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Remarks on homogeneous solitons of the G_2-Laplacian flow

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Abstract. We show the existence of expanding solitons of the G₂-Laplacian flow on non-solvable Lie groups, and we give the first example of a steady soliton that is not an extremally Ricci pinched G₂-structure.

1. Introduction

A G_2-structure on a 7-manifold M is said to be *closed* if the defining positive 3-form $\varphi \in \Omega^3_+(M)$ satisfies the equation $d\varphi = 0$. The intrinsic torsion of a closed G_2-structure can be identified with the unique 2-form τ for which $d *_\varphi \varphi = \tau \wedge \varphi = - *_\varphi \tau$, where $*_\varphi$ denotes the Hodge operator associated with the Riemannian metric g_φ and orientation dV_φ induced by φ (cf. [2]). When this *intrinsic torsion form τ* vanishes identically, the G_2-structure is called *torsion-free*.

A closed G_2-structure φ is called a *Laplacian soliton* if it satisfies the equation

$$\Delta_\varphi \varphi = \lambda \varphi + \mathscr{L}_X \varphi \tag{1}$$

for some real number λ and some vector field X on M, where Δ_φ denotes the Hodge Laplacian of the metric g_φ. It is known that a closed G_2-structure φ satisfies (1) if and only if the solution of the *Laplacian flow*

$$\begin{cases} \dfrac{\partial}{\partial t} \varphi(t) = \Delta_{\varphi(t)} \varphi(t), \\ d\varphi(t) = 0, \end{cases} \tag{2}$$

starting from it at $t = 0$ is self-similar (cf. [12, 14], and see the recent survey [13] for more information on the Laplacian flow (2)).

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Depending on the sign of λ, a Laplacian soliton is called *shrinking* ($\lambda < 0$), *steady* ($\lambda = 0$), or *expanding* ($\lambda > 0$). By [12, 14], on a compact manifold every Laplacian soliton which is not torsion-free must satisfy (1) with $\lambda > 0$ and $\mathscr{L}_X \varphi \neq 0$. The existence of non-trivial Laplacian solitons on compact manifolds is still an open problem.

The non-compact setting is less restrictive, and examples of Laplacian solitons of any type occur. Moreover, all examples obtained so far are given by seven-dimensional, connected, simply connected solvable Lie groups G endowed with a left-invariant closed G_2-structure φ satisfying (1) with respect to a special vector field X [3, 6, 9–11, 15]. In detail, the vector field X is defined by the one-parameter group of automorphisms $F_t \in \text{Aut}(G)$ such that $dF_t|_{1_G} = \exp(tD)$, where D is a derivation of the Lie algebra $\mathfrak{g}$ of G. According to [9], these Laplacian solitons are said to be *semi-algebraic*, and they are called *algebraic* if the adjoint of D with respect to the inner product g_φ on $\mathfrak{g}$ is also a derivation. Notice that, even if the Lie group G admits a co-compact discrete subgroup Γ, a (semi-)algebraic soliton does not define a Laplacian soliton on the compact quotient $\Gamma \backslash G$, as the vector field defined by F_t does not descend to $\Gamma \backslash G$.

The purpose of this note is to discuss new homogeneous examples of Laplacian solitons that differ in some aspects from those appearing in the literature.

In Section 3, we show that algebraic Laplacian solitons also exist on non-solvable Lie groups. This gives a further evidence of the difference between homogeneous Laplacian solitons and Ricci solitons, in addition to the results obtained in [9, 15]. Indeed, every homogeneous Ricci soliton is isometric to an algebraic soliton of the Ricci flow [7], and all known examples are isometric to a left-invariant algebraic Ricci soliton on a simply connected solvable Lie group (see [7, 8] for more details).

In Section 4, we focus on steady Laplacian solitons. Currently, all known examples [10, 11] are given by *extremally Ricci pinched* G_2-structures, namely closed G_2-structures φ whose intrinsic torsion form τ satisfies the equation $d\tau = \frac{1}{6} |\tau|^2 \varphi + \frac{1}{6} *_\varphi (\tau \wedge \tau)$. In fact, by [11] every left-invariant extremally Ricci pinched G_2-structure on a (necessarily solvable) Lie group is a steady algebraic soliton. Here, we show that the converse of this result does not hold. In detail, we give an example of a simply connected solvable Lie group endowed with a steady algebraic soliton that is not an extremally Ricci pinched G_2-structure. This example satisfies a further remarkable property: there exists a left-invariant vector field X on the Lie group for which $\Delta_\varphi \varphi = \mathscr{L}_X \varphi$. To our knowledge, this is the first example of a left-invariant closed G_2-structure satisfying the equation (1) with respect to a left-invariant vector field.

2. Algebraic Laplacian solitons on Lie groups

In this section, we briefly recall some known facts on left-invariant G_2-structures on Lie groups and on algebraic solitons of the G_2-Laplacian flow. We refer the reader to [4, 9] and the references therein for more details.

Let G be a seven-dimensional, connected, simply connected Lie group. It is well-known that there is a one-to-one correspondence between left-invariant G_2-structures on G and G_2-structures on the corresponding Lie algebra $\mathfrak{g}$.

Recall that a 3-form $\varphi \in \Lambda^3(\mathfrak{g}^*)$ defines a G_2-structure on $\mathfrak{g}$ if and only if the symmetric bilinear map

$$b_\varphi : \mathfrak{g} \times \mathfrak{g} \to \Lambda^7(\mathfrak{g}^*), \quad b_\varphi(u, v) := \frac{1}{6} \iota_u \varphi \wedge \iota_v \varphi \wedge \varphi,$$

satisfies the following conditions

(i) $\det(b_\varphi)^{1/9} \in \Lambda^7(\mathfrak{g}^*)$ is not zero;
(ii) the symmetric bilinear form $\det(b_\varphi)^{-1/9} b_\varphi : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is positive definite.

When this happens, the inner product and orientation induced by φ on $\mathfrak{g}$ are given by $g_\varphi = \det(b_\varphi)^{-1/9}\, b_\varphi$ and $dV_\varphi = \det(b_\varphi)^{1/9}$, respectively.

A G_2-structure φ on $\mathfrak{g}$ is *closed* if it satisfies the equation $d\varphi = 0$, where d denotes the Chevalley–Eilenberg differential of $\mathfrak{g}$. Using left multiplication, it is possible to extend a closed G_2-structure on $\mathfrak{g} \cong T_{1_G}G$ to a left-invariant closed G_2-structure on G.

Let φ be a closed G_2-structure on $\mathfrak{g}$, and let $\tau \in \Lambda^2(\mathfrak{g}^*)$ be the corresponding intrinsic torsion form. By [9, Proposition 4.5], the following conditions are equivalent

(i) there exist a real number λ and a derivation $D \in \mathrm{Der}(\mathfrak{g})$ for which

$$\mathrm{Ric}(g_\varphi) - \frac{1}{12}\,\mathrm{tr}(\tau_\varphi^2)\,\mathrm{Id} + \frac{1}{2}\,\tau_\varphi^2 = -\frac{\lambda}{3}\,\mathrm{Id} - \frac{1}{2}\left(D + D^*\right), \tag{3}$$

where $\mathrm{Ric}(g_\varphi) \in \mathrm{End}(\mathfrak{g})$ is the Ricci endomorphism of g_φ, $\tau_\varphi \in \mathrm{End}(\mathfrak{g})$ is the skew-symmetric endomorphism defined via the identity $\tau(\cdot,\cdot) = g_\varphi(\tau_\varphi\cdot,\cdot)$, and D^* is the adjoint of D with respect to g_φ;

(ii) the left-invariant closed G_2-structure induced by φ on G satisfies the equation

$$\Delta_\varphi\varphi = \lambda\varphi + \mathscr{L}_{X_D}\varphi,$$

where X_D is the vector field on G defined by the unique one-parameter group of automorphisms $F_t \in \mathrm{Aut}(G)$ such that $dF_t|_{1_G} = \exp(tD) \in \mathrm{Aut}(\mathfrak{g})$.

A closed G_2-structure φ satisfying any of the above conditions is called a *semi-algebraic soliton*, and it clearly defines a Laplacian soliton on G. In the case when D^* is also a derivation, φ is said to be an *algebraic soliton* (see [9] for more details).

3. Laplacian solitons on non-solvable Lie groups

In [4], we obtained the classification of seven-dimensional, unimodular, non-solvable Lie algebras admitting closed G_2-structures, showing that only four Lie algebras of this type exist, up to isomorphism. In this section, we give an example of an expanding algebraic soliton on two of them. Before doing this, we explain some conventions that we will use.

Given a Lie algebra $\mathfrak{g}$ of dimension n, we write its structure equations with respect to a basis of 1-forms $\{e^1,\dots,e^n\}$ by specifying the n-tuple $(de^1,\dots,de^n)$. Moreover, we use the shorthand $e^{ijk\cdots}$ to denote the wedge product $e^i \wedge e^j \wedge e^k \wedge \cdots$. Finally, we denote by $(e_1,\dots,e_n)$ the basis of $\mathfrak{g}$ with dual basis $(e^1,\dots,e^n)$, and we write the matrix associated with any endomorphism of $\mathfrak{g}$ with respect to this basis.

Example 1. Consider the one-parameter family of pairwise non-isomorphic, unimodular, non-solvable Lie algebras $\mathfrak{g}_\mu$ with the following structure equations

$$\mathfrak{g}_\mu = \left(e^{45}, -2\mu\,e^{27}, -2\,e^{37}, 2\,e^{15}, e^{14}, 2\left(1+\mu\right)e^{67}, 0\right), \quad -1 < \mu \le -\frac{1}{2}.$$

For each $\mu \in \left(-1, -\frac{1}{2}\right]$, $\mathfrak{g}_\mu$ is isomorphic to the decomposable Lie algebra $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{r}_{4,\mu,-1-\mu}$ appearing in [4, Main Theorem].

The following 3-form defines a closed G_2-structure on $\mathfrak{g}_\mu$

$$\varphi_\mu = -2\mu\,e^{127} + e^{347} + 2\left(1+\mu\right)e^{567} + e^{135} - e^{146} - e^{236} - e^{245}.$$

The inner product and the volume form induced by φ_μ on $\mathfrak{g}_\mu$ are given by

$$g_{\varphi_\mu} = \left(\frac{2\mu^2}{1+\mu}\right)^{1/3} \left[(e^1)^2 + (e^2)^2\right] + \left(\frac{1}{-4\mu(1+\mu)}\right)^{1/3} \left[(e^3)^2 + (e^4)^2\right]$$
$$+ \left(\frac{2(1+\mu)}{-\mu}\right)^{1/3} \left[(e^5)^2 + (e^6)^2\right] + \left(4\mu(1+\mu)\right)^{2/3} (e^7)^2,$$
$$dV_{\varphi_\mu} = \left(-4\mu(1+\mu)\right)^{1/3} e^{1234567},$$

and the intrinsic torsion form of φ_μ is

$$\tau_\mu = -2\left(\frac{2\mu^2}{1+\mu}\right)^{1/3} e^{12} + \left(\frac{2}{\mu+\mu^2}\right)^{2/3} e^{34} - 2\left(\frac{2(1+\mu)^2}{\mu}\right)^{2/3} e^{56}.$$

Using these data, it is possible to check that the equation (3) is satisfied for the following value of λ

$$\lambda = 2(\mu^2 + \mu + 1)\left(\frac{2}{\mu(1+\mu)}\right)^{2/3},$$

and the following symmetric derivation of $\mathfrak{g}_\mu$

$$D = -2 \operatorname{diag}\left(0, \left(\frac{4(1+\mu)}{\mu^2}\right)^{1/3}, (4\mu(1+\mu))^{1/3}, 0, 0, -\left(\frac{4\mu}{(1+\mu)^2}\right)^{1/3}, 0\right).$$

Since $-1 < \mu \le -\frac{1}{2}$, we have that $\lambda > 0$. Thus, φ_μ is an expanding algebraic soliton, and it gives rise to a left-invariant expanding Laplacian soliton on the simply connected non-solvable Lie group with Lie algebra $\mathfrak{g}_\mu$.

Example 2. Consider the one-parameter family of pairwise non-isomorphic, unimodular, non-solvable Lie algebras

$$\mathfrak{k}_\alpha = \left(-\alpha\, e^{17}, -2\, e^{35}, -e^{25} - e^{57}, \frac{\alpha}{2}\, e^{47} - e^{67}, e^{23} + e^{37}, e^{47} + \frac{\alpha}{2}\, e^{67}, 0\right), \quad \alpha > 0.$$

This family of Lie algebras is isomorphic to the family $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{r}'_{4,\alpha,-\alpha/2}$ appearing in [4, Main Theorem].

The 3-form

$$\varphi_\alpha = \frac{\alpha}{2}\left(e^{127} + e^{347} + e^{567}\right) + e^{135} - e^{146} - e^{236} - e^{245}$$

defines a closed G_2-structure on $\mathfrak{k}_\alpha$ inducing the inner product

$$g_{\varphi_\alpha} = \sum_{i=1}^6 (e^i)^2 + \frac{\alpha^2}{4}(e^7)^2,$$

and the volume form $dV_{\varphi_\alpha} = \frac{\alpha}{2}\, e^{1234567}$. The intrinsic torsion form of φ_α is

$$\tau_\alpha = 4\, e^{12} - 2\, e^{34} - 2\, e^{56}.$$

It is now possible to check that the equation (3) is satisfied for $\lambda = 6$ and for the symmetric derivation

$$D = \operatorname{diag}(-2,0,0,4,0,4,0) \in \operatorname{Der}(\mathfrak{k}_\alpha).$$

Thus, the 3-form φ_α is an algebraic soliton on $\mathfrak{k}_\alpha$, and it induces a left-invariant expanding Laplacian soliton on the simply connected non-solvable Lie group with Lie algebra $\mathfrak{k}_\alpha$.

4. A steady soliton that is not extremally Ricci pinched

A closed G_2-structure φ whose intrinsic torsion form τ satisfies the equation

$$d\tau = \frac{1}{6} |\tau|^2 \varphi + \frac{1}{6} *_\varphi (\tau \wedge \tau)$$

is called *extremally Ricci pinched* (*ERP* for short). This class of G_2-structures was introduced by Bryant in [2]. Homogeneous examples are discussed in [2, 5, 10, 11], and the existence of non-homogeneous examples has also been established [1]. In [5], we proved that the solution of the Laplacian flow (2) starting from an ERP G_2-structure exists for all times and stays ERP. However, on compact manifolds ERP G_2-structures cannot be steady solitons. This is not true anymore in the non-compact setting. Indeed, in the recent work [11] the authors proved that any left-invariant ERP G_2-structure on a simply connected Lie group is always a steady soliton. In this section, we show that the converse of this result does not hold.

Let us consider the solvable Lie algebra $\mathfrak{h}$ with the following structure equations

$$\mathfrak{h} = \left(0, 0, -e^{37}, e^{47}, 2\,e^{14} + e^{57}, -2\,e^{24} + e^{67}, 0\right).$$

This Lie algebra is not unimodular and it is isomorphic to the semidirect product $\mathbb{R} \ltimes \mathfrak{n}$, where $\mathbb{R} = \langle e_7 \rangle$ and $\mathfrak{n} = \langle e_1, \ldots, e_6 \rangle$ is a six-dimensional decomposable nilpotent Lie algebra.

The 3-form

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}$$

defines a closed G_2-structure on $\mathfrak{h}$ inducing the inner product $g_\varphi = \sum_{i=1}^{7} (e^i)^2$ and the volume form $dV_\varphi = e^{1234567}$. The intrinsic torsion form of φ is

$$\tau = 2\,e^{12} + 2\,e^{34} - 4\,e^{56},$$

and it satisfies

$$d\tau = \Delta_\varphi \varphi = -8\left(e^{146} + e^{245} - e^{567}\right).$$

We immediately see that φ is not ERP, since $*_\varphi (\tau \wedge \tau) = -16\left(e^{127} + e^{347}\right) + 8\,e^{567}$ and $|\tau|^2 = 24$.

The closed G_2-structure φ is a steady algebraic soliton, as it satisfies the equation (3) with $\lambda = 0$ and

$$D = \operatorname{diag}(0, 0, -4, 4, 4, 4, 0) \in \operatorname{Der}(\mathfrak{h}).$$

In particular, the left-invariant closed G_2-structure induced by φ on the simply connected solvable Lie group H with Lie algebra $\mathfrak{h}$ satisfies

$$\Delta_\varphi \varphi = \mathscr{L}_{X_D} \varphi.$$

This example has a further remarkable property, since there exists a left-invariant vector field X on H for which the left-invariant steady soliton φ satisfies the equation

$$\Delta_\varphi \varphi = \mathscr{L}_X \varphi = d(\iota_X \varphi).$$

In detail, X is the left-invariant vector field on H induced by the vector $-4\,e_7 \in \mathfrak{h}$.

References

[1] G. Ball, "Seven Dimensional Geometries with Special Torsion", PhD Thesis, Duke University, 2019.

[2] R. L. Bryant, "Some remarks on G_2-structures", in *Proceedings of Gökova Geometry-Topology Conference 2005*, International Press, 2006, p. 75-109.

[3] M. Fernández, A. Fino, A. Raffero, "Exact G_2-structures on unimodular Lie algebras", https://arxiv.org/abs/1904.11066v3, to appear in *Monatsh. Math.*, 2020.

[4] A. Fino, A. Raffero, "Closed G_2-structures on non-solvable Lie groups", *Rev. Mat. Complut.* **32** (2019), no. 3, p. 837-851.

[5] ———, "A class of eternal solutions to the G_2-Laplacian flow", https://arxiv.org/abs/1807.01128v2, to appear in *J. Geom. Anal.*, 2020.

[6] ——— , "Closed warped G_2-structures evolving under the Laplacian flow", *Ann. Sc. Norm. Super. Pisa, Cl. Sci.* **20** (2020), p. 315-348.

[7] M. Jablonski, "Homogeneous Ricci solitons are algebraic", *Geom. Topol.* **18** (2014), no. 4, p. 2477-2486.

[8] R. Lafuente, J. Lauret, "Structure of homogeneous Ricci solitons and the Alekseevskii conjecture", *J. Differ. Geom.* **98** (2014), no. 2, p. 315-347.

[9] J. Lauret, "Laplacian flow of homogeneous G_2-structures and its solitons", *Proc. Lond. Math. Soc.* **114** (2017), no. 3, p. 527-560.

[10] ——— , "Laplacian solitons: questions and homogeneous examples", *Differ. Geom. Appl.* **54** (2017), p. 345-360.

[11] J. Lauret, M. Nicolini, "Ricci pinched G_2-structures on Lie groups", https://arxiv.org/abs/1902.06375v3, to appear in *Commun. Anal. Geom.*, 2020.

[12] C. Lin, "Laplacian solitons and symmetry in G_2-geometry", *J. Geom. Phys.* **64** (2013), p. 111-119.

[13] J. D. Lotay, "Geometric Flows of G_2 Structures", To appear in a forthcoming volume of the Fields Institute Communications, entitled "Lectures and Surveys on G_2 manifolds and related topics".

[14] J. D. Lotay, Y. Wei, "Laplacian flow for closed G_2 structures: Shi-type estimates, uniqueness and compactness", *Geom. Funct. Anal.* **27** (2017), no. 1, p. 165-233.

[15] M. Nicolini, "Laplacian solitons on nilpotent Lie groups", *Bull. Belg. Math. Soc. Simon Stevin* **25** (2018), no. 2, p. 183-196.