ reduce into constants.

Example 17. The meromorphic function $w(z) = sec^2(\pi z)$ solves

$$w(z+1) - \frac{1}{4\pi^2}\cdot\left(\frac{w'(z)}{w(z)}\right)^2 = 1.$$

Thus, the conclusion "k must be one" is also invalid for one delay case.

2. Some lemmas

For the proofs of our theorems, we need some lemmas. The first one is the latest result on difference version of the logarithmic derivative lemma for meromrophic functions with minimal hypertype due to Zheng and Korhonen.

Lemma 18 ([5]). *Let w be a meromorphic function, if*

$$\limsup_{r\to\infty} \frac{\log^+ T(r,w)}{r} = 0,$$

then

$$m\left(r, \frac{w(z+c)}{w(z)}\right) = o\left(T(r,w)\right),$$

hold for a constant c as $r \notin E \to \infty$, where E is a subset of $[1,+\infty)$ with the zero upper density, that is

$$\overline{\mathrm{dens}}\, E = \limsup_{r\to\infty} \frac{1}{r}\int_{E\bigcap[1,r]} \mathrm{d}t = 0.$$

A differential-difference polynomial in $w(z)$ is defined by

$$P(z,w)=\sum_{l\in L} b_l(z)\,w(z)^{l_{0,0}}\,w(z+c_1)^{l_{1,0}}\cdots w(z+c_\nu)^{l_{\nu,0}}\,w'(z)^{l_{0,1}}\cdots w^{(\mu)}(z+c_\nu)^{l_{\nu,\mu}},$$

where $c_1,\dots,c_\nu$ are distinct complex nonzero constants, L is a finite index set consisting of elements of the form $l=(l_{0,0},\dots,l_{\nu,\mu})$ and the coefficients $b_l(z)$ are rational functions for z for all $l\in L$. By Lemma 18 and using the same proof, we can see that it is still true of the following lemma by Korhonen and Halburd [2].

Lemma 19. *Let $w(z)$ be a transcendental meromorphic solution of*

$$P(z,w)=0, \tag{11}$$

where $P(z,w)$ is a differential difference polynomial in w with rational coefficients, and let $a_1,\dots,a_m$ be rational functions satisfying $P(z,a_i)\not\equiv 0$ for all $i\in\{1,\dots,m\}$. If there exist $s>0$ and $\tau\in(0,1)$ such that

$$\sum_{i=1}^{m} n\left(r,\frac{1}{w-a_i}\right)\le k\tau n(r+s,w)+O(1), \tag{12}$$

then

$$\limsup_{r\to\infty}\frac{\log^+ T(r,f)}{r}>0.$$

Next, in this section, we will give several lemmas that will be used in subsequent sections. The Lemma 20 is about the degrees of $R(z,w(z))$ in equation (6). The Lemma 21 discusses the case when the right side of equation (6) reduces into a polynomial in w. Finally, the Lemma 22 gives the restriction on the denominator factors in the right side of equation (6).

Lemma 20. *Let $w(z)$ be a transcendental meromorphic solution of equation*

$$w(z+1)-w(z-1)+a(z)\left(\frac{w'(z)}{w(z)}\right)^k=R(z,w(z)),$$

where a is rational in z, R is rational in w with rational coefficients in z. If $\limsup_{r\to\infty}\frac{\log^+ T(r,f)}{r}=0$, then $\deg_w(R)\le 2k+2$. Furthermore, if $\deg_w(R)=2k+2$, then $\overline{N}\left(r,\frac{1}{w}\right)=T(r,w)+S(r,w)$ and $\overline{N}(r,w)=T(r,w)+S(r,w)$.

Proof. Taking the Nevanlinna characteristic function of both sides of (6) and applying Mohon'ko equality [6, Theorom 2.2.5] (for more details about Mohon'ko equality, readers could refer to [8, 12]), we have

$$\begin{aligned}T\left(r,w(z+1)-w(z-1)+a(z)\left(\frac{w'(z)}{w(z)}\right)^k\right)&=T(r,R(z,w))\\&=\deg_w(R)\,T(r,w)+O(\log r).\end{aligned}$$

Thus by using the Nevanlinna's lemma on the logarithmic derivative and its difference analogue (Lemma 18), it follows that

$$
\begin{aligned}
\deg_w(R)T(r,w) &\le T(r,w(z+1)-w(z-1)) + T\left(r,\left(\frac{w'(z)}{w(z)}\right)^k\right) + O(\log r)\\
&\le m(r,w(z+1)-w(z-1)) + N(r,w(z+1)-w(z-1))\\
&\quad + m\left(r,\left(\frac{w'(z)}{w(z)}\right)^k\right) + N\left(r,\left(\frac{w'(z)}{w(z)}\right)^k\right) + S(r,w)\\
&\le m\left(r,\frac{w(z+1)}{w(z)}\right) + m\left(r,\frac{w(z-1)}{w(z)}\right) + m(r,w)\\
&\quad + N(r,w(z+1)-w(z-1)) + kN\left(r,\frac{w'(z)}{w(z)}\right) + S(r,w)\\
&\le m(r,w) + N(r,w(z+1)-w(z-1)) + kN\left(r,\frac{w'(z)}{w(z)}\right) + S(r,w).
\end{aligned}
\tag{13}
$$

We can see that the logarithmic derivative $\frac{w'(z)}{w(z)}$ has a simple pole at $z = z_j$ if and only if w has a pole or a zero at $z = z_j$. So it immediately follows that

$$N\left(r,\frac{w'(z)}{w(z)}\right) \le \overline{N}(r,w) + \overline{N}\left(r,\frac{1}{w}\right).$$

On using [5] to obtain

$$
\begin{aligned}
N(r,w(z+1)-w(z-1)) &\le N(r,w(z+1)) + N(r,w(z-1))\\
&= 2N(r,w) + S(r,w).
\end{aligned}
$$

Therefore, inequality (13) becomes

$$\deg_w(R)T(r,w) \le m(r,w) + 2N(r,w) + k\overline{N}(r,w) + k\overline{N}(r,1/w) + S(r,w). \tag{14}$$

Therefore

$$
\begin{aligned}
(\deg_w(R)-1-2k)T(r,w) &\le \overline{N}\left(r,\frac{1}{w}\right) + S(r,w),\\
(\deg_w(R)-1-2k)T(r,w) &\le \overline{N}(r,w) + S(r,w),
\end{aligned}
\tag{15}
$$

which implies the conclusions of the lemma. □

Suppose that z_j is a zero or a pole of $w(z)$ of order p and that neither $a(z)$ nor any of the coefficient functions in $R(z,w)$ have a zero or a pole at z_j. We will also require that these coefficient functions do not have zeros or poles at points which are near to z_j (in this particular case, we take the form $z_j + l$ where l is a integer such that $-5 \le l \le 5$). We will call such a point z_j a *generic zero (or generic pole) of order p*. Since the coefficients are rational, when estimating the corresponding unintegrated counting functions, the contribution from the non-generic zeros (or poles) can be included in a bounded error term, leading to an error term of the type $O(\log r)$ in the integrated estimates involving $T(r,w)$. Therefore, we can only consider generic zeros (or generic poles) in similar situations.

Lemma 21. *Let w be a transcendental meromorphic solution of the equation*

$$w(z+1) - w(z-1) + a(z)\left(\frac{w'(z)}{w(z)}\right)^k = P(z,w(z)), \tag{16}$$

where $a(z)$ is a rational function and $P(z,w)$ is a polynomial in w and rational in z. If $\limsup_{r\to\infty}\frac{\log^+ T(r,w)}{r} = 0$, then $\deg_w(P) \le k$.

Proof. Assume that $\deg_w(P) \ge k+1$, and suppose first that w has either infinitely many zeros or poles (or both). In the first case, we divide three subcases as follows.

Case 1. Suppose firstly that w has a generic pole with order p at $z=z_j$. Obviously, $\deg_w(P)\cdot p\geq k+1>k$, then w has a pole of order at least $\deg_w(P)\cdot p$ at $z=z_j+1$ or $z=z_j-1$. Without loss of generality, we can assume that w takes such a pole at $z=z_j+1$. Similarly, from iterating equation (16) we obtain that w has a pole at $z=z_j+2$ of order at least $(\deg_w(P))^2\cdot p$, and a pole of order at least $(\deg_w(P))^3\cdot p)$ at $z=z_j+3$, and so on.

Case 2. Suppose next that w has a generic zero of order q at $z=z_j$, we can immediately know that $w(z+1)$ or $w(z-1)$ has a pole with order at least k at $z=z_j$. Without loss of generality, we can assume that $w(z+1)$ takes such a pole at $z=z_j$. By shifting (16) up we obtain that w has a pole at $z=z_j+2$ of order at least $\deg_w(P)\cdot k$ because the pole of the term $a(z+1)\left(\frac{w'(z)}{w(z)}\right)^k$ has order $k(<k+1\leq\deg_w(P)\cdot k)$, and a pole of order at least $(\deg_w(P))^2\cdot k)$ at $z=z_j+3$, and so on.

Thus, it follows from the above three subcases that

$$n(d+|z_j|,w)\geq(\deg_w(P))^d$$

for all $d\in\mathbb{N}$, and so

$$\begin{aligned}\limsup_{r\to\infty}\frac{\log^+T(r,w)}{r}&\geq\limsup_{r\to\infty}\frac{\log n(r,w)}{r}\\&\geq\limsup_{d\to\infty}\frac{\log n(d+|z_j|,w)}{d+|z_j|}\\&\geq\limsup_{d\to\infty}\frac{\log(\deg_w(P))^d}{d+|z_j|}\\&=\log(\deg_w(P))\\&\geq\log(k+1)>0.\end{aligned}$$

Therefore, this contradicts with $\limsup_{r\to\infty}\frac{\log^+T(r,w)}{r}=0$.

Suppose now that w has finitely many poles and zeros and that $\limsup_{r\to\infty}\frac{\log^+T(r,w)}{r}=0$ (which implies $\rho_2(w)\leq1$). Then by the Weierstrass factorization theorem [15], we have

$$w(z)=f(z)\cdot e^{g(z)}\tag{17}$$

where $f(z)$ is a rational function and $g(z)$ is a nonconstant entire function. By substituting (17) into (16), it follows that

$$f(z+1)\cdot e^{g(z+1)}-f(z-1)\cdot e^{g(z-1)}+a(z)\left(\frac{f'(z)}{f(z)}+g'(z)\right)^k=P\left(z,f(z)\cdot e^{g(z)}\right).\tag{18}$$

Now, by Lemma 18, that

$$T\left(r,e^{g(z+1)-g(z)}\right)=m\left(r,e^{g(z+1)-g(z)}\right)=m\left(r,\frac{e^{g(z+1)}}{e^{g(z)}}\right)=S(r,e^g),$$

and similarly

$$T\left(r,e^{g(z-1)-g(z)}\right)=m\left(r,e^{g(z-1)-g(z)}\right)=m\left(r,\frac{e^{g(z-1)}}{e^{g(z)}}\right)=S(r,e^g).$$

Hence, by writing (18) in the form

$$e^{g(z)}\left(f(z+1)\cdot e^{g(z+1)-g(z)}-f(z-1)\cdot e^{g(z-1)-g(z)}\right)+a(z)\left(\frac{f'(z)}{f(z)}+g'(z)\right)^k=P\left(z,f(z)\cdot e^{g(z)}\right)$$

and taking the Nevanlinna's characteristic from both sides, we arrive at the equation

$$\deg_w(P)T(r,e^g)=T(r,e^g)+S(r,e^g).$$

Since $\deg_w(P)\geq k+1\geq2$ by assumption, this implies that g is a constant. This is a contradiction. □

In what follows we use the notation $D(z_0,\tau)$ to denote an open disc of radius τ centered at $z_0 \in \mathbb{C}$. Also, ∞^l denotes a pole of w with order l. Similarly, 0^l and b_1+0^l denote a zero and a b_1-point of w, respectively, with order l. For instance, $w(z_0)=b_1(z_0)+0^l$ is a short notation for

$$w(z)=b_1(z_0)+c_0(z-z_0)^l+O((z-z_0)^{l+1})$$

for all $z\in D(z_0,\tau_0)$, where $c_0\neq 0$ and τ_0 is a sufficiently small constant.

Lemma 22. *Let w be a transcendental meromorphic solution of the equation*

$$w(z+1)-w(z-1)+a(z)\left(\frac{w'(z)}{w(z)}\right)^k=\frac{P(z,w(z))}{(w(z)-b_1(z))^m\breve{Q}(z,w(z))}, \tag{19}$$

where $a(z)$ and $b_1(z)$ are rational functions of z, $P(z,w)$ and $\breve{Q}(z,w)$ are polynomials in w with rational coefficients in z, and m is an integer greater than one. Furthermore we assume that $w-b_1$, $P(z,w)$ and $\breve{Q}(z,w)$ are pairwise co-prime. If $\limsup_{r\to\infty}\frac{\log^+T(r,w)}{r}=0$, then

$$m+\deg_w(\breve{Q})<\deg_w(P)<k+m+\deg_w(\breve{Q}).$$

Proof. We can transform (19) into $\Phi(z,w)=0$, where $\Phi(z,w)$ is a differential-difference polynomial in w with rational coefficients. Notice that $\Phi(z,b_1)\not\equiv 0$, even if $b_1\equiv 0$; thus the first condition of Lemma 19 is satisfied for b_1.

Suppose that z_j is a generic zero of $w-b_1$ of order p, then either $w(z+1)$ or $w(z-1)$ has a pole of order $q(\geq mp>p)$ at $z=z_j$. And we suppose without loss of generality that z_j+1 is such a pole. We divide two cases as follows.

Case 1. Assume that

$$\deg_w(P)\leq m+\deg_w(\breve{Q}). \tag{20}$$

Subcase 1.1: $q>k$. By shifting (19) up we can obtain $w(z_j+2)=\infty^k$ and $w(z_j+3)=\infty^q$. By continuing the iteration, it immediately follows that it may be $w(z_j+4)=b_1(z_j+4)$, and so it is at least in principle possible that $w(z_j+5)$ is a finite value. This can only happen if the order of the zero of $w-b_1$ at $z=z_j+4$ is $p'(=q/m\geq p)$. But even so, by considering the multiplicities of zeros and poles of $w-b_1$ in the set $z_j,z_j+1,\dots,z_j+4$, we find that there are $2q+k(>2q\geq mp+mp')$ poles of w for $p+p'$ zeros of $w-b_1$. This is the "worst case scenario", because in other situations the ratio of the generic zeros of $w-b_1$ and the generic poles of w is higher than this. By adding up the contribution from all point z_j to the corresponding counting functions, it follows that

$$n\left(r,\frac{1}{w-b_1}\right)\leq\frac{1}{m}n(r+4,w)+O(1).$$

Thus both conditions of Lemma 2.2 are satisfied, and so $\limsup_{r\to\infty}\frac{\log^+T(r,w)}{r}>0$. This is a contradiction.

Subcase 1.2: $q=k$. By shifting (19) up we can obtain $w(z_j+2)=\infty^k$. By continuing the iteration, it follows that it may be $w(z_j+3)=b_1(z_j+3)$, and so it is at least in principle possible that $w(z_j+4)$ is a finite value. This can only happen if the order of the zero of $w-b_1$ at $z=z_j+3$ is $p'(=k/m\geq p)$. But even so, by considering the multiplicities of zeros and poles of $w-b_1$ in the set $\{z_j,z_j+1,z_j+2,z_j+3\}$, we find that there are $q+k(\geq mp+mp')$ poles of w for $p+p'$ zeros of $w-b_1$. This is the "worst case scenario". By adding up the contribution from all point z_j to the corresponding counting functions, it follows that

$$n\left(r,\frac{1}{w-b_1}\right)\leq\frac{1}{m}n(r+3,w)+O(1).$$

Thus both conditions of Lemma 2.2 are satisfied, and so $\limsup_{r\to\infty}\frac{\log^+T(r,w)}{r}>0$. We also obtain a contradiction.

Subcase 1.3: $q < k$**.** By shifting (19) up we can obtain $w(z_j+2) = \infty^k$ and $w(z_j+3) = \infty^k$. Then the reduction is almost same as what we do in subcase "$q > k$" and we obtain a contradiction with $\limsup_{r\to\infty} \frac{\log^+ T(r,w)}{r} > 0$.

Case 2. Assume that

$$\deg_w(P) \ge k + m + \deg_w(\check{Q}). \tag{21}$$

Suppose again that z_j is a generic zero of $w - b_1$ of order p. Then, as in the case (20), w has a pole of order $q(\ge mp)$ at either z_j+1 or z_j-1, say z_j+1.

By shifting (19) up again we can obtain $w(z_j+2) = \infty^{q'}$ ($q' = (\deg_w(P) - m - \deg_w(\check{Q}))q \ge kq > k$), and so the only way that $w(z_j+4)$ can be finite is that $w - b_1$ has a zero at z_j+3 with multiplicities $p'(= q'/m)$ or $w(z_j+3)$ is a zero of $\check{Q}(z,w)$. Even if this were the case, we have found at least $mp + mp'$ poles, taking into account multiplicities, that correspond uniquely to at most $p + p'$ zeros of $w - b_1$. Therefore, we have

$$n\left(r, \frac{1}{w - b_1}\right) \le \frac{1}{m} n(r+3, w) + O(1)$$

by going through all zeros of $w - b_1$ in this way. Lemma 19 thus implies that $\limsup_{r\to\infty} \frac{\log^+ T(r,w)}{r} > 0$. We have a contradiction.

Therefore, we obtain the conclusion of this lemma. □

3. The proof of Theorem 4

If $\deg_w(Q) = 0$, it immediately follows $\deg_w(P) \le k$ from Lemma 21. Thus we obtain the conclusion (i).

We now consider the case $\deg_w(Q) \ge 1$. Assume w is a transcendental meromorphic solution of (6), with $\limsup_{r\to\infty} \frac{\log^+ T(r,w)}{r} = 0$. Then it immediately follows $\max\{\deg_w(P), \deg_w(Q)\} \le 2k+2$ from Lemma 20.

Assume that $Q(z,w)$ has one nonzero root at least. By Lemma 22, if $Q(z,w)$ has multiple roots, we obtain $\deg_w(Q) < \deg_w(P) < k + \deg_w(Q)$, then it immediately follows the conclusion (ii). So we can only consider the case when $Q(z,w)$ has simple roots.

Assume now that $\deg_w(Q) = 1$ (and thus without loss of generality we can assume $Q(z,w) = w - b_1$) and $\deg_w(P) > k+1$. It is obvious that b_1 is not a solution of (6). Let z_j be a generic zero of $w - b_1$ of order p. Then w has a pole of order at least p at z_j+1 or z_j-1. (If $b_1 \equiv 0$, we can not ensure this result is true, because it may in principle be possible that the pole of the right hand side of 6 at $z = z_j$ can be cancelled by the pole of the term $a(z)\left(\frac{w'(z)}{w(z)}\right)^k$ at $z = z_j$ in such a way that both $w(z_j+1)$ and $w(z_j-1)$ remain finite.) Without loss of generality we assume that z_j+1 is such a pole. Then z_j+2 is a pole of order at least $(k+1)p(\ge 2p)$ and z_j+3 is a pole of order at least $(k+1)^2 p(\ge 4p)$, and so on. Therefore, in this case we obtain

$$n\left(r, \frac{1}{w - b_1}\right) \le \frac{1}{3} n(r+2, w) + O(1).$$

So, Lemma 19 implies that $\limsup_{r\to\infty} \frac{\log^+ T(r,w)}{r} > 0$, a contradiction. Thus the assertion (i) of Theorem 4 holds.

Suppose now that the denominator of $R(z,w)$ has at least two non-zero rational roots for w as a function of z, say $b_1 \not\equiv 0$ and $b_2 \not\equiv 0$. Then we may write equation (6) in the form

$$w(z+1) - w(z-1) + a(z)\left(\frac{w'(z)}{w(z)}\right)^k = \frac{P(z,w(z))}{(w(z)-b_1(z))(w(z)-b_2(z))\widetilde{Q}(z,w(z))}, \tag{22}$$

where $P(z,w)\not\equiv 0$ and $\widetilde{Q}(z,w)\not\equiv 0$ are polynomials in w of at most degree $2k+2$ and $2k$ respectively, with no common factors. Then neither b_1 nor b_2 is a solution of (22), and so they satisfy the first condition of Lemma 19. Let $z=z_j$ be a generic zero of order p of $w-b_1$.

Now, by (22), it follows that either $w(z+1)$ or $w(z-1)$ has a pole at $z=z_j$ of order at least p. Without loss of generality we may assume that $w(z+1)$ has such a pole at $z=z_j$. Then, by shifting the equation (22), we have

$$\begin{aligned} w(z+2)-w(z)+a(z+1)\left(\frac{w'(z+1)}{w(z+1)}\right)^k \\ = \frac{P(z+1,w(z+1))}{(w(z+1)-b_1(z+1))(w(z+1)-b_2(z+1))\widetilde{Q}(z+1,w(z+1))}, \end{aligned} \tag{23}$$

which implies that $w(z+2)$ has a pole of order k provided

$$\deg_w(P)\le \deg_w(\widetilde{Q})+2. \tag{24}$$

We suppose first that (24) is valid, so it follows from Lemma 22 that $b_1\not\equiv b_2$. By iterating (22) one more step, we have

$$\begin{aligned} w(z+3)-w(z+1)+a(z+2)\left(\frac{w'(z+2)}{w(z+2)}\right)^k \\ = \frac{P(z+2,w(z+2))}{(w(z+2)-b_1(z+2))(w(z+2)-b_2(z+2))\widetilde{Q}(z+2,w(z+2))}. \end{aligned} \tag{25}$$

Now, we consider all generic zeros of both $w-b_1$ and $w-b_2$. Using the similar method in the proof of Lemma 22 for the subcases $p>k$, or $p=k$ or $p<k$, we can obtain

$$n\left(r,\frac{1}{w-b_1}\right)+n\left(r,\frac{1}{w-b_2}\right)\le n(r+4,w)+O(1). \tag{26}$$

Therefore the remaining condition of Lemma 19 is satisfied, so $\limsup_{r\to\infty}\frac{\log^+ T(r,w)}{r}>0$, a contradiction.

We consider now the case where the opposite inequality to (24) holds, i.e.,

$$\deg_w(P)>\deg_w(\widetilde{Q})+2.$$

If $\deg_w(\widetilde{Q})+2<\deg_w(P)\le\deg_w(\widetilde{Q})+2+k$, we can obtain the conclusion (ii). Now we assume the remain case

$$\deg_w(P)>\deg_w(\widetilde{Q})+2+k \tag{27}$$

and suppose that z_j is a generic zero of $w-b_1$ of order p. Then again, by (22) either $w(z+1)$ or $w(z-1)$ must have a pole at $z=z_j$ of order at least p, and we suppose as above that $w(z+1)$ has the pole at z_j. Then, it follows that $w(z+2)$ has a pole of order at least $(k+1)p\ge 2p$, and $w(z+3)$ has a pole of order at least $(k+1)^2p\ge 4p$. Then the reduction is similar to before one in the case $\deg_w(Q)=1$, we obtain the same result

$$n\left(r,\frac{1}{w-b_1}\right)\le\frac{1}{3}n(r+2,w)+O(1).$$

Lemma 19 therefore $\limsup_{r\to\infty}\frac{\log^+ T(r,w)}{r}>0$, a contradiction. In conclusion, the assertion (i) and (ii) are true.

4. The proof of Theorem 8

Let $z=z_j$ be an arbitrary generic zero of w. Then $\left(\frac{w'(z)}{w(z)}\right)^k$ has a pole with order k, and thus by (7) w has pole of order at least k at $z=z_j+1$ or $z=z_j-1$ (or both). Suppose first there is a pole at both points $z=z_j-1$ and $z=z_j+1$. Then, from (6) it immediately follows that there are poles with order at least k of $w(z)$ at $z=z_j+2$ and $z=z_j-2$. And at least in principle it might happen that $w(z_j+3)=0=w(z_j-3)$. Hence, at least $4k(\geq 4)$ poles of $w(z)$ which correspond to three zeros (ignoring multiplicity) of $w(z)$ and to on other zeros can be found in this case. Hence we can obtain

$$\overline{N}\left(r,\frac{1}{w}\right)\leq\frac{3}{4}T(r,w)+S(r,w)<\left(\frac{3}{4}+\epsilon\right)T(r,w)+S(r,w).$$

So this inequality is contradict with the inequality (8).

Now, we assume that there is a pole of $w(z)$ at only one of the points $z=z_j+1$ and $z=z_j-1$. we can suppose that $w(z)$ has a pole at $z=z_j+1$ (the case where the poles is at $z=z_1$ is completely analogous). If we let $N_1\left(r,\frac{1}{w}\right)$ denote the integrated function for the simple zeros of $w(z)$, then from (8), we can obtain the following inequality

$$N_1\left(r,\frac{1}{w}\right)\geq\left(\frac{1}{2}+\epsilon\right)T(r,w)+S(r,w).$$

For more details on the process of reasoning the above inequalities, please refer to [2]. Hence, there are at least "$\left(\frac{1}{2}+\epsilon\right)T(r,w)$" worth of simple zeros of $w(z)$. Thus, if we consider the case in which $w(z)$ has a simple zero at the point $z=z_j$, we can obtain

$$\begin{aligned} &L\in\mathbb{C},\qquad \alpha\in\mathbb{C}\setminus\{0\},\\ w(z-1)&=L+O(z-z_j),\\ w(z)&=\alpha(z-z_j)+O\left((z-z_j)^2\right),\\ w(z+1)&=-\frac{a(z)}{(z-z_j)^k}+O(1),\\ w(z+2)&=(-1)^{k+1}k^k\frac{a(z+1)}{(z-z_j)^k}+O(1).\end{aligned}\tag{28}$$

If $k>1$, we can find at least four poles of $w(z)$ correspond to at most three zeros (ignoring multiplicity) of $w(z)$ and to no other zeros (even if $w(z+3)=0$). Thus, this is contradict with the condition (8). Hence, k must reduce into 1, and Theorem 8 reduces into Theorem 2. As for the detailed proof of Theorem 2, readers can refer to [2].

5. The proof of Theorem 10

The proof process of this theorem is similar as that of Theorem 4. We only point out some differences in this section. We fist give some lemmas as follows.

Lemma 23. *Let $w(z)$ be a transcendental meromorphic solution of equation* (9)

$$w(z+1)+a(z)\left(\frac{w'(z)}{w(z)}\right)^k=R(z,w(z))=\frac{P(z,w(z))}{Q(z,w(z))},$$

where a is rational in z, R is rational in w with rational coefficients in z. If $\limsup_{r\to\infty}\frac{\log^+T(r,w)}{r}=0$, then $\deg_w(R)\leq 2k+1$. In addition, if $\deg_w(R)=2k+1$, then $\overline{N}(r,1/w)=T(r,w)+S(r,w)$ and $\overline{N}(r,w)=T(r,w)+S(r,w)$.

Proof. The proof of Lemma 23 is almost the same as the proof of Lemma 20. So we will not repeat for more details. □

Lemma 24. *Let w be a transcendental meromorphic solution of the equation*

$$w(z+1)+a(z)\left(\frac{w'(z)}{w(z)}\right)^k = P(z,w(z)), \tag{29}$$

where $a(z)$ is rational in z and $P(z,w)$ is a polynomial in w and rational in z. If $\limsup_{r\to\infty}\frac{\log^+ T(r,w)}{r}=0$, then $\deg_w(P)\le k$.

Proof. Assume that $\deg_w(P)\ge k+1$. And we suppose that w has either infinitely many zeros or poles (or both).

Case 1. Suppose firstly that w has a generic pole with order s at $z=z_j$. Then w has a pole of order $\deg_w(P)\cdot s$ at $z=z_j+1$. Similarly, from equation (29) we obtain that w has a pole at $z=z_j+2$ of order at least $\deg_w(P)\cdot s$, and a pole of order at least $\deg_w(P))^2\cdot s$ at $z=z_j+3$, and so on.

Case 2. Suppose next that w has a generic zero with order p at $z=z_j$. Then, $w(z)$ has a pole of order k at $z=z_j+1$. As we continue to iterate, we obtain that $w(z+2)$ has a pole of order $\deg_w(P)\cdot k$ at $z=z_j$ and $w(z+3)$ has a pole of order $(\deg_w(P))^2\cdot k$ at $z=z_j$, and so on.

Thus, by using the same operation as the previous proof in Lemma 21, we obtain a contradiction with $\limsup_{r\to\infty}\frac{\log^+ T(r,w)}{r}=0$.

For the case of finitely many zeros and poles for w, a contradiction is obtained by similar inference as in the proof of Lemma 21. □

Lemma 25. *Let w be a transcendental meromorphic solution of the equation*

$$w(z+1)+a(z)\left(\frac{w'(z)}{w(z)}\right)^k = \frac{P(z,w(z))}{(w(z)-b_1(z))^m\breve{Q}(z,w(z))}, \tag{30}$$

where $a(z)$ and $b_1(z)$ are rational functions of z, $P(z,w)$ and $\breve{Q}(z,w)$ are polynomials in w with rational coefficients in z, and m is an integer greater than one. Furthermore we assume that $w-b_1$, $P(z,w)$ and $\breve{Q}(z,w)$ are pairwise co-prime. If $\limsup_{r\to\infty}\frac{\log^+ T(r,w)}{r}=0$, then

$$m+\deg_w(\breve{Q})<\deg_w(P)<k+m+\deg_w(\breve{Q}).$$

Proof. Since the proof of Lemma 25 is almost the same as the proof of Lemma 22, we omit the details. □

Now we begin the proof of Theorem 10.

Proof of Theorem 10. In the case when $\deg_w(Q)=1$, we can assume that $\deg_w(P)\le 1$. Then we can write equation (9) in the form

$$w(z+1)+a(z)\left(\frac{w'(z)}{w(z)}\right)^k = \frac{P(z,w(z))}{w(z)-b(z)}, \tag{31}$$

where $b(z)$ is a rational function. Obviously, $b(z)$ is not a solution of (31), thus the first condition of Lemma 18 is satisfied for b. Let z_j be a generic zero of $w-b$ of order p. Obviously, $w(z+1)$ has a pole of order p at $z=z_j$. By iterating (31) one step to the right, we obtain

$$w(z+2)+a(z+1)\left(\frac{w'(z+1)}{w(z+1)}\right)^k = \frac{P(z+1,w(z+1))}{w(z+1)-b(z+1)}. \tag{32}$$

It follows immediately from (32) that $w(z+2)$ also has a pole of order p at $z=z_j$. If we continue the iteration, we can find that it is impossible to appear a zero of $w-b$ later. So we have

$$n\left(r,\frac{1}{w-b}\right)\le\frac{1}{2}n(r+2,w)+O(1). \tag{33}$$

Thus the second condition of Lemma 18 is satisfied, therefore $\limsup_{r\to\infty}\frac{\log^+ T(r,w)}{r}>0$, a contradiction.

Other cases are completely similar to the proof process of Theorem 4. We omit the details. $\square$

6. The proof of Theorem 15

Let $w(z)$ be a transcendental meromorphic solution of the equation (10), of minimal hypertype.

Suppose $w(z)$ has a generic zero at $z = z_j$ with order p. Then from the equation (10) we can obtain $w(z)$ has a pole with order k at the point $z = z_j + 1$. By iterating (10), we can obtain that $w(z)$ has a pole with order k at $z = z_j + 2$, and so on. Notice that $N(r, w(z+1)) = N(r, w) + S(r, w)$ under the assumption of minimal hypertype by [5]. Hence, by iterating infinitely many times we can obtain

$$N\left(r, \frac{1}{w}\right) \leq \frac{p}{km} N(r, w) + S(r, w) \leq \frac{p}{km} T(r, w) + S(r, w),$$

where m is a positive integer that can be sufficiently large. It immediately follows that

$$N\left(r, \frac{1}{w}\right) = S(r, w).$$

Now assume that $N(r, w) = S(r, w)$. Then by the Weierstrass factorization theorem we can write w as the form of $w(z) = f(z)e^{g(z)}$, where $f(z)$ is a rational function and $g(z)$ is nonconstant entire function. Submitting it into equation (10) gives

$$a(z)\left(\frac{f^{'}(z)}{f(z)} + g^{'}(z)\right)^{k} - b(z) = -f(z+1)e^{g(z+1)}.$$

By estimating the growth of both sides of the above equation, one can get a contradiction. Hence we have $N(r, w) \neq S(r, w)$.

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